Introducing Distance and Measurement in General Relativity: Changes for the Standard Tests and the Cosmological Large-Scale

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Relativistic motion in the gravitational field of a massive body is governed by the external metric of a spherically symmetric extended object. Consequently, any solution for the point-mass is inadequate for the treatment of such motions since it pertains to a fictitious object. I therefore develop herein the physics of the standard tests of General Relativity by means of the generalised solution for the field external to a sphere of incompressible homogeneous fluid.

1 Introduction

The orthodox treatment of physics in the vicinity of a massive body is based upon the Hilbert [1] solution for the point-mass, a solution which is neither correct nor due to Schwarzschild [2], as the latter is almost universally claimed.

In previous papers [3, 4] I derived the correct general solution for the point-mass and the point-charge in all their standard configurations, and demonstrated that the Hilbert solution is invalid. The general solution for the point-mass is however, inadequate for any real physical situation since the material point (and also the material point-charge) is a fictitious object, and so quite meaningless. Therefore, I avail myself of the general solution for the external field of a sphere of incompressible homogeneous fluid, obtained in a particular case by K. Schwarzschild [5] and generalised by myself [6] to,

$$ds^2 = \left[\frac{(\sqrt{C_n}-\alpha)}{\sqrt{C_n}}\right]dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n}-\alpha)}\right]\frac{{C'_n}^2}{4C_n}dr^2 - C_n(d\theta^2+\sin^2\theta d\varphi^2)\,, \tag{1}$$

$$C_n(r) = \left(\left|r-r_0\right|^n + \epsilon^n\right)^{\frac{2}{n}},$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3\left|\chi_a-\chi_0\right|\,,$$

$$R_{c_a} = \sqrt{\frac{3}{\kappa\rho_0}}\sin\left|\chi_a-\chi_0\right|\,,$$

$$\epsilon = \sqrt{\frac{3}{\kappa\rho_0}}\left\{\frac{3}{2}\sin^3\left|\chi_a-\chi_0\right| - \frac{9}{4}\cos\left|\chi_a-\chi_0\right|\left[\left|\chi_a-\chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a-\chi_0\right|\right]\right\}^{\frac{1}{3}},$$

$$r_0\in\Re,\; r\in\Re,\; n\in\Re^+,\; \chi_0\in\Re,\; \chi_a\in\Re,$$

$$\arccos\frac{1}{3} < |\chi_a-\chi_0| < \frac{\pi}{2}\,,$$

$$|r_a-r_0| \leqslant |r-r_0| < \infty\,,$$

where ρ_0 is the constant density of the fluid, k^2 is Gauss' gravitational constant, the sign a denotes values at the surface of the sphere, $|\chi-\chi_0|$ parameterizes the radius of curvature of the interior of the sphere centred arbitrarily at χ_0, $|r-r_0|$ is the coordinate radius in the spacetime manifold of Special Relativity which is a parameter space for the gravitational field external to the sphere centred arbitrarily at r_0.

To eliminate the infinite number of coordinate systems admitted by (1), I rewrite the said metric in terms of the only measurable distance in the gravitational field, i.e. the circumference G of a great circle, thus

$$ds^2 = \left(1-\frac{2\pi\alpha}{G}\right)dt^2 - \left(1-\frac{2\pi\alpha}{G}\right)^{-1}\frac{dG^2}{4\pi^2} - \frac{G^2}{4\pi^2}\left(d\theta^2+\sin^2\theta d\varphi^2\right)\,, \tag{2}$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a-\chi_0|\,,$$

$$2\pi\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a-\chi_0| \leqslant G < \infty\,,$$

$$\arccos\frac{1}{3} < |\chi_a-\chi_0| < \frac{\pi}{2}\,.$$

2 Distance and time

According to (1), if t is constant, a three-dimensional manifold results, having the line-element,

$$ds^2 = \left[\frac{\sqrt{C_n}}{(\sqrt{C_n}-\alpha)}\right]\frac{{C'_n}^2}{4C_n}dr^2 + C_n(d\theta^2+\sin^2\theta d\varphi^2)\,. \tag{3}$$

If $\alpha = 0$, (1) reduces to the line-element of flat spacetime,

$$ds^2 = dt^2 - dr^2 - |r - r_0|^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (4)$$
$$0 \leqslant |r - r_0| < \infty,$$

since then $r_a \equiv r_0$.

The introduction of matter makes $r_a \neq r_0$, owing to the extended nature of a real body, and introduces distortions from the Euclidean in time and distance. The value of α is effectively a measure of this distortion and therefore fixes the spacetime.

When $\alpha = 0$, the distance $D = |r - r_0|$ is the radius of a sphere centred at r_0. If $r_0 = 0$ and $r \geqslant 0$, then $D \equiv r$ and is then both a radius and a coordinate, as is clear from (4).

If r is constant in (3), then $C_n(r) = R_c^2$ is constant, and so (3) becomes,

$$ds^2 = R_c^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (5)$$

which describes a sphere of constant radius R_c embedded in Euclidean space. The infinitesimal tangential distances on (5) are simply,

$$ds = R_c \sqrt{d\theta^2 + \sin^2\theta d\varphi^2}.$$

When θ and φ are constant, (3) yields the proper radius,

$$R_p = \int \sqrt{\frac{\sqrt{C_n(r)}}{\sqrt{C_n(r)} - \alpha}} \, \frac{C_n'(r)}{2\sqrt{C_n(r)}} \, dr =$$
$$= \int \sqrt{\frac{\sqrt{C_n(r)}}{\sqrt{C_n(r)} - \alpha}} \, d\sqrt{C_n(r)}, \quad (6)$$

from which it clearly follows that the parameter r does not measure radial distances in the gravitational field.

Integrating (6) gives,

$$R_p(r) = \sqrt{\sqrt{C_n}(\sqrt{C_n} - \alpha)} + \alpha \ln \left| \sqrt{\sqrt{C_n}} + \sqrt{\sqrt{C_n} - \alpha} \right| + K,$$
$$K = \text{const},$$

which must satisfy the condition,

$$r \to r_a^{\pm} \Rightarrow R_p \to R_{p_a}^{+},$$

where r_a is the parameter value at the surface of the body and R_{p_a} the indeterminate proper radius of the sphere from outside the sphere. Therefore,

$$R_p(r) = R_{p_a} + \sqrt{\sqrt{C_n(r)} \left(\sqrt{C_n(r)} - \alpha \right)} -$$
$$- \sqrt{\sqrt{C_n(r_a)} \left(\sqrt{C_n(r_a} - \alpha \right)} + \quad (7)$$
$$+ \alpha \ln \left| \frac{\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}}{\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}} \right|,$$

which, by the use of (1) and (2), becomes

$$R_p(r) = R_{p_a} + \sqrt{\frac{G}{2\pi} \left(\frac{G}{2\pi} - \alpha \right)} -$$
$$- \sqrt{\sqrt{\frac{3}{\kappa\rho_0}} \sin |\chi_a - \chi_0| \left(\sqrt{\frac{3}{\kappa\rho_0}} \sin |\chi_a - \chi_0| - \alpha \right)} + \quad (8)$$
$$+ \alpha \ln \left| \frac{\sqrt{\frac{G}{2\pi}} + \sqrt{\frac{G}{2\pi} - \alpha}}{\sqrt{\sqrt{\frac{3}{\kappa\rho_0}} \sin |\chi_a - \chi_0|} + \sqrt{\sqrt{\frac{3}{\kappa\rho_0}} \sin |\chi_a - \chi_0| - \alpha}} \right|,$$
$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}} \sin^3 |\chi_a - \chi_0|.$$

According to (1), the proper time is related to the coordinate time by,

$$d\tau = \sqrt{g_{00}} \, dt = \sqrt{1 - \frac{\alpha}{\sqrt{C_n(r)}}} \, dt. \quad (9)$$

When $\alpha = 0$, $d\tau = dt$ so that proper time and coordinate time are one and the same in flat spacetime. With the introduction of matter, proper time and coordinate time are no longer the same. It is evident from (9) that both τ and t are finite and non-zero, since according to (1),

$$\frac{1}{9} < 1 - \frac{\alpha}{\sqrt{C_n(r_a)}} \leqslant 1 - \frac{\alpha}{\sqrt{C_n(r)}},$$

i.e.

$$\frac{1}{9} < \cos^2 |\chi_a - \chi_0| \leqslant 1 - \frac{\alpha}{\sqrt{C_n(r)}},$$

or

$$\frac{1}{3} dt \leqslant d\tau \leqslant dt,$$

since In the far field, according to (9),

$$\sqrt{C_n(r)} \to \infty \Rightarrow d\tau \to dt,$$

recovering flat spacetime asymptotically.

Therefore, if a body falls from rest from a point distant from the gravitating mass, it will reach the surface of the mass in a finite coordinate time and a finite proper time. According to an external observer, time does not stop at the surface of the body, where $dt = 3d\tau$, contrary to the orthodox analysis based upon the fictitious point-mass.

3 Radar sounding

Consider an observer in the field of a massive body. Let the observer have coordinates, $(r_1, \theta_0, \varphi_0)$. Let the coordinates of a small body located between the observer and the massive body along a radial line be $(r_2, \theta_0, \varphi_0)$. Let the observer emit a radar pulse towards the small body. Then by (1),

$$\left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right)dt^2 = \left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right)^{-1}\frac{C_n'^2(r)}{4C_n(r)}\,dr^2 = \\ = \left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right)^{-1} d\sqrt{C_n(r)}^{\,2},$$

so

$$\frac{d\sqrt{C_n(r)}}{dt} = \pm\left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right),$$

or

$$\frac{dr}{dt} = \pm\frac{2\sqrt{C_n(r)}}{C_n'(r)}\left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right).$$

The coordinate time for the pulse to travel to the small body and return to the observer is,

$$\Delta t = -\int_{\sqrt{C_n(r_1)}}^{\sqrt{C_n(r_2)}}\frac{d\sqrt{C_n}}{1-\frac{\alpha}{\sqrt{C_n}}} + \int_{\sqrt{C_n(r_2)}}^{\sqrt{C_n(r_1)}}\frac{d\sqrt{C_n}}{1-\frac{\alpha}{\sqrt{C_n}}} = \\ = 2\int_{\sqrt{C_n(r_2)}}^{\sqrt{C_n(r_1)}}\frac{d\sqrt{C_n}}{1-\frac{\alpha}{\sqrt{C_n}}}.$$

The proper time lapse is, according to the observer, by formula (1),

$$\Delta\tau = \sqrt{1-\frac{\alpha}{\sqrt{C_n}}}\,dt = 2\sqrt{1-\frac{\alpha}{\sqrt{C_n}}}\int_{\sqrt{C_n(r_2)}}^{\sqrt{C_n(r_1)}}\frac{d\sqrt{C_n}}{1-\frac{\alpha}{\sqrt{C_n}}} = \\ = 2\sqrt{1-\frac{\alpha}{\sqrt{C_n}}}\left(\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}+\alpha\ln\left|\frac{\sqrt{C_n(r_1)}-\alpha}{\sqrt{C_n(r_2)}-\alpha}\right|\right).$$

The proper distance between the observer and the small body is,

$$R_p = \int_{\sqrt{C_n(r_2)}}^{\sqrt{C_n(r_1)}}\sqrt{\frac{\sqrt{C_n}}{\sqrt{C_n}-\alpha}}\,d\sqrt{C_n} \\ = \sqrt{\sqrt{C_n(r_1)}\left(\sqrt{C_n(r_1)}-\alpha\right)} - \\ - \sqrt{\sqrt{C_n(r_2)}\left(\sqrt{C_n(r_2)}-\alpha\right)} + \\ + \alpha\ln\left|\frac{\sqrt{\sqrt{C_n(r_1)}}+\sqrt{\sqrt{C_n(r_1)}-\alpha}}{\sqrt{\sqrt{C_n(r_2)}}+\sqrt{\sqrt{C_n(r_2)}-\alpha}}\right|.$$

Then according to classical theory, the round trip time is

$$\Delta\bar{\tau} = 2R_p\,,$$

so $\Delta\tau \neq \Delta\bar{\tau}$.

If $\frac{\alpha}{\sqrt{C_n(r)}}$ is small for

$$\sqrt{C_n(r_2)} < \sqrt{C_n(r)} < \sqrt{C_n(r_1)}\,,$$

then

$$\Delta\tau \approx 2\Bigg[\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)} - \\ - \frac{\alpha\left(\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}\right)}{2\sqrt{C_n(r_1)}} + \alpha\ln\sqrt{\frac{\sqrt{C_n(r_1)}}{\sqrt{C_n(r_2)}}}\,\Bigg],$$

$$\Delta\bar{\tau} \approx 2\left[\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}+\frac{\alpha}{2}\ln\sqrt{\frac{\sqrt{C_n(r_1)}}{\sqrt{C_n(r_2)}}}\,\right].$$

Therefore,

$$\Delta\tau-\Delta\bar{\tau} \approx \alpha\Bigg[\ln\sqrt{\frac{\sqrt{C_n(r_1)}}{\sqrt{C_n(r_2)}}} - \\ - \frac{\left(\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}\right)}{\sqrt{C_n(r_1)}}\,\Bigg] = \\ = \alpha\left(\ln\sqrt{\frac{G_1}{G_2}}-\frac{G_1-G_2}{G_1}\right) = \\ = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a-\chi_0|\left(\ln\sqrt{\frac{G_1}{G_2}}-\frac{G_1-G_2}{G_1}\right), \tag{10}$$

$$G = G(r) = 2\pi\sqrt{C_n(D(r))}\,.$$

Equation (10) gives the time delay for a radar signal in the gravitational field.

4 Spectral shift

Let an emitter of light have coordinates $(t_E, r_E, \theta_E, \varphi_E)$. Let a receiver have coordinates $(t_R, r_R, \theta_R, \varphi_R)$. Let u be an affine parameter along a null geodesic with the values u_E and u_R at emitter and receiver respectively. Then,

$$\left(1-\frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{du}\right)^2 = \left(1-\frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{du}\right)^2 + \\ + C_n\left(\frac{d\theta}{du}\right)^2 + C_n\sin^2\theta\left(\frac{d\varphi}{du}\right)^2,$$

so

$$\frac{dt}{du} = \left[\left(1-\frac{\alpha}{\sqrt{C_n}}\right)^{-1}\bar{g}_{ij}\frac{dx^i}{du}\frac{dx^j}{du}\right]^{\frac{1}{2}},$$

where $\bar{g}_{ij} = -g_{ij}$. Then,

$$t_R - t_E = \int_{u_E}^{u_R} \left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1} \bar{g}_{ij} \frac{dx^i}{du} \frac{dx^j}{du} \right]^{\frac{1}{2}} du \,,$$

and so, for spatially fixed emitter and receiver,

$$t_R^{(1)} - t_E^{(1)} = t_R^{(2)} - t_E^{(2)} \,,$$

and therefore,

$$\Delta t_R = t_R^{(2)} - t_R^{(1)} = t_E^{(2)} - t_E^{(1)} = \Delta t_E \,. \tag{11}$$

Now by (1), the proper time is,

$$\Delta\tau_E = \sqrt{1 - \frac{\alpha}{\sqrt{C_n(r_E)}}} \,\Delta t_E \,,$$

and

$$\Delta\tau_R = \sqrt{1 - \frac{\alpha}{\sqrt{C_n(r_R)}}} \,\Delta t_R \,.$$

Then by (11),

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left[\frac{1 - \frac{\alpha}{\sqrt{C_n(r_R)}}}{1 - \frac{\alpha}{\sqrt{C_n(r_E)}}} \right]^{\frac{1}{2}} . \tag{12}$$

If z regular pulses of light are emitted, the emitted and received frequencies are,

$$\nu_E = \frac{z}{\Delta\tau_E} \,, \quad \nu_R = \frac{z}{\Delta\tau_R} \,,$$

so by (12),

$$\frac{\Delta\nu_R}{\Delta\nu_E} = \left[\frac{1 - \frac{\alpha}{\sqrt{C_n(r_E)}}}{1 - \frac{\alpha}{\sqrt{C_n(r_R)}}} \right]^{\frac{1}{2}} \approx$$

$$\approx 1 + \frac{\alpha}{2} \left(\frac{1}{\sqrt{C_n(r_R)}} - \frac{1}{\sqrt{C_n(r_E)}} \right),$$

whence,

$$\frac{\Delta\nu}{\nu_E} = \frac{\nu_R - \nu_E}{\nu_E} \approx \frac{\alpha}{2} \left(\frac{1}{\sqrt{C_n(r_R)}} - \frac{1}{\sqrt{C_n(r_E)}} \right) =$$

$$= \pi\alpha \left(\frac{1}{G_R} - \frac{1}{G_E} \right) =$$

$$= \pi \sqrt{\frac{3}{\kappa\rho_0}} \sin^3 |\chi_a - \chi_0| \left(\frac{1}{G_R} - \frac{1}{G_E} \right) .$$

5 Advance of the perihelia

Consider the Lagrangian,

$$\begin{aligned} L = &\frac{1}{2} \left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right) \left(\frac{dt}{d\tau}\right)^2 \right] - \\ &- \frac{1}{2} \left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1} \left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 \right] - \\ &- \frac{1}{2} \left[C_n \left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{d\tau}\right)^2 \right) \right] , \end{aligned} \tag{13}$$

where τ is the proper time. Restricting motion, without loss of generality, to the equatorial plane, $\theta = \frac{\pi}{2}$, the Euler-Lagrange equations for (13) are,

$$\begin{aligned} &\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1} \frac{d^2\sqrt{C_n}}{d\tau^2} + \frac{\alpha}{2C_n} \left(\frac{dt}{d\tau}\right)^2 - \\ &- \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-2} \frac{\alpha}{2C_n} \left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - \sqrt{C_n} \left(\frac{d\varphi}{d\tau}\right)^2 = 0 \,, \end{aligned} \tag{14}$$

$$\left(1 - \frac{\alpha}{\sqrt{C_n}}\right) \frac{dt}{d\tau} = \text{const} = K \,, \tag{15}$$

$$C_n \frac{d\varphi}{d\tau} = \text{const} = h \,, \tag{16}$$

and $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ becomes,

$$\begin{aligned} &\left(1 - \frac{\alpha}{\sqrt{C_n}}\right) \left(\frac{dt}{d\tau}\right)^2 - \\ &- \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1} \left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - C_n \left(\frac{d\varphi}{d\tau}\right)^2 = 1 \,. \end{aligned} \tag{17}$$

Rearrange (17) for,

$$\left(1 - \frac{\alpha}{\sqrt{C_n}}\right) \frac{\dot{t}^2}{\dot{\varphi}^2} - \left(1 - \frac{\alpha}{\sqrt{C_n}}\right) \left(\frac{d\sqrt{C_n}}{d\varphi}\right)^2 - C_n = \frac{1}{\dot{\varphi}^2} \,. \tag{18}$$

Substituting (15) and (16) into (18) gives,

$$\left(\frac{d\sqrt{C_n}}{d\varphi}\right)^2 + C_n \left(1 + \frac{C_n}{h^2}\right) \left(1 - \frac{\alpha}{\sqrt{C_n}}\right) - \frac{K^2}{h^2} C_n^2 = 0 \,.$$

Setting $u = \frac{1}{\sqrt{C_n}}$ reduces (18) to,

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = E + \frac{\alpha}{h^2} u + \alpha u^3 \,, \tag{19}$$

where $E = \frac{(K^2 - 1)}{h^2}$. The term αu^3 represents the general-relativistic perturbation of the Newtonian orbit.

Aphelion and perihelion occur when $\frac{du}{d\varphi} = 0$, so by (19),

$$\alpha u^3 - u^2 + \frac{\alpha}{h^2} u + E = 0 \,, \tag{20}$$

Let $u = u_1$ at aphelion and $u = u_2$ at perihelion, so $u_1 \leqslant u \leqslant u_2$. One then finds in the usual way that the angle $\Delta\varphi$ between aphelion and subsequent perihelion is,

$$\Delta\varphi = \left[1 + \frac{3\alpha}{4}\left(u_1 + u_2\right)\right]\pi\,.$$

Therefore, the angular advance ψ between successive perihelia is,

$$\psi = \frac{3\alpha\pi}{2}\left(u_1 + u_2\right) = \frac{3\alpha\pi}{2}\left(\frac{1}{\sqrt{C_n(r_1)}} + \frac{1}{\sqrt{C_n(r_2)}}\right) = \\ = 3\alpha\pi^2\left(\frac{1}{G_1} + \frac{1}{G_2}\right), \tag{21}$$

where G_1 and G_2 are the measurable circumferences of great circles at aphelion and at perihelion. Thus, to correctly determine the value of ψ, the values of the said circumferences must be ascertained by direct measurement. Only the circumferences are measurable in the gravitational field. The radii of curvature and the proper radii must be calculated from the circumference values.

If the field is weak, as in the case of the Sun, one may take $G \approx 2\pi r$, for r as an approximately "measurable" distance from the gravitating sphere to a spacetime event. In such a situation equation (21) becomes,

$$\psi \approx \frac{3\alpha\pi}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right). \tag{22}$$

In the case of the Sun, $\alpha \approx 3000$ m, and for the planet Mercury, the usual value of $\psi \approx 43$ arcseconds per century is obtained from (22). I emphasize however, that this value is a Euclidean approximation for a weak field. In a strong field equation (22) is entirely inappropriate and equation (21) must be used. Unfortunately, this means that accurate solutions cannot be obtained since there is no obvious way of obtaining the required circumferences in practise. This aspect of Einstein's theory seriously limits its utility. Since the relativists have not detected this limitation the issue has not previously arisen in general.

6 Deflection of light

In the case of a photon, equation (17) becomes,

$$\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{d\tau}\right)^2 - \\ - \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - C_n\left(\frac{d\varphi}{d\tau}\right)^2 = 1\,,$$

which leads to,

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = F + \alpha u^3\,. \tag{23}$$

Let the radius of curvature of a great circle at closest approach be $\sqrt{C_n(r_c)}$. Now when there is no mass present, (23) becomes

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = F\,,$$

and has solution,

$$u = u_c \sin\varphi \Rightarrow \sqrt{C_n(r_c)} = \sqrt{C_n(r)}\sin\varphi\,,$$

and

$$u_c^2 = \frac{1}{\sqrt{C_n(r_c)}} = F\,.$$

If

$$\sqrt{C_n(r)} \gg \alpha, \quad \sqrt{C_n(r)} > \sqrt{C_n(r_a)}\,,$$

and $u = u_c > u_a$ at closest approach, then

$$\frac{du}{d\varphi} = 0 \quad \text{at} \quad u = u_c\,,$$

so $F = u_c^2\,(1 - u_c\alpha)$, and (23) becomes,

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = u_c^2\,(1 - u_c\alpha) + \alpha u^3\,. \tag{24}$$

Equation (24) must have a solution close to flat spacetime, so let

$$u = u_c \sin\varphi + \alpha w(\varphi)\,.$$

Putting this into (24) and working to first order in α, gives

$$2\left(\frac{dw}{d\varphi}\right)\cos\varphi + 2w\sin\varphi = u_c^2\left(\sin^3\varphi - 1\right),$$

or

$$\frac{d}{d\varphi}\left(w\sec\varphi\right) = \frac{1}{2}\,u_c^2\left(\sec\varphi\tan\varphi - \sin\varphi - \sec^2\varphi\right),$$

and so,

$$w = \frac{1}{2}u_c^2\left(1 + \cos^2\varphi - \sin\varphi\right) + A\cos\varphi\,,$$

where A is an integration constant. If the photon originates at infinity in the direction $\varphi = 0$, then $w(0) = 0$, so $A = -u_c^2$, and

$$u = u_c\left(1 - \frac{1}{2}\,\alpha u_c\right)\sin\varphi + \frac{1}{2}\,\alpha u_c^2\,(1 - \cos\varphi)^2\,, \tag{25}$$

to first order in α. Putting $u = 0$ and $\varphi = \pi + \Delta\varphi$ into (25), then to first order in $\Delta\varphi$,

$$0 = -u_c\Delta\varphi + 2\alpha u_c^2\,,$$

so the angle of deflection is,

$$\Delta\varphi = 2\alpha u_c = \frac{2\alpha}{\sqrt{C_n(r_c)}} = \frac{2\alpha}{\left(\left|r_c - r_0\right|^n + \epsilon^n\right)^{\frac{1}{n}}} = \frac{4\pi\alpha}{G_c}\,,$$

$$G_c \geqslant G_a \,.$$

At a grazing trajectory to the surface of the body,

$$G_c = G_a = 2\pi\sqrt{C_n(r_a)}\,,$$

$$\sqrt{C_n(r_a)} = \sqrt{\frac{3}{\kappa\rho_0}}\,\sin\left|\,\chi_a - \chi_0\right|\,,$$

so then

$$\Delta\varphi = \frac{2\sqrt{\frac{3}{\kappa\rho_0}}\,\sin^3\left|\,\chi_a - \chi_0\right|}{\sqrt{\frac{3}{\kappa\rho_0}}\,\sin\left|\,\chi_a - \chi_0\right|} = 2\sin^2\left|\,\chi_a - \chi_0\right|. \qquad (26)$$

For the Sun [5],

$$\sin\left|\,\chi_a - \chi_0\right| \approx \frac{1}{500}\,,$$

so the deflection of light grazing the limb of the Sun is,

$$\Delta\varphi \approx \frac{2}{500^2} \approx 1.65''\,.$$

Equation (26) is an interesting and quite surprising result, for $\sin\left|\,\chi_a - \chi_0\right|$ gives the ratio of the "naturally measured" fall velocity of a free test particle falling from rest at infinity down to the surface of the spherical body, to the speed of light in vacuo. Thus,

> *the deflection of light grazing the limb of a spherical gravitating body is twice the square of the ratio of the fall velocity of a free test particle falling from rest at infinity down to the surface, to the speed of light in vacuo, i.e.,*

$$\Delta\varphi = 2\sin^2\left|\,\chi_a - \chi_0\right| = 2\left(\frac{v_a}{c}\right)^2 = \frac{4GM_g}{c^2 R_{c_a}}\,,$$

where R_{c_a} is the radius of curvature of the body, M_g the active mass, and G is the gravitational constant. The quantity v_a is the escape velocity,

$$v_a = \sqrt{\frac{2GM_g}{R_{c_a}}}\,.$$

7 Practical constraints and general comment

Owing to their invalid assumptions about the r-parameter [7], the relativists have not recognised the practical limitations associated with the application of General Relativity. It is now clear that the fundamental element of distance in the gravitational field is the circumference of a great circle, centred at the heart of an extended spherical body and passing through a spacetime event external thereto. Heretofore the orthodox theorists have incorrectly taken the r-parameter, not just as a radius in the gravitational field, but also as a *measurable* radius in the field. This is not correct. The only measurable distance in the gravitational field is the aforesaid circumference of a great circle, from which the radius of curvature $\sqrt{C_n(r)}$ and the proper radius $R_p(r)$ must be calculated, thus,

$$\sqrt{C_n(r)} = \frac{G}{2\pi}\,,$$

$$R_p(r) = \int \sqrt{-g_{11}}\; dr\,.$$

Only in the weak field, where the spacetime curvature is very small, can $\sqrt{C_n(r)}$ be taken approximately as the Euclidean value r, thereby making $R_p(r) \equiv \sqrt{C_n(r)} \equiv r$, as in flat spacetime. In a strong field this *cannot* be done. Consequently, the problem arises as to how to accurately measure the required great circumference? The correct determination, for example, of the circumferences of great circles at aphelion and perihelion seem to be beyond practical determination. Any method adopted for determining the required circumference must be completely independent of any Euclidean quantity since, other than the great circumference itself, only non-Euclidean distances are valid in the gravitational field, being determined by it. Therefore, anything short of physically measuring the great circumference will fail. Consequently, General Relativity, whether right or wrong as theories go, suffers from a serious practical limitation.

The value of the r-parameter is coordinate dependent and is rightly determined from the coordinate independent value of the circumference of the great circle associated with a spacetime event. One cannot obtain a circumference for the great circle of a given spacetime event, and hence the related radius of curvature and associated proper radius, from the specification of a coordinate radius, because the latter is not unique, being conditioned by arbitrary constants. The coordinate radius is therefore superfluous. It is for this reason that I completely eliminated the coordinate radius from the metric for the gravitational field, to describe the metric in terms of the only quantity that is measurable in the gravitational field — the great circumference (see also [6]). The presence of the r-parameter has proved misleading to the relativists. Stavroulakis [8, 9, 10] has also completely eliminated the r-parameter from the equations, but does not make use of the great circumference. His approach is formally correct, but rather less illuminating, because his resulting line element is in terms of the a quantity which is not measurable in the gravitational field. One cannot obtain an explicit expression for the great circumference in terms of the proper radius.

As to the cosmological large-scale, I have proved elsewhere [11] that General Relativity adds nothing to Special Relativity. Einstein's field equations do not admit of solutions when the cosmological constant is not zero, and they do not admit of the expanding universe solutions alleged by the relativists. The lambda "solutions" and the expanding universe "solutions" are the result of such a muddleheadedness that it is difficult to apprehend the kind of thoughtlessness that gave them birth. Since Special Relativity describes an empty world (no gravity) it cannot form a basis for any cosmology. This theoretical result is all the more interesting owing to its agreement with observation. Arp [12], for instance, has adduced considerable observational data which is consistent on the large-scale with a flat, infinite, non-expanding Universe in Heraclitian flux. Bearing in mind that both Special Relativity and General Relativity *cannot* yield a spacetime on the cosmological "large-scale", there is currently no theoretical replacement for Newton's cosmology, which accords with deep-space observations for a flat space, infinite in time and in extent. The all pervasive rolê given heretofore by the relativists to General Relativity, can be justified no longer. General Relativity is a theory of only *local* phenonomea, as is Special Relativity.

Another serious shortcoming of General Relativity is its current inability to deal with the gravitational interaction of two comparable masses. It is not even known if Einstein's theory admits of configurations involving two or more masses [13]. This shortcoming seems rather self evident, but apprently not so for the relativists, who routinely talk of black hole binary systems and colliding black holes (e.g. [14]), aside of the fact that no theory predicts the existence of black holes to begin with, but to the contrary, precludes them.

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Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

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