On the Vacuum Field of a Sphere of Incompressible Fluid

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The vacuum field of the point-mass is an unrealistic idealization which does not occur in Nature - Nature does not make material points. A more realistic model must therefore encompass the extended nature of a real object. This problem has also been solved for a particular case by K. Schwarzschild in his neglected paper on the gravitational field of a sphere of incompressible fluid. I revive Schwarzschild's solution and generalise it. The black hole is necessarily precluded. A body cannot undergo gravitational collapse to a material point.

1 Introduction

In my previous papers [1, 2] concerning the general solution for the point-mass I showed that the black hole is not consistent with General Relativity and owes its existence to a faulty analysis of the Hilbert [3] solution. In this paper I shall show that, along with the black hole, gravitational collapse to a point-mass is also untenable. This was evident to Karl Schwarzschild who, immediately following his derivation of his exact solution for the mass-point [4], derived a particular solution for an extended body in the form of a sphere of incompressible, homogeneous fluid [5]. This is also an idealization, and so too has its shortcomings, but represents a somewhat more plausable end result of gravitational collapse.

The notion that Nature makes material points, i. e. masses without extension, I view as an oxymoron. It is evident that there has been a confounding of a mathematical point with a material object which just cannot be rationally sustained. Einstein [6, 7] objected to the introduction of singularities in the field but could offer no viable alternative, even though Schwarzschild's extended body solution was readily at his hand.

The point-mass and the singularity are equivalent. Abrams [8] has remarked that singularities associated with a spacetime manifold are not uniquely determined until a boundary is correctly attached to it. In the case of the point-mass the source of the gravitational field is identified with a singularity in the manifold. The fact that the vacuum field for the point-mass is singular at a boundary on the manifold indicates that the point-mass does not occur in Nature. Oddly, the conventional view is that it embodies the material point. However, there exists no observational or experimental data supporting the idea of a point-mass or point-charge. I can see no way an electron, for instance, could be compressed into a material point-charge, which must occur if the point-mass is to be admitted. The idea of electron compression is meaningless, and therefore so is the point-mass. Eddington [9] has remarked in similar fashion concerning the electron, and relativistic degeneracy in general.

I regard the point-mass as a mathematical artifice and consider it in the fashion of a centre-of-mass, and therefore not as a physical object. In Newton's theory of gravitation, $r = 0$ is singular, and equivalently in Einstein's theory, the proper radius $R_p(r_0) \equiv 0$ is singular, as I have previously shown. Both theories therefore share the non-physical nature of the idealized case of the point-mass.

To obtain a model for a star and for the gravitational collapse thereof, it follows that the solution to Einstein's field equations must be built upon some manifold without boundary. In more recent years Stavroulakis [10, 11, 12] has argued the inappropriateness of the solutions on a manifold with boundary on both physical and mathematical grounds, and has derived a stationary solution from which he has concluded that gravitational collapse to a material point is impossible.

Utilizing Schwarzschild's particular solution I shall extend his result to a general solution for a sphere of incompressible fluid.

2 The general solution for Schwarzschild's incompressible sphere of fluid

At the surface of the sphere the required solution must maintain a smooth transition from the field outside the sphere to the field inside the sphere. Therefore, the metric for the interior and the metric for the exterior must attain the same value for the radius of curvature at the surface of the sphere. Furthermore, owing to the extended nature of the sphere, the exterior metric must take the form of the metric for the point-mass, but with a modified invariant containing the factors giving rise to the field, reflecting the non-pointlike nature of the source, thereby treating the source as a mass concentrated at the centre-of-mass of the sphere, just as in Newton's theory. Schwarzschild has achieved this in his particular case. He obtained the following metric for the field inside his sphere,

$$ds^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 dt^2 - \\ - \frac{3}{\kappa\rho_0}d\chi^2 - \frac{3\sin^2\chi}{\kappa\rho_0}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{1}$$

$$\sin\chi = \sqrt{\frac{\kappa\rho_0}{3}}\,\eta^{\frac{1}{3}}, \quad \eta = r^3 + \rho\,,$$

$$\rho = \left(\frac{\kappa\rho_0}{3}\right)^{\frac{-3}{2}}\left[\frac{3}{2}\sin^3\chi_a - \frac{9}{4}\cos\chi_a\left(\chi_a - \frac{1}{2}\sin 2\chi_a\right)\right],$$

$$\kappa = 8\pi k^2\,,$$

$$0 \leqslant \chi \leqslant \chi_a < \frac{\pi}{2},$$

where ρ_0 is the constant density of the fluid, k^2 Gauss' gravitational constant, and the subscript a denotes values at the surface of the sphere. Metric (1) is non-singular.

Schwarzschild's particular metric outside the sphere is,

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1}dR^2 - \\ - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2}$$

$$R^3 = r^3 + \rho\,, \quad \alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3\chi_a\,,$$

$$0 \leqslant \chi_a < \frac{\pi}{2}\,,$$

$$r_a \leqslant r < \infty\,.$$

Metric (2) is non-singular for an extended body.

In the case of the simple point-mass (i. e. non-rotating, no charge) I have shown elsewhere [13] that the general solution is,

$$ds^2 = \left[\frac{(\sqrt{C_n} - \alpha)}{\sqrt{C_n}}\right]dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n} - \alpha)}\right]\frac{C_n'^{\,2}}{4C_n}dr^2 - \\ - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{3}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}, \; \alpha = 2m\,,$$

$$n \in \Re^+\,, \; r \in \Re, \; r_0 \in \Re\,,$$

$$0 < |r - r_0| < \infty\,,$$

where n and r_0 are arbitrary.

Now Schwarzschild fixed his solution for $r_0 = 0$. I note that his equations, rendered herein as equations (1) and (2), can be easily generalised to an arbitrary $r_0 \in \Re$ and arbitrary $\chi_0 \in \Re$ by replacing his r and χ by $|r - r_0|$ and $|\chi - \chi_0|$ respectively. Furthermore, equation (3) must be modified to account for the extended configuration of the gravitating mass. Consequently, equation (1) becomes,

$$ds^2 = \left[\frac{3\cos\left|\chi_a - \chi_0\right| - \cos\left|\chi - \chi_0\right|}{2}\right]^2 dt^2 - \\ - \frac{3}{\kappa\rho_0}d\chi^2 - \frac{3\sin^2\left|\chi - \chi_0\right|}{\kappa\rho_0}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{4}$$

$$\sin\left|\chi - \chi_0\right| = \sqrt{\frac{\kappa\rho_0}{3}}\eta^{\frac{1}{3}}, \quad \eta = \left|r - r_0\right|^3 + \rho\,,$$

$$\rho = \left(\frac{\kappa\rho_0}{3}\right)^{\frac{-3}{2}}\left\{\frac{3}{2}\sin^3\left|\chi_a - \chi_0\right| - \right. \\ \left. - \frac{9}{4}\cos\left|\chi_a - \chi_0\right|\left[\left|\chi_a - \chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a - \chi_0\right|\right]\right\},$$

$$\kappa = 8\pi k^2\,, \; r_0 \in \Re\,, \; r \in \Re\,, \; \chi_a \in \Re\,, \; \chi_0 \in \Re\,,$$

$$0 \leqslant |\chi - \chi_0| \leqslant |\chi_a - \chi_0| < \frac{\pi}{2}\,,$$

and outside the sphere, equation (2) becomes,

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1}dR^2 - \\ - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{5}$$

$$R^3 = \left|r - r_0\right|^3 + \rho, \quad \alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3\left|\chi_a - \chi_0\right|\,,$$

$$n \in \Re^+\,, \; r_0 \in \Re\,, \; r \in \Re\,, \; \chi_0 \in \Re\,, \; \chi_a \in \Re\,,$$

$$0 \leqslant |\chi_a - \chi_0| < \frac{\pi}{2}\,,$$

$$|r_a - r_0| \leqslant |r - r_0| < \infty\,,$$

and outside the sphere, equation (3) becomes,

$$ds^2 = \left[\frac{(\sqrt{C_n} - \alpha)}{\sqrt{C_n}}\right]dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n} - \alpha)}\right]\frac{C_n'^{\,2}}{4C_n}dr^2 - \\ - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{6}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \epsilon^n\right)^{\frac{2}{n}}\,,$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3\left|\chi_a - \chi_0\right|\,,$$

$$\epsilon = \sqrt{\frac{3}{\kappa\rho_0}}\left\{\frac{3}{2}\sin^3\left|\chi_a - \chi_0\right| - \right. \\ \left. - \frac{9}{4}\cos\left|\chi_a - \chi_0\right|\left[\left|\chi_a - \chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a - \chi_0\right|\right]\right\}^{\frac{1}{3}},$$

$$r_0 \in \Re\,, \; r \in \Re\,, \; n \in \Re^+\,, \; \chi_0 \in \Re\,, \; \chi_a \in \Re\,,$$

$$|r_a - r_0| \leqslant |r - r_0| < \infty \,.$$

The general solution for the interior of the incompressible Schwarzschild sphere is given by (4), and (6) gives the general solution on the exterior of the sphere.

Consider the general form for a static metric for the gravitational field [13],

$$ds^2 = A(D)dt^2 - B(D)dD^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$D = |r - r_0|\,,$$

$$A, B, C > 0 \;\forall\; r \neq r_0 \,.$$

With respect to this metric I identify the real r-parameter, the radius of curvature, and the proper radius thus:

1. *The real r-parameter is the variable r.*
2. *The radius of curvature is $R_c = \sqrt{C(D)}$.*
3. *The proper radius is $R_p = \int \sqrt{B(D)}\, dD$.*

According to the foregoing, the proper radius of the sphere of incompressible fluid determined from *inside* the sphere is, from (4),

$$R_p = \int_{\chi_0}^{\chi_a} \sqrt{\frac{3}{\kappa\rho_0}}\,\frac{(\chi - \chi_0)}{|\chi - \chi_0|}\, d\chi = \sqrt{\frac{3}{\kappa\rho_0}}\, |\chi_a - \chi_0|\,. \tag{7}$$

The proper radius of the sphere cannot be determined from *outside* the sphere. According to (6) the proper radius to a spacetime event outside the sphere is,

$$\begin{aligned} R_p &= \int \sqrt{\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}}\, \frac{C_n'}{2\sqrt{C_n}}\, dr = \\ &= K + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} + \\ &+ \alpha \ln\left|\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}\right|, \end{aligned} \tag{8}$$

$$K = \text{const.}$$

At the surface of the sphere the proper radius from outside has some value R_{p_a}, for some value r_a of the parameter r. Therefore, at the surface of the sphere,

$$\begin{aligned} R_{p_a} &= K + \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} + \\ &+ \alpha \ln\left|\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}\right|. \end{aligned}$$

Solving for K,

$$\begin{aligned} K &= R_{p_a} - \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} - \\ &- \alpha \ln\left|\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}\right|. \end{aligned}$$

Substituting into (8) for K gives for the proper radius from outside the sphere,

$$\begin{aligned} R_p(r) &= R_{p_a} + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} - \\ &- \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} + \\ &+ \alpha \ln\left|\frac{\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}}{\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}}\right|. \end{aligned} \tag{9}$$

Then by (9), for $|r - r_0| \geqslant |r_a - r_0|$

$$|r - r_0| \to |r_a - r_0| \Rightarrow R_p \to R_{p_a}^+\,,$$

but R_{p_a} cannot be determined.

According to (4) the radius of curvature $R_c = P_a$ at the surface of the sphere is,

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}}\, \sin|\chi_a - \chi_0|\,. \tag{10}$$

Furthermore, inside the sphere,

$$\frac{G}{R_p} \leqslant 2\pi\,,$$

and

$$\lim_{\chi \to \chi_0^\pm} \frac{G}{R_p} = 2\pi\,,$$

where $G = 2\pi R_c$ is the circumference of a great circle.

But outside the sphere,

$$\frac{G}{R_p} \geqslant 2\pi\,,$$

with the equality only when $R_p \to \infty$.

The radius of curvature of (6) at the surface of the sphere is $\sqrt{C_n(r_a)}$ so,

$$\sqrt{C_n(r_a)} = \left(|r_a - r_0|^n + \epsilon^n\right)^{\frac{1}{n}}. \tag{11a}$$

At the surface of the sphere the measured circumference G_a of a great circle is,

$$G_a = 2\pi P_a = 2\pi\sqrt{C_n(r_a)}\,.$$

Therefore, at the surface of the sphere equations (10) and (11a) are equal,

$$\left(|r_a - r_0|^n + \epsilon^n\right)^{\frac{1}{n}} = \sqrt{\frac{3}{\kappa\rho_0}}\, \sin|\chi_a - \chi_0|\,, \tag{11b}$$

and so,

$$|r_a - r_0| = \left[\left(\frac{3}{\kappa\rho_0}\right)^{\frac{n}{2}} \sin^n|\chi_a - \chi_0| - \epsilon^n\right]^{\frac{1}{n}}. \tag{11c}$$

The variable r is just a *parameter* for the radial quantities R_p and R_c associated with (6). Similarly, χ is also a *parameter* for the radial quantities R_p and R_c associated with (4). I remark that r_0 and χ_0 are both *arbitrary*, and *independent* of one another, and that $|r - r_0|$ and $|\chi - \chi_0|$ do not of themselves denote radii in any direct way. The arbitrary values of the parameter "origins", r_0 and χ_0, are simply boundary points on r and χ respectively. Indeed, by (7), $R_p(\chi_0) \equiv 0$, and by (9), $R_p(r_a) \equiv R_{p_a}$, irrespective of the values of r_0, r_a, and χ_0. The centre-of-mass of the sphere of fluid is always located precisely at $R_p(\chi_0) \equiv 0$. Furthermore, $R_p(r)$ for $|r - r_0| < |r_a - r_0|$ has no meaning since inside the sphere (4) describes the geometry, not (6).

According to (11b), equation (9) can be written as,

$$R_p(r) = R_{p_a} + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} -$$

$$-\sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0|\left(\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0| - \alpha\right)} + \tag{12}$$

$$+\alpha\ln\left|\frac{\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}}{\sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0|} + \sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0| - \alpha}}\right|,$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0|\,.$$

Note that in (4), $|\chi - \chi_0|$ cannot grow up to $\frac{\pi}{2}$, so that Schwarzschild's sphere does not constitute the whole spherical space, which has a radius of curvature of $\sqrt{\frac{3}{\kappa\rho_0}}$. From (4) and (6),

$$\frac{\alpha}{P_a} = \sin^2|\chi_a - \chi_0|\,, \quad \alpha = \frac{\kappa\rho_0}{3}P_a^3\,. \tag{13}$$

The volume of the sphere is,

$$V = \left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\int_{\chi_0}^{\chi_a}\sin^2|\chi - \chi_0|\frac{(\chi - \chi_0)}{|\chi - \chi_0|}d\chi \times$$

$$\times\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi =$$

$$= 2\pi\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\left(|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right),$$

so the mass of the sphere is,

$$M = \rho_0 V = \frac{3}{4k^2}\sqrt{\frac{3}{\kappa\rho_0}}\left(|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right).$$

Schwarzschild [5] has also shown that the velocity of light inside his sphere of incompressible fluid is given by,

$$v_c = \frac{2}{3\cos\chi_a - \cos\chi}\,,$$

which generalises to,

$$v_c = \frac{2}{3\cos|\chi_a - \chi_0| - \cos|\chi - \chi_0|}\,. \tag{14}$$

At the centre $\chi = \chi_0$, so v_c reaches a maximum value there of,

$$v_c = \frac{2}{3\cos|\chi_a - \chi_0| - 1}\,,$$

Equation (14) is singular when $\cos|\chi_a - \chi_0| = \frac{1}{3}$, which means that there is a lower bound on the possible radii of curvature for spheres of incompressible, homogeneous fluid, which is, by (13) and (6),

$$P_{a\,(min)} = \frac{9}{8}\alpha = \sqrt{\frac{8}{3\kappa\rho_0}}\,, \tag{15a}$$

and consequently, by equation (11a),

$$|r_a - r_0|_{(min)} = \left[\left(\frac{9\alpha}{8}\right)^n - \epsilon^n\right]^{\frac{1}{n}} = \left[\left(\frac{8}{3\kappa\rho_0}\right)^{\frac{n}{2}} - \epsilon^n\right]^{\frac{1}{n}}, \tag{15b}$$

from which it is clear that a body cannot collapse to a material point.

From (13), a sphere of given gravitational mass $\frac{\alpha}{k^2}$, cannot have a radius of curvature, determined from outside, smaller than,

$$P_{a\,(min)} = \alpha\,,$$

so

$$|r_a - r_0|_{(min)} = [\alpha^n - \epsilon^n]^{\frac{1}{n}}\,,$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0|\,.$$

3 Kepler's 3rd Law for the sphere of incompressible fluid

There is no loss of generality in considering only the equatorial plane, $\theta = \frac{\pi}{2}$. Equation (6) then leads to the Lagrangian,

$$L = \frac{1}{2}\left[\left(\frac{\sqrt{C} - \alpha}{\sqrt{C}}\right)\dot{t}^2 - \left(\frac{\sqrt{C}}{\sqrt{C} - \alpha}\right)\left(\dot{\sqrt{C}}\right)^2 - C\dot{\varphi}^2\right],$$

where the dot indicates $\partial/\partial\tau$.

Let $R = \sqrt{C_n(r)}$. Then,

$$\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = \frac{R}{R-\alpha}\,\ddot{R} + \frac{\alpha}{2R^2}\,\dot{t}^2 - $$
$$- \frac{\alpha}{2\,(R-\alpha)}\,\dot{R}^2 - R\,\dot{\varphi}^2 = 0\,.$$

Now let $R = \text{const}$. Then,

$$\frac{\alpha}{2R^2}\,\dot{t}^2 = R\,\dot{\varphi}^2\,,$$

so

$$\omega^2 = \frac{\alpha}{2R^3} = \frac{\alpha}{2C^{\frac{3}{2}}} = \frac{\alpha}{2\left(\left|r-r_0\right|^n + \epsilon^n\right)^{\frac{3}{n}}}\,. \tag{16}$$

Equation (16) is Kepler's 3rd Law for the sphere of incompressible fluid.

Taking the near-field limit gives,

$$\omega_a^2 = \lim_{\left|r-r_0\right| \to \left|r_a - r_0\right|^+} \omega^2 = \frac{\alpha}{2\left(\left|r_a-r_0\right|^n + \epsilon^n\right)^{\frac{3}{n}}}\,.$$

According to (11b) and (10) this becomes,

$$\omega_a^2 = \frac{\alpha}{2\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\sin^3\left|\chi_a - \chi_0\right|} = \frac{\alpha}{2P_a^3}\,.$$

Finally, using (13),

$$\omega_a = \frac{\sin^3\left|\chi_a - \chi_0\right|}{\alpha\sqrt{2}}\,, \tag{17}$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\,\sin^3\left|\chi_a - \chi_0\right|\,.$$

In contrast, the limiting value of ω for the simple point-mass [4] is,

$$\omega_0 = \frac{1}{\alpha\sqrt{2}}\,,$$

$$\alpha = 2m\,.$$

When P_a is minimum, (17) becomes,

$$\omega_a^2 = \frac{16}{27\alpha}\,, \tag{18}$$

$$\alpha = \frac{16}{27}\sqrt{\frac{6}{\kappa\rho_0}}\,.$$

Clearly, equation (17) is an invariant,

$$\omega_a = \sqrt{\frac{\kappa\rho_0}{6}}\,.$$

4 Passive and active mass

The relationship between passive and active mass manifests, owing to the difference established by Schwazschild, between what he called "substantial mass" (passive mass) and the gravitational (i.e. active) mass. He showed that the former is larger than the latter.

Schwarzschild has shown that the substantial mass M is given by,

$$M = 2\pi\rho_0\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\left(\chi_a - \frac{1}{2}\sin 2\,\chi_a\right),$$
$$0 \leqslant \chi_a < \frac{\pi}{2}\,,$$

and the gravitational mass is,

$$m = \frac{\alpha c^2}{2G} = \frac{1}{2}\sqrt{\frac{3}{\kappa\rho_0}}\,\sin^3\chi_a = \frac{\kappa\rho_0}{6}P_a^3 = \frac{4\pi}{3}P_a^3\rho_0\,,$$

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}}\,\sin\chi_a\,,$$
$$0 \leqslant \chi_a < \frac{\pi}{2}\,.$$

I have generalised Schwarzschild's result to,

$$M = 2\pi\rho_0\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\left(\left|\chi_a - \chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a - \chi_0\right|\right),$$

$$\begin{aligned} m = \frac{\alpha c^2}{2G} = \frac{1}{2}\sqrt{\frac{3}{\kappa\rho_0}}\,\sin^3\left|\chi_a - \chi\rho_0\right| &= \\ = \frac{\kappa\rho_0}{6}P_a^3 = \frac{4\pi}{3}P_a^3\rho_0\,,& \end{aligned} \tag{19}$$

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}}\,\sin\left|\chi_a - \chi_0\right|\,,$$
$$0 \leqslant \left|\chi_a - \chi_0\right| < \frac{\pi}{2}\,,$$

where G is Newton's gravitational constant. Equation (19) is only formally the same as that for the Euclidean sphere, because the radius of curvature P_a is not a Euclidean quantity, and cannot be measured in the gravitational field.

The ratio between the gravitational mass and the substantial mass is,

$$\frac{m}{M} = \frac{2\sin^3\left|\chi_a - \chi_0\right|}{3\left(\left|\chi_a - \chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a - \chi_0\right|\right)}\,.$$

Schwarzschild has shown that the naturally measured fall velocity of a test particle, falling from rest at infinity down to the surface of the sphere of incompressible fluid is,

$$v_a = \sin\chi_a\,,$$

which I generalise to,

$$v_a = \sin\left|\chi_a - \chi_0\right| .$$

The quantity v_a is the escape velocity.

Therefore, as the escape velocity increases, the ratio $\frac{m}{M}$ decreases, owing to the increase in the mass concentration.

In the case of the fictitious point-mass,

$$\lim_{|\chi_a-\chi_0|\to 0}\left(\frac{m}{M}\right) = 1 .$$

However, according to equation (14), for an incompressible sphere of fluid,

$$\cos\left|\chi_a - \chi_0\right|_{min} = \frac{1}{3} ,$$

so

$$\left(\frac{m}{M}\right)_{max} \approx 0.609 .$$

Finally,

$$\text{as } \left|\chi_a - \chi_0\right| \to \frac{\pi}{2},\ \frac{m}{M} \to \frac{4}{3\pi} .$$

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

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