Non-Euclidean Geometry and Gravitation

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A great deal of misunderstandings and mathematical errors are involved in the currently accepted theory of the gravitational field generated by an isotropic spherical mass. The purpose of the present paper is to provide a short account of the rigorous mathematical theory and exhibit a new formulation of the problem. The solution of the corresponding equations of gravitation points out several new and unusual features of the stationary gravitational field which are related to the non-Euclidean structure of the space. Moreover it precludes the black hole from being a mathematical and physical notion.

1 Introduction

If the structure of the spacetime is actually non-Euclidean as is postulated by general relativity, then several non-Euclidean features will manifest themselves in the neighbourhoods of the sources of the gravitational field. So, a spherical distribution of matter will appear as a non-Euclidean ball and the concentric with it spheres will possess the structure of non-Euclidean spheres. Specifically, if this distribution of matter is isotropic, such a sphere will be characterised completely by its radius, say ρ, and its curvature radius which is a function of ρ, say $g(\rho)$, defining the area $4\pi(g(\rho))^2$ of the sphere as well as the length of circumference $2\pi g(\rho)$ of the corresponding great circles. It is then expected that the function $g(\rho)$ will play a significant part in the conception of the metric tensor related to the gravitational field of the spherical mass. Of course, in formulating the problem, we must distinguish clearly the radius ρ, which is introduced as a given length, from the curvature radius $g(\rho)$, the determination of which depends on the equations of gravitation. However the classical approach to the problem suppresses this distinction and assumes that the radius af the sphere is the unknown function $g(\rho)$. This glaring mistake underlies the pseudo-theorem of Birkhoff as well as the classical solutions, which have distorted the theory of the gravitational field.

Another glaring mistake of the classical approach to the problem is related to the topological space which underlies the definition of the metric tensor. The spatial aspect of the problem suggests to identify the centre of the spherical mass with the origin of the vector space $\mathbb{R}^3$ which is moreover considered with the product topology of three real lines. Regarding the time t, several assumptions suggest to consider it (or rather ct) as a variable describing the real line $\mathbb{R}$. It follows that the topological space pertaining to the considered situation is the space $\mathbb{R}\times\mathbb{R}^3$ equipped with the product topology of four real lines. This simple and clear algebraic and topological situation has been altered from the beginnings of general relativity by the introduction of the so-called polar coordinates of $\mathbb{R}^3$ which destroy the topological structure of $\mathbb{R}^3$ and replace it by the manifold with boundary $[0,+\infty[\times S^2$. The use of polar coordinates is allowed in the theory of integration, because the open set $]0,+\infty[\times]0,2\pi[\times]0,\pi[$, described by the point (r, ϕ, θ), is transformed diffeomorphically onto the open set

$$\mathbb{R}^3 - \{(x_1,x_2,x_3)\in\mathbb{R}^3;\ \ x_1 \geqslant 0,\ \ x_2 = 0\}$$

and moreover the half-plane

$$\{(x_1,x_2,x_3)\in\mathbb{R}^3;\ \ x_1 \geqslant 0,\ \ x_2 = 0\}$$

is of zero measure in $\mathbb{R}^3$. But in general relativity this half-plane cannot be omitted. Then by choosing two systems of geographic coordinates covering all of S^2, we define a C^∞ mapping of $[0,+\infty[\times S^2$ onto $\mathbb{R}^3$ transgressing the fundamental principle according to which only diffeomorphisms are allowed. In fact, this mapping is not even one-to-one: All of $\{0\}\times S^2$ is transformed into the origin of $\mathbb{R}^3$. This situation gives rise to inconsistent assertions. So, although the origin of $\mathbb{R}^3$ disappears in polar coordinates, the meaningless term "the origin $r=0$" is commonly used. Of course, the value $r=0$ does not define a point but the boundary $\{0\}\times S^2$ which is an abstract two-dimensional sphere without physical meaning. In accordance with the idea that the value $r=0$ defines the origin, the relativists introduce transformations of the form $r=h(\overline{r})$, $\overline{r}\geqslant 0$, in order to "change the origin". This extravagant idea goes back to Droste, who claims that by setting $r=\overline{r}+2\mu$, $\mu=\frac{km}{c^2}$, we define a "new radial coordinate $\overline{r}$" such that the sphere $r=2\mu$ reduces to the "new origin $\overline{r}=0$". Rosen [2] claims also that the transformation $r=\overline{r}+2\mu$ allows to consider a mass point placed at the origin $\overline{r}=0$! The same extravagant ideas are introduced in the definition of the so-called harmonic coordinates by Lanczos (1922) who begins by the introduction of the transformation $r=\overline{r}+\mu$ in order to define the "new radial coordinate $\overline{r}$".

The introduction of the manifold with boundary $[0,+\infty[\times S^2$ instead of $\mathbb{R}^3$, hence also the introduction of $\mathbb{R}\times[0,+\infty[\times S^2$ instead of $\mathbb{R}\times\mathbb{R}^3$, gives also rise to misunderstandings and mistakes regarding the space metrics and the spacetime metrics as well.

Given a C^∞ Riemannian metric on $\mathbb{R}^3$, its transform in polar coordinates is a C^∞ quadratic form on $[0,+\infty[\times S^2$, positive definite on $]0, +\infty[\times S^2$ and null on $\{0\} \times S^2$. (This is, in particular, true for the so-called metric of $\mathbb{R}^3$ in polar coordinates, namely $ds^2 = dr^2 + r^2 d\omega^2$ with $d\omega^2 = \sin^2\theta d\phi^2 + d\theta^2$ in the domain of validity of (ϕ, θ).) But the converse is not true. A C^∞ form on $[0, +\infty[\times S^2$ satisfying the above conditions is associated in general with a form on $\mathbb{R}^3$ presenting discontinuities at the origin of $\mathbb{R}^3$. So the C^∞ form $2dr^2 + r^2 d\omega^2$, conceived on $[0, +\infty[\times S^2$, results from a uniquely defined form on $\mathbb{R}^3$, namely

$$dx^2 + \frac{(xdx)^2}{\|x\|^2},$$

(here $dx^2 = dx_1^2 + dx_2^2 + dx_3^2$, $xdx = x_1dx_1 + x_2dx_2 + x_3dx_3$) which is discontinuous at $x = (0, 0, 0)$.

Now, given a C^∞ spacetime metric on $\mathbb{R} \times \mathbb{R}^3$, its transform in polar coordinates is a C^∞ form degenerating on the boundary $\mathbb{R} \times \{0\} \times S^2$. But the converse is not true. A C^∞ spacetime form on $\mathbb{R} \times [0, +\infty[\times S^2$ degenerating on the boundary $\mathbb{R} \times \{0\} \times S^2$ results in general from a spacetime form on $\mathbb{R} \times \mathbb{R}^3$ presenting discontinuities. For instance, the so-called Bondi metric

$$ds^2 = e^{2A}dt^2 + 2e^{A+B}dtdr - r^2 d\omega^2$$

where $A = A(t, r)$, $B = B(t, r)$, conceals singularities, because it results from a uniquely defined form on $\mathbb{R} \times \mathbb{R}^3$, namely

$$ds^2 = e^{2A}dt^2 + 2e^{A+B}\frac{(xdx)}{\|x\|}dt - dx^2 + \frac{(xdx)^2}{\|x\|}$$

which is discontinuous at $x = (0, 0, 0)$. It follows that the current practice of formulating problems with respect to $\mathbb{R} \times [0, +\infty[\times S^2$, instead of $\mathbb{R} \times \mathbb{R}^3$, gives rise to misleading conclusions. The problems must be always conceived with respect to $\mathbb{R} \times \mathbb{R}^3$.

2 SΘ(4)-invariant and Θ(4)-invariant tensor fields on $\mathbb{R} \times \mathbb{R}^3$.

The metric tensor is conceived naturally as a tensor field invariant by the action of the rotation group $SO(3)$. However, although $SO(3)$ acts naturally on $\mathbb{R}^3$, it does not the same on $\mathbb{R} \times \mathbb{R}^3$, and this is why we are led to introduce the group SΘ(4) consisting of the matrices

$$\begin{pmatrix} 1 & O_H \\ O_V & A \end{pmatrix}$$

with $O_H = (0, 0, 0)$, $O_V = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $A \in SO(3)$. We introduce also the group Θ(4) consisting of the matrices of the same form for which $A \in O(3)$. Obviously SΘ(4) is a subgroup of Θ(4).

With these notations, the metric tensor related to the isotropic distribution of matter is conceived as a SΘ(4)-invariant tensor field on $\mathbb{R} \times \mathbb{R}^3$. SΘ(4)-invariant tensor fields appear in several problems of relativity, so that it is convenient to study them in detail. Their rigorous theory appears in a previous paper [7] together with the theory of the *pure SΘ(4)-invariant tensor fields* which are not used in the present paper.

It is easily seen that a function $h(x_0, x_1, x_2, x_3)$ is SΘ(4)-invariant (or Θ(4)-invariant) if and only if it is of the form $f(x_0, \|x\|)$. Of course we confine ourselves to the case where $f(x_0, \|x\|)$ is C^∞ with respect to the coordinates x_0, x_1, x_2, x_3 on $\mathbb{R} \times \mathbb{R}^3$.

Proposition 2.1 *$f(x_0, \|x\|)$ is C^∞ on $\mathbb{R} \times \mathbb{R}^3$ if and only if the function $f(x_0, u)$ with $(x_0, u) \in \mathbb{R} \times [0, +\infty[$ is C^∞ on $\mathbb{R} \times [0, +\infty[$ and such that its derivatives of odd order with respect to u at $u = 0$ vanish.*

The functions satisfying these conditions constitute an algebra which will be denoted by Γ_0. As a corollary, we see that $f(x_0, \|x\|)$ belongs to Γ_0 if and only if the function $h(x_0, u)$ defined by setting

$$h(x_0, u) = h(x_0, -u) = f(x_0, u), \quad u \geqslant 0$$

is C^∞ on $\mathbb{R} \times \mathbb{R}$. It follows in particular that, if the function $f(x_0, \|x\|)$ belongs to Γ_0 and is strictly positive, then the functions $\frac{1}{f(x_0, \|x\|)}$ and $\sqrt{f(x_0, \|x\|)}$ belong also to Γ_0. Now, if $T(x_0, x)$, $x = (x_1, x_2, x_3)$, is an SΘ(4)-invariant (or Θ(4)-invariant) tensor field on $\mathbb{R} \times \mathbb{R}^3$, then, for every function $f \in \Gamma_0$, the tensor field $f(x_0, \|x\|)\, T(x_0, x)$ is also SΘ(4)-invariant (or Θ(4)-invariant). Consequently the set of SΘ(4)-invariant (or Θ(4)-invariant) tensor fields constitutes a Γ_0-module. In particular, we are interesting in the sub-module consisting of the covariant tensor fields of degree 2. The proof of the following proposition is given in the paper [7].

Proposition 2.2 *Let $T(x_0, x)$ be an SΘ(4)-invariant C^∞ covariant symmetric tensor field of degree 2 on $\mathbb{R} \times \mathbb{R}^3$. Then there exist four functions $q_{00} \in \Gamma_0$, $q_{01} \in \Gamma_0$, $q_{11} \in \Gamma_0$, $q_{22} \in \Gamma_0$ such that*

$$\begin{aligned} T(x_0, x) &= q_{00}(x_0, \|x\|)(dx_0 \otimes dx_0) + \\ &+ q_{01}(x_0, \|x\|)\big(dx_0 \otimes F(x) + F(x) \otimes dx_0\big) + \\ &+ q_{11}(x_0, \|x\|)\, E(x) + q_{22}(x_0, \|x\|)\big(F(x) \otimes F(x)\big), \end{aligned}$$

where $E(x) = \sum_1^3 (dx_i \otimes dx_i)$ and $F(x) = \sum_1^3 x_i dx_i$. Moreover $T(x_0, x)$ is Θ(4)-invariant.

So, the components $g_{\alpha\beta}$ of $T(x_0, x)$ are defined by means of the four functions q_{00}, q_{01}, q_{11}, q_{22} as follows

$$g_{00} = q_{00}, \qquad g_{0i} = g_{i0} = x_i q_{01},$$
$$g_{ii} = q_{11} + x_i^2 q_{22}, \qquad g_{ij} = x_i x_j q_{22},$$

where $i, j = 1, 2, 3$; $i \neq j$. Suppose now that the tensor field $T(x_0, x)$ is a metric tensor, namely a symmetric tensor field of signature $(+1, -1, -1, -1)$. Then we write it usually as a quadratic form

$$ds^2 = q_{00}\, dx_0^2 + 2q_{01}(xdx)\, dx_0 + q_{11} dx^2 + q_{22}(xdx)^2.$$

Since $x_0 = t$ is the time coordinate, we have $q_{00} = q_{00}(x_0, \|x\|) > 0$ for all $(x_0, x) \in \mathbb{R}\times\mathbb{R}^3$, so the function $f = f(x_0, \|x\|) = \sqrt{q_{00}(x_0, \|x\|)}$ is strictly positive and C^∞ on $\mathbb{R}\times\mathbb{R}^3$. Consequently the function $f_1 = \frac{q_{01}}{f}$ is also C^∞ on $\mathbb{R}\times\mathbb{R}^3$, namely a function belonging to Γ_0, and we can write the metric into the form

$$ds^2 = \big(f dt + f_1 (x dx)\big)^2 + q_{11} dx^2 + \big(q_{22} - f_1^2\big)(x dx)^2$$

which makes explicit the corresponding spatial (positive definite) metric $-q_{11} dx^2 - \big(q_{22} - f_1^2\big)(x dx)^2$ with $-q_{11} > 0$ and $-q_{11} - \big(q_{22} - f_1^2\big)\|x\|^2 > 0$ on $\mathbb{R}\times\mathbb{R}^3$. So we can introduce the strictly positive C^∞ functions

$$\ell_1 = \ell_1(t, \|x\|) = \sqrt{-q_{11}(t, \|x\|)}$$

and

$$\ell = \ell(t, \|x\|) = \sqrt{\ell_1^2 - \|x\|^2\big(q_{22} - f_1^2\big)}$$

which possess a clear geometrical meaning:

ℓ_1 serves to define the curvature radius $g(t,\rho) = g(t, \|x\|) = \|x\|\,\ell_1(t, \|x\|) = \rho\,\ell_1(t,\rho)$, $(\rho = \|x\|)$, of the non-Euclidean spheres centered at the origin of $\mathbb{R}^3$, whereas ℓ defines the element of length on the spatial radial geodesics.

Consequently it is very convenient to put the metric into a form exhibiting explicitly ℓ_1 and ℓ. This is obtained by remarking that the C^∞ function $q_{22} - f_1^2$ can be written as

$$\frac{\ell_1^2 - \ell^2}{\rho^2}\,.$$

Of course the last expression is C^∞ everywhere on account of the condition $\ell_1(t, 0) = \ell(t, 0)$ and the fact that $\ell_1 \in \Gamma_0$, $\ell \in \Gamma_0$. It follows that

$$ds^2 = \big(f dt + f_1 (x dx)\big)^2 - \ell_1^2 dx^2 - \frac{\ell^2 - \ell_1^2}{\rho^2}(x dx)^2 \quad (2.1)$$

or

$$ds^2 = f^2 dt^2 + 2 f f_1 (x dx)\, dt - \ell_1^2 dx^2 + \left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2\right)(x dx)^2 \quad (2.2)$$

with the components

$$g_{00} = f^2, \qquad g_{0i} = g_{i0} = x_i f f_1\,,$$

$$g_{ii} = -\ell_1^2 + x_i^2\left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2\right),$$

$$g_{ij} = x_i x_j\left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2\right), \qquad i, j = 1, 2, 3; \;\; i \neq j\,.$$

There are two significant functions which do not appear in (2. 1) and are not C^∞ on $\mathbb{R}\times\mathbb{R}^3$:

1. First the already considered curvature radius $g(t,\rho) = \rho\,\ell_1(t,\rho)$ of the non-Euclidean spheres centered at the origin;
2. Secondly the function $h(t,\rho) = \rho f_1(t,\rho)$ which appears in the equations defining the radial motions of photons outside the matter, namely the equations

$$(f dt + f_1 \rho\, d\rho)^2 = \ell^2 d\rho^2 \quad \text{or} \quad f dt + \rho f_1 d\rho = \pm\ell\, d\rho$$

which imply necessarily $|h| \leqslant \ell$ in order that both the ingoing and outgoing motions be possible [4]. In any case the condition $|h| \leqslant \ell$ must also be valid inside the matter in order that the nature of the variable t as time coordinate be preserved. Moreover h vanishes for $\rho = 0$.

Of course g and h are C^∞ with respect to $(t,\rho) \in \mathbb{R}\times[0, +\infty[$, but since $\rho = \|x\|$ is not differentiable at the origin, they are not differentiable on the subspace $\mathbb{R}\times\{(0,0,0)\}$ of $\mathbb{R}\times\mathbb{R}^3$. However, on account of their geometrical and physical significance, we introduce them in the computations remembering that, for any global solution on $\mathbb{R}\times\mathbb{R}^3$, the functions $\ell_1 = \frac{g}{\rho}$ and $f_1 = \frac{h}{\rho}$ appearing in (2.1) must be elements of the algebra Γ_0.

3 The Ricci tensor and the equations of gravitation

In order to obtain the equations of gravitation related to (2.1), we have first to introduce the Christoffel symbols and then compute the components of the Ricci tensor. At first sight the computations seem to be extremely complicated, but the $\Theta(4)$-invariance of the metric allows to obtain a great deal of simplification in accordance with the following proposition, the proof of which is given in the paper [8].

Proposition 3.1 **(a)** *The Christoffel symbols of the first kind as well as those of the second kind related to (2.2) are the components of a $\Theta(4)$-invariant tensor field*; **(b)** *The curvature tensor, the Ricci tensor, and the scalar curvature related to (2.2) are $\Theta(4)$-invariant*; **(c)** *If an energy-momentum tensor satisfies the equations of gravitation related to (2.2), it is $\Theta(4)$-invariant*.

Corollary 3.1. *The Christoffel symbols of the second kind related to (2.2) depend on ten C^∞ functions $B_\alpha = B_\alpha(t,\rho)$, $(\alpha = 0, 1, 2, \ldots, 9)$, as follows*:

$$\Gamma^0_{00} = B_0\,, \quad \Gamma^0_{0i} = \Gamma^0_{i0} = B_1 x_i\,, \quad \Gamma^i_{00} = B_2 x_i\,,$$

$$\Gamma^0_{ii} = B_3 + B_4 x_i^2, \qquad \Gamma^0_{ij} = \Gamma^0_{ji} = B_4 x_i x_j\,,$$

$$\Gamma^i_{i0} = \Gamma^i_{0i} = B_5 + B_6 x_i^2, \qquad \Gamma^i_{j0} = \Gamma^i_{0j} = B_6 x_i x_j\,,$$

$$\Gamma^i_{ii} = B_7 x_i^3 + (B_8 + 2B_9)\, x_i\,,$$

$$\Gamma^i_{jj} = B_7 x_i x_i^2 + B_8 x_i\,, \quad \Gamma^j_{ij} = \Gamma^j_{ji} = B_7 x_i x_j^2 + B_9 x_i\,,$$

$$\Gamma^i_{jk} = B_7 x_i x_j x_k\,, \qquad i, j, k = 1, 2, 3; \;\; j \neq k \neq i\,.$$

Regarding the Ricci tensor $R_{\alpha\beta}$, since it is symmetric and $\Theta(4)$-invariant, its components are defined, according to proposition 2.2, by four functions $Q_{00} = Q_{00}(t,\rho)$, $Q_{01} = Q_{01}(t,\rho)$, $Q_{11} = Q_{11}(t,\rho)$, $Q_{22} = Q_{22}(t,\rho)$ as follows:

$$R_{00} = Q_{00}\,, \quad R_{0i} = R_{i0} = Q_{01} x_i\,, \quad R_{ii} = Q_{11} + x_i^2 Q_{22}\,,$$

$$R_{ij} = x_i x_j Q_{22}\,, \qquad i, j = 1, 2, 3; \;\; i \neq j\,.$$

In the same way, an energy-momentum tensor $W_{\alpha\beta}$ satisfying the equations of gravitation related to (2.2) is defined by four functions of (t, ρ), say E_{00}, E_{01}, E_{11}, E_{22}:

$$W_{00} = E_{00}\,, \quad W_{0i} = x_i E_{01}\,, \quad W_{ii} = E_{11} + x_i^2 E_{22}\,,$$
$$W_{ij} = x_i x_j E_{22}\,, \qquad i, j = 1, 2, 3; \;\; i \neq j\,.$$

Moreover, since the scalar curvature $R = Q$ is $\Theta(4)$-invariant, it is a function of (t, ρ): $R = Q = Q(t, \rho)$.

It follows that the equations of gravitation (with cosmological constant -3λ)

$$R_{\alpha\beta} - \left(\frac{Q}{2} + 3\lambda\right) g_{\alpha\beta} + \frac{8\pi k}{c^4} W_{\alpha\beta} = 0$$

can be written from the outset as a system of four equations depending uniquely on (t, ρ):

$$Q_{00} - \left(\frac{Q}{2} + 3\lambda\right) f^2 + \frac{8\pi k}{c^4} E_{00} = 0\,,$$
$$Q_{01} - \left(\frac{Q}{2} + 3\lambda\right) f f_1 + \frac{8\pi k}{c^4} E_{01} = 0\,,$$
$$Q_{11} + \left(\frac{Q}{2} + 3\lambda\right) \ell_1^2 + \frac{8\pi k}{c^4} E_{11} = 0\,,$$
$$Q_{22} - \left(\frac{Q}{2} + 3\lambda\right) \left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2\right) + \frac{8\pi k}{c^4} E_{22} = 0\,.$$

Note that it is often convenient to replace the last equation by the equation

$$Q_{11} + \rho^2 Q_{22} - \left(\frac{Q}{2} + 3\lambda\right)\left(\rho^2 f_1^2 - \ell^2\right) + \frac{8\pi k}{c^4}\left(E_{11} + \rho^2 E_{22}\right) = 0\,.$$

In order to apply these equations to special situations, it is necessary to give the explicit expressions of Q_{00}, Q_{01}, Q_{11}, Q_{22} by means of the functions B_α, $(\alpha = 0, 1, 2, ..., 9)$, appearing in the Christoffel symbols. We recall the results of computation

$$Q_{00} = \frac{\partial}{\partial t}\left(3B_5 + \rho^2 B_6\right) - \rho\,\frac{\partial B_2}{\partial \rho} -$$
$$- B_2\left(3 + 4\rho^2 B_9 - \rho^2 B_1 + \rho^2 B_8 + \rho^2 B_7\right) -$$
$$- 3B_0 B_5 + 3B_5^2 + \rho^2 B_6\left(-B_0 + 2B_5 + \rho^2 B_6\right),$$

$$Q_{01} = \frac{\partial}{\partial t}\left(\rho^2 B_7 + B_8 + 4B_9\right) - \frac{1}{\rho}\frac{\partial B_5}{\partial \rho} - \rho\,\frac{\partial B_6}{\partial \rho} +$$
$$+ B_2\left(B_3 + \rho^2 B_4\right) - 2B_6\left(2 + \rho^2 B_9\right) - B_1\left(3B_5 + \rho^2 B_6\right),$$

$$Q_{11} = -\frac{\partial B_3}{\partial t} - \rho\frac{\partial B_8}{\partial \rho} - \left(B_0 + B_5 + \rho^2 B_6\right) B_3 +$$
$$+ \left(1 - \rho^2 B_8\right)\left(B_1 + \rho^2 B_7 + B_8 + 2B_9\right) - 3B_8\,,$$

$$Q_{22} = -\frac{\partial B_4}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(B_1 + B_8 + 2B_9\right) + B_1^2 + B_8^2 -$$
$$- 2B_9^2 - 2B_1 B_9 + 2B_3 B_6 + \left(-B_0 - B_5 + \rho^2 B_6\right) B_4 +$$
$$+ \left(-3 + \rho^2\left(-B_1 + B_8 - 2B_9\right)\right) B_7\,.$$

4 Stationary vacuum solutions

The radial motion of the isotropic spherical distribution of matter generates a non-stationary (dynamical) gravitational field extending beyond the boundary in the exterior space. This field is defined by non-stationary $\Theta(4)$-invariant vacuum solutions of the equations of gravitation and exhibits essential and unusual features related to the propagation of gravitation. Several problems related to it are not yet clarified. But, in any case, in order to establish and understand the dynamical solutions, a previous knowledge of the stationary solutions is necessary. This is why, in the sequel we confine ourselves to the simple problems related to the stationary vacuum solutions. So we suppose that we have a stationary metric

$$ds^2 = \left(f dt + f_1\,(x dx)\right)^2 - \ell_1^2 dx^2 - \frac{\ell^2 - \ell_1^2}{\rho^2}\,(x dx)^2\,, \quad (4.1)$$

where $f = f(\rho)$, $f_1 = f_1(\rho)$, $\ell_1 = \ell_1(\rho)$, $\ell = \ell(\rho)$.

Of course, we have also to take into account the significant functions

$$h = h(\rho) = \rho f_1(\rho)\,, \qquad g = g(\rho) = \rho \ell_1(\rho)\,,$$

which are not differentiable at the origin $(0, 0, 0)$. Every half-line issuing from the origin, $x_1 = \alpha_1\rho$, $x_2 = \alpha_2\rho$, $x_3 = \alpha_3\rho$ (where $0 \leqslant \rho < +\infty$ and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$) is a geodesic of the spatial metric $\ell_1^2 dx^2 + \frac{\ell^2 - \ell_1^2}{\rho^2}\,(x dx)^2$ so that the geodesic distance δ of the origin from the point $x = (x_1, x_2, x_3)$ is defined by the integral

$$\delta = \int_0^\rho \ell(u) du\,, \qquad \rho = \|x\|.$$

As already noticed, the function $\ell(\rho)$, where $0 \leqslant \rho < +\infty$, is strictly positive, but it cannot be arbitrarily given. Suppose, for instance, that

$$\ell(\rho) = \frac{\epsilon}{\rho^2}\,, \qquad \epsilon = \text{const} > 0$$

on $[1, +\infty[$. Then the geodesic distance $\delta = \int_0^1 \ell(u) du + \int_1^\rho \frac{\epsilon}{u^2} du = \int_0^1 \ell(u) du + \epsilon - \frac{\epsilon}{\rho}$ tends to the finite value $\int_0^1 \ell(u) du + \epsilon$ as $\rho \to \infty$, which cannot be physically accepted. Consequently the positive function $\ell(\rho)$ is allowable only if the integral $\int_0^\rho \ell(u) du$ tends to $+\infty$ as $\rho \to +\infty$.

This being said, it is easy to see that the functions $B_\alpha = B_\alpha(\rho)$, $(\alpha = 0, 1, \ldots, 9)$, occurring in the Christoffel symbols resulting from the stationary metric (4.1) are defined by the following formulae:

$$B_0 = -\frac{hf'}{\ell^2}\,, \quad B_1 = \frac{f'}{\rho f} - \frac{h^2 f'}{\rho f \ell^2}\,,$$
$$B_2 = \frac{f f'}{\rho \ell^2}\,, \quad B_3 = \frac{hgg'}{\rho^2 f \ell^2}\,,$$
$$B_4 = \frac{hf'}{\rho^2 f^2} - \frac{h^3 f'}{\rho^2 f^2 \ell^2} + \frac{h'}{\rho^2 f} - \frac{h\ell'}{\rho^2 f \ell} - \frac{hgg'}{\rho^4 f \ell^2}\,,$$

$$B_5 = 0, \qquad B_6 = \frac{hf'}{\rho^2 \ell^2},$$
$$B_7 = \frac{h^2 f'}{\rho^3 f \ell^2} + \frac{\ell'}{\rho^3 \ell} + \frac{gg'}{\rho^5 \ell^2} - \frac{2g'}{\rho^3 g} + \frac{1}{\rho^4},$$
$$B_8 = \frac{1}{\rho^2} - \frac{gg'}{\rho^3 \ell^2}, \qquad B_9 = -\frac{1}{\rho^2} + \frac{g'}{\rho g}.$$

Then inserting these expressions in the formulae brought out at the end of the previous section, we find the functions

$$Q_{00} = f\left(-\frac{f''}{\ell^2} + \frac{f'\ell'}{\ell^3} - \frac{2f'g'}{\ell^2 g}\right), \qquad g = \rho\ell_1,$$
$$Q_{01} = \frac{h}{\rho f} Q_{00}, \qquad h = \rho f_1,$$
$$Q_{11} = \frac{1}{\rho^2}\left(-1 + \frac{g'^2}{\ell^2} + \frac{gg''}{\ell^2} - \frac{\ell' gg'}{\ell^3} + \frac{f'gg'}{f\ell^2}\right),$$
$$Q_{11} + \rho^2 Q_{22} = \frac{f''}{f} + \frac{2g''}{g} - \frac{f'\ell'}{f\ell} - \frac{2\ell' g'}{\ell g} + \frac{h^2}{f^2} Q_{00},$$

which are everywhere valid, namely outside as well as inside the matter, Specifically, by using them, we can establish the gravitational equations outside the matter with electromagnetic field and cosmological constant. However, in the present short account, our purpose is to put forward the most significant elementary facts, and this is why we confine ourselves to the pure gravitational field outside the matter without cosmological constant. Then $Q = R = 0$, $\lambda = 0$, so that $Q_{00} = 0$, $Q_{01} = 0$, $Q_{11} = 0$, $Q_{11} + \rho^2 Q_{22} = 0$. Since $Q_{00} = 0$ implies $Q_{01} = 0$, we have finally the following three equations

$$-f'' + \frac{f'\ell'}{\ell} - \frac{2f'g'}{g} = 0, \tag{4.2}$$
$$-1 + \frac{g'^2}{\ell^2} + \frac{gg''}{\ell^2} - \frac{\ell' gg'}{\ell^3} + \frac{f'gg'}{f\ell^2} = 0, \tag{4.3}$$
$$f'' + \frac{2fg''}{g} - \frac{f'\ell'}{\ell} - \frac{2f\ell' g'}{\ell g} = 0, \tag{4.4}$$

By adding (4.2) to (4.4) we obtain

$$\frac{f'g'}{f} = g'' - \frac{\ell' g'}{\ell} \tag{4.5}$$

and inserting this expression of $\frac{f'g'}{f}$ into (4.3), we find the equation

$$-1 + \frac{g'^2}{\ell^2} + \frac{2gg''}{\ell^2} - \frac{2\ell' gg'}{\ell^3} = 0$$

which implies $\frac{d}{d\rho}\left(-g + \frac{gg'^2}{\ell^2}\right) = 0$ so that

$$-g + \frac{gg'^2}{\ell^2} = -2A = \text{const.} \tag{4.6}$$

On the other hand (4.5) can be written as $(f\ell)'g' = (f\ell)\, g''$ whence $\frac{d}{d\rho}\left(\frac{g'}{f\ell}\right) = 0$ and

$$f\ell = cg', \quad (\text{where } c = \text{const}). \tag{4.7}$$

The equations (4.6) and (4.7) define the general stationary solution outside the matter. The function h does not appear in them, but it is not empty of physical meaning as is usually believed. It occurs in the problem as a function satisfying the condition $|h| \leqslant \ell$. The different allowable choices of h correspond to different significations of the time coordinate.

Proposition 4.1. *If $A = 0$, the solution defined by (4.6) and (4.7) is a pseudo-Euclidean metric (or, better, a family of pseudo-Euclidean metrics).*

Proof. On account of $A = 0$, (4.6) implies $g' = \ell$ and next (4.7) gives $f = c$. Referring to (4.1) and setting $\int_0^\rho v f(v) dv = \alpha(\rho)$, we have

$$d\alpha(\rho) = \rho f_1(\rho) d\rho = f_1(\rho) x dx$$

and

$$f(\rho) dt + f_1(\rho) x dx = d\big(ct + \alpha(\rho)\big),$$

which suggests the transformation $\tau = t + \frac{\alpha(\rho)}{c}$. On the other hand, since $\ell = g' = (\rho\ell_1)' = \rho\ell_1' + \ell_1$, we have

$$\ell_1^2 dx^2 + \frac{\ell^2 - \ell_1^2}{\rho^2}(x dx)^2 = \ell_1^2 dx^2 + 2\ell_1\ell_1' \frac{(xdx)^2}{\|x\|} + \ell_1'^2 (xdx)^2 =$$
$$= \left(\ell_1 dx_1 + x_1\ell_1' \frac{xdx}{\rho}\right)^2 + \left(\ell_1 dx_2 + x_2\ell_1' \frac{xdx}{\rho}\right)^2 +$$
$$+ \left(\ell_1 dx_3 + x_3\ell_1' \frac{xdx}{\rho}\right)^2 = \big(d(\ell_1 x_1)\big)^2 + \big(d(\ell_1 x_2)\big)^2 + \big(d(\ell_1 x_3)\big)^2$$

so that by setting $y_1 = \ell_1 x_1$, $y_2 = \ell_1 x_2$, $y_3 = \ell_1 x_3$, we obtain the metric in the standard pseudo-Euclidean form $ds^2 = c^2 d\tau^2 - (dy_1^2 + dy_2^2 + dy_3^2)$. In the sequel we give up this trivial case and assume $A \neq 0$.

5 Punctual sources of the gravitational field do not exist

(4.6) is a first order differential equation with respect to the unknown function $g = g(\rho)$, so that its general solution depends on an arbitrary constant. But (4.6) contains already the constant A and moreover the function $\ell = \ell(\rho)$ which is not given. Consequently the general solution of (4.6) contains two constants. Moreover, it seems that it depends on the function $\ell(\rho)$, namely that to every allowable function $\ell(\rho)$ there corresponds a solution of (4.6) depending on two constants. However, we can prove that the function $\ell(\rho)$ is not actually involved in the general solution of (4.6).

Since the geodesic distance $\delta = \int_0^\rho \ell(u) du = \beta(\rho)$ is a strictly increasing function of ρ tending to $+\infty$ as $\rho \to +\infty$, the inverse function $\rho = \gamma(\delta)$ is also a strictly increasing function of δ tending to $+\infty$ as $\delta \to +\infty$. Consequently $g(\rho)$ can be considered as a function of δ:

$$G(\delta) = g\big(\gamma(\delta)\big).$$

It follows that the determination of $G(\delta)$ as a function of the geodesic distance δ, which possesses an intrinsic meaning with respect to the stationary metric, allows its definition with respect to any other radial coordinate depending diffeomorphically on δ.

Now, since $\delta = \beta\left(\gamma(\delta)\right)$, we have $1 = \frac{d\beta}{d\rho}\frac{d\rho}{d\delta} = \ell(\rho)\gamma'(\delta)$ and $G' = G'(\delta) = g'(\rho)\gamma'(\delta) = \frac{g'(\rho)}{\ell(\rho)}$, so that the equation (4.6) takes the form $-G + GG'^2 = -2A$ or

$$GG'^2 = G - 2A \tag{5.1}$$

which does not contain the function ℓ.

Regarding (4.7), it is obviously replaced by the equation

$$F = cG'$$

with $F = F(\delta) = f\left(\gamma(\delta)\right)$. The functions F and G are related to a stationary metric which results from the stationary metric (4.1) by the introduction of the new space coordinates:

$$y_i = \frac{\delta}{\rho}\,x_i = \frac{\beta(\rho)}{\rho}\,x_i\,, \tag{5.2}$$

where $i=1,2,3$; $\|y\|=\delta$; $\|x\|=\rho$. This transformation is C^∞ everywhere, even at the origin, because the function $B(\rho) = \frac{\beta(\rho)}{\rho}$ (where $B(0)=\ell(0)$) belongs to the algebra Γ_0. In fact, since $\beta'(\rho) = \ell(\rho)$, we have $\beta(\rho) = \rho\int_0^1 \beta'(\rho u)du = \rho\int_0^1 \ell(\rho u)du$ and

$$B(\rho) = \int_0^1 \ell(\rho u)du\,,$$

consequently $B^{(2m+1)}(\rho) = \int_0^1 \ell^{(2m+1)}(\rho u)\,u^{2m+1}du$ and since $\ell\in\Gamma_0$ implies $\ell^{(2m+1)}(0) = 0$, we obtain

$$B^{(2m+1)}(0) = 0\,, \qquad (m = 0,1,2,3,\dots)$$

and, from proposition 2.1, it follows that $B \in \Gamma_0$.

The inverse of (5.2) is defined by the equations

$$x_i = \Delta(\delta)\,y_i\,, \qquad i = 1,2,3\,, \tag{5.3}$$

where $\Delta(\delta) = \frac{\rho}{\beta(\rho)} = \frac{\gamma(\delta)}{\delta}$. Since $\gamma(\delta) = \delta\int_0^1 \gamma'(\delta u)du = \delta\int_0^1 \frac{du}{\ell(\gamma(\delta u))}$, it can be shown by induction that the function $\Delta(\delta) = \frac{\gamma(\delta)}{\delta} = \int_0^1 \frac{du}{\ell(\gamma(\delta u))}$ is an element of the algebra Γ_0, so that (5.3) is universally valid. A simple computation gives

$$xdx = \sum_1^3 x_i\,dx_i = \frac{\gamma\gamma'}{\delta}\,(ydy)\,,$$

$$dx^2 = \sum_1^3 dx_i^2 = \left(\frac{\gamma'^2}{\delta^2} - \frac{\gamma^2}{\delta^4}\right)(ydy)^2 + \frac{\gamma^2}{\delta^2}dy^2$$

so that, by setting $F(\delta)=f\left(\gamma(\delta)\right)$, $F_1(\delta)=f_1\left(\gamma(\delta)\right)\frac{\gamma(\delta)\gamma'(\delta)}{\delta}$, $L_1(\delta) = \ell_1\left(\gamma(\delta)\right)\frac{\gamma(\delta)}{\delta}$, $L(\delta) = \ell\left(\gamma(\delta)\right)\gamma'(\delta) = 1$, we obtain the transformed metric

$$ds^2 = \left(Fdt + F_1(ydy)\right)^2 - \left(L_1^2dy^2 + \frac{1-L_1^2}{\delta^2}(ydy)^2\right) \tag{5.4}$$

which is related to the geodesic distance $\delta = \|y\|$ and the functions F and G. Instead of $h(\rho)$, we have now the function $H = H(\delta) = \delta F_1(\delta)$, and moreover the invariant curvature radius of the spheres $\delta = \text{const.}$ is given by the function

$$G = G(\delta) = \delta L_1(\delta)\,.$$

Before solving the equation (5.1), we can anticipate that the values of the solution $G(\delta)$ do not cover the whole half-line $[0,+\infty[$ or, possibly, the whole open half-line $]0,+\infty[$, because by taking a sequence of positive values $\delta_n \to 0$, we have $G(\delta_n)\to 0$ and then the equation (5.1) implies $A=0$ contrary to our assumption $A\neq 0$. (This conclusion follows also from (4.6), because $g(0)=0$ implies $A=0$.) So, we are led to anticipate that the values of the solution $G(\delta)$ cover a half-line $[\alpha,+\infty[$ with $\alpha>0$. This important property, which implies that the source of the field cannot be reduced to a point, will be verified by the explicit expression of the solution.

Now, since $G(\delta)\geqslant \alpha > 0$ and $G - 2A \geqslant 0$ according to (5.1), the function $G(\delta)$ is obtained by the equation

$$\frac{dG}{d\delta} = \sqrt{1 - \frac{2A}{G}}$$

and since $\sqrt{1-\frac{2A}{G}} > 0$, $G(\delta)$ is a strictly increasing function of δ. Moreover $G(\delta)$ can not remain bounded because $\frac{dG}{d\delta}\to 1$ as $G\to+\infty$.

Technically, we have first to obtain the inverse function $\delta = P(G)$ by integrating the equation

$$\frac{d\delta}{dG} = \frac{1}{\sqrt{1-\frac{2A}{G}}}$$

which implies also that $\delta = P(G)$ is a strictly increasing and not bounded function of G. Now, we introduce an auxiliary fixed positive length ξ which will not appear in the final result, but it is needed in order to carry out correctly the computations. In fact, since G, A, $G-2A$ represent also lengths, the ratios $\frac{G}{\xi}$, $\frac{G-2A}{\xi}$ are dimensionless, so that we can introduce the logarithm

$$\ln\left(\sqrt{\frac{G}{\xi}} + \sqrt{\frac{G-2A}{\xi}}\right)$$

and since $\frac{d}{dG}\left(\sqrt{G\,(G-2A)} + 2A\ln\left(\sqrt{\frac{G}{\xi}} + \sqrt{\frac{G-2A}{\xi}}\right)\right) = \frac{1}{\sqrt{1-\frac{2A}{G}}}$ the preceding equation gives $\delta = P(G)$,

$$\delta = B + \sqrt{G(G-2A)} + 2A\ln\left(\sqrt{\frac{G}{\xi}} + \sqrt{\frac{G-2A}{\xi}}\right) \tag{5.5}$$

where $B = \text{const.}$ It follows that

$$\frac{\delta}{G(\delta)} = \frac{P(G)}{G} = \frac{B}{G} + \sqrt{1-\frac{2A}{G}} + \frac{2A}{G}\ln\left(\sqrt{\frac{G}{\xi}} + \sqrt{\frac{G-2A}{\xi}}\right)$$

and since we have $\frac{2A}{G}\ln\left(\sqrt{\frac{G}{\xi}} + \sqrt{\frac{G-2A}{\xi}}\right) = \frac{2A}{G}\ln\sqrt{\frac{G}{\xi}} + \frac{2A}{G}\ln\left(1 + \sqrt{1-\frac{2A}{G}}\right) \to 0$ as $G\to+\infty$ we have

$$\frac{\delta}{G(\delta)} = 1 + \epsilon(\delta), \qquad \frac{G(\delta)}{\delta} = \frac{1}{1+\epsilon(\delta)}$$

with $\epsilon(\delta) \to 0$ as $\delta \to +\infty$. This property allows to determine the constant A by using the so-called Newtonian approximation of the metric (5.4) for the great values of the distance δ. Classically this approximation is referred to the static metric, namely to the case where $F_1 = 0$. We have already seen that $|\delta F_1(\delta)| \leqslant 1$, but this condition does not imply that $\delta F_1(\delta)$ possesses a limit as $\delta \to +\infty$. So we accept the condition $F_1(\delta) = 0$ for the derivation of the Newtonian approximation, without forgetting that we have to do with a specific choice of F_1 used for convenience in the case of a special problem.

This being said, the Newtonian approximation is obtained by setting $\epsilon(\delta) = 0$ and $F_1 = 0$. Then since $F = cG' = c\sqrt{1-\frac{2A}{G}} = c\sqrt{1-\frac{2A}{\delta}-\frac{2A\epsilon(\delta)}{\delta}}$, $1 - L_1^2 = 1 - \left(\frac{1}{1+\epsilon(\delta)}\right)^2$, and $\frac{\|y\|}{\delta} = 1$, we get the form

$$ds^2 = c^2\left(1 - \frac{2A}{\delta}\right)dt^2 - dy^2$$

which, by means of a known argument, leads to identify $\frac{c^2 A}{\delta}$ with $\frac{km}{\delta}$, whence $A = \frac{km}{c^2} = \mu$.

Since $G - 2A \geqslant 0$, we have $G(\delta) \geqslant 2\mu$, so that, as anticipated, $G(\delta)$ possesses the strictly positive greatest lower bound 2μ, which, as we see, is independent of the second constant B appearing in the solution (5.5). It follows that the strictly increasing function $G(\delta)$ appears as an implicit function defined by the equation

$$\delta = B + \sqrt{G\,(G-2\mu)} + 2\mu \ln\left(\sqrt{\frac{G}{\xi}} + \sqrt{\frac{G-2\mu}{\xi}}\right).$$

The greatest lower bound 2μ is obtained for $\delta = B + 2\mu\ln\sqrt{\frac{2\mu}{\xi}}$ and this is why it is convenient to introduce, instead of B, the constant $\delta_0 = B + 2\mu\ln\sqrt{\frac{2\mu}{\xi}}$, which allows to write $\delta = \delta_0 + \sqrt{G\,(G-2\mu)} + 2\mu\ln\left(\sqrt{\frac{G}{2\mu}} + \sqrt{\frac{G}{2\mu}-1}\right)$ or

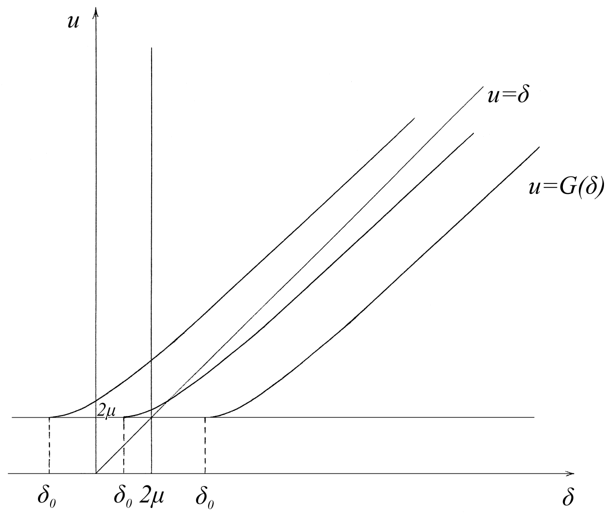
$$\delta = \delta_0 + \int_{2\mu}^{G} \frac{du}{\sqrt{1-\frac{2\mu}{u}}}, \qquad G = G(\delta) \geqslant 2\mu$$

which does not contain the auxiliary length ξ. The solution is completed by the determination of the function

$$F = cG' = c\sqrt{1 - \frac{2\mu}{G\,(\delta)}}\,.$$

As far as $H(\delta) = \delta F_1(\delta)$ is concerned, we repeat that it is introduced simply as a C^∞ function vanishing for $\delta = 0$ and satisfying the condition $|H(\delta)| \leqslant 1$.

What about the new constant δ_0? From the mathematical point of view, negative values of δ_0 are not excluded. So, we distinguish the following cases (see Figure):



(a) $\delta_0 < 0$. Then the values of $G(\delta)$ for $\delta_0 \leqslant \delta < 0$ are meaningless physically, because $G(\delta)$ is conceived on $[0, +\infty[$. But the value $\delta = 0$ is also excluded because

$$\int_{2\mu}^{G(0)} \frac{du}{\sqrt{1-\frac{2\mu}{u}}} = -\delta_0 > 0$$

implies $G(0) > 2\mu$ contrary to the geometrical condition $G(0) = 0$. Consequently there exists a constant $\delta_1 > 0$ (the radius of the considered distribution of matter) such that only the restriction of $G(\delta)$ to $[\delta_1, +\infty[$ is taken into account.

(b) $\delta_0 = 0$. Then

$$\int_{2\mu}^{G(0)} \frac{du}{\sqrt{1-\frac{2\mu}{u}}} = 0$$

so that $G(0) = 2\mu$ contrary to the geometrical condition $G(0) = 0$. Consequently the solution is valid, as previously, on a half-line $[\delta_1, +\infty[$ with $\delta_1 > 0$.

(c) $\delta_0 > 0$. Then $G(\delta_0) = 2\mu$, $F(\delta_0) = 0$, so that the metric degenerates for $\delta = \delta_0$. A degenerate metric does not possess physical meaning. Consequently, there exists a constant $\delta_1 > \delta_0$ (the radius of the sphere bounding the matter) such that the solution is physically valid only on the half-line $[\delta_1, +\infty[$.

Whatever the case may be, the vacuum solution is not defined for $\delta < \delta_1$. In other words, the ball $\|y\| \leqslant \delta_1$ is occupied by matter, so that the source of the field cannot be reduced to a point. The constant δ_0 is related to a boundary condition, namely the value of the curvature radius of the sphere bounding the matter. In fact, if δ_1 is the radius of this sphere, and the value $G(\delta_1)$ is known, then the value δ_0 is easily obtained:

$$\delta_0 = \delta_1 - \sqrt{G(\delta_1)\,(G(\delta_1) - 2\mu)} - 2\mu\ln\left(\sqrt{\frac{G(\delta_1)}{2\mu}} + \sqrt{\frac{G(\delta_1)}{2\mu} - 1}\right).$$

However, it is difficult, even impossible, to obtain $G(\delta_1)$ by direct measurements. So the value δ_0 is to be found indirectly by taking into account the phenomena induced by δ_0. This problem will be treated in another paper.

The most impressive characteristic of the solution is perhaps the non-Euclidean structure of the space and specifically the strong non-Euclidean properties in the neighbourhood of the origin. If the theory is applicable to the elementary particles, then strong deviations from the Euclidean geometry are to be expected in the world of microphysics. Together with the new geometrical ideas, the solution brings about an improvement of the law of gravitation in accordance with Poincaré's prediction: "Il est difficile de ne pas supposer que la loi véritable contient des termes complémentaires qui deviendraient sensibles aux petites distances" [1]. In fact, the Newton potential

$$-\frac{km}{\delta}$$

is an approximation of the more accurate expression

$$-\frac{km}{G(\delta)}$$

which depends on the curvature radius $G(\delta)$. There is a significant discrepancy between the two formulae. Although, as shown earlier, the ratio $\frac{G(\delta)}{\delta}$ converges to 1, the difference

$$\delta - G(\delta) = P(G) - G = \delta_0 + \\ + 2\mu \ln\left(\sqrt{\frac{G}{2\mu}} + \sqrt{\frac{G}{2\mu} - 1}\right) - \frac{2\mu}{1 + \sqrt{1 - \frac{2\mu}{G}}}$$

tends to $+\infty$ as $\delta \to +\infty$. Moreover $G(\delta)$ depends not only on the radius δ, but also on the constant δ_0. Of course, the distinction between Newton's theory and Einstein's theory does not reduce to the distinction between δ and $G(\delta)$. Einstein's theory provides a new entity, namely a spacetime metric.

A last question regards the "boundary conditions at infinity". Classically it is required that the metric admit as limit form the standard pseudo-Euclidean metric as $\delta \to +\infty$. Since, as already remarked, $\delta F_1(\delta)$ does not possess a limit as $\delta \to +\infty$, this requirement presupposes that $F_1 = 0$, namely that we are dealing with a static metric. Then the metric can be written as

$$ds^2 = c^2\left(1 - \frac{2\mu}{G(\delta)}\right)dt^2 - \\ - \left(\left(\frac{G(\delta)}{\delta}\right)^2 dy^2 + \frac{1}{\delta^2}\left(1 - \left(\frac{G(\delta)}{\delta}\right)^2\right)(y dy)^2\right)$$

and since $G(\delta) \to +\infty$, $\frac{G(\delta)}{\delta} \to 1$, $\frac{\|y\|}{\delta} = 1$, we find, in fact, "at infinity" the standard pseudo-Euclidean form

$$ds^2 = c^2 dt^2 - dy^2.$$

Note that, if we introduce the so-called polar coordinates, this conclusion fails. In fact, then we have the form

$$ds^2 = c^2\left(1 - \frac{2\mu}{G(\delta)}\right)dt^2 - \left(d\delta^2 + \left(G(\delta)\right)^2\left(\sin^2\theta d\phi^2 + d\theta^2\right)\right)$$

which does not possess a limit form as $\delta \to +\infty$.

6 Black holes do not exist

The pseudo-theory of black holes appeared as a consequence of misunderstandings and mathematical errors brought out in detail in the papers [3, 5, 6]. We emphasize that the so-called "horizon" does not represent an observable value of the curvature radius $G(\delta)$. According to the established rigorous solution, 2μ is the greatest lower bound of the vacuum solution $G(\delta)$ and is defined for a certain value δ_0 of the new constant. If $\delta_0 \leqslant 0$ there exists no real sphere with the curvature radius 2μ, and the physically valid part of the solution is defined for $\delta \geqslant \delta_1$, where δ_1 is a strictly positive value such that $G(\delta_1) > 2\mu$. On the other hand, if $\delta_0 > 0$, the degeneracy of the metric for $\delta = \delta_0$ implies that the corresponding sphere lies inside the matter, so that the vacuum solution is valid for $\delta \geqslant \delta_1$ where $\delta_1 > \delta_0$ and $G(\delta_1) > 2\mu$. Whatever the case may be, the notion of black hole is inconceivable.

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