On Geometric Probability, Holography, Shilov Boundaries and the Four Physical Coupling Constants of Nature

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By recurring to Geometric Probability methods, it is shown that the coupling constants, $\alpha_{EM}; \alpha_W; \alpha_C$ associated with Electromagnetism, Weak and the Strong (color) force are given by the *ratios of the ratios* of the measures of the Shilov *boundaries* $Q_2 = S^1 \times RP^1$; $Q_3 = S^2 \times RP^1$; S^5, respectively, with respect to the ratios of the measures $\mu[Q_5]/\mu_N[Q_5]$ associated with the $5D$ conformally compactified real Minkowski spacetime $\bar{M}_5$ that has the same topology as the Shilov boundary Q_5 of the 5 complex-dimensional poly-disc D_5. The homogeneous symmetric complex domain $D_5 = SO(5,2)/SO(5) \times SO(2)$ corresponds to the conformal relativistic *curved* 10 real-dimensional phase space $\mathcal{H}^{10}$ associated with a particle moving in the $5D$ Anti de Sitter space AdS_5. The geometric coupling constant associated to the gravitational force can also be obtained from the ratios of the measures involving Shilov boundaries. We also review our derivation of the observed vacuum energy density based on the geometry of de Sitter (Anti de Sitter) spaces.

1 The fine structure constant and Geometric Probability

Geometric Probability [21] is the study of the probabilities involved in geometric problems, e. g., the distributions of length, area, volume, etc. for geometric objects under stated conditions. One of the most famous problem is the Buffon's Needle Problem of finding the probability that a needle of length l will land on a line, given a floor with equally spaced parallel lines a distance d apart. The problem was first posed by the French naturalist Buffon in 1733. For $l < d$ the probability is

$$P = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{l|\cos(\theta)|}{d} = \frac{4l}{2\pi d} \int_0^{\pi/2} \cos(\theta) = \frac{2l}{\pi d} = \frac{2ld}{\pi d^2}\,. \tag{1}$$

Hence, the Geometric Probability is essentially the *ratio* of the areas of a rectangle of length $2d$, and width l and the area of a circle of radius d. For $l > d$, the solution is slightly more complicated [21]. The Buffon needle problem provides with a numerical experiment that determines the value of π empirically. Geometric Probability is a vast field with profound connections to Stochastic Geometry.

Feynman long ago speculated that the fine structure constant may be related to π. This is the case as Wyler found long ago [1]. We will based our derivation of the fine structure constant based on Feynman's physical interpretation of the electron's charge as the probability amplitude that an electron emits (or absorbs) a photon. The clue to evaluate this probability within the context of Geometric Probability theory is provided by the electron self-energy diagram. Using Feynman's rules, the self-energy $\Sigma(p)$ as a function of the electron's incoming (outgoing) energy-momentum p_μ is given by the integral involving the photon and electron propagator along the internal lines

$$-i\Sigma(p) = (-ie)^2 \times \times \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu \, \frac{i}{\gamma^\rho(p_\rho - k_\rho) - m} \, \frac{-ig_{\mu\nu}}{k^2}\, \gamma^\nu. \tag{2}$$

The integral is taken with respect to the values of the photon's energy-momentum k^μ. By inspection one can see that the electron self-energy is proportional to the fine structure constant $\alpha_{EM} = e^2$, the square of the probability amplitude (in natural units of $\hbar = c = 1$) and physically represents the electron's emission of a virtual photon (off-shell, $k^2 \neq 0$) of energy-momentum k_ρ at a given moment, followed by an absorption of this virtual photon at a later moment.

Based on this physical picture of the electron self-energy graph, we will evaluate the Geometric Probability that an electron emits a photon at $t = -\infty$ (infinite past) and reabsorbs it at a much later time $t = +\infty$ (infinite future). The off-shell (virtual) photon associated with the electron self-energy diagram *asymptotically* behaves on-shell at the very moment of emission ($t = -\infty$) and absorption ($t = +\infty$). However, the photon can remain off-shell in the intermediate region between the moments of emission and absorption by the electron.

The topology of the boundaries (at conformal infinity) of the past and future light-cones are spheres S^2 (the celestial sphere). This explains why the (Shilov) boundaries are essential mathematical features to understand the geometric derivation of all the coupling constants. In order to describe the physics at infinity we will recur to Penrose's ideas [10] of conformal compactifications of Minkowski spacetime by attaching the light-cones at conformal infinity. Not unlike the one-point compactification of the complex plane by adding the points at infinity leading to the Gauss-Riemann sphere. The conformal group leaves the light-cone fixed and it does not alter the causal properties of spacetime despite the rescalings of the metric. The topology of the conformal compactification of real Minkowski spacetime $\bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$ is precisely the same as the topology of the Shilov boundary Q_4 of the 4 complex-dimensional poly-disc D_4. The action of the discrete group Z_2 amounts to an antipodal identification of the future null infinity $\mathcal{I}^+$ with the past null infinity $\mathcal{I}^-$; and the antipodal identification of the past timelike infinity i^- with the future timelike infinity, i^+, where the electron emits, and absorbs the photon, respectively.

Shilov boundaries of homogeneous (symmetric spaces) complex domains, G/K [7, 8, 9] are not the same as the ordinary topological boundaries (except in some special cases). The reason being that the action of the isotropy group K of the origin is not necesarily *transitive* on the ordinary topological boundary. Shilov boundaries are the minimal subspaces of the ordinary topological boundaries which implement the Maldacena-'T Hooft-Susskind *holographic* principle [13] in the sense that the holomorphic data in the interior (bulk) of the domain is fully determined by the holomorphic data on the Shilov boundary. The latter has the property that the maximum modulus of any holomorphic function defined on a domain is attained at the Shilov boundary.

For example, the poly-disc D_4 of 4 complex dimensions is an 8 real-dim Hyperboloid of constant negative scalar curvature that can be identified with the conformal relativistic *curved* phase space associated with the electron (a particle) moving in a $4D$ Anti de Sitter space AdS_4. The poly-disc is a Hermitian symmetric homogeneous coset space associated with the $4D$ conformal group $SO(4,2)$ since $D_4 = SO(4,2)/SO(4)\times SO(2)$. Its Shilov boundary $Shilov$ $(D_4) = Q_4$ has precisely the *same* topology as the $4D$ conformally compactified real Minkowski spacetime $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$. For more details about Shilov boundaries, the conformal group, future tubes and holography we refer to the article by Gibbons [12] and [7, 16].

In order to define the Geometric Probability associated with this process of the electron's emission of a photon at i^- ($t=-\infty$), followed by an absorption at i^+ ($t=+\infty$), we must take into account the important fact that the photon is on-shell $k^2=0$ *asymptotically* (at $t=\pm\infty$), but it can move off-shell $k^2 \neq 0$ in the intermediate region which is represented by the *interior* of the conformally compactified real Minkowski spacetime $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$.

Denoting by $\hat{\mu}[Q_4]$ the measure-density (the measure-current) whose *flux* through the future and past celestial spheres S^2 (associated with the future/past light-cones) at timelike infinity i^+, i^-, respectively, is $V(S^2)\hat{\mu}[Q_4]$. The *net* flux through the two celestial spheres S^2 at timelike infinity $i^\pm$ requires an overall factor of 2 giving then the value of $2V(S^2)\hat{\mu}[Q_4]$. The Geometric Probability is defined by the ratio of the measures associated with the celestial spheres S^2 at i^+, i^- timelike infinity, where the photon moves on-shell, relative to the measure of the full *interior* region of $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$, where the photon can move off-shell, as it propagates from i^- to i^+:

$$\alpha = \frac{2V(S^2)\,\hat{\mu}[Q_4]}{\mu[Q_4]}. \tag{3}$$

The ratio $(\hat{\mu}[Q_4]/\mu[Q_4]\ )$ can be re-written in terms of the ratios of the normalized measures of

$$\bar{M}_5 = Q_5 = Shilov\ [D_5] = S^4 \times S^1/Z_2 = S^4 \times RP^1, \tag{4}$$

namely, in terms of the normalized measures of the conformally compactified $5D$ Minkowski spacetime. This is achieved as follows [4]

$$\frac{\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{1}{V(S^4)}\,\frac{\mu_N[Q_5]}{\mu[Q_5]}, \tag{5}$$

resulting from the embeddings (inmersions) of $D_4 \to D_5$.

The origin of the factor $V(S^4)$ in the r. h. s of (5), as one goes from the ratio of measures in Q_4 to the ratio of the measures in Q_5, is due to the reduction from the action of the isotropy group of the origin $SO(5)\times SO(2)$ on Q_5, to the action of the isotropy group of the origin $SO(4)\times SO(2)$ on Q_4, furnishing an overall reduction factor of $V[SO(5)/SO(4)] = V(S^4)$. The 5 complex-dimensional poly-disc $D_5 = SO(5,2)/SO(5)\times SO(2)$ is the 10 real-dim Hyperboloid $\mathcal{H}^{10}$ corresponding to the conformal relativistic curved phase space of a particle moving in $5D$ Anti de Sitter Space AdS_5. This picture is also consistent with the Kaluza-Klein compactification procedure of obtaining $4D$ EM from pure Gravity in $5D$. The $\mathcal{H}^{10}$ can be embedded in the 11-dim pseudo-Euclidean $R^{9,2}$ space, with two-time like directions. This is where 11-dim lurks into our construction.

Next we turn to the Hermitian metric on D_5 constructed by Hua [8] which is $SO(5,2)$-invariant and is based on the Bergmann kernel [15] involving a crucial normalization factor of $1/V(D_5)$. However, the standard normalized measure $\mu_N[Q_5]$ based on the Poisson kernel and involving a normalization factor of $1/V(Q_5)$ is *not* invariant under the full group $SO(5,2)$. It is only invariant under the isotropy group of the origin $SO(5)\times SO(2)$. In order to construct an invariant measure on Q_5 under the full group $SO(5,2)$ one requires to introduce a crucial factor related to the Jacobian measure involving the action of the conformal group $SO(5,2)$ on the full bulk domain D_5. As explained by [4] one has:

$$\frac{\mu_N[Q_5]}{\mu[Q_5]} = \frac{1}{V(Q_5)}\,||\mathcal{J}_C^{-1}|| =$$
$$= \frac{1}{V(Q_5)}\,\sqrt{||\mathcal{J}_C^{-1}(\mathcal{J}_C^{*})^{-1}||} = \frac{1}{V(Q_5)}\,\sqrt{||\mathcal{J}_R^{-1}||} =$$

$$= \frac{1}{V(Q_5)} \sqrt{\sqrt{|\det g|^{-1}}} = \frac{1}{V(Q_5)} [|\det(g)|]^{-\frac{1}{4}} = = \frac{1}{V(Q_5)} [V(D_5)]^{\frac{1}{4}}, \quad (6)$$

the z dependence of the complex Jacobian is no longer explicit because the determinant of the $SO(5,2)$ matrices is unity.

This explains very clearly the origins of the factor $[V(D_5)]^{\frac{1}{4}}$ in Wyler's formula for the fine structure constant [1]. This reduction factor of $V(Q_5)$ is in this case given by $V(D_5)^{\frac{1}{4}}$. As we shall see below, the power of $\frac{1}{4}$ is related to the inverse of the $\dim(S^4) = 4$. This summarizes, briefly, the role of Bergmann kernel [15] in the construction by Hua [8], and adopted by Wyler [1], of the Hermitian metric of a bounded homogenous (symmetric) complex domain. To sum up, we must perform the reduction from $V(Q_5) \to V(Q_5)/V(D_5)^{\frac{1}{4}}$ in the construction of the normalized measure $\mu_N[Q_5]$. This approach is very different than the interpretation given by Smith [3] and later adopted by Smilga [5].

Hence, the Geometric Probability ratio becomes

$$\frac{\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{1}{V(S^4)} \frac{\mu_N[Q_5]}{\mu[Q_5]} = = \frac{1}{V(S^4)} \frac{1}{V(Q_5)} [\ V(D_5)\ ]^{\frac{1}{4}} \equiv \frac{1}{\alpha_G}. \quad (7a)$$

This last ratio, for reasons to be explained below, is nothing but the inverse of the geometric coupling strength of gravity, $1/\alpha_G$. The relationship to the gravitational constant is based on the definition of the coupling appearing in the Einstein-Hilbert Lagrangian $(R/16\pi G)$, as follows

$$(16\pi G)(m^2_{Planck}) \equiv \alpha_{EM}\ \alpha_G = 8\pi \ \Rightarrow$$

$$G = \frac{1}{16\pi} \frac{8\pi}{m^2_{Planck}} = \frac{1}{2m^2_{Planck}} \ \Rightarrow \quad (7b)$$

$$G m^2_{proton} = \frac{1}{2} \left(\frac{m_{proton}}{m_{Planck}} \right)^2 \sim 5.9\times 10^{-39},$$

and in natural units $\hbar = c = 1$ yields the physical force strength of Gravity at the Planck Energy scale 1.22×10^{19} GeV. The Planck mass is obtained by equating the Schwarzschild radius $2Gm_{Planck}$ to the Compton wavelength $1/m_{Planck}$ associated with the mass; where $m_{Planck}\sqrt{2} = 1.22\times10^{19}$ GeV and the proton mass is 0.938 GeV. Some authors define the Planck mass by absorbing the factor of $\sqrt{2}$ inside the definition of $m_{Planck} = 1.22\times10^{19}$ GeV.

The role of the conformal group in Gravity in these expressions (besides the holographic bulk/boundary AdS/CFT duality correspondence [13]) stems from the MacDowell-Mansouri-Chamseddine-West formulation of Gravity based on the conformal group $SO(3,2)$ which has the same number of 10 generators as the $4D$ Poincare group. The $4D$ vielbein e^a_μ which gauges the spacetime translations is identified with the $SO(3,2)$ generator $A^{[a5]}_\mu$, up to a crucial scale factor R, given by the size of the Anti de Sitter space (de Sitter space) throat. It is known that the Poincare group is the Wigner-Inonu group contraction of the de Sitter Group $SO(4,1)$ after taking the throat size $R = \infty$. The spin-connection ω^{ab}_μ that gauges the Lorentz transformations is identified with the $SO(3,2)$ generator $A^{[ab]}_\mu$. In this fashion, the e^a_μ, ω^{ab}_μ are encoded into the $A^{[mn]}_\mu$ $SO(3,2)$ gauge fields, where m, n run over the group indices $1,2,3,4,5$. A word of caution, Gravity is a gauge theory of the full diffeomorphisms group which is infinite-dimensional and which includes the translations. Therefore, strictly speaking gravity is not a gauge theory of the Poincare group. The Ogiovetsky theorem shows that the diffeomorphisms algebra in $4D$ can be generated by an infinity of *nested* commutators involving the $GL(4,R)$ and the $4D$ Conformal Group $SO(4,2)$ generators.

In [17] we have shown why the MacDowell-Mansouri-Chamseddine-West formulation of Gravity, with a cosmological constant and a topological Gauss-Bonnet invariant term, can be obtained from an action inspired from a BF-Chern-Simons-Higgs theory based on the conformal $SO(3,2)$ group. The AdS_4 space is a natural vacuum of the theory. The vacuum energy density was derived to be the geometric-mean between the UV Planck scale and the IR throat size of de Sitter (Anti de Sitter) space. Setting the throat size to coincide with the future horizon scale (of an accelerated de Sitter Universe) given by the Hubble scale (today) R_H, the geometric mean relationship yields the observed value of the vacuum energy density $\rho \sim (L_P)^{-2}(R_H)^{-2} = (L_P)^{-4} \times (L^2_P/R^2_H) \sim 10^{-122} M^4_{Planck}$. Nottale [23] gave a different argument to explain the small value of ρ based on Scale Relativistic arguments. It was also shown in [17] why the Euclideanized AdS_{2n} spaces are $SO(2n-1,2)$ instantons solutions of a non-linear sigma model obeying a double self duality condition.

Therefore, the Geometric Probability α_{EM} for an electron to emit a photon at $t = -\infty$ and to absorb it at $t = +\infty$ agrees with the Wyler's celebrated expression for the fine structure constant

$$\alpha_{EM} = \frac{2V(S^2)\hat{\mu}[Q_4]}{\mu[Q_4]} = (8\pi) \frac{1}{V(S^4)} \frac{1}{V(Q_5)} \times \times [V(D_5)]^{\frac{1}{4}} = \frac{9}{8\pi^4} \left(\frac{\pi^5}{2^4\times 5!} \right)^{\frac{1}{4}} = \frac{1}{137.03608}, \quad (8)$$

after one inserts the values of the volumes:

$$V(D_5) = \frac{\pi^5}{2^4\times 5!}, \quad V(Q_5) = \frac{8\pi^3}{3}, \quad V(S^4) = \frac{8\pi^2}{3}. \quad (9)$$

In general

$$V(D_n) = \frac{\pi^n}{2^{n-1} n!}, \quad V(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad (10a)$$

$$V(Q_n)=V(S^{n-1}\times RP^1)=V(S^{n-1})\times V(RP^1) = = \frac{2\pi^{n/2}}{\Gamma(n/2)}\times\pi = \frac{2\pi^{(n+2)/2}}{\Gamma(n/2)}\,. \qquad (10b)$$

Objections were raised to Wyler's original expression by Robertson [2]. One of them was that the hyperboloids (discs) are not compact and whose volumes diverge since the Lobachevsky metric diverges on the boundaries of the poly-discs. Gilmore explained [2] why one requires to use the Euclideanized regularized volumes as Wyler did. Furthermore, in order to resolve the scaling problems of Wyler's expression, Gilmore showed why it is essential to use dimensionless volumes by setting the throat sizes of the Anti de Sitter hyperboloids to $r=1$, because this is the only choice for r where all elements in the bounded domains are also coset representatives, and therefore, amount to honest group operations. Hence the so-called scaling objections against Wyler raised by Robertson were satisfactory solved by Gilmore [2].

The question as to *why* the value of α_{EM} obtained in Wyler's formula is precisely the value of α_{EM} observed at the *scale* of the Bohr radius a_B, has not been solved, to my knowledge. The Bohr radius is associated with the ground (most stable) state of the Hydrogen atom [3]. The spectrum generating group of the Hydrogen atom is well known to be the conformal group $SO(4,2)$ due to the fact that there are two conserved vectors, the angular momentum and the Runge-Lentz vector. After quantization, one has two commuting $SU(2)$ copies $SO(4) = SU(2)\times SU(2)$. Thus, it makes physical sense why the Bohr-scale should appear in this construction. Bars [14] has studied the many physical applications and relationships of many seemingly distinct models of particles, strings, branes and twistors, based on the (super) conformal groups in diverse dimensions. In particular, the relevance of two-time physics in the formulation of M, F, S theory has been advanced by Bars for some time. The Bohr radius corresponds to an energy of $137.036\times2\times13.6$ eV $\sim 3.72\times10^3$ eV. It is well known that the Rydberg scale, the Bohr radius, the Compton wavelength of electron, and the classical electron radius are all related to each other by a successive scaling in products of α_{EM}.

2 The fiber bundle interpretation of the Wyler formula

Having found Wyler's expression from Geometric Probability, we shall present a Fiber Bundle interpretation of the Wyler expression by starting with a Fiber bundle E over the base curved-space $D_5 = SO(5,2)/SO(5)\times SO(2)$. The subgroup $H{=}SO(5)$ of the isotropy group $K{=}SO(5)\times SO(2)$ acts on the Fibers $F=S^4$ (the internal symmetry space). Locally, and only locally, the Fiber bundle E is the product $D_5\times S^4$. However, this is *not* true globally. On the Shilov boundary Q_5, the restriction of the Fiber bundle E to the Shilov boundary Q_5 is written by $E|_{Q_5}$ and *locally* is the product of $Q_5\times S^4$, but this is *not* true globally. For this reason one has that the volume $V(E|_{Q_5})\neq V(Q_5\times S^4) = V(Q_5)\times V(S^4)$. But instead, $V(E|_{Q_5}) = V(S^4)\times (V(Q_5)/V(D_5)^{1/4})$.

This is the reasoning behind the construction of the quantity $\hat\mu[Q_4]/\mu[Q_4]$ that has the units of a density. Its inverse $\mu[Q_4]/\hat\mu[Q_4]$ is the volume associated with the restriction of the Fiber Bundle E to the Shilov boundary Q_5: $V(E|_{Q_5}) = V(S^4)\times(V(Q_5)/V(D_5)^{1/4})$.

The reason why one embeds $D_4\to D_5$ and $Q_4\to Q_5$ is because the space $Q_4 = S^3\times RP^1$ is *not* large enough to implement the action of the $SO(5)$ group, the compact version of the Anti de Sitter Group $SO(3,2)$ that is required in the MacDowell-Mansouri-Chamseddine-West formulation of Gravity. However, the space $Q_5 = S^4\times RP^1$ is large enough to implement the action of $SO(5)$ via the internal symmetry space $S^4 = SO(5)/SO(4)$. This justifies the embedding procedure of $D_4\to D_5$. This Fiber Bundle interpretation is not very different from Smith's interpretation [3]. Following the Fiber Bundle interpretation of the volume $V(E|_{Q_5}) = V(S^4)\times(V(Q_5)/V(D_5)^{1/4})$, we will now prove why

$$2\,V(S^2) = \frac{\mu(S^1)}{\hat\mu(S^1)} = 8\pi\,. \qquad (11)$$

The space S^1 is associated with the $U(1)$ group action and naturally encodes the $U(1)$ gauge invariance linked to Electromagnetism (EM). The result of eq-(11) is what will allow us to *define* α_{EM} as the *ratio of the ratios* of suitable measures in S^1 and Q_4, respectively,

$$\alpha_{EM} = \frac{2V(S^2)\;\hat\mu[Q_4]}{\mu[Q_4]} = \frac{(\mu(S^1)/\hat\mu(S^1))}{(\mu[Q_4]/\hat\mu[Q_4])}\,. \qquad (12)$$

We may notice that $S^1\equiv Q_1$ (very special case) since the circle is both the Shilov and ordinary topological boundary of the disc D_1. However, $Q_2\equiv S^1\times S^1/Z_2 = S^1\times RP^1$. Once again, we will write the ratio of the measures in $Q^1 = S^1$ in terms of the ratio of the normalized measures in Q^2 via the reduction from $S^1\times S^1/Z_2$ to S^1. This requires the embedding (inmersion) of $D_1\to D_2$ in order to construct the measures on D_1, Q_1 as induced from the measures in D_2, Q_2 resulting from the embedding (inmersion):

$$\frac{\hat\mu(S^1)}{\mu(S^1)} = \frac{\hat\mu(Q_1)}{\mu(Q_1)} = \frac{1}{V(S^1/Z_2)}\frac{\mu_N[Q_2]}{\mu[Q_2]} = = \frac{1}{V(S^1/Z_2)}\;\frac{1}{(V(Q_2)/V(D_2)}\,. \qquad (13)$$

Notice that $\hat\mu(S^1)$ as explained before is a measure-density on S^1. Likewise, $\hat\mu(Q_4)$ was a measure-density on Q_4. We should not confuse these measure-densities with the normalized measures in one-higher dimension.

By inserting the values of the measures and using

$$V(S^1/Z_2) = V(RP^1) = \pi\,, \quad V(D_2) = \frac{\pi^2}{2{\times}2!}\,, \qquad V(Q^2) = \frac{2\pi^2}{\Gamma(1)} = 2\pi^2, \tag{14}$$

it yields then

$$\frac{\mu(S^1)}{\hat{\mu}(S^1)} = (2\pi^2)\,(\pi)\,\frac{1}{(\pi^2/2{\times}2!)} = 8\pi = 2\,V(S^2) \tag{15}$$

as claimed. Therefore, $2V(S^2) = \mu(S^1)/\hat{\mu}(S^1) = 8\pi$ is the crucial factor appearing in Wyler's formula which admits a natural Geometric probability explanation which is very different from the different interpretations provided in [3, 4, 5].

The Fiber Bundle interpretation associated with the $U(1) \sim SO(2)$ group is the following. The Fiber bundle E is defined over the curved space $D_2 = SO(2,2)/SO(2) \times SO(2)$. The subgroup $H = SO(2) \sim U(1)$ of the isotropy group $K = SO(2) \times SO(2)$ acts on the fibers identified with the symmetry space S^1 (where the $U(1)$ group acts). The Fiber bundle E locally can be written as $D_2 \times S^1$ but not globally. The restriction of the Fiber bundle E to the Shilov boundary $Q_2 = S^1 \times S^1/Z_2 = S^1 \times RP^1$ is $E|_{Q_2}$ and locally can be written as $Q_2 \times S^1$, but *not* globally. This is why the volume $V(E|_{Q_2}) \neq V(Q_2) \times V(S_1)$ but instead it equals $(V(Q_2)/V(D_2)) \times V(S^1/Z_2) = 2V(S^2) = 8\pi$.

Concluding, the Geometric Probability that an electron emits a photon at $t = -\infty$ and absorbs it at $t = +\infty$ is given by the *ratio* of the *ratios* of measures, and it agrees with Wheeler's ideas that one must normalize the couplings with respect to the geometric coupling strength of Gravity:

$$\alpha_{EM} = \frac{2V(S^2)\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{(\mu(S^1)/\hat{\mu}(S^1))}{(\mu[Q_4]/\hat{\mu}[Q_4])} = = (8\pi)\frac{1}{V(S^4)}\frac{1}{V(Q_5)}[V(D_5)]^{\frac{1}{4}} = \frac{1}{137.03608}\,. \tag{16}$$

The second important conclusion that can be *derived* from Geometric Probability theory is the general numerical values of the exponents s_n appearing in the factors $V(D_n)^{s_n}$. The normalization factor $V(Q_5)/V(D_5)^{1/4}$ in the construction of the ratio of measures $\mu_N[Q_5]/\mu[Q_5]$ involves in this case powers of the type $V(D_5)^{1/4}$. The power of $\frac{1}{4}$ is related to the inverse of the $\dim(S^4) = 4$ (the internal symmetry space $SO(5)/SO(4)$). From eq-(13) we learnt that the reduction factor of $V(Q^2)/V(D_2)$ was $V(D_2)$; i. e. the exponent is unity. The power of *unity* is related to the inverse of the $\dim(S^1/Z_2) = 1$. Thus, the arguments based on Geometric Probability leads to normalized measures by factors of $V(Q_n)/V(D_n)^{s_n}$ and whose exponents s_n are given by the *inverse* of the dimensions of the internal symmetry spaces $s_n = (\dim(S^{n-1}))^{-1}$. There is a different interpretation of these factors $V(D_n)^{s_n}$ given by Smith [3]. In general, for other homogeneous complex domains, this power is given by the inverse of the dimension of the internal symmetry space.

3 The weak and strong coupling constants from Geometric Probability

We turn now to the derivation of the other coupling constants. The Fiber Bundle picture of the previous section is essential in our construction. The Weak and the Strong geometric coupling constant strength, defined as the probability for a particle to emit and later absorb a $SU(2), SU(3)$ gauge boson, respectively, can both be obtained by using the main formula derived from Geometric Probability after one identifies the suitable homogeneous domains and their Shilov boundaries to work with. We will show why the weak and strong couplings are given by

$$\alpha_{Weak} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(\mu[Q_4]/\hat{\mu}[Q_4])} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{\alpha_G} = = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(8\pi/\alpha_{EM})}\,, \tag{17}$$

and

$$\alpha_{Color} = \frac{(\mu[S^4]/\hat{\mu}[S^4])}{(\mu[Q_4]/\hat{\mu}[Q_4]} = \frac{(\mu[S^4]/\hat{\mu}[S^4])}{\alpha_G} = = \frac{(\mu[S^4]/\hat{\mu}[S^4])}{(8\pi/\alpha_{EM})}\,. \tag{18}$$

At this point we must emphasize that we define α_{weak}, α_{color} as g_w^2, g_c^2 instead of the conventional $(g_w^2/4\pi)$, $(g_c^2/4\pi)$ definitions used in the Renormalization Group program. The Shilov boundary of (D_2) is $Q_2 = S^1 \times RP^1$ but is not large enough to accommodate the action of the isospin group $SU(2)$. One needs a Fiber Bundle over $D_3 = = SO(3,2)/SO(3) \times SO(2)$ whose subgroup $H = SO(3)$ of the isotropy group $K = SO(3) \times SO(2)$ acts on the internal symmetry space S^2 (the fibers). Since the coset space $SU(2)/U(1)$ is a double-cover of the S^2 as one goes from the $SO(3)$ action to the $SU(2)$ action one must take into account an extra factor of 2. This is the reason why one jumps to one-dimension higher from Q_2 to $Q_3 = S^2 \times RP^1$, because the coset $SU(2)/U(1)$ is a double-cover of the sphere $S^2 = SO(3)/SO(2)$ and can accommodate the action of the $SU(2)$ group.

By following the same procedure as above, i. e. by rewriting the ratio of the measures $(\hat{\mu}[Q_2]/\mu[Q_2])$ in terms of the ratio of the measures $(\mu_N[Q_3]/\mu[Q_3])$ via the embeddings of $D_2 \to D_3$, one has

$$(\hat{\mu}[Q_2]/\mu[Q_2]) = \frac{1}{V(SU(2)/U(1))}\,\frac{\mu_N[Q_3]}{\mu[Q_3]}. \tag{19}$$

Notice that because $SU(2)$ is a $2-1$ covering of the $SO(3)$, this implies that the measure

$$V(SU(2)/U(1)) = 2V(SO(3)/U(1)) = 2V(S^2) = 8\pi\,. \tag{20}$$

As indicated above, because the dimension of the internal symmetry space is $\dim(S^2) = 2$, the construction of the normalized measure $\mu_N[Q_3]$ will require a reduction of $V(Q_3)$ by a factor of $V(D_3)$ raised to the power of $(\dim(S^2))^{-1} = \frac{1}{2}$:

$$\frac{\mu_N[Q_3]}{\mu[Q_3]} = \frac{1}{V(Q_3)/V(D_3)^{1/2}} = \frac{1}{V(Q_3)}\, V(D_3)^{1/2}. \quad (21)$$

Therefore, the ratio of the measures is

$$\frac{\hat{\mu}[Q_2]}{\mu[Q_2]} = \frac{1}{2V(S^2)}\,\frac{1}{V(Q_3)}\, V(D_3)^{1/2}, \quad (22)$$

whose Fiber Bundle interpretation is that the volume of the Fiber Bundle over D_3, but restricted to the Shilov boundary Q_3, and whose structure group is $SU(2)$ (the double cover of $SO(3)$), is $V(E|_{Q_3}) = 2V(S^2) \times (V(Q_3)/V(D_3)^{1/2})$. Thus, that the Geometric probability expression is

$$\alpha_{Weak} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(\mu[Q_4]/\hat{\mu}[Q_4])} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(8\pi/\alpha_{EM})} = \\ = 2V(S^2)V(Q_3)\frac{1}{V(D_3)^{1/2}}\frac{\alpha_{EM}}{8\pi} = 0.2536, \quad (23)$$

that corresponds to the weak geometric coupling constant α_W at an energy of the order of

$$E = M = 146 \text{ GeV} \sim \sqrt{M^2_{W_+} + M^2_{W_-} + M^2_Z}\,, \quad (24)$$

after we have inserted the expressions

$$V(S^2) = 4\pi\,, \quad V(Q_3) = 4\pi^2\,, \quad V(D_3) = \frac{\pi^3}{24}\,, \quad (25a)$$

into the formula (23). The relationship to the Fermi coupling G_{Fermi} goes as follows (after indentifying the energy scale $E = M = 146$ GeV):

$$G_F \equiv \frac{\alpha_W}{M^2} \Rightarrow G_F\, m^2_{proton} = \left(\frac{\alpha_W}{M^2}\right) m^2_{proton} = \\ = 0.2536 \times \left(\frac{m_{proton}}{146 \text{ GeV}}\right)^2 \sim 1.04{\times}10^{-5} \quad (25b)$$

in very good agreement with experimental observations.

Once more, it is unknown why the value of α_{Weak} obtained from Geometric Probability corresponds to the energy scale related to the W_+, W_-, Z_0 boson mass, after spontaneous symmetry breaking.

Finally, we shall derive the value of α_{Color} from eq-(18). Since S^4 is not large enough to accommodate the action of the color group $SU(3)$ one needs to work with one-dimension higher S^5 , that can be interpreted as the boundary of the $6D$ Ball $B_6 = SU(4)/U(3) = SU(4)/SU(3) \times U(1)$. Thus, the $SU(3)$ group is part of the isotropy group $K = SU(3) \times U(1)$ that defines the coset space B_6. In this special case the Shilov and ordinary topological boundaries of B_6 coincide with S^5 [3]. Hence, following the same procedures as above, the ratio of the measures in S^4 (boundary of B_5) can be re-written in terms of the ratio of the measures in S^5 (boundary of B_6) via the embeddings of $B_5 \to B_6$ as follows:

$$\frac{\hat{\mu}[S^4]}{\mu[S^4]} = \frac{1}{V(S^4)}\frac{\mu_N[S^5]}{\mu[S^5]} = \frac{1}{V(S^4)}\frac{1}{V(S^5)/V(B_6)^{1/4}} = \\ = \frac{1}{V(S^4)}\frac{1}{V(S^5)}\, V(B_6)^{1/4}, \quad (26)$$

since the exponent of the reduction factor $V(B_6)^{1/4}$ is given by $(\dim(S^4))^{-1} = \frac{1}{4}$. Notice, again, that $\hat{\mu}[S^4]$ is the measure-density in S^4 and must not be confused with the normalized measures.

Therefore, one arrives at

$$\alpha_{Color} = V(S^4)\, V(S^5)\,\frac{1}{V(B_6)^{1/4}}\,\frac{\alpha_{EM}}{8\pi} = 0.6286, \quad (27)$$

that corresponds to the strong coupling constant at an energy related to the pion masses [3]:

$$E = 241 \text{ MeV} \sim \sqrt{m^2_{\pi^+} + m^2_{\pi^-} + m^2_{\pi^0}} \quad (28)$$

and where we have used the expressions:

$$V(S^4) = \frac{8\pi^2}{3}\,, \quad V(S^5) = 4\pi^3\,, \quad V(B_6) = \frac{\pi^3}{6}\,. \quad (29)$$

The pions are the known lightest quark/antiquark pairs that feel the strong interaction [3]. For a detailed analysis of volumes of compact manifolds (coset spaces) see [24].

Once again, it is unknown why the value of α_{Color} obtained from Geometric Probability (28) corresponds to the energy scale related to the masses of the three pions [3]. Masses of the fundamental particles were derived in [3] based on the definitions that mass is the probability amplitude for a particle to change direction.

To conclude, by defining the geometric coupling constants $\alpha = g^2$ as the Geometric Probability to emit (and later absorb) a gauge boson, all the three geometric coupling constants, $\alpha_{EM}; \alpha_{Weak}; \alpha_{Color}$ are given by the *ratios of the ratios* of the measures of the Shilov *boundaries* $Q_2 = S^1 \times RP^1$; $Q_3 = S^2 \times RP^1$; S^5, respectively, with respect to the ratios of the measures $\mu[Q_5]/\mu_N[Q_5]$ associated with the $5D$ conformally compactified real Minkowski spacetime $\bar{M}_5$ that has the same topology as the Shilov boundary Q_5 of the 5 complex-dimensional poly-disc D_5. The latter corresponds to the conformal relativistic *curved* 10 real-dimensional phase space $\mathcal{H}^{10}$ associated with a particle moving in the $5D$ Anti de Sitter space AdS_5. The ratios of particle masses, like the proton to electron mass ratio $m_p/m_e \sim 6\pi^5$ has also been calculated using the volumes of homogeneous bounded domains [3, 4].

It is not known whether this procedure would work for Grand Unified Theories based on the groups

$$SU(5), SO(10), E_6, E_7, E_8\,. \qquad (30)$$

Beck [6] has obtained all the Standard Model parameters by studying the numerical minima (and zeros) of certain potentials associated with the Kaneko coupled two-dim lattices based on Stochastic Quantization methods. The results above and by Smith [3] are analytical rather than being numerical [6] and it is not clear if there is any relationship between these two approaches. Noyes has proposed an iterated numerical hierarchy based on Mersenne primes $M_p = 2^p - 1$ for certain values of $p =$ primes [18] and obtained many numerical values for the physical parameters. Pitkanen has developed methods to calculate the physical masses recurring to a p-adic hierarchy of scales based on Mersenne primes [19].

An important connection between anomaly cancellation in string theory and perfect even numbers was found in [22]. These are numbers which can be written in terms of sums of its divisors, including unity, like $6 = 1 + 2 + 3$, and are of the form $P(p) = \frac{1}{2}\,2^p(2^p - 1)$ if, and only if, $2^p - 1$ is a Mersenne prime. Not all values of $p =$ prime yields primes. The number $2^{11} - 1$ is not a Mersenne prime, for example. The number of generators of the anomaly free groups $SO(32)$, $E_8 \times E_8$ of the 10-dim superstring is 496 which is an even perfect number. Another important group related to the unique tadpole-free bosonic string theory is the $SO(2^{13}) = SO(8192)$ group related to the bosonic string compactified on the $E_8 \times SO(16)$ lattice. The number of generators of $SO(8192)$ is an even perfect number since $2^{13} - 1$ is a Mersenne prime. For an introduction to p-adic numbers in Physics and String theory see [20]. A lot more work needs to be done to be able to answer the question: Is all this just a mere numerical coincidence or is it design?

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