Hydrodynamic Covariant Symplectic Structure from Bilinear Hamiltonian Functions

Salvatore Capozziello*, Salvatore De Martino†, Stephan Tzenov†

*Dipartimento di Scienze Fisiche and INFN sez. di Napoli, Università "Federico II" di Napoli,
Complesso Universitario di Monte S. Angelo Edificio N, via Cinthia 45, I-80126 Napoli, Italy
E-mail: capozziello@na.infn.it

†Dipartimento di Fisica "E.R. Caianiello" and INFN sez. di Napoli, Università di Salerno,
Via S. Allende, I-84081 Baronissi (SA), Italy
E-mail: demartino@sa.infn.it, tzenov@sa.infn.it

Starting from generic bilinear Hamiltonians, constructed by covariant vector, bivector or tensor fields, it is possible to derive a general symplectic structure which leads to holonomic and anholonomic formulations of Hamilton equations of motion directly related to a hydrodynamic picture. This feature is gauge free and it seems a deep link common to all interactions, electromagnetism and gravity included. This scheme could lead toward a full canonical quantization.

1 Introduction

It is well known that a self-consistent quantum field theory of space-time (quantum gravity) has not been achieved, up to now, using standard quantization approaches. Specifically, the request of general coordinate invariance (one of the main features of General Relativity) gives rise to unescapable troubles in understanding the dynamics of gravitational field. In fact, for a physical (non-gravitational) field, one has to assign initially the field amplitudes and their first time derivatives, in order to determine the time development of such a field considered as a dynamical entity. In General Relativity, these quantities are not useful for dynamical determination since the metric field $g_{\alpha\beta}$ can evolve at any time simply by a general coordinate transformation. No change of physical observables is the consequence of such an operation since it is nothing else but a relabelling under which the theory is invariant. This apparent "shortcoming" (from the quantum field theory point of view) means that it is necessary a separation of metric degrees of freedom into a part related to the true dynamical information and a part related only to the coordinate system. From this viewpoint, General Relativity is similar to classical Electromagnetism: the coordinate invariance plays a role analogous to the electromagnetic gauge invariance and in both cases (Lorentz and gauge invariance) introduces redundant variables in order to insure the maintenance of transformation properties. However, difficulties come out as soon as one try to disentangle dynamical from gauge variables. This operation is extremely clear in Electromagnetism while it is not in General Relativity due to its intrinsic non-linearity. A determination of independent dynamical modes of gravitational field can be achieved when the theory is cast into a canonical form involving the minimal number of degrees of freedom which specify the state of the system. The canonical formalism is essential in quantization program since it leads directly to Poisson bracket relations among conjugate variables. In order to realize it in any fundamental theory, one needs first order field equations in time derivatives (Hamilton-like equations) and a $(3+1)$-form of dynamics where time has been unambiguously singled out. In General Relativity, the program has been pursued using the first order Palatini approach [6], where metric $g_{\alpha\beta}$ is taken into account independently of affinity connections $\Gamma^{\gamma}_{\alpha\beta}$ (this fact gives rise to first order field equations) and the so called ADM formalism [7] where $(3+1)$-dimensional notation has led to the definition of gravitational Hamiltonian and time as a conjugate pair of variables. However, the genuine fundament of General Relativity, the covariance of all coordinates without the distinction among space and time, is impaired and, despite of innumerable efforts, the full quantization of gravity has not been achieved up to now. The main problems are related to the lack of a well-definite Hilbert space and a quantum concept of measure for $g_{\alpha\beta}$. An extreme consequence of this lack of full quantization for gravity could be related to the dynamical variables: very likely, the true variables could not be directly related to metric but to something else as, for example, the connection $\Gamma^{\gamma}_{\alpha\beta}$. Despite of this lack, a covariant symplectic structure can be identified also in the framework of General Relativity and then also this theory could be equipped with the same features of other fundamental theories. This statement does not still mean that the identification of a symplectic structure immediately leads to a full quantization but it could be a useful hint toward it.

The aim of this paper is to show that a prominent role in the identification of a covariant symplectic structure is

played by bilinear Hamiltonians which have to be conserved. In fact, taking into account generic Hamiltonian invariants, constructed by covariant vectors, bivectors or tensors, it is possible to show that a symplectic structure can be achieved in any case. By specifying the nature of such vector fields (or, in general, tensor invariants), it gives rise to intrinsically symplectic structure which is always related to Hamilton-like equations (and a Hamilton-Jacobi-like approach is always found). This works for curvature invariants, Maxwell theory and so on. In any case, the only basic assumption is that conservation laws (in Hamiltonian sense) have to be identi-fied in the framework of the theory.

The layout of the paper is the following. In Sec.II, we give the generalities on the symplectic structure and the canonical description of mechanics. Sec.III is devoted to the discussion of symplectic structures which are also generally covariant. We show that a covariant analogue of Hamilton equations can be derived from covariant vector (or tensor) fields in holonomic and anholonomic coordinates. In Sec. IV, the covariant symplectic structure is casted into the hydro-dynamic picture leading to the recovery of the covariant Hamilton equations. Sec.V is devoted to applications, dis-cussion and conclusions.

2 Generalities on the Symplectic Structure and the Ca-nonical description

In order to build every fundamental theory of physics, it is worth selecting the symplectic structure of the manifold on which such a theory is formulated. This goal is achieved if suitable symplectic conjugate variables and even-dimensional vector spaces are chosen. Furthermore, we need an anti-symmetric, covariant tensor which is non-degenerate.

We are dealing with a symplectic structure if the couple

$$\{\mathbf{E_{2n}}, \mathbf{w}\}, \tag{1}$$

is defined, where $\mathbf{E_{2n}}$ is a vector space and the tensor $\mathbf{w}$ on $\mathbf{E_{2n}}$ associates scalar functions to pairs of vectors, that is

$$[\mathbf{x}, \mathbf{y}] = \mathbf{w}(\mathbf{x}, \mathbf{y}), \tag{2}$$

which is the *antiscalar* product. Such an operation satisfies the following properties

$$[\mathbf{x}, \mathbf{y}] = -[\mathbf{y}, \mathbf{x}] \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbf{E_{2n}} \tag{3}$$

$$[\mathbf{x}, \mathbf{y} + \mathbf{z}] = [\mathbf{x}, \mathbf{y}] + [\mathbf{xz}] \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{E_{2n}}, \tag{4}$$

$$a[\mathbf{x}, \mathbf{y}] = [a\mathbf{x}, \mathbf{y}] \qquad \forall a \in R, \ \mathbf{x}, \mathbf{y} \in \mathbf{E_{2n}} \tag{5}$$

$$[\mathbf{x}, \mathbf{y}] = 0 \qquad \forall y \in \mathbf{E_{2n}} \ \Rightarrow \ \mathbf{x} = 0 \tag{6}$$

$$[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] + [\mathbf{y}, [\mathbf{z}, \mathbf{x}]] + [\mathbf{z}, [\mathbf{x}, \mathbf{y}]] = 0. \tag{7}$$

The last one is the Jacobi cyclic identity.

If $\{\mathbf{e}_i\}$ is a vector basis in $\mathbf{E_{2n}}$, the antiscalar product is completely singled out by the matrix elements

$$w_{ij} = [\mathbf{e}_i, \mathbf{e}_j], \tag{8}$$

where $\mathbf{w}$ is an antisymmetric matrix with determinant differ-ent from zero. Every antiscalar product between two vectors can be expressed as

$$[\mathbf{x}, \mathbf{y}] = w_{ij}\, x^i y^j, \tag{9}$$

where x^i and y^j are the vector components in the given basis.

The form of the matrix $\mathbf{w}$ and the relation (9) become considerably simpler if a canonical basis is taken into account for $\mathbf{w}$. Since $\mathbf{w}$ is an antisymmetric non-degenerate tensor, it is always possible to represent it through the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{10}$$

where I is a $(n \times n)$ unit matrix. Every basis where $\mathbf{w}$ can be represented through the form (10) is a *symplectic basis*. In other words, the symplectic bases are the canonical bases for any antisymmetric non-degenerate tensor $\mathbf{w}$ and can be characterized by the following conditions:

$$[\mathbf{e}_i, \mathbf{e}_j] = 0, \quad [\mathbf{e}_{n+i}, \mathbf{e}_{n+j}] = 0, \quad [\mathbf{e}_i, \mathbf{e}_{n+j}] = \delta_{ij}, \tag{11}$$

which have to be verified for every pair of values i and j ranging from 1 to n.

Finally, the expression of the antiscalar product between two vectors, in a symplectic basis, is

$$[\mathbf{x}, \mathbf{y}] = \sum_{i=1}^{n} \left(x^{n+i} y^i - x^i y^{n+i} \right), \tag{12}$$

and a symplectic transformation in $\mathbf{E_{2n}}$ leaves invariant the antiscalar product

$$\mathbf{S}[\mathbf{x}, \mathbf{y}] = [\mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{y})] = [x, y]. \tag{13}$$

It is easy to see that standard Quantum Mechanics satis-fies such properties and so it is endowed with a symplectic structure.

On the other hand a standard canonical description can be sketched as follows. For example, the relativistic Lagrangian of a charged particle interacting with a vector field $A(q; s)$ is

$$\mathcal{L}(q, u; s) = \frac{mu^2}{2} - eu \cdot A(q; s), \tag{14}$$

where the scalar product is defined as

$$z \cdot w = z_\mu w^\mu = \eta_{\mu\nu} z^\mu w^\nu, \tag{15}$$

and the signature of the Minkowski spacetime is the usual one with

$$z_\mu = \eta_{\mu\nu} z^\nu, \qquad \widehat{\eta} = \operatorname{diag}(1, -1, -1, -1). \tag{16}$$

Furthermore, the contravariant vector u^μ with components $u = (u^0, u^1, u^2, u^3)$ is the four-velocity

$$u^\mu = \frac{dq^\mu}{ds}. \qquad (17)$$

The canonical conjugate momentum π^μ is defined as

$$\pi^\mu = \eta^{\mu\nu} \frac{\partial \mathcal{L}}{\partial u^\nu} = m u^\mu - e A^\mu, \qquad (18)$$

so that the relativistic Hamiltonian can be written in the form

$$\mathcal{H}(q, \pi; s) = \pi \cdot u - \mathcal{L}(q, u; s). \qquad (19)$$

Suppose now that we wish to use any other coordinate system x^α as Cartesian, curvilinear, accelerated or rotating one. Then the coordinates q^μ are functions of the x^α, which can be written explicitly as

$$q^\mu = q^\mu(x^\alpha). \qquad (20)$$

The four-vector of particle velocity u^μ is transformed according to the expression

$$u^\mu = \frac{\partial q^\mu}{\partial x^\alpha} \frac{dx^\alpha}{ds} = \frac{\partial q^\mu}{\partial x^\alpha} v^\alpha, \qquad (21)$$

where

$$v^\mu = \frac{dx^\mu}{ds}. \qquad (22)$$

is the transformed four-velocity expressed in terms of the new coordinates. The vector field A^μ is also transformed as a vector

$$\mathcal{A}^\mu = \frac{\partial x^\mu}{\partial q^\alpha} A^\alpha. \qquad (23)$$

In the new coordinate system x^α the Lagrangian (14) becomes

$$\mathcal{L}(x, v; s) = g_{\mu\nu} \left[\frac{m}{2} v^\mu v^\nu - e v^\mu \mathcal{A}^\nu(x; s) \right], \qquad (24)$$

where

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial q^\mu}{\partial x^\alpha} \frac{\partial q^\nu}{\partial x^\beta}. \qquad (25)$$

The Lagrange equations can be written in the usual form

$$\frac{d}{ds} \left(\frac{\partial \mathcal{L}}{\partial v^\lambda} \right) - \frac{\partial \mathcal{L}}{\partial x^\lambda} = 0. \qquad (26)$$

In the case of a free particle (no interaction with an external vector field), we have

$$\frac{d}{ds} (g_{\lambda\mu} v^\mu) - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} v^\mu v^\nu = 0. \qquad (27)$$

Specifying the covariant velocity v_λ as

$$v_\lambda = g_{\lambda\mu} v^\mu, \qquad (28)$$

and using the well-known identity for connections $\Gamma^\alpha_{\mu\nu}$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} + \Gamma^\alpha_{\lambda\nu} g_{\alpha\mu}, \qquad (29)$$

we obtain

$$\frac{D v_\lambda}{Ds} = \frac{d v_\lambda}{ds} - \Gamma^\mu_{\lambda\nu} v^\nu v_\mu = 0. \qquad (30)$$

Here $D v_\lambda / Ds$ denotes the covariant derivative of the covariant velocity v_λ along the curve $x^\nu(s)$. Using Eqs. (28) and (29) and the fact that the affine connection $\Gamma^\lambda_{\mu\nu}$ is symmetric in the indices μ and ν, we obtain the equation of motion for the contravariant vector v^λ

$$\frac{D v^\lambda}{Ds} = \frac{d v^\lambda}{ds} + \Gamma^\lambda_{\mu\nu} v^\mu v^\nu = 0. \qquad (31)$$

Before we pass over to the Hamiltonian description, let us note that the generalized momentum p_μ is defined as

$$p_\mu = \frac{\partial \mathcal{L}}{\partial v^\mu} = m g_{\mu\nu} v^\nu, \qquad (32)$$

while, from Lagrange equations of motion, we obtain

$$\frac{d p_\mu}{ds} = \frac{\partial \mathcal{L}}{\partial x^\mu}. \qquad (33)$$

The transformation from $(x^\mu, v^\mu; s)$ to $(x^\mu, p_\mu; s)$ can be accomplished by means of a Legendre transformation, and instead of the Lagrangian (24), we consider the Hamilton function

$$\mathcal{H}(x, p; s) = p_\mu v^\mu - \mathcal{L}(x, v; s). \qquad (34)$$

The differential of the Hamiltonian in terms of x, p and s is given by

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial x^\mu} dx^\mu + \frac{\partial \mathcal{H}}{\partial p_\mu} dp_\mu + \frac{\partial \mathcal{H}}{\partial s} ds. \qquad (35)$$

On the other hand, from Eq.(34), we have

$$d\mathcal{H} = v^\mu dp_\mu + p_\mu dv^\mu - \frac{\partial \mathcal{L}}{\partial v^\mu} dv^\mu - \frac{\partial \mathcal{L}}{\partial x^\mu} dx^\mu - \frac{\partial \mathcal{L}}{\partial s} ds. \qquad (36)$$

Taking into account the defining Eq.(32), the second and the third term on the right-hand-side of Eq.(36) cancel out. Eq.(33) can be further used to cast Eq.(36) into the form

$$d\mathcal{H} = v^\mu dp_\mu - \frac{d p_\mu}{ds} dx^\mu - \frac{\partial \mathcal{L}}{\partial s} ds, \qquad (37)$$

Comparison between Eqs.(35) and (37) yields the Hamilton equations of motion

$$\frac{dx^\mu}{ds} = \frac{\partial \mathcal{H}}{\partial p_\mu}, \qquad\qquad \frac{d p_\mu}{ds} = -\frac{\partial \mathcal{H}}{\partial x^\mu}, \qquad (38)$$

where the Hamiltonian is given by

$$\mathcal{H}(x, p; s) = \frac{g^{\mu\nu}}{2m} p_\mu p_\nu + \frac{e}{m} p_\mu \mathcal{A}^\mu. \qquad (39)$$

In the case of a free particle, the Hamilton equations can be written explicitly as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{g^{\mu\nu}}{m} p_\nu, \qquad \frac{\mathrm{d}p_\lambda}{\mathrm{d}s} = -\frac{1}{2m} \frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu p_\nu. \tag{40}$$

To obtain the equations of motion we need the expression

$$\frac{\partial g^{\mu\nu}}{\partial x^\lambda} = -\Gamma^\mu_{\lambda\alpha} g^{\alpha\nu} - \Gamma^\nu_{\lambda\alpha} g^{\alpha\mu}, \tag{41}$$

which can be derived from the obvious identity

$$\frac{\partial}{\partial x^\lambda}(g^{\mu\alpha} g_{\alpha\nu}) = 0, \tag{42}$$

and Eq.(29). From the second of Eqs. (40), we obtain

$$\frac{\mathrm{D}p_\lambda}{\mathrm{D}s} = \frac{\mathrm{d}p_\lambda}{\mathrm{d}s} - \Gamma^\mu_{\lambda\nu} v^\nu p_\mu = 0, \tag{43}$$

similar to equation (30). Differentiating the first of the Hamilton equations (40) with respect to s and taking into account equations (41) and (43), we again arrive to the equation for the geodesics (31).

Let us now show that on a generic curved (torsion-free) manifolds the Poisson brackets are conserved. To achieve this result, we need the following identities

$$g^{\mu\nu} = g^{\nu\mu} = \eta^{\alpha\beta} \frac{\partial x^\mu}{\partial q^\alpha} \frac{\partial x^\nu}{\partial q^\beta}, \tag{44}$$

$$\frac{\partial^2 x^\lambda}{\partial q^\alpha \partial q^\beta} = -\Gamma^\lambda_{\mu\nu} \frac{\partial x^\mu}{\partial q^\alpha} \frac{\partial x^\nu}{\partial q^\beta}, \tag{45}$$

■ To prove (45), we differentiate the obvious identity

$$\frac{\partial x^\lambda}{\partial q^\rho} \frac{\partial q^\rho}{\partial x^\nu} = \delta^\lambda_\nu. \tag{46}$$

As a result, we find

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial q^\rho} \frac{\partial^2 q^\rho}{\partial x^\mu \partial x^\nu} = -\frac{\partial q^\rho}{\partial x^\nu} \frac{\partial q^\sigma}{\partial x^\mu} \frac{\partial^2 x^\lambda}{\partial q^\rho \partial q^\sigma}. \blacksquare \tag{47}$$

The next step is to calculate the fundamental Poisson brackets in terms of the variables (x^μ, p_ν), initially defined using the canonical variables (q^μ, π_ν) according to the relation

$$[U, V] = \frac{\partial U}{\partial q^\mu} \frac{\partial V}{\partial \pi_\mu} - \frac{\partial V}{\partial q^\mu} \frac{\partial U}{\partial \pi_\mu}, \tag{48}$$

where $U(q^\mu, \pi_\nu)$ and $V(q^\mu, \pi_\nu)$ are arbitrary functions. Making use of Eqs.(18) and (21), we know that the variables

$$q^\mu \quad \Leftrightarrow \quad \pi_\mu = mu_\mu = m\eta_{\mu\nu} u^\nu = m\eta_{\mu\nu} \frac{\partial q^\nu}{\partial x^\alpha} v^\alpha, \tag{49}$$

form a canonical conjugate pair. Using Eq.(32), we would like to check whether the variables

$$x^\mu \quad \Leftrightarrow \quad p_\mu = mg_{\mu\nu} v^\nu = g_{\mu\nu} \eta^{\alpha\lambda} \pi_\lambda \frac{\partial x^\nu}{\partial q^\alpha}, \tag{50}$$

form a canonical conjugate pair. We have

$$[U, V] = \left[\frac{\partial U}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial q^\mu} + \frac{\partial U}{\partial p_\sigma} \eta^{\beta\lambda} \pi_\lambda \frac{\partial}{\partial q^\mu} \left(g_{\sigma\nu} \frac{\partial x^\nu}{\partial q^\beta} \right) \right] \times$$
$$\times \frac{\partial V}{\partial p_\alpha} g_{\alpha\chi} \eta^{\rho\mu} \frac{\partial x^\chi}{\partial q^\rho} -$$
$$- \left[\frac{\partial V}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial q^\mu} + \frac{\partial V}{\partial p_\sigma} \eta^{\beta\lambda} \pi_\lambda \frac{\partial}{\partial q^\mu} \left(g_{\sigma\nu} \frac{\partial x^\nu}{\partial q^\beta} \right) \right] \times$$
$$\times \frac{\partial U}{\partial p_\alpha} g_{\alpha\chi} \eta^{\rho\mu} \frac{\partial x^\chi}{\partial q^\rho}. \tag{51}$$

The first and the third term on the right-hand-side of Eq.(51) can be similarly manipulated as follows

$$\textbf{I-st term} = \frac{\partial U}{\partial x^\alpha} \frac{\partial V}{\partial p_\beta} g_{\beta\chi} \eta^{\rho\mu} \frac{\partial x^\chi}{\partial q^\rho} \frac{\partial x^\alpha}{\partial q^\mu} =$$
$$= g_{\beta\chi} g^{\chi\alpha} \frac{\partial U}{\partial x^\alpha} \frac{\partial V}{\partial p_\beta} = \frac{\partial U}{\partial x^\alpha} \frac{\partial V}{\partial p_\alpha}, \tag{52}$$

$$\textbf{III-rd term} = -\frac{\partial V}{\partial x^\alpha} \frac{\partial U}{\partial p_\alpha}. \tag{53}$$

Next, we manipulate the second term on the right-hand-side of Eq.(51). We obtain

$$\textbf{II-nd term} = \frac{\partial U}{\partial p_\sigma} \frac{\partial V}{\partial p_\alpha} g_{\alpha\chi} \eta^{\rho\mu} \frac{\partial x^\chi}{\partial q^\rho} \eta^{\beta\lambda} \pi_\lambda \times$$
$$\times \left[g_{\sigma\nu} \frac{\partial^2 x^\nu}{\partial q^\mu \partial q^\beta} + \frac{\partial x^\nu}{\partial q^\beta} \frac{\partial g_{\sigma\nu}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial q^\mu} \right] =$$
$$= \frac{\partial U}{\partial p_\sigma} \frac{\partial V}{\partial p_\alpha} g_{\alpha\chi} \eta^{\rho\mu} \frac{\partial x^\chi}{\partial q^\rho} \eta^{\beta\lambda} \pi_\lambda \times$$
$$\times \left[-g_{\sigma\nu} \Gamma^\nu_{\gamma\delta} \frac{\partial x^\gamma}{\partial q^\mu} \frac{\partial x^\delta}{\partial q^\beta} + \frac{\partial x^\delta}{\partial q^\beta} \frac{\partial x^\gamma}{\partial q^\mu} (\Gamma^\nu_{\gamma\sigma} g_{\nu\delta} + \Gamma^\nu_{\gamma\delta} g_{\nu\sigma}) \right] =$$
$$= \frac{\partial U}{\partial p_\sigma} \frac{\partial V}{\partial p_\alpha} g_{\alpha\chi} g^{\chi\gamma} \eta^{\beta\lambda} \pi_\lambda \frac{\partial x^\delta}{\partial q^\beta} g_{\nu\delta} \Gamma^\nu_{\gamma\sigma} =$$
$$= \frac{\partial U}{\partial p_\sigma} \frac{\partial V}{\partial p_\beta} g_{\mu\nu} \eta^{\alpha\lambda} \pi_\lambda \frac{\partial x^\nu}{\partial q^\alpha} \Gamma^\mu_{\beta\sigma} = \Gamma^\lambda_{\mu\nu} p_\lambda \frac{\partial U}{\partial p_\nu} \frac{\partial V}{\partial p_\mu}. \tag{54}$$

The fourth term is similar to the second one but with U and V interchanged

$$\textbf{IV-th term} = -\Gamma^\lambda_{\mu\nu} p_\lambda \frac{\partial U}{\partial p_\mu} \frac{\partial V}{\partial p_\nu}. \tag{55}$$

In the absence of torsion, the affine connection $\Gamma^\lambda_{\mu\nu}$ is symmetric with respect to the lower indices, so that the second and the fourth term on the right-hand-side of Eq.(51) cancel each other. Therefore,

$$[U, V] = \frac{\partial U}{\partial x^\mu} \frac{\partial V}{\partial p_\mu} - \frac{\partial V}{\partial x^\mu} \frac{\partial U}{\partial p_\mu}, \tag{56}$$

which means that the fundamental Poisson brackets are conserved. On the other hand, this implies that the variables $\{x^\mu, p_\nu\}$ are a canonical conjugate pair.

As a final remark, we have to say that considering a generic metric $g_{\alpha\beta}$ and a connection $\Gamma^{\alpha}_{\mu\nu}$ is related to the fact that we are passing from a Minkowski-flat spacetime (local inertial reference frame) to an accelerated reference frame (curved spacetime). In what follows, we want to show that a generic bilinear Hamiltonian invariant, which is conformally conserved, gives always rise to a canonical symplectic structure. The specific theory is assigned by the vector (or tensor) fields which define the Hamiltonian invariant.

3 A symplectic structure compatible with general covariance

The above considerations can be linked together leading to a more general scheme where a covariant symplectic structure is achieved. Summarizing, the main points which we need are: (i) an even-dimensional vector space $\mathbf{E_{2n}}$ equipped with an antiscalar product satisfying the algebra (3)-(7); (ii) generic vector fields defined on such a space which have to satisfy the Poisson brackets; (iii) first-order equations of motion which can be read as Hamilton-like equations; (iv) general covariance which has to be preserved.

Such a program can be pursued by taking into account covariant and contravariant vector fields. In fact, it is possible to construct the Hamiltonian invariant

$$\mathcal{H} = V^{\alpha}V_{\alpha}\,, \tag{57}$$

which is a scalar quantity satisfying the relation

$$\delta\mathcal{H} = \delta(V^{\alpha}V_{\alpha}) = 0\,, \tag{58}$$

being δ a spurious variation due to the transport. It is worth stressing that the vectors V^{α} and V_{α} are not specified and the following considerations are completely general. Eq.(57) is a so called "*already parameterized*" invariant which can constitute the "density" of a parameterized action principle where the time coordinate is not distinguished *a priori* from the other coordinates [8, 9].

Let us now take into account the intrinsic variation of V^{α}. On a generic curved manifold, we have

$$DV^{\alpha} = dV^{\alpha} - \delta V^{\alpha} = \partial_{\beta}V^{\alpha}dx^{\beta} - \delta V^{\alpha}, \tag{59}$$

where D is the intrinsic variation, d the total variation and δ the spurious variation due to the transport on the curved manifold. The spurious variation has a very important meaning since, in General Relativity, if such a variation for a given quantity is equal to zero, this means that the quantity is conserved. From the definition of covariant derivative, applied to the contravariant vector, we have

$$DV^{\alpha} = \partial_{\beta}V^{\alpha}dx^{\beta} + \Gamma^{\alpha}_{\sigma\beta}V^{\sigma}dx^{\beta}, \tag{60}$$

and

$$\nabla_{\beta}V^{\alpha} = \partial_{\beta}V^{\alpha} + \Gamma^{\alpha}_{\sigma\beta}V^{\sigma}, \tag{61}$$

and then

$$\delta V^{\alpha} = -\Gamma^{\alpha}_{\sigma\beta}V^{\sigma}dx^{\beta}. \tag{62}$$

Analogously, for the covariant derivative applied to the covariant vector,

$$DV_{\alpha} = dV_{\alpha} - \delta V_{\alpha} = \partial_{\beta}V_{\alpha}dx^{\beta} - \delta V_{\alpha}\,, \tag{63}$$

and then

$$DV_{\alpha} = \partial_{\beta}V_{\alpha}dx^{\beta} - \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}dx^{\beta}, \tag{64}$$

and

$$\nabla_{\beta}V_{\alpha} = \partial_{\beta}V_{\alpha} - \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}\,. \tag{65}$$

The spurious variation is now

$$\delta V_{\alpha} = \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}dx^{\beta}. \tag{66}$$

Developing the variation (58), we have

$$\delta\mathcal{H} = V_{\alpha}\delta V^{\alpha} + V^{\alpha}\delta V_{\alpha}\,, \tag{67}$$

and

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = V_{\alpha}\frac{\delta V^{\alpha}}{dx^{\beta}} + V^{\alpha}\frac{\delta V_{\alpha}}{dx^{\beta}}, \tag{68}$$

which becomes

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = \frac{\delta V^{\alpha}}{dx^{\beta}}\frac{\partial\mathcal{H}}{\partial V^{\alpha}} + \frac{\delta V_{\alpha}}{dx^{\beta}}\frac{\partial\mathcal{H}}{\partial V_{\alpha}}, \tag{69}$$

being

$$\frac{\partial\mathcal{H}}{\partial V^{\alpha}} = V_{\alpha}, \qquad \frac{\partial\mathcal{H}}{\partial V_{\alpha}} = V^{\alpha}. \tag{70}$$

From Eqs.(62) and (66), it is

$$\frac{\delta V^{\alpha}}{dx^{\beta}} = -\Gamma^{\alpha}_{\sigma\beta}V^{\sigma} = -\Gamma^{\alpha}_{\sigma\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right), \tag{71}$$

$$\frac{\delta V_{\alpha}}{dx^{\beta}} = \Gamma^{\sigma}_{\alpha\beta}V_{\sigma} = \Gamma^{\sigma}_{\alpha\beta}\left(\frac{\partial\mathcal{H}}{\partial V^{\sigma}}\right), \tag{72}$$

and substituting into Eq.(69), we have

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = -\Gamma^{\alpha}_{\sigma\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\alpha}}\right) + \Gamma^{\sigma}_{\alpha\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\alpha}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\sigma}}\right), \tag{73}$$

and then, since α and σ are mute indexes, the expression

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = \left(\Gamma^{\alpha}_{\sigma\beta} - \Gamma^{\alpha}_{\sigma\beta}\right)\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\alpha}}\right) \equiv 0\,, \tag{74}$$

is identically equal to zero. In other words, $\mathcal{H}$ is absolutely conserved, and this is very important since the analogy with a canonical Hamiltonian structure is straightforward. In fact, if, as above,

$$\mathcal{H} = \mathcal{H}(p, q) \tag{75}$$

is a classical generic Hamiltonian function, expressed in the canonical phase-space variables $\{p, q\}$, the total variation (in

a vector space $\mathbf{E_{2n}}$ whose dimensions are generically given by p_i and q_j with $i, j = 1, ..., n$) is

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q} dq + \frac{\partial \mathcal{H}}{\partial p} dp, \qquad (76)$$

and

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial q} \dot{q} + \frac{\partial \mathcal{H}}{\partial p} \dot{p} =$$

$$= \frac{\partial \mathcal{H}}{\partial q} \frac{\partial \mathcal{H}}{\partial p} - \frac{\partial \mathcal{H}}{\partial p} \frac{\partial \mathcal{H}}{\partial q} \equiv 0, \qquad (77)$$

thanks to the Hamilton canonical equations

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}. \qquad (78)$$

Such a canonical approach holds also in our covariant case if we operate the substitutions

$$V^\alpha \longleftrightarrow p \qquad V_\alpha \longleftrightarrow q \qquad (79)$$

and the canonical equations are

$$\frac{\delta V^\alpha}{dx^\beta} = -\Gamma^\alpha_{\sigma\beta} \left(\frac{\partial \mathcal{H}}{\partial V_\sigma} \right) \quad \longleftrightarrow \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}, \qquad (80)$$

$$\frac{\delta V_\alpha}{dx^\beta} = \Gamma^\sigma_{\alpha\beta} \left(\frac{\partial \mathcal{H}}{\partial V^\sigma} \right) \quad \longleftrightarrow \quad \frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}. \qquad (81)$$

In other words, starting from the (Hamiltonian) invariant (57), we have recovered a covariant canonical symplectic structure. The variation (67) may be seen as the generating function $\mathcal{G}$ of canonical transformations where the generators of $q-$, $p-$ and $t-$changes are dealt under the same standard.

At this point, some important remarks have to be done. The covariant and contravariant vector fields can be also of different nature so that the above fundamental Hamiltonian invariant can be generalized as

$$\mathcal{H} = W^\alpha V_\alpha, \qquad (82)$$

or, considering scalar smooth and regular functions, as

$$\mathcal{H} = f(W^\alpha V_\alpha), \qquad (83)$$

or, in general

$$\mathcal{H} = f\left(W^\alpha V_\alpha, B^{\alpha\beta} C_{\alpha\beta}, B^{\alpha\beta} V_\alpha V'_\beta, \dots \right), \qquad (84)$$

where the invariant can be constructed by covariant vectors, bivectors and tensors. Clearly, as above, the identifications

$$W^\alpha \longleftrightarrow p \qquad V_\alpha \longleftrightarrow q \qquad (85)$$

hold and the canonical equations are

$$\frac{\delta W^\alpha}{dx^\beta} = -\Gamma^\alpha_{\sigma\beta} \left(\frac{\partial \mathcal{H}}{\partial V_\sigma} \right), \quad \frac{\delta V_\alpha}{dx^\beta} = \Gamma^\sigma_{\alpha\beta} \left(\frac{\partial \mathcal{H}}{\partial W^\sigma} \right). \qquad (86)$$

Finally, conservation laws are given by

$$\frac{\delta \mathcal{H}}{dx^\beta} = \left(\Gamma^\alpha_{\sigma\beta} - \Gamma^\alpha_{\sigma\beta} \right) \left(\frac{\partial \mathcal{H}}{\partial V_\sigma} \right) \left(\frac{\partial \mathcal{H}}{\partial W^\alpha} \right) \equiv 0. \qquad (87)$$

In our picture, this means that the canonical symplectic structure is assigned in the way in which covariant and contravariant vector fields are related. However, if the Hamiltonian invariant is constructed by bivectors and tensors, equations (86) and (87) have to be generalized but the structure is the same. It is worth noticing that we never used the metric field but only connections in our derivations.

These considerations can be made independent of the reference frame if we define a suitable system of unitary vectors by which we can pass from holonomic to anholonomic description and viceversa. We can define the reference frame on the event manifold $\mathcal{M}$ as vector fields $e_{(k)}$ in event space and dual forms $e^{(k)}$ such that vector fields $e_{(k)}$ define an orthogonal frame at each point and

$$e^{(k)}\left(e_{(l)} \right) = \delta^{(k)}_{(l)}. \qquad (88)$$

If these vectors are unitary, in a Riemannian 4-spacetime are the standard *vierbiens* [5].

If we do not limit this definition of reference frame by orthogonality, we can introduce a *coordinate reference frame* $(\partial_\alpha, ds^\alpha)$ based on vector fields tangent to line $x^\alpha = $ const. Both reference frames are linked by the relations

$$e_{(k)} = e^\alpha_{(k)} \partial_\alpha; \qquad e^{(k)} = e^{(k)}_\alpha dx^\alpha. \qquad (89)$$

From now on, Greek indices will indicate holonomic coordinates while Latin indices between brackets, the anholonomic coordinates (*vierbien* indices in 4-spacetimes). We can prove the existence of a reference frame using the orthogonalization procedure at every point of spacetime. From the same procedure, we get that coordinates of frame smoothly depend on the point. The statement about the existence of a global reference frame follows from this. A smooth field on time-like vectors of each frame defines congruence of lines that are tangent to this field. We say that each line is a world line of an observer or a *local reference frame*. Therefore a reference frame is a set of local reference frames. The *Lorentz transformation* can be defined as a transformation of a reference frame

$$x'^\alpha = f\left(x^0, x^1, x^2, x^3, \dots, x^n \right), \qquad (90)$$

$$e'^\alpha_{(k)} = A^\alpha_\beta B^{(l)}_{(k)} e^\beta_{(l)}, \qquad (91)$$

where

$$A^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x'^\beta}, \qquad \delta_{(i)(l)} B^{(i)}_{(j)} B^{(l)}_{(k)} = \delta_{(j)(k)}. \qquad (92)$$

We call the transformation A^α_β the holonomic part and transformation $B^{(l)}_{(k)}$ the anholonomic part.

A vector field V has two types of coordinates: *holonomic coordinates* V^{α} relative to a coordinate reference frame and *anholonomic coordinates* $V^{(k)}$ relative to a reference frame. For these two kinds of coordinates, the relation

$$V^{(k)} = e_{\alpha}^{(k)} V^{\alpha}, \tag{93}$$

holds. We can study parallel transport of vector fields using any form of coordinates. Because equations (90) and (91) are linear transformations, we expect that parallel transport in anholonomic coordinates has the same form as in holonomic coordinates. Hence we write

$$DV^{\alpha} = dV^{\alpha} + \Gamma_{\beta\gamma}^{\alpha} V^{\beta} dx^{\gamma}, \tag{94}$$

$$DV^{(k)} = dV^{(k)} + \Gamma_{(l)(p)}^{(k)} V^{(l)} dx^{(p)}. \tag{95}$$

Because DV^{α} is also a tensor, we get

$$\Gamma_{(l)(p)}^{(k)} = e_{(l)}^{\alpha} e_{(p)}^{\beta} e_{\gamma}^{(k)} \Gamma_{\alpha\beta}^{\gamma} + e_{(l)}^{\alpha} e_{(p)}^{\beta} \frac{\partial e_{\alpha}^{(k)}}{\partial x^{\beta}}. \tag{96}$$

Eq.(96) shows the similarity between holonomic and anholonomic coordinates. Let us introduce the symbol $\partial_{(k)}$ for the derivative along the vector field $e_{(k)}$

$$\partial_{(k)} = e_{(k)}^{\alpha} \partial_{\alpha}. \tag{97}$$

Then Eq.(96) takes the form

$$\Gamma_{(l)(p)}^{(k)} = e_{(l)}^{\alpha} e_{(p)}^{\beta} e_{\gamma}^{(k)} \Gamma_{\alpha\beta}^{\gamma} + e_{(l)}^{\alpha} \partial_{(p)} e_{\alpha}^{(k)}. \tag{98}$$

Therefore, when we move from holonomic coordinates to anholonomic ones, the connection also transforms the way similarly to when we move from one coordinate system to another. This leads us to the model of anholonomic coordinates. The vector field $e_{(k)}$ generates lines defined by the differential equations

$$e_{(l)}^{\alpha} \frac{\partial \tau}{\partial x^{\alpha}} = \delta_{(l)}^{(k)}, \tag{99}$$

or the symbolic system

$$\frac{\partial \tau}{\partial x^{(l)}} = \delta_{(l)}^{(k)}. \tag{100}$$

Keeping in mind the symbolic system (100), we denote the functional τ as $x^{(k)}$ and call it the anholonomic coordinate. We call the regular coordinate holonomic. Then we can find derivatives and get

$$\frac{\partial x^{(k)}}{\partial x^{\alpha}} = \delta_{\alpha}^{(k)}. \tag{101}$$

The necessary and sufficient conditions to complete the integrability of system (101) are

$$\omega_{(k)(l)}^{(i)} = e_{(k)}^{\alpha} e_{(l)}^{\beta} \left(\frac{\partial e_{\alpha}^{(i)}}{\partial x^{\beta}} - \frac{\partial e_{\beta}^{(i)}}{\partial x^{\alpha}} \right) = 0, \tag{102}$$

where we introduced the anholonomic object $\omega_{(k)(l)}^{(i)}$.

Therefore each reference frame has n vector fields

$$\partial_{(k)} = \frac{\partial}{\partial x^{(k)}} = e_{(k)}^{\alpha} \partial_{\alpha}, \tag{103}$$

which have the commutators

$$\begin{aligned}
\left[\partial_{(i)}, \partial_{(j)} \right] &= \left(e_{(i)}^{\alpha} \partial_{\alpha} e_{(j)}^{\beta} - e_{(j)}^{\alpha} \partial_{\alpha} e_{(i)}^{\beta} \right) e_{\beta}^{(m)} \partial_{(m)} = \\
&= e_{(i)}^{\alpha} e_{(j)}^{\beta} \left(-\partial_{\alpha} e_{\beta}^{(m)} + \partial_{\beta} e_{\beta}^{(m)} \right) \partial_{(m)} = \omega_{(i)(j)}^{(m)} \partial_{(m)}. \tag{104}
\end{aligned}$$

For the same reason, we introduce the forms

$$dx^{(k)} = e^{(k)} = e_{\beta}^{(k)} dx^{\beta}, \tag{105}$$

and a differential of this form is

$$\begin{aligned}
d^2 x^{(k)} &= d\left(e_{\alpha}^{(k)} dx^{\alpha} \right) = \left(\partial_{\beta} e_{\alpha}^{(k)} - \partial_{\alpha} e_{\beta}^{(k)} \right) dx^{\alpha} \wedge dx^{\beta} = \\
&= -\omega_{(k)(l)}^{(m)} dx^{(k)} \wedge dx^{(l)}. \tag{106}
\end{aligned}$$

Therefore when $\omega_{(k)(l)}^{(i)} \neq 0$, the differential $dx^{(k)}$ is not an exact differential and the system (101), in general, cannot be integrated. However, we can consider meaningful objects which model the solution. We can study how the functions $x^{(i)}$ changes along different lines. The functions $x^{(i)}$ is a natural parameter along a flow line of vector fields $e_{(i)}$. It is defined along any line.

All the above results can be immediately achieved in holonomic and anholonomic formalism considering the equation

$$\mathcal{H} = W^{\alpha} V_{\alpha} = W^{(k)} V_{(k)}, \tag{107}$$

and the analogous ones. This means that the results are independent of the reference frame and the symplectic covariant structure always holds.

4 The hydrodynamic picture

In order to further check the validity of the above approach, we can prove that it is always consistent with the hydrodynamic picture (see also [10] for details on hydrodynamic covariant formalism).

Let us define a phase space density $f(x, p; s)$ which evolves according to the Liouville equation

$$\frac{\partial f}{\partial s} + \frac{1}{m} \frac{\partial}{\partial x^{\mu}} (g^{\mu\nu} p_{\nu} f) - \frac{1}{2m} \frac{\partial}{\partial p_{\lambda}} \left(\frac{\partial g^{\mu\nu}}{\partial x^{\lambda}} p_{\mu} p_{\nu} f \right) = 0. \tag{108}$$

Next we define the density $\varrho(x; s)$, the covariant current velocity $v_{\mu}(x; s)$ and the covariant stress tensor $\mathcal{P}_{\mu\nu}(x; s)$ according to the relations

$$\varrho(x; s) = mn \int d^4 p f(x, p; s), \tag{109}$$

$$\varrho(x;s)v_\mu(x;s) = n \int \mathrm{d}^4 p\, p_\mu f(x,p;s)\,, \qquad (110)$$

$$\mathcal{P}_{\mu\nu}(x;s) = \frac{n}{m} \int \mathrm{d}^4 p\, p_\mu p_\nu f(x,p;s)\,. \qquad (111)$$

It can be verified, by direct substitution, that a solution to the Liouville Eq.(108) of the form

$$f(x,p;s) = \frac{1}{mn}\varrho(x;s)\delta^4[p_\mu - mv_\mu(x;s)]\,, \qquad (112)$$

leads to the equation of continuity

$$\frac{\partial\varrho}{\partial s} + \frac{\partial}{\partial x^\mu}(g^{\mu\nu}v_\nu\varrho) = 0\,, \qquad (113)$$

and to the equation for balance of momentum

$$\frac{\partial}{\partial s}(\varrho v_\mu) + \frac{\partial}{\partial x^\lambda}(g^{\lambda\alpha}\mathcal{P}_{\alpha\mu}) + \frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}\mathcal{P}_{\alpha\beta} = 0\,. \qquad (114)$$

Taking into account the fact that for the particular solution (112), the stress tensor, as defined by Eq.(111), is given by the expression

$$\mathcal{P}_{\mu\nu}(x;s) = \varrho v_\mu v_\nu\,, \qquad (115)$$

we obtain the final form of the hydrodynamic equations

$$\frac{\partial\varrho}{\partial s} + \frac{\partial}{\partial x^\mu}(\varrho v^\mu) = 0\,, \qquad (116)$$

$$\frac{\partial v_\mu}{\partial s} + v^\lambda\left(\frac{\partial v_\mu}{\partial x^\lambda} - \Gamma^\nu_{\mu\lambda}v_\nu\right) = \frac{\partial v_\mu}{\partial s} + v^\lambda\nabla_\lambda v_\mu = 0\,. \qquad (117)$$

It is straightforward to see that, through the substitution $v_\mu \to V_\mu$, Eq.(72) is immediately recovered along a geodesic, that is our covariant symplectic structure is consistent with a hydrodynamic picture. It is worth noting that if $\frac{\partial v_\mu}{\partial s}$ in Eq.(117), the motion is not geodesic. The meaning of this term different from zero is that an extra force is acting on the system.

5   Applications, Discussion and Conclusions

Many applications of the previous results can be achieved specifying the nature of vector (or tensor) fields which define the Hamiltonian conserved invariant $\mathcal{H}$. Considerations in General Relativity and Electromagnetism are particularly interesting at this point. Let us take into account the Riemann tensor $R^\rho_{\sigma\mu\nu}$. It comes out when a given vector V^ρ is transported along a closed path on a generic curved manifold. It is

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma\,, \qquad (118)$$

where ∇_μ is the covariant derivative. We are assuming a Riemannian $\mathbf{V}_n$ manifold as standard in General Relativity. If connection is not symmetric, an additive torsion field comes out from the parallel transport.

Clearly, the Riemann tensor results from the commutation of covariant derivatives and it can be expressed as the sum of two commutators

$$R^\rho_{\sigma\mu\nu} = \partial_{[\mu}\Gamma^\rho_{\nu]\sigma} + \Gamma^\rho_{\lambda[\mu}\Gamma^\lambda_{\nu]\sigma}\,. \qquad (119)$$

Furthermore, (anti) commutation relations and cyclic identities (in particular Bianchi's identities) hold for the Riemann tensor [5].

All these straightforward considerations suggest the presence of a symplectic structure whose elements are covariant and contravariant vector fields, V^α and V_α, satisfying the properties (3)-(7). In this case, the dimensions of vector space $\mathbf{E}_{2n}$ are assigned by V^α and V_α. It is important to notice that such properties imply the connections (Christoffel symbols) and not the metric tensor.

As we said, the invariant (57) is a generic conserved quantity specified by the choice of V^α and V_α. If

$$V^\alpha = \frac{dx^\alpha}{ds}\,, \qquad (120)$$

is a 4-velocity, with $\alpha = 0, 1, 2, 3$, immediately, from Eq.(80), we obtain the equation of geodesics of General Relativity,

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,. \qquad (121)$$

On the other hand, being

$$\delta V^\alpha = R^\alpha_{\beta\mu\nu}V^\beta dx_1^\mu dx_2^\nu\,, \qquad (122)$$

the result of the transport along a closed path, it is easy to recover the geodesic deviation considering the geodesic (121) and the infinitesimal variation ξ^α with respect to it, i. e.

$$\frac{d^2(x^\alpha+\xi^\alpha)}{ds^2} + \Gamma^\alpha_{\mu\nu}(x+\xi)\frac{d(x^\mu+\xi^\mu)}{ds}\frac{d(x^\nu+\xi^\nu)}{ds} = 0, \qquad (123)$$

which gives, through Eq.(119),

$$\frac{d^2\xi^\alpha}{ds^2} = R^\alpha_{\mu\lambda\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\lambda. \qquad (124)$$

Clearly the symplectic structure is due to the fact that the Riemann tensor is derived from covariant derivatives either as

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma\,, \qquad (125)$$

or

$$[\nabla_\mu, \nabla_\nu]V_\rho = R^\sigma_{\mu\nu\rho}V_\sigma\,. \qquad (126)$$

In other words, fundamental equations of General Relativity are recovered from our covariant symplectic formalism.

Another interesting choice allows to recover the standard Electromagnetism. If $V^\alpha = A^\alpha$, where A^α is the vector potential and the Hamiltonian invariant is

$$\mathcal{H} = A^\alpha A_\alpha\,, \qquad (127)$$

it is straightforward, following the above procedure, to obtain, from the covariant Hamilton equations, the electromagnetic tensor field

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \nabla_{[\alpha} A_{\beta]} \,, \qquad (128)$$

and the Maxwell equations (in a generic empty curved spacetime)

$$\nabla^\alpha F_{\alpha\beta} = 0 \,, \qquad \nabla_{[\alpha} F_{\lambda\beta]} = 0 \,. \qquad (129)$$

The standard Lorentz gauge is

$$\nabla^\alpha A_\alpha = 0 \,, \qquad (130)$$

and electromagnetic wave equation is easily recovered.

In summary, a covariant, symplectic structure can be found for every Hamiltonian invariant which can be constructed by covariant vectors, bivectors and tensor fields. In fact, any theory of physics has to be endowed with a symplectic structure in order to be formulated at a fundamental level.

We pointed out that curvature invariants of General Relativity can show such a feature and, furthermore, they can be recovered from Hamiltonian invariants opportunely defined. Another interesting remark deserves the fact that, starting from such invariants, covariant and contravariant vector fields can be read as the configurations q^i and the momenta p_i of classical Hamiltonian dynamics so then the Hamilton-like equations of motion are recovered from the application of covariant derivative to both these vector fields. Besides, the approach can be formulated in a holonomic and anholonomic representations, once vector fields (or tensors in general) are represented in *vierbien* or coordinate–frames. This feature is essential to be sure that general covariance and symplectic structure are conserved in any case.

Specifying the nature of vector fields, we select the particular theory. For example, if the vector field is the 4-velocity, we obtain geodesic motion and geodesic deviation. If the vector is the vector potential of Electromagnetism, Maxwell equations and Lorentz gauge are recovered. The scheme is independent of the nature of vector field and, in our opinion, it is a strong hint toward a unifying view of basic interactions, gravity included.

Acknowledgements

References

1. Sakurai J. J. Modern Quantum Mechanics. Revised Edition, Addison-Wesley Publ. Co., New York, 1994.

2. Itzykson C., Zuber J. B. Quantum field theory. McGraw-Hill, Singapore, 1980.

3. Kaku M. Quantum field theory. Oxford Univ. Press, Oxford, 1993.

4. Birrell N., Davies P. C. Quantum fields in curved space. Cambridge Univ., 1984.

5. Landau L. D., Lifshitz E. M. Theorie du Champs. Mir, Moscow, 1960.

6. Palatini A. *Rend. Circ. Mat. Palermo*, 1919, v. 43, 203.

7. Misner C. W., Thorne K. S., and Wheeler J. W. Gravitation. W. H. Freeman and Company, New York, 1970.

8. Lanczos C. The variational principles of mechanics. Toronto Univ. Press, 1949.

9. Schwinger E. *Phys. Rev.*, 1951, v. 82, 914; *Phys. Rev.*, 1953, v. 91, 713.

10. Ferapontov E. V., Pavlov M. V. nlin.SI/0212026, 2002.