On the Gravitational Field of a Pulsating Source

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Because of the pseudo-theorem of Birkhoff, the important problem related to the dynamical gravitational field of a non-stationary spherical mass is ignored by the relativists. A clear formulation of this problem appears in the paper [5], which deals also with the establishment of the appropriate form of the spacetime metric. In the present paper we establish the corresponding equations of gravitation and bring out their solutions.

1 Introduction

As is shown in the paper [5], the propagation of gravitation from a spherical pulsating source is governed by a function $\pi(t,\rho)$, termed *propagation function*, satisfying the following conditions

$$\frac{\partial\pi(t,\rho)}{\partial t} > 0, \quad \frac{\partial\pi(t,\rho)}{\partial\rho} \leqslant 0, \quad \pi(t,\sigma(t)) = t,$$

where $\sigma(t)$ denotes the time-dependent radius of the sphere bounding the matter. The propagation function is not uniquely defined. Any function fulfilling the above conditions characterizes the propagation of gravitation according to the following rule: If the gravitational disturbance reaches the sphere $\|x\| = \rho$ at the instant t, then $\tau = \pi(t,\rho)$ is the instant of its radial emission from the entirety of the sphere bounding the matter. Among the infinity of possible choices of $\pi(t,\rho)$, we distinguish principally the one identified with the time coordinate, namely the propagation function giving rise to the *canonical* $\Theta(4)$*-invariant metric*

$$ds^2 = \left(f(\tau,\rho)d\tau + \ell(\tau,\rho)\,\frac{x\,dx}{\rho}\right)^2 - \\ -\left[\left(\frac{g(\tau,\rho)}{\rho}\right)^2 dx^2 + \left(\big(\ell(\tau,\rho)\big)^2 - \left(\frac{g(\tau,\rho)}{\rho}\right)^2\right)\frac{(x\,dx)^2}{\rho^2}\right] \tag{1.1}$$

(here τ denotes the time coordinate instead of the notation u used in the paper [5]).

Any other $\Theta(4)$-invariant metric results from (1.1) if we replace τ by a conveniently chosen propagation function $\pi(t,\rho)$. Consequently the general form of a $\Theta(4)$-invariant metric outside the matter can be written as

$$ds^2 = \left[\left(f\big(\pi(t,\rho),\rho\big)\frac{\partial\pi(t,\rho)}{\partial t}\right)dt + \\ + \left(f\big(\pi(t,\rho),\rho\big)\frac{\partial\pi(t,\rho)}{\partial\rho} + \ell\big(\pi(t,\rho),\rho\big)\right)\frac{x\,dx}{\rho}\right]^2 - \\ -\left[\left(\frac{g\big(\pi(t,\rho),\rho\big)}{\rho}\right)^2 dx^2 + \left(\Big(\ell\big(\pi(t,\rho),\rho\big)\Big)^2 - \\ -\left(\frac{g\big(\pi(t,\rho),\rho\big)}{\rho}\right)^2\right)\frac{(x\,dx)^2}{\rho^2}\right]. \tag{1.2}$$

The equations of gravitation related to (1.2) are very complicated, but we do not need to write them explicitly, because the propagation function occurs in them as an arbitrary function. So their solution results from that of the equations related to (1.1) if we replace τ by a general propagation function $\pi(t,\rho)$. It follows that the investigation of the $\Theta(4)$-invariant gravitational field must by based on the canonical metric (1.1). The metric (1.2) indicates the dependence of the gravitational field upon the general propagation function $\pi(t,\rho)$, but it is of no interest in dealing with specific problems of gravitation for the following reason. Each allowable propagation function is connected with a certain conception of time, so that the infinity of allowable propagation functions introduces an infinity of definitions of time with respect to the general $\Theta(4)$-invariant metric. This is why the notion of time involved in (1.2) is not clear.

On the other hand, the notion of time related to the canonical metric, although unusual, is uniquely defined and conceptually easily understandable.

This being said, from now on we will confine ourselves to the explicit form of the canonical metric, namely

$$ds^2 = \big(f(\tau,\rho)\big)^2 d\tau^2 + 2f(\tau,\rho)\,\ell(\tau,\rho)\frac{(x\,dx)}{\rho}\,d\tau - \\ - \left(\frac{g(\tau,\rho)}{\rho}\right)^2 dx^2 + \left(\frac{g(\tau,\rho)}{\rho}\right)^2\frac{(x\,dx)^2}{\rho^2} \tag{1.3}$$

which brings out its components:

$$g_{00} = \big(f(\tau,\rho)\big)^2, \qquad g_{0i} = f(\tau,\rho)\,\ell(\tau,\rho)\frac{x_i}{\rho},$$

$$g_{ii} = -\left(\frac{g(\tau,\rho)}{\rho}\right)^2 + \left(\frac{g(\tau,\rho)}{\rho}\right)^2\frac{x_i^2}{\rho^2},$$

$$g_{ij} = \left(\frac{g(\tau,\rho)}{\rho}\right)^2\frac{x_i x_j}{\rho^2}, \qquad (i,j = 1,2,3;\ i \neq j).$$

Note that, since the canonical metric, on account of its own definition, is conceived outside the matter, we have not to bother ourselves about questions of differentiability on the subspace $\mathbb{R}\times\{(0,0,0)\}$ of $\mathbb{R}\times\mathbb{R}^3$. It will be always understood that the spacetime metric is defined for $(\tau,\rho)\in\overline{U}$, $\rho = \|x\|$, $\overline{U}$ being the closed set $\{(\tau,\rho)\in\mathbb{R}^2 | \rho \geq \sigma(\tau)\}$.

2 Summary of auxiliary results

We recall that the Christoffel symbols of second kind related to a given $\Theta(4)$-invariant spacetime metric [3] are the components of a $\Theta(4)$-invariant tensor field and depend on ten functions $B_\alpha = B_\alpha(t,\rho)$, $(\alpha = 0,1,\ldots,9)$, according to the following formulae

$$\Gamma^0_{00} = B_0,\ \ \Gamma^0_{0i} = \Gamma^0_{i0} = B_1 x_i\,, \quad \Gamma^i_{00} = B_2 x_i\,,$$

$$\Gamma^0_{ii} = B_3 + B_4 x_i^2\,,\ \ \Gamma^0_{ij} = \Gamma^0_{ji} = B_4 x_i x_j\,,$$

$$\Gamma^i_{i0} = \Gamma^i_{0i} = B_5 + B_6 x_i^2\,,\ \ \Gamma^i_{j0} = \Gamma^i_{0j} = B_6 x_i x_j\,,$$

$$\Gamma^i_{ii} = B_7 x_i^3 + (B_8 + 2B_9)x_i\,,$$

$$\Gamma^i_{jj} = B_7 x_i x_j^2 + B_8 x_i\,, \quad \Gamma^j_{ij} = \Gamma^j_{ji} = B_7 x_i x_j^2 + B_9 x_i\,,$$

$$\Gamma^i_{jk} = B_7 x_i x_j x_k\,, \quad (i,j,k = 1,2,3;\ i \neq j \neq k \neq i).$$

We recall also that the corresponding Ricci tensor is a symmetric $\Theta(4)$-invariant tensor defined by four functions Q_{00}, Q_{01}, Q_{11}, Q_{22}, the computation of which is carried out by means of the functions B_α occurring in the Christoffel symbols:

$$Q_{00} = \frac{\partial}{\partial t}(3B_5 + \rho^2 B_6) - \rho\frac{\partial B_2}{\partial\rho} - B_2(3 + 4\rho^2 B_9 - \rho^2 B_1 + \rho^2 B_8 + \rho^2 B_7) - 3B_0 B_5 + 3B_5^2 + \rho^2 B_6(-B_0 + 2B_5 + \rho^2 B_6)\,,$$

$$Q_{01} = \frac{\partial}{\partial t}(\rho^2 B_7 + B_8 + 4B_9) - \frac{1}{\rho}\frac{\partial B_5}{\partial\rho} - \rho\frac{\partial B_6}{\partial\rho} + B_2(B_3 + \rho^2 B_4) - 2B_6(2 + \rho^2 B_9) - B_1(3B_5 + \rho^2 B_6)\,,$$

$$Q_{11} = -\frac{\partial B_3}{\partial t} - \rho\frac{\partial B_8}{\partial\rho} - (B_0 + B_5 + \rho^2 B_6)B_3 + (1 - \rho^2 B_8)(B_1 + \rho^2 B_7 + B_8 + 2B_9) - 3B_8\,,$$

$$Q_{22} = -\frac{\partial B_4}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial\rho}(B_1 + B_8 + 2B_9) + B_1^2 + B_8^2 - 2B_9^2 - 2B_1 B_9 + 2B_3 B_6 + (-B_0 - B_5 + \rho^2 B_6)B_4 + \big(-3 + \rho^2(-B_1 + B_8 - 2B_9)\big)B_7\,.$$

3 The Ricci tensor related to the canonical metric (1.3)

In order to find out the functions B_α, $(\alpha = 0,1,\ldots,9)$, resulting from the metric (1.3), we have simply to write down the explicit expressions of the Christoffel symbols Γ^0_{00}, Γ^0_{01}, Γ^1_{00}, Γ^0_{11}, Γ^1_{01}, Γ^1_{12}, Γ^1_{22}, thus obtaining

$$B_0 = \frac{1}{f}\frac{\partial f}{\partial\tau} + \frac{1}{\ell}\frac{\partial\ell}{\partial\tau} - \frac{1}{\ell}\frac{\partial f}{\partial\rho}\,, \qquad B_1 = 0\,,$$

$$B_2 = -\frac{f}{\rho\ell^2}\frac{\partial\ell}{\partial\tau} + \frac{f}{\rho\ell^2}\frac{\partial f}{\partial\rho}\,,$$

$$B_3 = \frac{g}{\rho^2 f\ell}\frac{\partial g}{\partial\rho}\,, \qquad B_4 = -\frac{g}{\rho^4 f\ell}\frac{\partial g}{\partial\rho}\,,$$

$$B_5 = \frac{1}{g}\frac{\partial g}{\partial\tau}\,, \qquad B_6 = \frac{1}{\rho^2\ell}\frac{\partial f}{\partial\rho} - \frac{1}{\rho^2 g}\frac{\partial g}{\partial\tau}\,,$$

$$B_7 = -\frac{g}{\rho^5 f\ell}\frac{\partial g}{\partial\tau} + \frac{1}{\rho^3 f}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^4} + \frac{g}{\rho^5\ell^2}\frac{\partial g}{\partial\rho} + \frac{1}{\rho^3\ell}\frac{\partial\ell}{\partial\rho} - \frac{2}{\rho^3 g}\frac{\partial g}{\partial\rho}\,,$$

$$B_8 = \frac{g}{\rho^3 f\ell}\frac{\partial g}{\partial\tau} + \frac{1}{\rho^2} - \frac{g}{\rho^3\ell^2}\frac{\partial g}{\partial\rho}\,,$$

$$B_9 = -\frac{1}{\rho^2} + \frac{1}{\rho g}\frac{\partial g}{\partial\rho}\,.$$

The conditions $B_1 = 0$, $B_3 + \rho^2 B_4 = 0$ imply several simplifications. Moreover an easy computation gives

$$Q_{11} + \rho^2 Q_{22} = 2\rho\frac{\partial B_9}{\partial\rho} - 2(1 + \rho^2 B_9)(B_8 + B_9 + \rho^2 B_7) + 4B_9.$$

Replacing now everywhere the functions B_α, $(\alpha = 0,1,\ldots,9)$, by their expressions, we obtain the four functions defining the Ricci tensor.

Proposition 3.1 *The functions Q_{00}, Q_{01}, Q_{11}, Q_{22} related to (1.3) are defined by the following formulae.*

$$\begin{aligned} Q_{00} = {} & \frac{1}{\ell}\frac{\partial^2 f}{\partial\tau\partial\rho} - \frac{f}{\ell^2}\frac{\partial^2 f}{\partial\rho^2} + \frac{f}{\ell^2}\frac{\partial^2\ell}{\partial\tau\partial\rho} + \frac{2}{g}\frac{\partial^2 g}{\partial\tau^2} - \\ & - \frac{f}{\ell^3}\frac{\partial\ell}{\partial\tau}\frac{\partial\ell}{\partial\rho} + \frac{f}{\ell^3}\frac{\partial f}{\partial\rho}\frac{\partial\ell}{\partial\rho} + \frac{2f}{\ell^2 g}\frac{\partial\ell}{\partial\tau}\frac{\partial g}{\partial\rho} - \\ & - \frac{2f}{\ell^2 g}\frac{\partial f}{\partial\rho}\frac{\partial g}{\partial\rho} - \frac{2}{fg}\frac{\partial f}{\partial\tau}\frac{\partial g}{\partial\tau} - \frac{2}{\ell g}\frac{\partial\ell}{\partial\tau}\frac{\partial g}{\partial\tau} + \\ & + \frac{2}{\ell g}\frac{\partial f}{\partial\rho}\frac{\partial g}{\partial\tau} - \frac{1}{f\ell}\frac{\partial f}{\partial\tau}\frac{\partial f}{\partial\rho}\,, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \rho Q_{01} = {} & \frac{\partial}{\partial\tau}\left(\frac{1}{f\ell}\frac{\partial(f\ell)}{\partial\rho}\right) - \frac{\partial}{\partial\rho}\left(\frac{1}{\ell}\frac{\partial f}{\partial\rho}\right) + \\ & + \frac{2}{g}\frac{\partial^2 g}{\partial\tau\partial\rho} - \frac{2}{\ell g}\frac{\partial f}{\partial\rho}\frac{\partial g}{\partial\rho}\,, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \rho^2 Q_{11} = {} & -1 - \frac{2g}{f\ell}\frac{\partial^2 g}{\partial\tau\partial\rho} + \frac{g}{\ell^2}\frac{\partial^2 g}{\partial\rho^2} - \frac{2}{f\ell}\frac{\partial g}{\partial\tau}\frac{\partial g}{\partial\rho} - \\ & - \frac{g}{\ell^3}\frac{\partial\ell}{\partial\rho}\frac{\partial g}{\partial\rho} + \frac{1}{\ell^2}\left(\frac{\partial g}{\partial\rho}\right)^2 + \frac{g}{f\ell^2}\frac{\partial f}{\partial\rho}\frac{\partial g}{\partial\rho}\,, \end{aligned} \tag{3.3}$$

$$Q_{11} + \rho^2 Q_{22} = \frac{2}{g}\left(\frac{\partial^2 g}{\partial \rho^2} - \frac{\partial g}{\partial \rho}\frac{1}{f\ell}\frac{\partial (f\ell)}{\partial \rho}\right). \tag{3.4}$$

Note that from (3.1) and (3.2) we deduce the following useful relation

$$\ell Q_{00} - f\rho\, Q_{01} = \frac{2\ell}{g}\frac{\partial^2 g}{\partial \tau^2} + \frac{2f}{\ell g}\frac{\partial \ell}{\partial \tau}\frac{\partial g}{\partial \rho} - \\ - \frac{2\ell}{fg}\frac{\partial f}{\partial \tau}\frac{\partial g}{\partial \tau} - \frac{2}{g}\frac{\partial \ell}{\partial \tau}\frac{\partial g}{\partial \tau} + \frac{2}{g}\frac{\partial f}{\partial \rho}\frac{\partial g}{\partial \tau} - \frac{2f}{g}\frac{\partial^2 g}{\partial \tau \partial \rho}. \tag{3.5}$$

4 Reducing the system of the equations of gravitation

In order to clarify the fundamental problems with a minimum of computations, we will assume that the spherical source is not charged and neglect the cosmological constant. The charge of the source and the cosmological constant do not add difficulties in the discussion of the main problems, so that they may be considered afterwards.

Of course, the equations of gravitation outside the pulsating source are obtained by writing simply that the Ricci tensor vanishes, namely

$$Q_{00} = 0, \quad Q_{01} = 0, \quad Q_{11} = 0, \quad Q_{11} + \rho^2 Q_{22} = 0.$$

The first equation $Q_{00} = 0$ is to be replaced by the equation

$$\ell Q_{00} - f\rho\, Q_{01} = 0$$

which, on account of (3.5), is easier to deal with.

This being said, in order to investigate the equations of gravitation, we assume that the dynamical states of the gravitational field alternate with the stationary ones without diffusion of gravitational waves.

We begin with the equation $Q_{11} + \rho^2 Q_{22} = 0$, which, on account of (3.4), can be written as

$$\frac{\partial}{\partial \rho}\left(\frac{1}{f\ell}\frac{\partial g}{\partial \rho}\right) = 0$$

so that

$$\frac{\partial g}{\partial \rho} = \beta f \ell$$

where β is a function depending uniquely on the time τ.

Let us consider a succession of three intervals of time,

$$[\tau_1, \tau_2] \quad]\tau_2, \tau_3[\quad [\tau_3, \tau_4],$$

such that the gravitational field is stationary during $[\tau_1, \tau_2]$ and $[\tau_3, \tau_4]$ and dynamical during $]\tau_2, \tau_3[$.

When τ describes $[\tau_1, \tau_2]$ and $[\tau_3, \tau_4]$, the functions f, ℓ, g depend uniquely on ρ, so that β reduces then necessarily to a constant, which, according to the known theory of the stationary vacuum solutions, equals $\frac{1}{c}$, c being the classical constant (which, in the present situation, does not represent the velocity of propagation of light in vacuum). It follows that, if β depends effectively on τ during $]\tau_2, \tau_3[$, then it appears as a boundary condition at finite distance, like the radius and the curvature radius of the sphere bounding the matter. However, we cannot conceive a physical situation related to such a boundary condition. So we are led to assume that β is a universal constant, namely $\frac{1}{c}$, keeping this value even during the dynamical states of the gravitational field. However, before accepting finally the universal constancy of β, it is convenient to investigate the equations of gravitation under the assumption that β depends effectively on time during the interval $]\tau_2, \tau_3[$.

We first prove that $\beta = \beta(\tau)$ does not vanish in $]\tau_2, \tau_3[$. We argue by contradiction, assuming that $\beta(\tau_0) = 0$ for some value $\tau_0 \in]\tau_2, \tau_3[$. Then $\frac{\partial g}{\partial \rho}$ and $\frac{\partial^2 g}{\partial \rho^2} = \beta \frac{\partial (f\ell)}{\partial \rho}$ vanish for $\tau = \tau_0$, whereas $\frac{\partial^2 g}{\partial \tau \partial \rho} = (f\ell)\beta' + \beta \frac{\partial (f\ell)}{\partial \tau}$ reduces to $(f\ell)\beta'(\tau_0)$ for $\tau = \tau_0$. Consequently the equation $\rho^2 Q_{11} = 0$ reduces to the condition $1 + 2g\beta'(\tau_0) = 0$ whence $\beta'(\tau_0) < 0$ (since $g > 0$). It follows that $\beta(\tau)$ is strictly decreasing on a certain interval $[\tau_0 - \varepsilon, \tau_0 + \varepsilon] \subset]\tau_2, \tau_3[$, $\varepsilon > 0$, so that $\beta(\tau) < 0$ for every $\tau \in]\tau_0, \tau_0 + \varepsilon]$. Let τ_{00} be the least upper bound of the set of values $\tau \in]\tau_0 + \varepsilon, \tau_3[$ for which $\beta(\tau) = 0$ (This value exists because $\beta(\tau) = \frac{1}{c} > 0$ on $[\tau_3, \tau_4]$). Then $\beta(\tau_{00}) = 0$ and $\beta(\tau) > \tau_{00}$ for $\tau > \tau_{00}$. But, according to what has just been proved, the condition $\beta(\tau_{00}) = 0$ implies that $\beta(\tau) < 0$ on a certain interval $]\tau_{00}, \tau_{00} + \eta]$, $\eta > 0$, giving a contradiction. It follows that the function $\beta(\tau)$ is strictly positive on $]\tau_2, \tau_3[$, hence also on any interval of non-stationarity, and since $\beta(\tau) = \frac{1}{c}$ on the intervals of stationarity, it is strictly positive everywhere. Consequently we are allowed to introduce the inverse function $\alpha = \alpha(\tau) = \frac{1}{\beta(\tau)}$ and write

$$f\ell = \alpha \frac{\partial g}{\partial \rho} \tag{4.1}$$

and

$$f = \frac{\alpha}{\ell}\frac{\partial g}{\partial \rho}. \tag{4.2}$$

Inserting this expression of f into the equation $\rho^2 Q_{11} = 0$ and then multiplying throughout by $\frac{\partial g}{\partial \rho}$, we obtain an equation which can be written as

$$\frac{\partial}{\partial \rho}\left(-\frac{2g}{\alpha}\frac{\partial g}{\partial \tau} + \frac{g}{\ell^2}\left(\frac{\partial g}{\partial \rho}\right)^2 - g\right) = 0$$

whence

$$-\frac{2g}{\alpha}\frac{\partial g}{\partial \tau} + \frac{g}{\ell^2}\left(\frac{\partial g}{\partial \rho}\right)^2 - g = -2\mu = \text{function of } \tau,$$

and

$$\frac{\partial g}{\partial \tau} = \frac{\alpha}{2}\left(-1 + \frac{2\mu}{g} + \frac{1}{\ell^2}\left(\frac{\partial g}{\partial \rho}\right)^2\right). \tag{4.3}$$

It follows that

$$\frac{\partial^2 g}{\partial \tau \partial \rho} = \alpha\left(-\frac{\mu}{g^2}\frac{\partial g}{\partial \rho} - \frac{1}{\ell^3}\frac{\partial \ell}{\partial \rho}\left(\frac{\partial g}{\partial \rho}\right)^2 + \frac{1}{\ell^2}\frac{\partial g}{\partial \rho}\frac{\partial^2 g}{\partial p^2}\right) \tag{4.4}$$

and

$$\frac{\partial^3 g}{\partial\tau\partial\rho^2} = \alpha\left(\frac{2\mu}{g^3}\left(\frac{\partial g}{\partial\rho}\right)^2 - \frac{\mu}{g^2}\frac{\partial^2 g}{\partial\rho^2} + \right.$$
$$+\frac{3}{\ell^4}\left(\frac{\partial\ell}{\partial\rho}\right)^2\left(\frac{\partial g}{\partial\rho}\right)^2 - \frac{1}{\ell^3}\frac{\partial^2\ell}{\partial\rho^2}\left(\frac{\partial g}{\partial\rho}\right)^2 - \tag{4.5}$$
$$\left. -\frac{4}{\ell^3}\frac{\partial\ell}{\partial\rho}\frac{\partial g}{\partial\rho}\frac{\partial^2 g}{\partial\rho^2} + \frac{1}{\ell^2}\left(\frac{\partial^2 g}{\partial\rho^2}\right)^2 + \frac{1}{\ell^2}\frac{\partial g}{\partial\rho}\frac{\partial^3 g}{\partial\rho^3}\right).$$

On the other hand, since $f\ell = \alpha\frac{\partial g}{\partial\rho}$, the expression (3.2) is transformed as follows

$$\rho\, Q_{01} = \frac{1}{\left(\frac{\partial g}{\partial\rho}\right)^2}\left(\frac{\partial g}{\partial\rho}\frac{\partial^3 g}{\partial\tau\partial\rho^2} - \frac{\partial^2 g}{\partial\rho^2}\frac{\partial^2 g}{\partial\tau\partial\rho}\right) +$$
$$+\alpha\left(-\frac{3}{\ell^4}\left(\frac{\partial\ell}{\partial\rho}\right)^2\frac{\partial g}{\partial\rho} + \frac{1}{\ell^3}\frac{\partial^2\ell}{\partial\rho^2}\frac{\partial g}{\partial\rho} + \right.$$
$$\left. +\frac{3}{\ell^3}\frac{\partial\ell}{\partial\rho}\frac{\partial^2 g}{\partial\rho^2} - \frac{1}{\ell^2}\frac{\partial^3 g}{\partial\rho^3} - \frac{2\mu}{g^3}\frac{\partial g}{\partial\rho}\right)$$

and replacing in it $\frac{\partial^2 g}{\partial\tau\partial\rho}$ and $\frac{\partial^3 g}{\partial\tau\partial\rho^2}$ by their expressions (4.4) and (4.5), we find $\rho\, Q_{01} = 0$. Consequently the equation of gravitation $\rho\, Q_{01} = 0$ is verified. It remains to examine the equation $\ell Q_{00} - f\rho\, Q_{01} = 0$. We need some preliminary computations. First we consider the expression of $\frac{\partial^2 g}{\partial\tau^2}$ resulting from the derivation of (4.3) with respect to τ, and then replacing in it $\frac{\partial g}{\partial\tau}$ and $\frac{\partial^2 g}{\partial\tau\partial\rho}$ by their expressions (4.3) and (4.4), we obtain

$$2\frac{\partial^2 g}{\partial\tau^2} = -\frac{d\alpha}{d\tau} + 2\frac{d\alpha}{d\tau}\frac{\mu}{g} + \frac{1}{\ell^2}\frac{d\alpha}{d\tau}\left(\frac{\partial g}{\partial\rho}\right)^2 +$$
$$+\frac{2\alpha}{g}\frac{d\mu}{d\tau} - \frac{2\alpha^2\mu^2}{g^3} + \frac{\alpha^2\mu}{g^2} - \frac{2\alpha}{\ell^3}\frac{\partial\ell}{\partial\tau}\left(\frac{\partial g}{\partial\rho}\right)^2 - \tag{4.6}$$
$$-3\frac{\alpha^2\mu}{\ell^2 g^2}\left(\frac{\partial g}{\partial\rho}\right)^2 - \frac{2\alpha^2}{\ell^5}\frac{\partial\ell}{\partial\rho}\left(\frac{\partial g}{\partial\rho}\right)^3 + \frac{2\alpha^2}{\ell^4}\left(\frac{\partial g}{\partial\rho}\right)^2\frac{\partial^2 g}{\partial\rho^2}.$$

Next, because of (4.2), we have

$$\frac{\partial f}{\partial\rho} = -\frac{\alpha}{\ell^2}\frac{\partial\ell}{\partial\rho}\frac{\partial g}{\partial\rho} + \frac{\alpha}{\ell}\frac{\partial^2 g}{\partial\rho^2} \tag{4.7}$$

and

$$\frac{\partial f}{\partial\tau} = \frac{1}{\ell}\frac{d\alpha}{d\tau}\frac{\partial g}{\partial\rho} - \frac{\alpha}{\ell^2}\frac{\partial\ell}{\partial\tau}\frac{\partial g}{\partial\rho} + \frac{\alpha}{\ell}\frac{\partial^2 g}{\partial\tau\partial\rho}.$$

Lastly taking into account (4.4), we obtain

$$\frac{\partial f}{\partial\tau} = \frac{1}{\ell}\frac{d\alpha}{d\tau}\frac{\partial g}{\partial\rho} - \frac{\alpha}{\ell^2}\frac{\partial\ell}{\partial\tau}\frac{\partial g}{\partial\rho} - \frac{\alpha^2\mu}{\ell g^2}\frac{\partial g}{\partial\rho} -$$
$$-\frac{\alpha^2}{\ell^4}\frac{\partial\ell}{\partial\rho}\left(\frac{\partial g}{\partial\rho}\right)^2 + \frac{\alpha^2}{\ell^3}\frac{\partial g}{\partial\rho}\frac{\partial^2 g}{\partial\rho^2}. \tag{4.8}$$

Now inserting (4.2), (4.3), (4.4), (4.6), (4.7), (4.8) into (3.5), we obtain, after cancelations, the very simple expression

$$\ell Q_{00} - f\rho\, Q_{01} = \frac{2\alpha\ell}{g^2}\frac{d\mu}{d\tau}.$$

Consequently the last equation of gravitation, namely $\ell Q_{00} - f\rho\, Q_{01} = 0$, implies that $\frac{d\mu}{d\tau} = 0$, namely that μ reduces to a constant.

Finally the system of the equations of gravitation is reduced to a system of two equations, namely (4.1) and (4.3), where μ is a constant valid whatever is the state of the field, and α is a strictly positive function of time reducing to the constant c during the stationary states of the field. As already remarked, if α depends effectively on τ during the dynamical states, then it plays the part of a boundary condition the origin of which is indefinable. The following reasoning, which is allowed according to the principles of General Relativity, corroborates the idea that α must be taken everywhere equal to c.

Since $\alpha(\tau) > 0$ everywhere, we can introduce the new time coordinate

$$u = \frac{1}{c}\int_{\tau_0}^{\tau}\alpha(v)dv$$

which amounts to a change of coordinate in the sphere bounding the matter. The function

$$\psi(\tau) = \frac{1}{c}\int_{\tau_0}^{\tau}\alpha(v)dv$$

being strictly increasing, its inverse $\tau = \varphi(u)$ is well defined and $\varphi' = \frac{1}{\psi'} = \frac{c}{\alpha}$. Instead of $\ell(\tau,\rho)$ and $g(\tau,\rho)$ we have now the functions $L(u,\rho) = \ell(\varphi(u),\rho)$ and $G(u,\rho) = g(\varphi(u),\rho)$,

Moreover, since $f d\tau = f\varphi' du$, $f(\tau,\rho)$ is replaced by the function $F(u,\rho) = \varphi'(u) f(\varphi(u),\rho) = \frac{c}{\alpha} f(\varphi(u),\rho)$.

It follows that

$$FL = \varphi' f\ell = \frac{c}{\alpha}\,\alpha\frac{\partial g}{\partial\rho} = c\frac{\partial G}{\partial\rho} \tag{4.9}$$

and

$$\frac{\partial G}{\partial u} = \frac{\partial g}{\partial\tau}\frac{d\tau}{du} = \frac{\alpha}{2}\left(-1 + \frac{2\mu}{g} + \frac{1}{\ell^2}\left(\frac{\partial g}{\partial\rho}\right)^2\right)\frac{c}{\alpha} =$$
$$= \frac{c}{2}\left(-1 + \frac{2\mu}{G} + \frac{1}{L^2}\left(\frac{\partial G}{\partial\rho}\right)^2\right). \tag{4.10}$$

Writing again $f(\tau,\rho)$, $\ell(\tau,\rho)$, $g(\tau,\rho)$ respectively instead of $F(u,\rho)$, $L(u,\rho)$, $G(u,\rho)$, we see that the equations (4.9) and (4.10) are rewritten as

$$f\ell = c\frac{\partial g}{\partial\rho} \tag{4.11}$$

$$\frac{\partial g}{\partial\tau} = \frac{c}{2}\left(-1 + \frac{2\mu}{g} + \frac{1}{\ell^2}\left(\frac{\partial g}{\partial\rho}\right)^2\right). \tag{4.12}$$

So (4.1) and (4.3) preserve their form, but the function α is now replaced by the constant c. Finally we are allowed to dispense with the function α and deal subsequently with the equations (4.11) and (4.12).

5 Stationary and non-stationary solutions

If the field is stationary during a certain interval of time, then the derivative $\frac{\partial g}{\partial \tau}$ vanishes on this interval. The converse is also true. In order to clarify the situation, consider the succession of three intervals of time $]\tau_1, \tau_2[$, $[\tau_2, \tau_3]$, $]\tau_3, \tau_4[$ such that $]\tau_1, \tau_2[$ and $]\tau_3, \tau_4[$ be maximal intervals of non-stationarity, and $\frac{\partial g}{\partial \tau} = 0$ on $[\tau_2, \tau_3]$. Then we have on $[\tau_2, \tau_3]$ the equation

$$-1 + \frac{2\mu}{g} + \frac{1}{\ell^2}\left(\frac{\partial g}{\partial \rho}\right)^2 = 0$$

from which it follows that ℓ does not depend either on τ. On account of (4.11), this property is also valid for f. Consequently the vanishing of $\frac{\partial g}{\partial \tau}$ on $[\tau_2, \tau_3]$ implies the establishment of a stationary state.

During the stationary state we are allowed to introduce the radial geodesic distance

$$\delta = \int_0^\rho \ell(v)\,dv$$

and investigate subsequently the stationary equations in accordance with the exposition appearing in the paper [4]. Since

$$\delta = \beta(\rho)$$

is a strictly increasing function of ρ, the inverse function $\rho = \gamma(\delta)$ is well defined and allows to consider as function of δ every function of ρ. In particular the curvature radius $G(\delta) = g(\gamma(\delta))$ appears as a function of the geodesic distance δ and gives rise to a complete study of the stationary field. From this study it follows that the constant μ equals $\frac{km}{c^2}$ and that the solution $G(\delta)$ possesses the greatest lower bound 2μ. Moreover $G(\delta)$ is defined by the equation

$$\int_{2\mu}^{G} \frac{du}{\sqrt{1 - \frac{2\mu}{u}}} = \delta - \delta_0 \tag{5.1}$$

where δ_0 is a new constant unknown in the classical theory of gravitation. This constant is defined by means of the radius δ_1 and the curvature radius $\zeta_1 = G(\delta_1)$ of the sphere bounding the matter:

$$\delta_0 = \delta_1 - \sqrt{G(\delta_1)(G(\delta_1) - 2\mu)} - \\ - 2\mu \ln\left(\sqrt{\frac{G(\delta_1)}{2\mu}} + \sqrt{\frac{G(\delta_1)}{2\mu} - 1}\right).$$

So the values δ_1 and $\zeta_1 = G(\delta_1)$ constitute the boundary conditions at finite distance. Regarding $F = F(\delta) = f(\gamma(\delta))$, it is defined by means of G:

$$F = cG' = c\sqrt{1 - \frac{2\mu}{G}}, \quad (G \geqslant 2\mu).$$

The so obtained solution does not extend beyond the interval $[\tau_2, \tau_3]$ and even its validity for $\tau = \tau_2$ and $\tau = \tau_3$ is questionable. The notion of radial geodesic distance does not make sense in the intervals of non-stationarity such as $]\tau_1, \tau_2[$ and $]\tau_3, \tau_4[$. Then the integral

$$\int_0^\rho \ell(\tau, v)\,dv$$

depends on the time τ and does not define an invariant length. As a way out of the difficulty we confine ourselves to the consideration of the radical coordinate related to the manifold itself, namely $\rho = \|x\|$.

Regarding the curvature radius $\zeta(\tau)$, it is needed in order to conceive the solution of the equations of gravitation. The function $g(\tau, \rho)$ must be so defined that $g(\tau, \sigma(\tau)) = \zeta(\tau)$. The functions $\sigma(\tau)$ and $\zeta(\tau)$ are the boundary conditions at finite distance for the non-stationary field. They are not directly connected with the boundary conditions of the stationary field defined by means of the radial geodesic distance.

6 On the non-stationary solutions

According to very strong arguments summarized in the paper [2], the relation $g \geqslant 2\mu$ is always valid outside the matter whatever is the state of the field. This is why the first attempt to obtain dynamical solutions was based on an equation analogous to (5.1), namely

$$\int_{2\mu}^{g} \frac{du}{\sqrt{1 - \frac{2\mu}{u}}} = \gamma(\tau, \rho)$$

where $\gamma(\tau, \rho)$ is a new function satisfying certain conditions. This idea underlies the results presented briefly in the paper [1]. However the usefulness of introduction of a new function is questionable. It is more natural to deal directly with the functions f, ℓ, g involved in the metric. In any case we have to do with two equations, namely (4.11) and (4.12), so that we cannot expect to define completely the three unknown functions. Note also that, even in the considered stationary solution, the equation (5.1) does not define completely the function G on account of the new unknown constant δ_0. In the general case there is no way to define the function $g(\tau, \rho)$ by means of parameters and simpler functions. The only available equation, namely (4.12), a partial differential equation including the unknown function $\ell(\tau, \rho)$, is, in fact, intractable. As a way out of the difficulties, we propose to consider the function $g(\tau, \rho)$ as a new entity required by the non-Euclidean structure involved in the dynamical gravitational field. In the present state of our knowledge, we confine ourselves to put forward the main features of $g(\tau, \rho)$ in the closed set

$$\overline{U} = \{(\tau, \rho) \in \mathbb{R}^2 | \rho \geqslant \sigma(\tau)\}.$$

Since the vanishing of f or ℓ would imply the degeneracy of the spacetime metric, these two functions are necessarily strictly positive on $\overline{U}$. Then from the equation (4.11) it follows that

$$\frac{\partial g(\tau,\rho)}{\partial\rho} > 0 \tag{6.1}$$

on the closed set $\overline{U}$. On the other hand, since (4.12) can be rewritten as

$$\frac{2}{c}\frac{\partial g}{\partial\tau} + 1 - \frac{2\mu}{g} = \frac{1}{\ell^2}\left(\frac{\partial g}{\partial\rho}\right)^2$$

we have also

$$\frac{2}{c}\frac{\partial g}{\partial\tau} + 1 - \frac{2\mu}{g} > 0 \tag{6.2}$$

on the closed set $\overline{U}$. Now, on account of (6.1) and (6.2), the equations (4.11) and (4.12) define uniquely the functions f and ℓ by means of g:

$$f = c\sqrt{\frac{2}{c}\frac{\partial g}{\partial\tau} + 1 - \frac{2\mu}{g}} \tag{6.3}$$

$$\ell = \frac{\partial g/\partial\rho}{\sqrt{\frac{2}{c}\frac{\partial g}{\partial\tau} + 1 - \frac{2\mu}{g}}}\,. \tag{6.4}$$

It is now obvious that the curvature radius $g(\tau,\rho)$ plays the main part in the conception of the gravitational field. Although it has nothing to do with coordinates, the relativists have reduced it to a so-called radial coordinate from the beginnings of General Relativity. This glaring mistake has given rise to intolerable misunderstandings and distorted completely the theory of the gravitational field.

Let $]\tau_1,\tau_2[$ be a maximal bounded open interval of non-stationarity. Then $\frac{\partial g}{\partial\tau} = 0$ for $\tau=\tau_1$ and $\tau=\tau_2$, but $\frac{\partial g}{\partial\tau} \neq 0$ on an open dense subset of $]\tau_1,\tau_2[$. So $\frac{\partial g}{\partial\tau}$ appears as a gravitational wave travelling to infinity, and it is natural to assume that $\frac{\partial g}{\partial\tau}$ tends uniformly to zero on $[\tau_1,\tau_2]$ as $\rho\to+\infty$. Of course the behaviour of $\frac{\partial g}{\partial\tau}$ depends on the boundary conditions which do not appear in the obtained general solution. They are to be introduced in accordance with the envisaged problem. In any case the gravitational disturbance plays the fundamental part in the conception of the dynamical gravitation, but the state of the field does not follow always a simple rule.

In particular, if the gravitational disturbance vanishes during a certain interval of time $[\tau_1,\tau_2]$, the function $g(\tau,\rho)$ does not depend necessarily only on ρ during $[\tau_1,\tau_2]$. In other words, the gravitational field does not follow necessarily the Huyghens principle contrary to the solutions of the classical wave equation in $\mathbb{R}^3$.

We deal briefly with the case of *a Huyghens type field*, namely a $\Theta(4)$-invariant gravitational field such that the vanishing of the gravitational disturbance on a time interval implies the establishment of a universal stationary state. Then the time is involved in the curvature radius by means of the boundary conditions $\sigma(\tau)$, $\zeta(\tau)$, so that $g(\tau,\rho)$ is in fact a function of $(\sigma(\tau),\zeta(\tau),\rho)$: $g(\sigma(\tau),\zeta(\tau),\rho)$. The corresponding expressions for f and ℓ result from (6.3) and (6.4):

$$f = c\sqrt{\frac{2}{c}\left(\frac{\partial g}{\partial\sigma}\,\sigma'(\tau) + \frac{\partial g}{\partial\zeta}\,\zeta'(\tau)\right) + 1 - \frac{2\mu}{g}}$$

$$\ell = \frac{\frac{\partial g}{\partial\rho}}{\sqrt{\frac{2}{c}\left(\frac{\partial g}{\partial\sigma}\,\sigma'(\tau) + \frac{\partial g}{\partial\zeta}\,\zeta'(\tau)\right) + 1 - \frac{2\mu}{g}}}$$

where g denotes $g(\sigma(\tau),\zeta(\tau),\rho)$.

If $\sigma'(\tau)=\zeta'(\tau)=0$ during an interval of time, the boundary conditions $\sigma(\tau)$, $\zeta(\tau)$ reduce to positive constants σ_0, ζ_0 on this interval, so that the curvature radius defining the stationary states depends on the constants $\sigma_0,\zeta_0 : g(\sigma_0,\zeta_0,\rho)$. It is easy to write down the conditions satisfied by $g(\sigma_0,\zeta_0,\rho)$, considered as function of three variables.

Submitted on June 12, 2007
Accepted on June 13, 2007

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