Single Photon Experiments and Quantum Complementarity

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Single photon experiments have been used as one of the most striking illustrations of the apparently nonclassical nature of the quantum world. In this review we examine the mathematical basis of the principle of complementarity and explain why the Englert-Greenberger duality relation is not violated in the configurations of Unruh and of Afshar.

1 Introduction

In classical physics if we have two distinct physical states $\psi_1 \neq \psi_2$ of a physical system and we know that $\psi_1 \,\mathrm{OR}\, \psi_2$ is a *true* statement we can easily deduce that $\psi_1 \,\mathrm{XOR}\, \psi_2$ is a true statement too. In Quantum Mechanics however we encounter a novel possibility for *quantum coherent superposition*. It has been verified in numerous experiments that a qubit can be prepared in a linear combination of two orthogonal states, and this parallel existence in the quantum realm, in the form $\psi_1 \,\mathrm{AND}\, \psi_2$, is what requires caution when we draw conclusions from a given set of premises — the truth of $\psi_1 \,\mathrm{OR}\, \psi_2$ now does not lead to the truth of $\psi_1 \,\mathrm{XOR}\, \psi_2$.* If a qubit at point x is in a state $\psi_1 \mathrm{XOR} \psi_2$ then ψ_1 and ψ_2 are called *distinguishable states*. Logically, if the qubit at point x is in a state $\psi_1 \mathrm{XNOR} \psi_2$ the two states ψ_1 and ψ_2 will be *indistinguishable*. From the requirement for mathematical consistency it follows that two states ψ_1 and ψ_2 cannot be both *distinguishable* and *indistinguishable* at the same time.

The concept of distinguishability is intimately tied with the notion of quantum complementarity. While the quantum amplitudes evolve linearly according to the Schrödinger equation, the physical observables are obtained from the underlying quantum amplitudes through nonlinearity prescribed by Born's rule.

Thus if quantum states $\psi_1(x) \neq 0$ and $\psi_2(x) \neq 0$ are *indistinguishable* at a point x (coherent superposition), that is $\psi_1(x) \,\mathrm{AND}\, \psi_2(x)$, the probability distribution (observed intensity) is given by $P = |\psi_1(x) + \psi_2(x)|^2$. The *density matrix* of the setup is a pure type one, $\hat{\rho} = \begin{pmatrix} |\psi_1|^2 & \psi_1\psi_2^* \\ \psi_2\psi_1^* & |\psi_2|^2 \end{pmatrix}$, and $\hat{\rho} = \hat{\rho}^2$ holds. The two quantum states do quantum mechanically interfere. In Hilbert space the two functions are not orthogonal and the overlap integral is not *zero* (Vedral [12]):

ψ_1	ψ_2	XOR output	ψ_1	ψ_2	XNOR output
0	0	0	0	0	1
0	1	1	0	1	0
1	0	1	1	0	0
1	1	0	1	1	1

Table 1: Distinguishable –vs– indistinguishable states

$$\int \psi_1^*(x)\, \psi_2(x)\, dx \neq 0\,. \tag{1}$$

Alternatively, if quantum states $\psi_1(x)$ and $\psi_2(x)$ are *distinguishable* at a point x (incoherent superposition), that is $\psi_1(x) \,\mathrm{XOR}\, \psi_2(x)$, then the probability distribution is given by $\mathcal{P} = |\psi_1(x)|^2 + |\psi_2(x)|^2$. The *(reduced) density matrix* is mixed type one, $\hat{\rho} = \begin{pmatrix} |\psi_1|^2 & 0 \\ 0 & |\psi_2|^2 \end{pmatrix}$, and $\hat{\rho} \neq \hat{\rho}^2$. The two quantum states do not quantum mechanically interfere but just sum classically. In Hilbert space the two functions are orthogonal and the overlap integral is *zero*:

$$\int \psi_1^*(x)\, \psi_2(x)\, dx = 0\,. \tag{2}$$

The observable value given by $\mathcal{P}$ should not necessarily describe an incoherently superposed state. It might as well describe a fictious statistical average of two single amplitude experiments in which either only $\psi_1(x)$ or only $\psi_2(x)$ participates. In this case however $\psi_1(x)$ and $\psi_2(x)$ should be separately normalized to 1, and as elements in the main diagonal of the density matrix must be taken the *statistical probabilities* defining the mixture (Zeh [14]).

Next, despite the fact that qubits generally might take more than one path in a coherent superposition (Feynman and Hibbs [7]), we will still show that the "which way" claims ("*welcher weg*", in German) can be derived rigourously within the quantum mechanical formalism. The "which way" claim will be defined as an existent *one-to-one correspondence (bijection) between elements of two sets* (typically input state and observable).

*Such a direct interpretation of the AND gate as having corresponding quantum coherent superposed reality is consistent with the prevailing view among working physicists that resembles Everett's many worlds interpretation (MWI) of Quantum Mechanics in many ways (Tegmark and Wheeler [11]). However, the reality of quantum superposition is not a characteristic feature only of MWI. The transactional interpretation (TI) proposed by Cramer [4] and quantum gravity induced objective reduction (OR) proposed by Penrose [8] both admit of the reality of superposed quantum waves, respectively superposed space-times.

$$\overset{\mathfrak{L}_1}{[|\psi\rangle]} \to \overset{\text{path 1}}{\left[i\frac{1}{\sqrt{2}}|\psi_1\rangle\right]} + \overset{\text{path 2}}{\left[\frac{1}{\sqrt{2}}|\psi_2\rangle\right]} \to \overset{\text{path 3}}{\left[-\frac{1}{\sqrt{2}}|\psi_1\rangle\right]} + \overset{\text{path 4}}{\left[i\frac{1}{\sqrt{2}}|\psi_2\rangle\right]} \to \overset{\mathfrak{D}_1}{\left[-\frac{1}{2}|\psi_1\rangle - \frac{1}{2}|\psi_2\rangle\right]} + \overset{\mathfrak{D}_2}{\left[-i\frac{1}{2}|\psi_1\rangle + i\frac{1}{2}|\psi_2\rangle\right]} \quad (3)$$

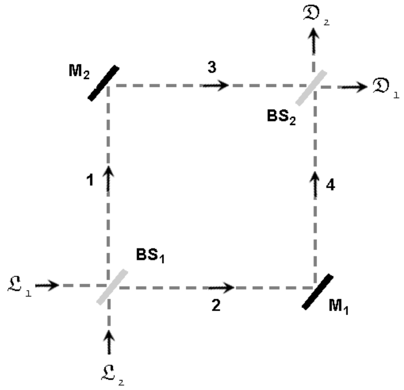

Fig. 1: *Mach-Zehnder interferometer*. Incoming photon at $\mathfrak{L}_1$ quantum mechanically *self-interferes* in order to produce its own full cancelation at detector $\mathfrak{D}_2$ and recover itself entirely at detector $\mathfrak{D}_1$. The opposite holds for the photon entering at $\mathfrak{L}_2$. Legend: BS, beam splitter, M, fully silvered mirror.

2 The Mach-Zehnder interferometer

In order to illustrate the "which way" concept let us introduce the *Mach-Zehnder interferometer*, from which more complicated interferometers can be built up. The setup is symmetric and contains two half-silvered and two fully silvered mirrors positioned at angle $\frac{\pi}{4}$ to the incoming beam (Fig. 1). The action of the beam splitter (half-silvered mirror) will be such as to transmit forward without phase shift $\frac{1}{\sqrt{2}}\psi$ of the incoming quantum amplitude ψ, while at the same time reflects perpendicularly in a coherent superposition $i\frac{1}{\sqrt{2}}\psi$ of it. The action of the fully silvered mirrors will be such as to reflect perpendicularly all of the incoming amplitude ψ with a phase shift of $\frac{\pi}{2}$, which is equivalent to multiplying the state by $e^{i\frac{\pi}{2}} = i$ (Elitzur and Vaidman [6]; Vedral [12]).

In this relatively simple setup it can be shown that a photon entering at $\mathfrak{L}_1$ will always be detected by detector $\mathfrak{D}_1$, whilst a photon entering at $\mathfrak{L}_2$ will always be detected by detector $\mathfrak{D}_2$. It is observed that the photon quantum mechanically *destructively self-interferes* at one of the detectors, whilst it quantum mechanically *constructively self-interferes* at the other detector, creating a one-to-one correspondence between the entry point and the exit point in the Mach-Zehnder interferometer.

Let the incoming amplitude Ψ at $\mathfrak{L}_1$ be normalized so that $|\Psi|^2 = 1$. The evolution of the wave package in the interferometer branches is described by formula (3), where $|\psi_1\rangle$ refers to passage along path 1 and $|\psi_2\rangle$ refers to passage along path 2.

Since the two interferometer paths are indistinguishable one easily sees that at $\mathfrak{D}_1$ one gets *constructive quantum interference*, while at $\mathfrak{D}_2$ one gets *destructive quantum interference*. The inverse will be true if the photon enters at $\mathfrak{L}_2$. Therefore we have established a one-to-one correspondence (bijection) between the entry points and detector clicks. The *indistinguishability* of ψ_1 and ψ_2 allows for *quantum self-interference* of Ψ at the detectors. Insofar as we don't specify which path of the interferometer has been traversed, allow quantum interference of amplitudes at the exit gates coming from both interferometer paths, so $\psi_1 \,\text{AND}\, \psi_2$ (*indistinguishable* ψ_1 and ψ_2), we will maintain the one-to-one correspondence between entry points and detectors (*distinguishable* $\mathfrak{D}_1$ and $\mathfrak{D}_2$).

If we however block one of the split beams ψ_1 or ψ_2, or we label ψ_1 and ψ_2, e.g. by different polarization filters, **V** (vertical polarization) and **H** (horizontal polarization), we will lose the quantum interference at the exit gates and the one-to-one correspondence between entry points and exit points will be lost. Thus we have encountered the phenomenon of *complementarity.* We can determine which of the interferometer paths has been taken by the photon, hence $\psi_1 \,\text{XOR}\, \psi_2$ (*distinguishable* ψ_1 and ψ_2), and destroy the one-to-one correspondence between entry points and exit gates (*indistinguishable* $\mathfrak{D}_1$ and $\mathfrak{D}_2$). A photon entering at $\mathfrak{L}_1$ (or $\mathfrak{L}_2$) will not self-interfere and consequently could be detected by either of the detectors with probability of $\frac{1}{2}$.

Thus we have shown that quantum coherent superposition of photon paths itself does not preclude the possibility for one to establish one-to-one correspondence (bijection) between two observables (entry and exit points). However, it will be shown that the bijection $\mathfrak{L}_1 \to \mathfrak{D}_1$, $\mathfrak{L}_2 \to \mathfrak{D}_2$ is valid for the discussed mixed case in which we have input $\mathfrak{L}_1 \,\text{XOR}\, \mathfrak{L}_2$, yet might not be true in the case where the input points $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are in quantum coherent superposition ($\mathfrak{L}_1 \,\text{AND}\, \mathfrak{L}_2$) as is the case in Unruh's setup.

3 Unruh's interferometer

Unruh's thought experiment is an arrangement that tries to create a more understandable version of Afshar's experiment, which will be discussed later. Unruh's interferometer is essentially a multiple pass interferometer with two elementary building blocks of the Mach-Zehnder type. In Fig. 2 each arm of the interferometer is labelled with a number, and a photon enters at $\mathfrak{L}_1$.

Application of Feynman's *sum over histories* approach leads us to the correct quantum mechanical description of the experiment. Expression (4) is *Dirac's ket notation* for the quantum states evolving in the interferometer arms.

$$\begin{aligned}
&\overset{\mathfrak{L}_1}{\big[|\psi\rangle\big]} \to \overset{\text{path 1}}{\left[i\,\frac{1}{\sqrt{2}}|\psi_1\rangle\right]} + \overset{\text{path 2}}{\left[\frac{1}{\sqrt{2}}|\psi_2\rangle\right]} \to \overset{\text{path 3}}{\left[-\frac{1}{\sqrt{2}}|\psi_1\rangle\right]} + \overset{\text{path 4}}{\left[i\,\frac{1}{\sqrt{2}}|\psi_2\rangle\right]} \to \\
&\to \overset{\text{path 5}}{\left[-i\,\frac{1}{2}|\psi_1\rangle + i\,\frac{1}{2}|\psi_2\rangle\right]} + \overset{\text{path 6}}{\left[-\frac{1}{2}|\psi_1\rangle - \frac{1}{2}|\psi_2\rangle\right]} \to \overset{\text{path 7}}{\left[\frac{1}{2}|\psi_1\rangle - \frac{1}{2}|\psi_2\rangle\right]} + \overset{\text{path 8}}{\left[-i\,\frac{1}{2}|\psi_1\rangle - i\,\frac{1}{2}|\psi_2\rangle\right]} \to \\
&\to \overset{\mathfrak{D}_1}{\left[\left[\frac{1}{\sqrt{8}}|\psi_1\rangle - \frac{1}{\sqrt{8}}|\psi_2\rangle\right] + \left[\frac{1}{\sqrt{8}}|\psi_1\rangle + \frac{1}{\sqrt{8}}|\psi_2\rangle\right]\right]} + \overset{\mathfrak{D}_2}{\left[\left[i\,\frac{1}{\sqrt{8}}|\psi_1\rangle - i\,\frac{1}{\sqrt{8}}|\psi_2\rangle\right] + \left[-i\,\frac{1}{\sqrt{8}}|\psi_1\rangle - i\,\frac{1}{\sqrt{8}}|\psi_2\rangle\right]\right]}
\end{aligned} \tag{4}$$

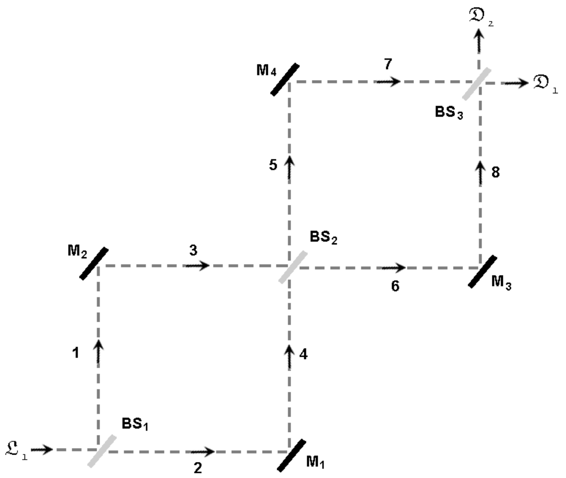

Fig. 2: Unruh's version of a multiple pass interferometer setup that captures the essence of Afshar's experiment. It is composed of two elementary building blocks described in the text, and the incoming photon at $\mathfrak{L}_1$ has an equal chance to end either at $\mathfrak{D}_1$, or at $\mathfrak{D}_2$.

3.1 Unruh's "which way" claim

Unruh obstructed path 1 and correctly argues that the photons coming from the source that pass the first half-silvered mirror and take path 2 (that is they are not reflected to be absorbed by the obstruction located in path 1) will all reach detector $\mathfrak{D}_2$. These are exactly 50% of the initial photons. The explanation is the one provided in the analysis of the Mach-Zehnder interferometer. So Unruh shows that there is a one-to-one corespondence between path 2 and detector $\mathfrak{D}_2$ when path 1 is blocked. Similarly he argues that in the inverted setup with the obstruction in path 2, all the photons that take path 1 (that is they are not absorbed by the obstruction located in path 2) will reach detector $\mathfrak{D}_1$. This suggests a one-to-one correspondence between path 1 and detector $\mathfrak{D}_1$ when path 2 is blocked.

Note at this stage that Unruh investigates a statistical mixture of two single path experiments. Therefore the case is $\psi_1\,\mathrm{XOR}\,\psi_2$, both paths ψ_1 and ψ_2 are *distinguishable* because of the *existent obstruction*, and ψ_1 and ψ_2 do not quantum *cross-interfere* with each other in the second block of the interferometer (in the first block they are separated spatially, in the second branch they are separated temporally). Thus in the mixed setup there is a one-to-one correspondence between paths and exit gates due to the *distinguishability* of ψ_1 and ψ_2, that is, there is no quantum interference between ψ_1 and ψ_2 in the second building block of Unruh's interferometer.

Unruh then unimpedes both paths ψ_1 and ψ_2, and considering the statistical mixture of the two single path experiments argues that photons that end up at detector $\mathfrak{D}_1$ have taken path ψ_1, while those ending at detector $\mathfrak{D}_2$ come from path ψ_2. The logic is that the second building block of the interferometer has both of its arms open, and the one-to-one correspondence is a result of *self-interference* of ψ_1 and *self-interference* of ψ_2 respectively.

The problem now is to secure the conclusion that "which way" information in the form of a one-to-one correspondence between paths ψ_1 and ψ_2 and the two detectors still "remains" when both paths 1 and 2 are unimpeded? The only way to justify the existence of the bijection is to take the following two statements as axioms: (i) ψ_1 and ψ_2 do not *quantum cross-interfere* with each other; (ii) ψ_1 and ψ_2 *can only quantum self-interfere*. Concisely written together, both statements reduce to one logical form, $\psi_1\,\mathrm{XOR}\,\psi_2$ i.e. ψ_1 and ψ_2 are orthogonal states. Thus Unruh's "which way" statement when both paths of the interferometer are unimpeded is equivalent to the statement that the density matrix of the photons at the detectors is a *mixed one*. Thus stated Unruh's "which way" claim, which is mathematically equivalent with the claim for a mixed state density matrix of the setup, is subject to experimental test. Quantum mechanically one may perform experiments to find whether or not two incoming beams are quantum coherent (pure state) or incoherent (mixed state). Hence Unruh's thesis is experimentally disprovable, and in order to keep true his thesis Unruh must immunize it against experimental test by postulating that one cannot experimentally distinguish the mixed state from the pure state. Otherwise one may decide to let the two beams (led away from the detectors) cross each other. If an interference pattern is build up then one will have experimental verification that the density matrix of the setup is not of the *mixed type* ($\psi_1\,\mathrm{XOR}\,\psi_2$, $\hat{\rho} \neq \hat{\rho}^2$), but one of *pure type* ($\psi_1\,\mathrm{AND}\,\psi_2$, $\hat{\rho} = \hat{\rho}^2$). It is not conventional to think that the mixed state cannot be experimentally distinguished from the pure state, and that is why Unruh's "which way" claim for the double path coherent setup is incorrect. One notices however that if each of the paths 1 and 2 is labelled by different polarization filters, e.g. **V** and **H**, then the density matrix of the setup will be a *mixed one* (incoherent superposition in the second interferometer block), and the "which way" claim will be correct because the different polarizations will convert ψ_1 and ψ_2 into orthogonal states. If the two beams lead away from the detectors and cross, they will not produce an interference pattern.

3.2 Correct "no which way" thesis

We have already shown that if one argues that there is "which way" correspondence, he must accept that ψ_1 and ψ_2 are *distinguishable*, and hence that they will not be able to cross-interfere at arms 5–8 of the interferometer.

Now we will show the opposite; that postulating "unmeasured destructive interference" in arms 5 and 7 of the interferometer, regardless of the fact that the interference is not measured, is sufficient to erase completely the "which way" information. Postulating quantum interference in arms 5–8 is equivalent to postulating *indistinguishability* (quantum coherent superposition) of ψ_1 and ψ_2, which is equivalent to saying that ψ_1 and ψ_2 can annihilate each other.

The quantum amplitude at $\mathfrak{D}_1$ is:

$$\mathfrak{D}_1 : \left[\frac{1}{\sqrt{8}}|\psi_1\rangle - \frac{1}{\sqrt{8}}|\psi_2\rangle\right] + \left[\frac{1}{\sqrt{8}}|\psi_1\rangle + \frac{1}{\sqrt{8}}|\psi_2\rangle\right]. \quad (5)$$

The first two members in the expression have met each other earlier, so they annihilate each other. What remains is $\frac{1}{\sqrt{8}}|\psi_1\rangle + \frac{1}{\sqrt{8}}|\psi_2\rangle$ and when squared gives $\frac{1}{2}|\Psi|^2$, where ψ_1 and ψ_2 contribute equally to the observed probability of detecting a photon. Now is clear why one cannot hold consistently both the existence of "which way" one-to-one correspondence and existent but undetected interference at paths 5 and 6.

- If one postulates $\psi_1\,\mathrm{XOR}\,\psi_2$ then $\frac{1}{\sqrt{8}}|\psi_2\rangle - \frac{1}{\sqrt{8}}|\psi_2\rangle$ will interfere at the exit and the resulting observable intensity $\frac{1}{2}|\Psi|^2$ will come from squaring $\frac{1}{\sqrt{8}}|\psi_1\rangle + \frac{1}{\sqrt{8}}|\psi_1\rangle$ i.e. only from path 1.
- If one postulates $\psi_1\,\mathrm{AND}\,\psi_2$ then $\frac{1}{\sqrt{8}}|\psi_1\rangle - \frac{1}{\sqrt{8}}|\psi_2\rangle$ will interfere first, and the resulting observable intensity $\frac{1}{2}|\Psi|^2$ will come from squaring $\frac{1}{\sqrt{8}}|\psi_1\rangle + \frac{1}{\sqrt{8}}|\psi_2\rangle$ i.e. both paths 1 and 2.

The "mixing of the two channels" at $\mathfrak{D}_2$ is analogous.

$$\mathfrak{D}_2 : \left[i\frac{1}{\sqrt{8}}|\psi_1\rangle - i\frac{1}{\sqrt{8}}|\psi_2\rangle\right] + \left[-i\frac{1}{\sqrt{8}}|\psi_1\rangle - i\frac{1}{\sqrt{8}}|\psi_2\rangle\right]. \quad (6)$$

- If one postulates $\psi_1\,\mathrm{XOR}\,\psi_2$ then $i\frac{1}{\sqrt{8}}|\psi_1\rangle - i\frac{1}{\sqrt{8}}|\psi_1\rangle$ will interfere at the exit and the obtained observable intensity $\frac{1}{2}|\Psi|^2$ will come from squaring $-i\frac{1}{\sqrt{8}}|\psi_2\rangle - i\frac{1}{\sqrt{8}}|\psi_2\rangle$ i.e. only from path 2.
- If one postulates $\psi_1\,\mathrm{AND}\,\psi_2$ then $i\frac{1}{\sqrt{8}}|\psi_1\rangle - i\frac{1}{\sqrt{8}}|\psi_2\rangle$ will interfere first, and the obtained observable intensity $\frac{1}{2}|\Psi|^2$ will come from squaring of $-i\frac{1}{\sqrt{8}}|\psi_1\rangle - i\frac{1}{\sqrt{8}}|\psi_2\rangle$ i.e. both paths 1 and 2.

3.3 Inconsistent interpretation: "which way" + pure state density matrix

It has been suggested in web blogs and various colloquia, that only measurement of the interference at arms 5–8 disturbs the "which way" interpretation, and if the destructive quantum interference is not measured it can peacefully coexist with the "which way" claim. Mathematically formulated the claim is that there is "which way" one-to-one correspondence between paths 1 and 2, and $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively, while at the same time the whole setup is described by a pure state density matrix. Afshar [1–3] claims an equivalent statement for his setup insisting on a "which way" + pure state density matrix.

We will *prove* that assuming a "which way" + pure state density matrix leads to mathematical inconsistency. In order to show *where the inconsistency arises* we should rewrite the expressions of the quantum amplitudes at the two detectors in a fashion where each of the wavefunctions ψ_1 and ψ_2 is written as a superposition of its own branches $|\psi_{15}\rangle, |\psi_{16}\rangle$ and $|\psi_{25}\rangle, |\psi_{26}\rangle$, respectively, where the second subscript 5 or 6 denotes a branch in the second building block of Unruh's interferometer:

$$\mathfrak{D}_1 : \frac{1}{\sqrt{8}}|\psi_{15}\rangle - \frac{1}{\sqrt{8}}|\psi_{25}\rangle + \frac{1}{\sqrt{8}}|\psi_{16}\rangle + \frac{1}{\sqrt{8}}|\psi_{26}\rangle \quad (7)$$

$$\mathfrak{D}_2 : i\frac{1}{\sqrt{8}}|\psi_{15}\rangle - i\frac{1}{\sqrt{8}}|\psi_{25}\rangle - i\frac{1}{\sqrt{8}}|\psi_{16}\rangle - i\frac{1}{\sqrt{8}}|\psi_{26}\rangle. \quad (8)$$

From the "which way" claim it follows that the contributions to the final intensity (squared amplitude) detected at $\mathfrak{D}_1$ or $\mathfrak{D}_2$ must come from ψ_1 or ψ_2 only. This is possible *if and only if* the individual branches 5 or 6 of each function are *indistinguishable*, so that the claim mathematically yields quantum destructive interference (annihilation) between ψ_{15} and ψ_{16}, and between ψ_{25} and ψ_{26}, respectively.

However to postulate at the same time that the density matrix is a pure type one i.e. there is "*undetected negative quantum cross-interference*" at branch 5 between ψ_1 and ψ_2 (self-interference of Ψ) is equivalent to saying that paths 5 and 6 are *distinguishable*. We have arrived at a logical inconsistency.

Paths 5 and 6 cannot be both *distinguishable* and *indistinguishable* for the quantum state Ψ — this is what the complementarity principle says.

Due to basic arithmetic axiom ψ_{15} cannot entirely annihilate both ψ_{16} and ψ_{25}. Thus the complementarity principle itself is a manifestation of the underlying mathematical formalism and one ends up with an XOR bifurcation of two inconsistent with each other outcomes. The two alternative outcomes do not "complement" each other instead they logically exclude each other.

We have therefore proved that within standard Quantum Mechanics one cannot claim both "which way" and pure state of the density matrix at the same time. Whether the quantum interference at branch 5 is measured or not does not matter. Its consistent postulation is sufficient to rule out the "which way information".

3.4 Retrospective reconstructions and complementarity

Now notice that arguing that photons possess "which way" information implies that the photon density matrix at detectors is that of a *mixed type*. We have denoted the quantum amplitude through path 1 by ψ_1, and the quantum amplitude through path 2 by ψ_2. Therefore when we *retrospectively reconstruct* the photon probability distribution function we should use the correct complementarity rule $\mathcal{P} = |\psi_1|^2 + |\psi_2|^2$, and we must logically and consistently argue that there is no negative interference at path 5 — simply, we do not just add ψ_1 to ψ_2 but first square each of those amplitudes. Basically, if the two paths ψ_1 and ψ_2 are *distinguishable*, then the interference terms must be *zero*, and the (reduced) density matrix will be of *mixed type* i.e. one with off-diagonal elements being *zeroes*. To accept that there is "which way" information is equivalent to accepting that the setup with both paths unobstructed is a statistical mixture of the two single path setups with obstructions so the complementarity rule for making retrospective predictions is $\mathcal{P} = |\psi_1|^2 + |\psi_2|^2$. This alternative formulation of the principle of complementarity is in a form of instruction as to how to make the correct retrospective reconstruction of a *mixed state* setup — it says that *mixed state setups* should be retrospectively reconstructed with $\mathcal{P} = |\psi_1|^2 + |\psi_2|^2$ distribution.

However, if the beams along paths 1 and 2 interfere so that no photons are expected along path 5, the setup is a "no which way" *pure state* setup. In this case the retrospective photon probability distribution should be calculated as $P = |\psi_1 + \psi_2|^2$. Thus the alternative formulation of the principle of complementarity in a form of instruction as to how to make the correct retrospective reconstruction of *pure state* setup is — *pure state setups* should be retrospectively reconstructed with the $P = |\psi_1 + \psi_2|^2$ distribution.

Taken together the above two instructions provide a clear idea of complementarity — one cannot retrospectively recover a given setup with both types of probability distributions $\mathcal{P} = |\psi_1|^2 + |\psi_2|^2$ and $P = |\psi_1 + \psi_2|^2$ at the same time, because otherwise you will produce a mathematical inconsistency.

One sees that, in some special cases for a given point x both probability distributions coincide, so $\mathcal{P}(x) = P(x)$, and if one observes only the point x the choice of how to retrospectively reconstruct the setup might be tricky. It is unwise to retrospectively reconstruct a *pure state* setup with $\mathcal{P} = |\psi_1|^2 + |\psi_2|^2$ probability distribution. One will not arrive at a direct experimental contradiction if he looks only within the region where $\mathcal{P}(x) = P(x)$. Yet, any measurement outside this region will reveal the improper retrospective reconstruction.

4 Afshar's setup

In Afshar's setup, light generated by a laser passes through two closely spaced circular pinholes. After the dual pinholes, a lens refocuses the light so that *each image of a pinhole* is received by a separate photo-detector. Considering a mixture of single pinhole trials Afshar argues that a photon that goes through *pinhole 1* impinges only on detector $\mathfrak{D}_1$, and similarly, if it goes through *pinhole 2* impinges only on detector $\mathfrak{D}_2$. Exactly as in Unruh's setup, Afshar investigates a statistical mixture $\psi_1 \,\mathrm{XOR}\, \psi_2$ and after that draws *non sequitur* conclusions for the $\psi_1 \,\mathrm{AND}\, \psi_2$ setup. Thus according to Afshar, there is a one-to-one correspondence between pinholes and the corresponding images even when the light coherently passes through both pinholes. While in classical optics this is a straightforward conclusion, in quantum coherent setups we will shortly prove that each image of a pinhole in the coherent dual pinhole setup is counter-intuitively assembled by light coming from both pinholes at once.

Afshar [1, 2] claimed (erroneously) that Unruh's setup (originally intended to disprove Afshar's reasoning) is not equivalent to Afshar's setup, and therefore that the *"plane constructed by Unruh has no wings"*. At first glance one might argue that in Afshar's setup at *image 1* comes only quantum amplitude from *pinhole 1*, and *zero* amplitude from *pinhole 2*, and at *image 2* comes amplitude from *pinhole 2* and *zero* from *pinhole 1*. The putative difference between Unruh's setup and Afshar's setup at first glance seems to be this:

- *Afshar's setup: image 1*: $\frac{1}{\sqrt{2}}\psi_1 + 0 \times \psi_2$ and *image 2*: $\frac{1}{\sqrt{2}}\psi_2 + 0 \times \psi_1$. The *zero* looks "physically unstructured", not a result of negative interference of positive and negative amplitudes contributed from the alternative pinhole.
- *Unruh's setup:* $\mathfrak{D}_1$: $\frac{1}{\sqrt{2}}\psi_1 + \left[\frac{1}{\sqrt{8}}\psi_2 - \frac{1}{\sqrt{8}}\psi_2\right]$ and $\mathfrak{D}_2$: $\frac{1}{\sqrt{2}}\psi_2 + \left[\frac{1}{\sqrt{8}}\psi_1 - \frac{1}{\sqrt{8}}\psi_1\right]$. In this case the *zero* manifests "with physical structure", and is a result of negative interference of positive and negative amplitudes contributed from the alternative path.

If one shows that the "no which way" proof applied to Unruh's setup is not applicable to Afshar's setup, he will also show that Unruh's plane is indeed without wings. If in contrast, one can prove that in Afshar's setup the zero pinhole amplitude contribution at the opposite image is gene-

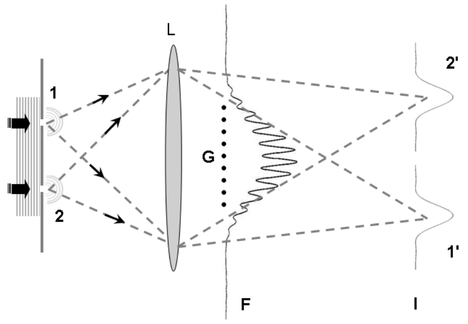

Fig. 3: Action of a lens in a dual pinhole setup — pinholes 1 and 2 create two peak images, $1'$ and $2'$, F denotes the focal plane of the lens, I denotes the image planes of the lens, G is the grid that can be used to verify the existence of an interference pattern in the coherent setup when both pinholes are open. The image is released under GNU free documentation licence by A. Drezet.

rated by *negative quantum interference*, he will show that Unruh's setup is completely equivalent to Afshar's setup. Thus our criticism of Afshar will be the same as in Unruh's case — logical fallacy and mathematical error in claiming both *pure state* and "which way".

It will now be shown that Afshar's setup is equivalent to Unruh's setup. In brief Afshar has dual pinholes, a lens, and detectors that record photons streaming away from the pinhole images created at the image plane of the lens (Afshar [3]). Analogously to Unruh's setup one closes *pinhole 1* and sees that light goes only to *image 2*, then closes *pinhole 2* and sees that light goes only to *image 1*. One may, analogously to Unruh's setup, *inconsistently postulate* "which way" + *pure state density matrix*. However, one should note that, in the single pinhole experiments, at the image plane of the lens the *zero* light intensity outside the central Airy disc of the pinhole image is a result of *destructive quantum interference*. There are many faint higher order maxima and minima outside the central Airy disc resulting from quantum interference. In order for the two pinhole images to be *resolvable** the image of the second pinhole must be outside the central Airy disc, and located in the first negative Airy ring of the first pinhole image (or further away). Therefore in the case of open *pinhole 2* at *image 1* there are destructively interfering quantum amplitudes contributed by *pinhole 2* because *image 1* resides in an Airy minimum of *image 2*. In contrast at *image 2* the waves from *pinhole 2* will constructively interfere. If both pinholes are open and some of the waves coming from *pinhole 1* cross-interfere with waves coming from *pinhole 2* in the space before the lens, there will remain a contribution by *pinhole 2* at *image 1* that will compensate exactly the decrease of quantum waves contributed by *pinhole 1* at *image 1*. Now one has to "choose" which amplitudes will annihilate, and which will remain to be squared. If one postulates the existent interference in the space before the lens (or after the lens as is the case at the focal plane of the lens) then the annihilation between ψ_1 and ψ_2 at the dark fringes will be equivalent to the interference at path 5 of Unruh's setup, and the final observed intensities at the detectors cannot be claimed to come only from one of the pinholes. Thus Afshar is wrong to say that *"Unruh's plane is without wings"*. Afshar's setup is equivalent to Unruh's setup. The treatment of complementarity is analogous. In the case with both pinholes open there is no "which way" information in Afshar's experiment. Counter-intuitively *each image of a pinhole* is assembled from light coming by half from both pinholes.

An exact calculation is adduced by Qureshi [9] where he shows that the quantum state at the overlap region where the dark interference fringes are detected can be written as

$$\begin{aligned}\psi(y,t) = a\,C(t)\,e^{-\frac{y^2+y_0^2}{\Omega(t)}}\left[\cosh\frac{2yy_0}{\Omega(t)} + \sinh\frac{2yy_0}{\Omega(t)}\right] + \\ + b\,C(t)\,e^{-\frac{y^2+y_0^2}{\Omega(t)}}\left[\cosh\frac{2yy_0}{\Omega(t)} - \sinh\frac{2yy_0}{\Omega(t)}\right],\end{aligned} \tag{9}$$

where $C(t) = \frac{1}{(\pi/2)^{1/4}\sqrt{\epsilon + 2i\hbar t/m\epsilon}}$, $\Omega(t) = \epsilon^2 + 2i\hbar t/m$, a is the amplitude contribution from *pinhole 1*, b is the amplitude contribution from *pinhole 2*, ϵ is the width of the wave-packets, $2y_0$ is the slit separation.

For Afshar's setup $a = b = \frac{1}{\sqrt{2}}$ so the sinh terms cancel out at the dark fringes and what is left is

$$\begin{aligned}\psi(y,t) = \frac{1}{2}\,a\,C(t)\left[e^{-\frac{(y-y_0)^2}{\Omega(t)}} + e^{-\frac{(y+y_0)^2}{\Omega(t)}}\right] + \\ + \frac{1}{2}\,b\,C(t)\,e^{-\frac{y+y_0^2}{\Omega(t)}}\left[e^{-\frac{(y-y_0)^2}{\Omega(t)}} + e^{-\frac{(y+y_0)^2}{\Omega(t)}}\right].\end{aligned} \tag{10}$$

If a lens is used, after the interference has occurred, to direct the $e^{-\frac{(y-y_0)^2}{\Omega(t)}}$ part into one detector and the part $e^{-\frac{(y+y_0)^2}{\Omega(t)}}$ into the other detector, one easily sees that the amplitudes from each slit evolve into a superposition of two parts that go to both detectors. Note that the coefficient of the part from a slit to each of the detectors becomes exactly $\frac{1}{\sqrt{8}}$ as we have obtained via analysis of Unruh's setup.

5 Englert-Greenberger duality relation

Afshar claimed he has violated the Englert-Greenberger duality relation $V^2 + D^2 \leqslant 1$, where V stands for *visibility* and D stands for *distinguishability* and are defined as:

$$D = \frac{\big||\psi_1|^2 - |\psi_2|^2\big|}{|\psi_1|^2 + |\psi_2|^2}\,, \tag{11}$$

$$V = \frac{2|\psi_1||\psi_2|}{|\psi_1|^2 + |\psi_2|^2}\,. \tag{12}$$

*One should cautiously note that *resolvable* images of a pinhole is not equivalent with *distinguishable* pinholes. *Resolvable* means that the two images of a pinhole are separated and not fused into a single spot. The *distinguishability* of the pinholes has to be proven by existent bijection between an image and a pinhole.

Since the duality relation is a mathematically true statement (theorem) then it cannot be disproved by experiment and certainly means that Afshar's arguments, through which he violates the duality relation, are inconsistent. Indeed the calculation of V and D depends on the principle of complementarity and distinguishability of the states ψ_1 and ψ_2. The calculation of V and D in Unruh's and Afshar's setup is different for pure state and mixed state setups.

5.1 Mixed state setup

In view of the foregoing explanation for Unruh's claim with *mixed density matrix*, the calculation simply yields $D = 1$ and $V = 0$. This will be true if we label the paths by different polarization filters, or if we investigate a statistical mixture of two single path/slit experiments.

$\mathfrak{D}_1$: $|\psi_1| = \frac{1}{\sqrt{2}}$, $|\psi_2| = 0$,

$\mathfrak{D}_2$: $|\psi_1| = 0$, $|\psi_2| = \frac{1}{\sqrt{2}}$.

Thus the two paths 1 and 2 are *distinguishable* and they *do not quantum mechanically cross-interfere*. This cannot be said for the quantum coherent setup with both paths/slits unimpeded.

5.2 Pure state setup

The correct analysis of Unruh's and Afshar's setup suggests a *pure state density matrix*, and amplitudes for each of the exit gates being $|\psi_1| = |\psi_2| = \frac{1}{\sqrt{8}}$. Thus one gets $D = 0$ and $V = 1$:

$\mathfrak{D}_1$: $|\psi_1| = \frac{1}{\sqrt{8}}$, $|\psi_2| = \frac{1}{\sqrt{8}}$,

$\mathfrak{D}_2$: $|\psi_1| = \frac{1}{\sqrt{8}}$, $|\psi_2| = \frac{1}{\sqrt{8}}$.

The two paths 1 and 2 are *indistinguishable*, and they *quantum mechanically cross-interfere*.

6 Conclusions

It is wrongly believed that the lens at the image plane always provides "which way" information (Afshar [1, 2]; Drezet [5]). However we have shown that Afshar's analysis is inconsistent, and that the distinguishability and visibility in Afshar's setup are erroneously calculated by Afshar *and colleagues* [3]. The two peak image at the image plane in Afshar's setup, even without wire grid in the path of the photons, is an interference pattern and does not provide any "which way" information. Exact calculations for the lens setup have been performed by Qureshi [9] and Reitzner [10], showing that once the two paths interfere the interference cannot be undone, and the "which way" information cannot be regained. The probability distribution can look like the one in a mixed setup, but the retrospective reconstruction of the setup for times before the detector click must be done with interfering waves which do not carry the "which way" information. Afshar's mathematics is inconsistent, hence Afshar's setup does not disprove MWI, or any other rival interpretation of Quantum Mechanics that opposes the standard Copenhagen paradigm.

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