Some Remarks on Ricci Flow and the Quantum Potential

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We indicate some formulas connecting Ricci flow and Perelman entropy to Fisher information, differential entropy, and the quantum potential. There is a known relation involving the Schroedinger equation in a Weyl space where the Weyl-Ricci curvature is proportional to the quantum potential. The quantum potential in turn is related to Fisher information which is given via the Perelman entropy functional arising from a differential entropy under Ricci flow. These relations are written out and seem to suggest connections between quantum mechanics and Ricci flow.

1 Formulas involving Ricci flow

Certain aspects of Perelman's work on the Poincaré conjecture have applications in physics and we want to suggest a few formulas in this direction; a fuller exposition will appear in a book in preparation [8]. We go first to [13, 24–28, 33, 39] and simply write down a few formulas from [28, 39] here with minimal explanation. Thus one has Perelman's functional ($\dot{\mathcal{R}}$ is the Riemannian Ricci curvature)

$$\mathfrak{F} = \int_M (\dot{\mathcal{R}} + |\nabla f|^2)\exp(-f)dV \tag{1.1}$$

and a so-called Nash entropy $(\mathbf{1A})$ $N(u) = \int_M u\log(u)dV$ where $u = \exp(-f)$. One considers Ricci flows with $\delta g \sim \partial_t g = h$ and for $(\mathbf{1B})$ $\Box^* u = -\partial_t u - \Delta u + \dot{\mathcal{R}} u = 0$ (or equivalently $\partial_t f + \Delta f - |\nabla f|^2 + \dot{\mathcal{R}} = 0$) it follows that $\int_M \exp(-f)dV = 1$ is preserved and $\partial_t N = \mathfrak{F}$. Note the Ricci flow equation is $\partial_t g = -2Ric$. Extremizing $\mathfrak{F}$ via $\delta\mathfrak{F} \sim \sim \partial_t \mathfrak{F} = 0$ involves $Ric + Hess(f) = 0$ or $R_{ij} + \nabla_i\nabla_j f = 0$ and one knows also that

$$\left.\begin{aligned} \partial_t N &= \int_M (|\nabla f|^2 + \dot{\mathcal{R}})\exp(-f)\,dV = \mathfrak{F};\\ \partial_t \mathfrak{F} &= 2\int_M |Ric + Hess(f)|^2 \exp(-f)\,dV. \end{aligned}\right. \tag{1.2}$$

2 The Schrödinger equation and WDW

Now referring to [3–5, 7–12, 15, 16, 18–23, 29–32, 35–38, 40] for details we note first the important observation in [39] that $\mathfrak{F}$ is in fact a Fisher information functional. Fisher information has come up repeatedly in studies of the Schrödinger equation (SE) and the Wheeler-deWitt equation (WDW) and is connected to a differential entropy correspondingto the Nash entropy above (cf. [4, 7, 18, 19]). The basic ideas involve (using 1-D for simplicity) a quantum potential Q such that $\int_M PQ\,dx \sim \mathfrak{F}$ arising from a wave function $\psi = R\exp(iS/\hbar)$ where $Q = -(\hbar^2/2m)(\Delta R/R)$ and $P \sim |\psi|^2$ is a probability density. In a WDW context for example one can develop a framework

$$\left.\begin{aligned} &Q = cP^{-1/2}\,\partial(GP^{1/2});\\ &\int QP = c\int P^{1/2}\,\partial(GP^{1/2})\,\mathfrak{D}h\,dx \;\to\\ &\to -c\int \partial P^{1/2}\,G\partial P^{1/2}\,\mathfrak{D}h\,dx \end{aligned}\right\} \tag{2.1}$$

where G is an expression involving the deWitt metric $G_{ijk\ell}(h)$. In a more simple minded context consider a SE in 1-D $i\hbar\partial_t\psi = -(\hbar^2/2m)\,\partial_x^2\psi + V\psi$ where $\psi = R\exp(iS/\hbar)$ leads to the equations

$$\left.\begin{aligned} &S_t + \frac{1}{2m}S_x^2 + Q + V = 0;\\ &\partial_t R^2 + \frac{1}{m}(R^2 S_x)_x = 0:\;\; Q = -\frac{\hbar^2}{2m}\frac{R_{xx}}{R}. \end{aligned}\right\} \tag{2.2}$$

In terms of the exact uncertainty principle of Hall and Reginatto (see [21, 23, 34] and cf. also [4, 6, 7, 31, 32]) the quantum Hamiltonian has a Fisher information term $c\int dx\,(\nabla P\cdot\nabla P/2mP)$ added to the classical Hamiltonian (where $P = R^2 \sim |\psi|^2$) and a simple calculation gives

$$\begin{aligned} \int PQ\,d^3x &\sim -\frac{\hbar^2}{8m}\int\left[2\Delta P - \frac{1}{P}|\nabla P|^2\right]d^3x = \\ &= \frac{\hbar^2}{8m}\int\frac{1}{P}|\nabla P|^2\,d^3x\,. \end{aligned} \tag{2.3}$$

In the situation of (2.1) the analogues to Section 1 involve $(\partial \sim \partial_x)$

$$\left.\begin{aligned} &P \sim e^{-f};\qquad P' \sim P_x \sim -f'e^{-f};\\ &Q \sim e^{f/2}\,\partial(G\partial e^{-f/2});\;\; PQ \sim e^{-f/2}\,\partial(G\partial e^{-f/2});\\ &\int PQ \to -\int\partial e^{-f/2}\,G\partial e^{-f/2} \sim -\int\partial P^{1/2}G\partial P^{1/2}. \end{aligned}\right\} \tag{2.4}$$

In the context of the SE in Weyl space developed in [1, 2, 4, 7, 10, 11, 12, 35, 36, 40] one has a situation $|\psi|^2 \sim R^2 \sim \sim P \sim \hat{\rho} = \rho/\sqrt{g}$ with a Weyl vector $\vec{\phi} = -\nabla\log(\hat{\rho})$ and a quantum potential

$$Q \sim -\frac{\hbar^2}{16m}\left[\dot{\mathcal{R}} + \frac{8}{\sqrt{\hat{\rho}}}\frac{1}{\sqrt{g}}\,\partial_i\left(\sqrt{g}\,g^{ik}\partial_k\sqrt{\hat{\rho}}\right)\right] = = -\frac{\hbar^2}{16m}\left[\dot{\mathcal{R}} + \frac{8}{\sqrt{\hat{\rho}}}\,\Delta\sqrt{\hat{\rho}}\right] \tag{2.5}$$

(recall $\mathrm{div}\,\mathrm{grad}(U) = \Delta U = (1/\sqrt{g})\,\partial_m(\sqrt{g}\,g^{mn}\partial_n U)$. Here the Weyl-Ricci curvature is $(\mathbf{2A})$ $\mathcal{R} = \dot{\mathcal{R}} + \mathcal{R}_w$ where

$$\mathcal{R}_w = 2|\vec{\phi}|^2 - 4\nabla\cdot\vec{\phi} = 8\,\frac{\Delta\sqrt{\hat{\rho}}}{\sqrt{\hat{\rho}}} \tag{2.6}$$

and $Q = -(\hbar^2/16m)\,\mathcal{R}$. Note that

$$-\nabla\cdot\vec{\phi} \sim -\Delta\log(\hat{\rho}) \sim -\frac{\Delta\hat{\rho}}{\hat{\rho}} + \frac{|\nabla\hat{\rho}|^2}{\hat{\rho}^2} \tag{2.7}$$

and for $\exp(-f) = \hat{\rho} = u$

$$\int \hat{\rho}\,\nabla\cdot\vec{\phi}\,dV = \int\left[-\Delta\hat{\rho} + \frac{|\nabla\hat{\rho}|^2}{\hat{\rho}}\right]dV \tag{2.8}$$

with the first term in the last integral vanishing and the second providing Fisher information again. Comparing with Section 1 we have analogues $(\mathbf{2B})$ $G \sim (R + |\vec{\phi}|^2)$ with $\vec{\phi} = = -\nabla\log(\hat{\rho}) \sim \nabla f$ to go with (2.4). Clearly $\hat{\rho}$ is basically a probability concept with $\int\hat{\rho}\,dV = 1$ and Quantum Mechanics (QM) (or rather perhaps Bohmian mechanics) seems to enter the picture through the second equation in (2.2), namely $(\mathbf{2C})$ $\partial_t\hat{\rho} + (1/m)\,\mathrm{div}\,(\hat{\rho}\,\nabla S) = 0$ with $p = mv = \nabla S$, which must be reconciled with (**1B**) (i.e. $(1/m)\,\mathrm{div}(u\nabla S) = = \Delta u - \dot{\mathcal{R}}u$). In any event the term $G = \dot{\mathcal{R}} + |\vec{\phi}|^2$ can be written as $(\mathbf{2D})$ $\dot{\mathcal{R}} + \mathcal{R}_w + (|\vec{\phi}|^2 - \mathcal{R}_w) = \alpha Q + (4\nabla\cdot\vec{\phi} - |\vec{\phi}|^2)$ which leads to $(\mathbf{2E})$ $\mathfrak{F} \sim \alpha\int_M QP\,dV + \beta\int|\vec{\phi}|^2 P dV$ putting Q directly into the picture and suggesting some sort of quantum mechanical connection.

REMARK 2.1. We mention also that Q appears in a fascinating geometrical role in the relativistic Bohmian format following [3, 15, 37, 38] (cf. also [4, 7] for survey material). Thus e.g. one can define a quantum mass field via

$$\mathfrak{M}^2 = m^2\exp(Q) \sim m^2(1+Q);$$
$$Q \sim \frac{-\hbar^2}{c^2m^2}\frac{\Box(\sqrt{\rho})}{\sqrt{\rho}} \sim \frac{\alpha}{6}\,\mathcal{R}_w \tag{2.9}$$

where ρ refers to an appropriate mass density and $\mathfrak{M}$ is in fact the Dirac field β in a Weyl-Dirac formulation of Bohmian quantum gravity. Further one can change the 4-D Lorentzian metric via a conformal factor $\Omega^2 = \mathfrak{M}^2/m^2$ in the form $\tilde{g}_{\mu\nu} = = \Omega^2 g_{\mu\nu}$ and this suggests possible interest in Ricci flows etc. in conformal Lorentzian spaces (cf. here also [14]). We refer to [3, 15] for another fascinating form of the quantum potential as a mass generating term and intrinsic self energy. ■

NOTE. Publication information for items below listed by archive numbers can often be found on the net listing. ■

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