The Extended Relativity Theory in Clifford Spaces: Reply to a Review by W. A. Rodrigues, Jr.

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In a review, W. A. Rodrigues, Jr., wrote that we confused vector and affine spaces, and that we misunderstood the concept of curvature. We reply to those comments, and point out, that in our paper there was an explicit expression for the curvature of a connection. Therefore we were quite aware — contrary to what asserted the reviewer — that the curvature of a manifold has nothing to do with a choice of a frame field which, of course, even in a flat manifold can be position dependent.

In 2005 we published a paper entitled *The Extended Relativity Theory in Clifford Spaces* [1] which was reviewed by W.A. Rodrigues, Jr. [2]. A good review, even if critical, is always welcome, provided that the criticism is correct and relevant. Unfortunately the reviewer produced some statements which need a reply. He wrote:

> "Two kinds of Clifford spaces are introduced in their paper, flat and curved. According to their presentation, which is far from rigorous by any mathematical standard, we learn that flat Clifford space is a vector space, indeed the vector space of a Clifford algebra of real vector space $\mathbb{R}^D$ equipped with a metric of signature $P+Q=D$. As such the authors state that the coordinates of Clifford space are noncommutative Clifford-valued quantities. It is quite obvious for a mathematician that the authors confuse a vector space with an affine space. This is clear when we learn their definition of a curved Clifford space, which is a 16-dimensional manifold where the tangent vectors are position dependent and at any point are generators of a Clifford algebra $C_{P,Q}$. The authors, as is the case of many physicists, seem not to be aware that the curvature of a manifold has to do with the curvature of a connection that we may define on such a manifold, and has nothing to do with the fact that we may choose even in flat manifold a section of the frame bundle consisting of vectors that depend on the coordinates of the manifold points in a given chart of the maximal atlas of the manifold."

When introducing flat C-space we just said that the Clifford-valued polyvector denotes the position of a point in a manifold, called Clifford space, or C-space. It is a common practice to consider coordinates, e. g., four coordinates x^μ, $\mu = 0, 1, 2, 3$, of a point $\mathcal{P}$ of a flat spacetime as components of a radius vector from a chosen point $\mathcal{P}_0$ ("the origin") to $\mathcal{P}$. If we did not provide at this point a several pages course on vector and affine spaces, this by no means implies that we were not aware of a distinction of the two kinds of spaces. That position in flat spacetime is described by radius vector is so common that we do not need to provide any further explanation in this respect. Our paper is about *physics* and not mathematics. We just *use* the well established mathematics. Of course a spacetime manifold (including a flat one) is not the same space as a vector space, but, choosing an "origin" in spacetime, to every point there corresponds a vector, so that there is a one-to-one correspondence between the two spaces. This informal description is true, regardless of the fact that there exist corresponding rigorous, formal, mathematical descriptions (to be found in many textbooks on physics and mathematics).

The correspondence between points and vectors does no longer hold in a curved space, at least not according to the standard wisdom practiced in the textbooks on differential geometry. However, there exists an alternative approach adopted by Hestenes and Sobcyk in their book [3], according to which even the points of a curved space are described by vectors. Moreover, there is yet another possibility, described in refs. [4, 5], which employs vector fields $a^\mu(x)\gamma_\mu$ in a curved space $\mathcal{M}$, where γ_μ, $\mu = 0, 1, 2, \ldots, n-1$, are the coordinate basis vector fields. At every point $\mathcal{P}$ the vectors $\gamma_\mu|_{\mathcal{P}}$ span a tangent space $T_{\mathcal{P}}\mathcal{M}$ which is a vector space.

A particular case can be such that in a given coordinate system* we have $a^\mu(x) = x^\mu$. Then at every point $\mathcal{P} \in \mathcal{M}$, the object $x(\mathcal{P}) = x^\mu(\mathcal{P})\gamma_\mu(\mathcal{P})$ is a tangent vector. So we have one-to-one correspondence between the points $\mathcal{P}$ of $\mathcal{M}$ and the tangent vectors $x(\mathcal{P}) = x^\mu(\mathcal{P})\gamma_\mu(\mathcal{P})$, shortly $x^\mu\gamma_\mu$. The set of objects $x(\mathcal{P})$ for all point $\mathcal{P}$ in a region $R \subset \mathcal{M}$ we call the coordinate vector field [4]. So although the manifold is curved, every point in it can be described by a tangent

*"Coordinate system" or simply "coordinates" is an abbreviation for "the coordinates of the manifold points in a given chart of the maximal atlas of the manifold".

vector at that point, the components of the tangent vector being equal to the coordinates of that point.* Those tangent vectors $x^\mu(\mathcal{P})\gamma_\mu(\mathcal{P})$ are now "house numbers" assigned to a point $\mathcal{P}$. We warn the reader not to confuse the tangent vector $x(\mathcal{P})$ at a point $\mathcal{P}$ with the vector pointing from $\mathcal{P}_0$ (a coordinate origin) to a point $\mathcal{P}$, a concept which is ill defined in a curved manifold.

Analogous holds for Clifford space C. It is a manifold whose points $\mathcal{E}$ can be physically interpreted as extended "events". One possible way to describe those points is by means of a *polyvector field* $A(X) = A^M(X)\gamma_M(X) = X^M\gamma_M$, where $\gamma_M|_\mathcal{E}$, $M = 1, 2, \ldots, 2^n$, are tangent polyvectors that at every point $\mathcal{E} \in C$ span a Clifford algebra. At a given point $\mathcal{E} \in C$ it may hold [6, 5]

$$\gamma_M = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r}\,, \qquad r = 0, 1, 2, \ldots, n \quad (1)$$

i. e., γ_M are defined as wedge product of vectors γ_μ, $\mu = 0, 1, 2, \ldots, n-1$. The latter property cannot hold in a general curved Clifford space [6, 5]. In refs. [7, 1] we considered a particular subclass of curved C-spaces, for which it does hold.

If we choose a particular point $\mathcal{E}_0 \in C$, to which we assign coordinates $X^M(\mathcal{E}_0) = 0$, then we have a correspondence between points $\mathcal{E} \in C$ and Clifford numbers $X^M(\mathcal{E})\,\gamma_M(\mathcal{E})$. In this sense one has to understand the sentence of ref. [1]:

> "An element of C-space is a Clifford number, called also polyvector or Clifford aggregate which we now write in the form $X = s\gamma + x^\mu\gamma_\mu + x^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + \ldots$"

Therefore, a more correct formulation would be, e. g.,

> "An element of C-space is an extended event $\mathcal{E}$, to which one can assign a Clifford number, called polyvector, $X^M(\mathcal{E})\gamma_M(\mathcal{E}) \equiv X^M\gamma_M$."

together with an explanation in the sense as given above. So a rigorous formulation is, not that an element of C-space *is* a Clifford number, but that to a point of C-space there *corresponds* a Clifford number, and that this holds for all points within a domain $\Omega \subset C$ corresponding to a given chart of the maximal atlas of C.

On the one hand we have a 2^n-dimensional manifold $C \equiv \{\mathcal{E}\}$ of points (extended events) $\mathcal{E}$, and on the other hand the 2^n-dimensional space $\{X(\mathcal{E})\}$ of Clifford numbers $X(\mathcal{E}) = X^M(\mathcal{E})\,\gamma\,(\mathcal{E})$ for $\mathcal{E} \in \Omega \subset C$. The latter space $\{X(\mathcal{E})\}$, of course, is not a Clifford algebra. It is a subspace of 2×2^n-dimensional tangent bundle TC of the manifold C. At every point $\mathcal{E} \in C$ there is a also another subspace of TC, namely the 2^n-dimensional tangent space $T_\mathcal{E}C$, which is a Clifford algebra $\mathcal{C}_n$. Since there is a one-to-one correspondence between the spaces $\{X(\mathcal{E})\}$ and $\{\mathcal{E}\}$, the space $\{X(\mathcal{E})\}$ can be used for description of the space $\{\mathcal{E}\}$.

It is true that physicists are often sloppy with mathematical formulations and usage of language, but it is also true that mathematicians often read *physics papers* superficially and see misconceptions, "errors", erroneous mathematical statements, etc., instead of trying to figure out the true content behind an informal (and therefore necessarily imprecise) description, whose emphasis is on physics and not mathematics.

A culmination is when the reviewer writes

> "The authors, as is the case of many physicists, seem not to be aware that curvature... has nothing to do with the fact that we may choose even in flat manifold a section of the frame bundle consisting of vectors that depend on coordinates of the manifold points..."

That curvature has nothing to do with coordinate transformations† is clear to everybody who has ever studied the basis of general relativity. Everyone who has a good faith that the author(s) of a paper have a minimal level of competence would interpret a text such as [1]

> "In flat C-space the basis polyvectors γ_M are constant. In a curved C-space this is no longer true. Each γ_M is a function of C-space coordinates X^M..."

according to

> "In flat C-space one can always find coordinates‡ in which γ_M are constant. In a curved C-space this is no longer true. Each γ_M depends on position in C-space." Or equivalently, "Each γ_M is a function of the C-space coordinates".

However, even our formulation as it stands in ref. [1] makes sense within the context in which we first consider flat space in which we *choose* a constant frame field, i. e., constant basis polyvectors. We denote the latter polyvectors as γ_M. If we then deform§ the flat space into a curved one, then the *same* (poly)vector fields γ_M in general can no longer be independent of position. In this sense the formulation as it stands in our paper is quite correct.

We then define a *connection* on our manifold C, and the corresponding *curvature* (see eqs. (77), (78) of ref. [1]). That

*If we change coordinate system, then $a^\mu(x)\gamma_\mu = x^\mu\gamma_\mu = a'^\mu(x')\gamma'_\mu(x')$, with $a'^\mu = a^\nu(x)(\partial x'^\mu/\partial x^\nu) = x^\nu(\partial x'^\mu/\partial x^\nu)$. In another coordinate system S' one can then take another vector field, such that $b^\mu(x') = x'^\mu$. Let us stress that $b^\mu(x') = x'^\mu$ is a different field from $a'^\mu(x')$, therefore the reader should not think that we say $x'^\mu = x^\nu(\partial x'^\mu/\partial x^\nu)$ which is, of course, wrong. What we say is $a'^\mu(x') = (\partial x'^\mu/\partial x^\nu)a^\nu(x)$, where, in particular, $a^\nu(x) = x^\nu$.

†For instance, in flat spacetime one can introduce a curvilinear coordinate system of coordinates, like the use of polar coordinates in the plane and spherical coordinates in $\mathbb{R}^3$. However, the introduction of a curvilinear coordinate system does not convert the original flat space into a curved one. And vice versa, one can introduce a non-Euclidean metric (non-flat metric) on a two-dim flat surface, for example, like the hyperbolic Lobachevsky metric of constant negative scalar curvature.

‡We renounce to use here the lengthy formulation provided by the reviewer. Usage of the term "coordinates" is sufficient, and it actually means "coordinates of the manifold point in a given chart of the maximal atlas of the manifold".

§This is easy to imagine, if we consider a flat surface embedded in a higher dimensional space, and then deform the surface. In general, we may deform the surface so that is is curved not only extrinsically, but also intrinsically.

the reviewer reproaches us of being ignorant of the fact that the curvature of a manifold has to do with the curvature of a connection is therefore completely out of place, to say at least.

Finally, let us mention that in the review of another paper [8] the same reviewer ascribed to one of us (M.P) an incorrect mathematical statement. But I was quite aware of the well known fact that Clifford algebras associated with vector spaces of different signatures (p, q), with $p+q=n$, are not all isomorphic (in the sense as stated, e. g., in the book by Porteous [9]). What I discussed in that paper was something different. This should be clear from my description, therefore I did not explicitly warn the reader about the difference (although I was aware of the danger that at superficial reading some people might believe me of committing an error). However, in subsequent ref. [1] we did warn the reader about the possibility of such a confusion.

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