Ricci Flow and Quantum Theory

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We show how Ricci flow is related to quantum theory via Fisher information and the quantum potential.

1 Introduction

In [9, 13, 14] we indicated some relations between Weyl geometry and the quantum potential, between conformal general relativity (GR) and Dirac-Weyl theory, and between Ricci flow and the quantum potential. We would now like to develop this a little further. First we consider simple Ricci flow as in [35, 49]. Thus from [35] we take the Perelman entropy functional as $(\mathbf{1A})$ $\mathfrak{F}(g,f) = \int_M (|\nabla f|^2 + R)\exp(-f)dV$ (restricted to f such that $\int_M \exp(-f)dV = 1$) and a Nash (or differential) entropy via $(\mathbf{1B})$ $N(u) = \int_M u\log(u)dV$ where $u = \exp(-f)$ (M is a compact Riemannian manifold without boundary). One writes $dV = \sqrt{\det(g)}\prod dx^i$ and shows that if $g \to g + \mathrm{sh}\,(g, h \in \mathcal{M} = Riem(M))$ then $(\mathbf{1C})$ $\partial_s \det(g)|_{s=0} = g^{ij}h_{ij}\det(g) = (Tr_g h)\det(g)$. This comes from a matrix formula of the following form $(\mathbf{1D})$ $\partial_s \det(A+B)|_{s=0} = (A^{-1}:B)\det(A)$ where $A^{-1}:B = a^{ij}b_{ji} = a^{ij}b_{ij}$ for symmetric B (a^{ij} comes from A^{-1}). If one has Ricci flow $(\mathbf{1E})$ $\partial_s g = -2Ric$ (i.e. $\partial_s g_{ij} = -2R_{ij}$) then, considering $h \sim -2Ric$, one arrives at $(\mathbf{1F})$ $\partial_s dV = -RdV$ where $R = g^{ij}R_{ij}$ (more general Ricci flow involves $(\mathbf{1G})$ $\partial_t g_{ik} = -2(R_{ik} + \nabla_i\nabla_k\phi)$). We use now t and s interchangeably and suppose $\partial_t g = -2Ric$ with $u = \exp(-f)$ satisfying $\square^* u = 0$ where $\square^* = -\partial_t - \Delta + R$. Then $\int_M \exp(-f)dV = 1$ is preserved since $(\mathbf{1H})$ $\partial_t \int_M u dV = \int_M (\partial_s u - Ru)dV = -\int_M \Delta u dV = 0$ and, after some integration by parts,

$$\partial_t N = \int_M \left[\partial_t u(\log(u)+1)dV + u\log(u)\partial_t dV\right] = \int_M (|\nabla f|^2 + R)e^{-f}dV = \mathfrak{F}. \tag{1.1}$$

In particular for $R \geqslant 0$, N is monotone as befits an entropy. We note also that $\square^* u = 0$ is equivalent to $(\mathbf{1I})$ $\partial_t f = -\Delta f + |\nabla f|^2 - R$.

It was also noted in [49] that $\mathfrak{F}$ is a Fisher information functional (cf. [8, 10, 24, 25]) and we showed in [13] that for a given 3-D manifold M and a Weyl-Schrödinger picture of quantum evolution based on [42, 43] (cf. also [4, 5, 6, 8, 9, 10, 11, 12, 16, 17, 51]) one can express $\mathfrak{F}$ in terms of a quantum potential Q in the form $(\mathbf{1J})$ $\mathfrak{F} \sim \alpha\int_M QPdV + \beta\int_M |\vec{\phi}|^2 PdV$ where $\vec{\phi}$ is a Weyl vector and P is a probability distribution associated with a quantum mass density $\hat{\rho} \sim |\psi|^2$. There will be a corresponding Schrödinger equation (SE) in a Weyl space as in [10, 13] provided there is a phase S (for $\psi = |\psi|\exp(iS/\hbar)$) satisfying $(\mathbf{1K})$ $(1/m)\mathrm{div}(P\nabla S) = \Delta P - RP$ (arising from $\partial_t\hat{\rho} - \Delta\hat{\phi} = -(1/m)\mathrm{div}(\hat{\rho}\nabla S)$ and $\partial_t\hat{\rho} + \Delta\hat{\rho} - R\hat{\rho} = 0$ with $\hat{\rho} \sim P \sim u \sim |\psi|^2$). In the present work we show that there can exist solutions S of $(\mathbf{1K})$ and this establishes a connection between Ricci flow and quantum theory (via Fisher information and the quantum potential). Another aspect is to look at a relativistic situation with conformal perturbations of a 4-D semi-Riemannian metric g based on a quantum potential (defined via a quantum mass). Indeed in a simple minded way we could perhaps think of a conformal transformation $\hat{g}_{ab} = \Omega^2 g_{ab}$ (in 4-D) where following [14] we can imagine ourselves immersed in conformal general relativity (GR) with metric $\hat{g}$ and $(\mathbf{1L})$ $\exp(Q) \sim \mathfrak{M}^2/m^2 = \Omega^2 = \hat{\phi}^{-1}$ with $\beta \sim \mathfrak{M}$ where β is a Dirac field and Q a quantum potential $Q \sim (\hbar^2/m^2c^2)(\square_g\sqrt{\rho})/\sqrt{\rho})$ with $\rho \sim |\psi^2|$ referring to a quantum matter density. The theme here (as developed in [14]) is that Weyl-Dirac action with Dirac field β leads to $\beta \sim \mathfrak{M}$ and is equivalent to conformal GR (cf. also [8, 10, 36, 45, 46, 47] and see [28] for ideas on Ricci flow gravity).

REMARK 1.1. For completeness we recall (cf. [10, 50]) for $\mathfrak{L}_G = (1/2\chi)\sqrt{-g}\,R$

$$\delta\mathfrak{L} = \frac{1}{2\chi}\left[R_{ab} - \frac{1}{2}g_{ab}R\right]\sqrt{-g}\,\delta g^{ab} + \frac{1}{2\chi}g^{ab}\sqrt{-g}\,\delta R_{ab}\,. \tag{1.2}$$

The last term can be converted to a boundary integral if certain derivatives of g_{ab} are fixed there. Next following [7, 9, 14, 27, 38, 39, 40] the Einstein frame GR action has the form

$$S_{GR} = \int d^4x\sqrt{-g}\,(R - \alpha(\nabla\psi)^2 + 16\pi L_M) \tag{1.3}$$

(cf. [7]) whose conformal form (conformal GR) is

$$\hat{S}_{GR} = \int d^4x\sqrt{-\hat{g}}\,e^{-\psi} \times \left[\hat{R} - \left(\alpha - \frac{3}{2}\right)(\hat{\nabla}\psi)^2 + 16\pi e^{-\psi}L_M\right] = \int d^4x\sqrt{-g}\left[\hat{\phi}\hat{R} - \left(\alpha - \frac{3}{2}\right)\frac{(\hat{\nabla}\hat{\phi})^2}{\hat{\phi}} + 16\pi\hat{\phi}^2 L_M\right], \tag{1.4}$$

where $\hat{g}_{ab} = \Omega^2 g_{ab}$, $\Omega^2 = \exp(\psi) = \phi$, and $\hat{\phi} = \exp(-\psi) = \phi^{-1}$. If we omit the matter Lagrangians, and set $\lambda = \frac{3}{2} - \alpha$, (1.4) becomes for $\hat{g}_{ab} \to g_{ab}$

$$\tilde{S} = \int d^4x \sqrt{-g}\, e^{-\psi} \big[R + \lambda(\nabla\psi)^2\big]. \tag{1.5}$$

In this form on a 3-D manifold M we have **exactly** the situation treated in [10, 13] with an associated SE in Weyl space based on (**1K**). ■

2 Solution of (1K)

Consider now $(\mathbf{1K})$ $(1/m)\mathrm{div}(P\nabla S) = \Delta P - RP$ for $P \sim \hat{\rho} \sim |\psi|^2$ and $\int P\sqrt{|g|}d^3x = 1$ (in 3-D we will use here $\sqrt{|g|}$ for $\sqrt{-g}$). One knows that $\mathrm{div}(P\nabla S) = P\Delta S + \nabla P \cdot \nabla S$ and

$$\left.\begin{aligned} &\Delta\psi = \frac{1}{\sqrt{|g|}}\partial_m\big(\sqrt{|g|}\nabla\psi\big), \quad \nabla\psi = g^{mn}\partial_n\psi \\ &\int_M \mathrm{div}\mathbf{V}\sqrt{|g|}\,d^3x = \int_{\partial M}\mathbf{V}\cdot\mathbf{ds} \end{aligned}\right\} \tag{2.1}$$

(cf. [10]). Recall also $\int P\sqrt{|g|}\,d^3x = 1$ and

$$\left.\begin{aligned} &Q \sim -\frac{\hbar^2}{8m}\left[\left(\frac{\nabla P}{P}\right)^2 - 2\left(\frac{\Delta P}{P}\right)\right] \\ &<Q>_\psi = \int PQ\,d^3x \end{aligned}\right\}. \tag{2.2}$$

Now in 1-D an analogous equation to (**1K**) would be $(\mathbf{3A})$ $(PS')' = P' - RP = F$ with solution determined via

$$\begin{aligned} &PS' = P' - \int RP + c \Rightarrow \\ &\Rightarrow S' = \partial_x\log(P) - \frac{1}{P}\int RP + cP^{-1} \Rightarrow \\ &\Rightarrow S = \log(P) - \int\frac{1}{P}\int RP + c\int P^{-1} + k\,, \end{aligned} \tag{2.3}$$

which suggests that solutions of (**1K**) do in fact exist in general. We approach the general case in Sobolev spaces à la [1, 2, 15, 22]. The volume element is defined via $\eta = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$ (where $n = 3$ for our purposes) and $* : \wedge^p M \to \wedge^{n-p}M$ is defined via

$$\left.\begin{aligned} &(*\alpha)_{\lambda_{p+1}\cdots\lambda_n} = \frac{1}{p!}\eta_{\lambda_1\cdots\lambda_n}\alpha^{\lambda_1\cdots\lambda_p} \\ &(\alpha,\beta) = \frac{1}{p!}\alpha_{\lambda_1\cdots\lambda_p}\beta^{\lambda_1\cdots\lambda_p} \end{aligned}\right\}, \tag{2.4}$$

$*1 = \eta$; $**\alpha = (-1)^{p(n-p)}\alpha$; $*\eta = 1$; $\alpha \wedge (*\beta) = (\alpha,\beta)\eta$. One writes now $<\alpha,\beta> = \int_M(\alpha,\beta)\eta$ and, for (Ω,ϕ) a local chart we have $(\mathbf{2A})$ $\int_M f dV = \int_{\phi(\Omega)}(\sqrt{|g|}f)\circ\phi^{-1}\prod dx^i$ $(\sim \int_M f\sqrt{|g|}\prod dx^i)$. Then one has $(\mathbf{2B})$ $<d\alpha,\gamma> = <\alpha,\delta\gamma>$ for $\alpha \in \wedge^p M$ and $\gamma \in \wedge^{p+1}M$ where the codifferential δ on p-forms is defined via $(\mathbf{2C})$ $\delta = (-1)^p *^{-1} d*$. Then $\delta^2 = d^2 = 0$ and $\Delta = d\delta + \delta d$ so that $\Delta f = \delta df = -\nabla^\nu\nabla_\nu f$. Indeed for $\alpha \in \wedge^p M$

$$(\delta\alpha)_{\lambda_1,\cdots,\lambda_{p-1}} = -\nabla^\gamma\alpha_{\gamma,\lambda_1,\cdots,\lambda_{p-1}} \tag{2.5}$$

with $\delta f = 0$ $(\delta : \wedge^p M \to \wedge^{p-1}M)$. Then in particular $(\mathbf{2D})$ $<\Delta\phi,\phi> = <\delta d\phi,\phi> = <d\phi,d\phi> = \int_M \nabla^\nu\phi\nabla_\nu\phi\eta$.

Now to deal with weak solutions of an equation in divergence form look at an operator $(\mathbf{2E})$ $Au = -\nabla(a\nabla u) \sim (-1/\sqrt{|g|})\,\partial_m(\sqrt{|g|}\,ag^{mn}\nabla_n u) = -\nabla_m(a\nabla^m u)$ so that for $\phi \in \mathcal{D}(M)$

$$\begin{aligned} \int_M Au\phi dV &= -\int\big[\nabla_m(ag^{mn}\nabla_n u)\big]\phi dV = \\ &= \int ag^{mn}\nabla_n u\,\nabla_m\phi dV = \int a\nabla^m u\,\nabla_m\phi dV. \end{aligned} \tag{2.6}$$

Here one imagines M to be a complete Riemannian manifold with Soblev spaces $H_0^1(M) \sim H^1(M)$ (see [1, 3, 15, 26, 29, 48]). The notation in [1] is different and we think of $H^1(M)$ as the space of L^2 functions u on M with $\nabla u \in L^2$ and H_0^1 means the completion of $\mathcal{D}(M)$ in the H^1 norm $||u||^2 = \int_M[|u|^2 + |\nabla u|^2]dV$. Following [29] we can also assume $\partial M = \emptyset$ with M connected for all M under consideration. Then let $H = H^1(M)$ be our Hilbert space and consider the operator $A(S) = -(1/m)\nabla(P\nabla S)$ with

$$B(S,\psi) = \frac{1}{m}\int P\,\nabla^m S\,\nabla_m\psi\,dV \tag{2.7}$$

for $S,\psi \in H_0^1 = H^1$. Then $A(S) = RP - \Delta P = F$ becomes $(\mathbf{2F})$ $B(S,\psi) = <F,\psi> = \int F\psi\,dV$ and one has $(\mathbf{2G})$ $|B(S,\psi)| \leqslant c\,||S||_H\,||\psi||_H$ and $|B(S,S)| = \int P(\nabla S)^2\,dV$. Now $P \geqslant 0$ with $\int PdV = 1$ but to use the Lax-Milgram theory we need here $|B(S,S)| \geqslant \beta||S||_H^2$ $(H = H^1)$. In this direction one recalls that in Euclidean space for $\psi \in H_0^1(\mathbf{R}^3)$ there follows $(\mathbf{2H})$ $||\psi||_{L^2}^2 \leqslant c\,||\nabla\psi||_{L^2}^2$ (Friedrich's inequality — cf. [48]) which would imply $||\psi||_H^2 \leqslant (c+1)||\nabla\psi||_{L^2}^2$. However such Sobolev and Poincaré-Sobolev inequalities become more complicated on manifolds and (**2H**) is in no way automatic (cf. [1, 29, 48]). However we have some recourse here to the definition of P, namely $P = \exp(-f)$, which basically is a conformal factor and $P > 0$ unless $f \to \infty$. One heuristic situation would then be to assume $(\mathbf{2I})$ $0 < \epsilon \leqslant P(x)$ on M (and since $\int\exp(-f)dV = 1$ with $dV = \sqrt{|g|}\prod_1^3 dx^i$ we must then have $\epsilon\int dV \leqslant 1$ or $\mathrm{vol}(M) = \int_M dV \leqslant (1/\epsilon)$). Then from (**2G**) we have $(\mathbf{2J})$ $|B(S,S)| \geqslant \epsilon||(\nabla S)^2||$ and for any $\kappa > 0$ it follows: $|B(S,S)| + \kappa||S||_{L^2}^2 \geqslant \min(\epsilon,\kappa)||S||_{H^1}^2$. This means via Lax-Milgram that the equation

$$A(S) + \kappa S = -\frac{1}{m}\nabla(P\nabla S) + \kappa S = F = RP - \Delta P \tag{2.8}$$

has a unique weak solution $S \in H^1(M)$ for any $\kappa > 0$ (assuming $F \in L^2(M)$). Equivalently $(\mathbf{2K})$ $-\frac{1}{m}[P\Delta S + (\nabla P)(\nabla S)] + \kappa S = F$ has a unique weak solution $S \in H^1(M)$. This is close but we cannot put $\kappa = 0$. A different approach following from remarks in [29], pp. 56–57 (corrected in [30], p. 248), leads to an heuristic weak solution of (**1K**). Thus from a result of Yau [53] if M is a complete simply connected 3-D differential manifold with sectional curvature $K < 0$ one has for $u \in \mathcal{D}(M)$

$$\int_M |\psi| dV \leqslant (2\sqrt{-K})^{-1} \int_M |\nabla\psi| dV \Rightarrow$$
$$\Rightarrow \int_M |\psi|^2 dV \leqslant c \int_M |\nabla\psi|^2 dV. \tag{2.9}$$

Hence (**2H**) holds and one has $\|\psi\|^2_{H^1} \leqslant (1+c)\|\nabla\psi\|^2$. Morever if M is bounded and simply connected with a reasonable boundary ∂M (e.g. weakly convex) one expects $(\mathbf{2L})$ $\int_M |\psi|^2 dV \leqslant c \int_M |\nabla\psi|^2 dV$ for $\psi \in \mathcal{D}(M)$ (cf. [41]). In either case $(\mathbf{2M})$ $|B(S,S)| \geqslant \epsilon\|(\nabla S)^2\| \geqslant (c+1)^{-1}\epsilon\|S\|^2_{H^1_0}$ and this leads via Lax-Milgram again to a sample result

THEOREM 2.1. Let M be a bounded and simply connected 3-D differential manifold with a reasonable boundary ∂M. Then there exists a unique weak solution of (**1K**) in $H^1_0(M)$.

REMARK 2.1. One must keep in mind here that the metric is changing under the Ricci flow and assume that estimates involving e.g. K are considered over some time interval. ■

REMARK 2.2. There is an extensive literature concerning eigenvalue bounds on Riemannian manifolds and we cite a few such results. Here $I_\infty(M) \sim \inf_\Omega(A(\partial\Omega)/V(\Omega))$ where Ω runs over (connected) open subsets of M with compact closure and smooth boundary (cf. [18, 19]). Yau's result is $I_\infty(M) \geqslant 2\sqrt{-K}$ (with equality for the 3-D hyperbolic space) and Cheeger's result involves follows $|\nabla\phi\|_{L^2} \geqslant (1/2)I_\infty(M)\|\phi\|_{L^2} \geqslant \sqrt{-K}\|\phi\|_{L^2}$. There are many other results where e.g. $\lambda_1 \geqslant c(\mathrm{vol}(M))^{-2}$ for M a compact 3-D hyperbolic manifold of finite volume (see [21, 34, 44] for this and variations). There are also estimates for the first eigenvalue along a Ricci flow in [33, 37] and estimates of the form $\lambda_1 \geqslant 3K$ for closed 3-D manifolds with Ricci curvature $R \geqslant 2K$ $(K > 0)$ in [32, 33]. In fact Ling obtains $\lambda_1 \geqslant K + (\pi^2/\tilde{d}^2)$ where $\tilde{d}$ is the diameter of the largest interior ball in nodal domains of the first eigenfunction. There are also estimates $\lambda_1 \geqslant (\pi^2/d^2)$ $(d = \mathrm{diam}(M),\ R \geqslant 0)$ in [31, 52, 54] and the papers of Ling give an excellent survey of results, new and old, including estimates of a similar kind for the first Dirichlet and Neumann eigenvalues. ■

Submitted on November 02, 2007
Accepted on November 19, 2007

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