Gravitation and Electricity

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The equations of gravitation together with the equations of electromagnetism in terms of the General Theory of Relativity allow to conceive an interdependence between the gravitational field and the electromagnetic field. However the technical difficulties of the relevant problems have precluded from expressing clearly this interdependence. Even the simple problem related to the field generated by a charged spherical mass is not correctly solved. In the present paper we reexamine from the outset this problem and propose a new solution.

1 Introduction

Although gravitation and electromagnetism are distinct entities, the principles of General Relativity imply that they affect each other. In fact, the equations of electromagnetism, considered in the spacetime of General Relativity, depend on the gravitational tensor, so that the electromagnetic field depends necessarily on the gravitational potentials. On the other hand, the electromagnetism is involved in the equations of gravitation by means of the corresponding energy-momentum tensor, so that the gravitational potentials depend necessarily on the electromagnetic field. It follows that, in order to bring out the relationship between gravitation and electromagnetism, we must consider together the equations of electromagnetism, which depend on the gravitational tensor, and the equations of gravitation, which depend on the electromagnetic potentials. So we have to do with a complicated system of equations, which are intractable in general. Consequently it is very difficult to bring out in explicit form the relationship between gravitation and electromagnetism. However the problem can be rigorously solved in the case of the field (gravitational and electric) outside a spherical charged mass. The classical solution of this problem, the so-called Reissner-Nordström metric, involves mathematical errors which distort the relationship between gravitational and electric field. In dealing with the derivation of this metric, H. Weyl notices that "For the electrostatic potential we get the same formula as when the gravitation is disregarded" [5], without remarking that this statement includes an inconsistency: The electrostatic potential without gravitation is conceived in the usual spacetime, whereas the gravitation induces a non-Euclidean structure affecting the metrical relations and, in particular, those involved in the definition of the electrostatic potential. The correct solution shows, in fact, that the electrostatic potential depends on the gravitational tensor.

In the present paper we reexamine from the outset the problem related to the joint action of the gravitation and electromagnetism which are generated by a spherical charged source. We assume that the distribution of matter and charges is such that the corresponding spacetime metric is $S\Theta(4)$-invariant (hence also $\Theta(4)$-invariant), namely a spacetime metric of the following form [3, 4]

$$ds^2 = f^2 dx_0^2 + 2ff_1(xdx)dx_0 - \ell_1^2 dx^2 + \left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2\right)(xdx)^2, \tag{1.1}$$

(where $f = f(x_0, \|x\|)$, $f_1 = f_1(x_0, \|x\|)$, $\ell_1 = \ell_1(x_0, \|x\|)$, $\ell = \ell(x_0, \|x\|)$, $\rho = \|x\|$).

It is useful to write down the components of (1.1):

$$g_{00} = f^2, \quad g_{0i} = g_{i0} = x_i f f_1 ,$$

$$g_{ii} = -\ell_1^2 + \left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2\right)x_i^2 ,$$

$$g_{ij} = \left(\frac{\ell_1^2 - \ell^2}{\rho^2} + f_1^2\right)x_i x_j , \qquad (i, j = 1, 2, 3; i \neq j) ,$$

the determinant of which equals $-f^2\ell^2\ell_1^4$. Then an easy computation gives the corresponding contravariant components:

$$g^{00} = \frac{\ell^2 - \rho^2 f_1^2}{f^2\ell^2} , \quad g^{0i} = g^{i0} = x_i \frac{f_1}{f\ell^2} ,$$

$$g^{ii} = -\frac{1}{\ell_1^2} - \frac{1}{\rho_2}\left(\frac{1}{\ell^2} - \frac{1}{\ell_1^2}\right)x_i^2 ,$$

$$g^{ij} = -\frac{1}{\rho^2}\left(\frac{1}{\ell^2} - \frac{1}{\ell_1^2}\right)x_i x_j , \qquad (i, j = 1, 2, 3; i \neq j) .$$

Regarding the electromagnetic field, with respect to (1.1), it is defined by a skew-symmetrical $S\Theta(4)$-invariant tensor field of degree 2 which may be expressed either by its covariant components

$$\sum V_{\alpha\beta}\, dx_\alpha \otimes dx_\beta , \quad (V_{\alpha\beta} = -V_{\beta\alpha}) ,$$

or by its contravariant components

$$\sum V^{\alpha\beta} \frac{\partial}{\partial x_\alpha} \otimes \frac{\partial}{\partial x_\beta} , \quad (V^{\alpha\beta} = -V^{\beta\alpha}) .$$

2 Electromagnetic field outside a spherical charged source. Vanishing of the magnetic field

According to a known result [2], the skew-symmetrical $S\Theta(4)$-invariant tensor field $\sum V_{\alpha\beta} dx_\alpha \otimes dx_\beta$ is the direct sum of the following two tensor fields:

(a) A $\Theta(4)$-invariant skew-symmetrical tensor field

$$q(x_0, \|x\|)(dx_0 \otimes F(x) - F(x) \otimes dx_0)\,,$$

$$\left(F(x) = \sum_{i=1}^{3} x_i dx_i\right),$$

which represents the electric field with components

$$\left.\begin{aligned} &V_{01} = -V_{10} = qx_1\,, \quad V_{02} = -V_{20} = qx_2\,, \\ &V_{03} = -V_{30} = qx_3\,. \end{aligned}\right\}; \qquad (2.1)$$

(b) A purely $S\Theta(4)$-invariant skew-symmetrical tensor field

$$\begin{aligned} q_1(x_0, \|x\|)\big[&x_1(dx_2 \otimes dx_3 - dx_3 \otimes dx_2) + \\ &+ x_2(dx_3 \otimes dx_1 - dx_1 \otimes dx_3) + \\ &+ x_3(dx_1 \otimes dx_2 - dx_2 \otimes dx_1)\big]\,, \end{aligned}$$

which represents the magnetic field with components

$$\left.\begin{aligned} &V_{23} = -V_{32} = q_1x_1\,, \quad V_{31} = -V_{13} = q_1x_2\,, \\ &V_{12} = -V_{21} = q_1x_3\,. \end{aligned}\right\}. \qquad (2.2)$$

Since the metric (1.1) plays the part of a fundamental tensor, we can introduce the contravariant components of the skew-symmetrical tensor field $\sum V_{\alpha\beta} dx_\alpha \otimes dx_\beta$ with respect to (1.1).

Proposition 2.1 *The contravariant components of the $S\Theta(4)$-invariant skew-symmetrical tensor field $\sum V_{\alpha\beta} dx_\alpha \otimes dx_\beta$ are defined by the following formulae:*

$$V^{01} = -V^{10} = -\frac{qx_1}{f^2\ell^2}\,, \quad V^{02} = -V^{20} = -\frac{qx_2}{f^2\ell^2}\,,$$

$$V^{03} = -V^{30} = -\frac{qx_3}{f^2\ell^2}\,,$$

$$V^{23} = -V^{32} = \frac{q_1x_1}{\ell_1^4}\,, \quad V^{31} = -V^{13} = \frac{q_1x_2}{\ell_1^4}\,,$$

$$V^{12} = -V^{21} = \frac{q_1x_3}{\ell_1^4}\,.$$

Proof. The componets V^{01} and V^{23}, for instance, result from the obvious formulae

$$\begin{aligned} V^{01} = \sum g^{0\alpha} g^{1\beta} V_{\alpha\beta} &= (g^{00}g^{11} - g^{01}g^{10})V_{01} + \\ &+ (g^{00}g^{12} - g^{02}g^{10})V_{02} + (g^{00}g^{13} - g^{03}g^{10})V_{03} + \\ &+ (g^{02}g^{13} - g^{03}g^{12})V_{23} + (g^{03}g^{11} - g^{01}g^{13})V_{31} + \\ &+ (g^{01}g^{12} - g^{02}g^{11})V_{12} \end{aligned}$$

and

$$\begin{aligned} V^{23} = \sum g^{2\alpha} g^{3\beta} V_{\alpha\beta} &= (g^{20}g^{31} - g^{21}g^{30})V_{01} + \\ &+ (g^{20}g^{32} - g^{22}g^{30})V_{02} + (g^{20}g^{33} - g^{23}g^{30})V_{03} + \\ &+ (g^{22}g^{33} - g^{23}g^{32})V_{23} + (g^{23}g^{31} - g^{21}g^{33})V_{31} + \\ &+ (g^{21}g^{32} - g^{22}g^{31})V_{12} \end{aligned}$$

after effectuating the indicated operations.

Proposition 2.2 *The functions $q = q(x_0, \rho)$, $q_1 = q_1(x_0, \rho)$, $(x_0 = ct,\ \rho = \|x\|)$, defining the components (2.1) and (2.2) outside the charged spherical source are given by the formulae*

$$q = \frac{\varepsilon f \ell}{\rho^3 \ell_1^2}\,, \qquad q_1 = \frac{\varepsilon_1}{\rho^3}\,,$$

$$(\varepsilon = \text{const},\ \varepsilon_1 = \text{const.})$$

(The equations of the electromagnetic field are to be considered together with the equations of gravitation, and since these last are inconsistent with a punctual source, there exists a length $\alpha > 0$ such that the above formulae are valid only for $\rho \geqslant \alpha$.)

Proof. Since outside the source there are neither charges nor currents, the components (2.1), (2.2) are defined by the classical equations

$$\frac{\partial V_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial V_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial V_{\gamma\alpha}}{\partial x_\beta} = 0\,, \qquad (2.3)$$

$$\big(x_0 = ct,\ (\alpha, \beta, \gamma) \in \{(0,1,2), (0,2,3), (0,3,1), (1,2,3)\}\big),$$

$$\sum_{\beta=0}^{3} \frac{\partial}{\partial x_\beta}\left(\sqrt{-G}\, V^{\alpha\beta}\right) = 0\,, \qquad (2.4)$$

$$\big(\alpha = 0, 1, 2, 3;\ \ G = -f^2\ell^2\ell_1^4\big).$$

Taking $(\alpha, \beta, \gamma) = (0, 1, 2)$, we have, on account of (2.3),

$$\frac{\partial(qx_1)}{\partial x_2} + \frac{\partial(q_1x_3)}{\partial x_0} - \frac{\partial(qx_2)}{\partial x_1} = 0$$

and since

$$\frac{\partial q}{\partial x_i} = \frac{\partial q}{\partial \rho}\frac{x_i}{\rho}\,, \quad (i = 1, 2, 3)\,,$$

we obtain

$$x_1x_2\frac{\partial q}{\partial \rho} - x_2x_1\frac{\partial q}{\partial \rho} + x_3\frac{\partial q_1}{\partial x_0} = 0\,,$$

whence $\dfrac{\partial q_1}{\partial x_0} = 0$, so that q_1 depends only on ρ, $q_1 = q_1(\rho)$.

On the other hand, taking $(\alpha, \beta, \gamma) = (1, 2, 3)$, the equation (2.3) is written as

$$\frac{\partial(q_1x_3)}{\partial x_3} + \frac{\partial(q_1x_1)}{\partial x_1} + \frac{\partial(q_1x_2)}{\partial x_2} = 0\,,$$

whence $3q_1 + \rho q_1' = 0$, so that $3\rho^2 q_1 + \rho^3 q_1' = 0$ or $(\rho^3 q_1)' = 0$ and $\rho^3 q_1 = \varepsilon_1 = \text{const}$ or $q_1 = \dfrac{\varepsilon_1}{\rho^3}$.

Consider now the equation (2.4) with $\alpha = 1$. Since $G = -f^2\ell^2\ell_1^4$,

$$V^{11} = 0\,,\;\; V^{10} = \frac{qx_1}{f^2\ell^2}\,,\;\; V^{12} = \frac{q_1x_3}{\ell_1^4}\,,\;\; V^{13} = -\frac{q_1x_2}{\ell_1^4}\,,$$

we have

$$\frac{\partial}{\partial x_0}\left(\frac{q\ell_1^2}{f\ell}\,x_1\right) + \frac{\partial}{\partial x_2}\left(\frac{q_1f\ell}{\ell_1^2}\,x_3\right) - \frac{\partial}{\partial x_3}\left(\frac{q_1f\ell}{\ell_1^2}\,x_2\right) = 0\,.$$

Because of

$$\frac{\partial}{\partial x_2}\left(\frac{q_1f\ell}{\ell_1^2}\,x_3\right) = \frac{x_3x_2}{\rho}\frac{\partial}{\partial\rho}\left(\frac{q_1f\ell}{\ell_1^2}\right) = \frac{\partial}{\partial x_3}\left(\frac{q_1f\ell}{\ell_1^2}\,x_2\right),$$

we obtain

$$x_1\frac{\partial}{\partial x_0}\left(\frac{q\ell_1^2}{f\ell}\right) = 0$$

so that $\frac{q\ell_1^2}{f\ell}$ depends only on ρ: $\frac{q\ell_1^2}{f\ell} = \varphi(\rho)$.

Now the equation (2.4) with $\alpha = 0$ is written as

$$\frac{\partial}{\partial x_1}(x_1\varphi(\rho)) + \frac{\partial}{\partial x_2}(x_2\varphi(\rho)) + \frac{\partial}{\partial x_3}(x_3\varphi(\rho)) = 0\,,$$

whence $3\varphi(\rho) + \rho\varphi'(\rho) = 0$ and $3\rho^2\varphi(\rho) + \rho^3\varphi'(\rho) = 0$ or $(\rho^3\varphi(\rho))' = 0$.

Consequently $\rho^3\varphi(\rho) = \varepsilon = \text{const}$ and $q = \frac{\varepsilon f\ell}{\rho^3\ell_1^2}$.

The meaning of the constants ε and ε_1:

Since the function q occurs in the definition of the electric field (2.1), it is natural to identify the constant ε with the electric charge of the source. Does a similar reasoning is applicable to the case of the magnetic field (2.2)? In other words, does the constant ε_1 represents a magnetic charge of the source? This question is at first related to the case where $\varepsilon = 0$, $\varepsilon_1 \neq 0$, namely to the case where the spherical source appears as a magnetic monopole. However, although the existence of magnetic monopoles is envisaged sometimes as a theoretical possibility, it is not yet confirmed experimentally. Accordingly we are led to assume that $\varepsilon_1 = 0$, namely that the purely $S\Theta(4)$-invariant magnetic field vanishes. So we have to do only with the electric field (2.1), which, on account of $q = \frac{\varepsilon f\ell}{\rho^3\ell_1^2}$, depends on the gravitational tensor (contrary to Weyl's assertion).

3 Equations of gravitation outside the charged source

We recall that, if an electromagnetic field

$$\sum V_{\alpha\beta}dx_\alpha\otimes dx_\beta\,,\quad (V_{\alpha\beta} = -V_{\beta\alpha})$$

is associated with a spacetime metric

$$\sum g_{\alpha\beta}dx_\alpha\otimes dx_\beta\,,$$

then it gives rise to an energy-momentum tensor

$$\sum W_{\alpha\beta}dx_\alpha\otimes dx_\beta$$

defined by the formulae

$$W_{\alpha\beta} = \frac{1}{4\pi}\left(\frac{1}{4}\,g_{\alpha\beta}\sum V_{\gamma\delta}V^{\gamma\delta} - \sum V_{\alpha\delta}V_\beta^{\cdot\,\delta}\right). \quad (3.1)$$

In the present situation, the covariant and contravariant components $V_{\gamma\delta}$ and $V^{\gamma\delta}$ are already known. So it remains to compute the mixed components

$$V_\beta^{\cdot\,\delta} = \sum g^{\gamma\delta}V_{\beta\gamma} = -\sum g^{\delta\gamma}V_{\gamma\beta} = -V^\delta_{\cdot\,\beta}\,.$$

Taking into account the vanishing of the magnetic field, an easy computation gives

$$V_0^{\cdot\,0} = \frac{\rho^2qf_1}{f\ell^2}\,,\quad V_0^{\cdot\,k} = -\frac{qx_k}{\ell^2}\,,\quad (k=1,2,3)\,,$$

$$V_k^{\cdot\,0} = -\frac{\ell^2-\rho^2f_1^2}{f^2\ell^2}\,qx_k\,,\qquad (k=1,2,3)\,,$$

$$V_k^{\cdot\,k} = -\frac{qf_1}{f\ell^2}\,x_k^2\,,\qquad (k=1,2,3)\,,$$

$$V_2^{\cdot\,3} = -\frac{qf_1}{f\ell^2}\,x_2x_3 = V_3^{\cdot\,2}\,,$$

$$V_3^{\cdot\,1} = -\frac{qf_1}{f\ell^2}\,x_3x_1 = V_1^{\cdot\,3}\,,$$

$$V_1^{\cdot\,2} = -\frac{qf_1}{f\ell^2}\,x_1x_2 = V_2^{\cdot\,1}\,.$$

It follows that

$$\sum V_{\gamma\delta}V^{\gamma\delta} = -\frac{2\rho^2q^2}{f^2\ell^2}\,,$$

$$\sum V_{0\delta}V_0^{\cdot\,\delta} = -\frac{\rho^2q^2}{\ell^2}\,,$$

$$\sum V_{0\delta}V_1^{\cdot\,\delta} = -\frac{\rho^2f_1q^2}{f\ell^2}\,x_1\,,$$

$$\sum V_{1\delta}V_2^{\cdot\,\delta} = \frac{\ell^2-\rho^2f_1^2}{f^2\ell^2}\,q^2x_1x_2\,,$$

$$\sum V_{1\delta}V_1^{\cdot\,\delta} = \frac{\ell^2-\rho^2f_1^2}{f^2\ell^2}\,q^2x_1^2\,,$$

and then the formula (3.1) gives the components W_{00}, W_{01}, W_{11}, W_{12} of the energy-momentum tensor. The other components are obtained simply by permuting indices.

Proposition 3.1 *The energy-momentum tensor associated with the electric field (2.1) is a $\Theta(4)$-invariant tensor defined by the following formulae*

$$W_{00} = E_{00}\,,\qquad W_{0i} = W_{i0} = x_iE_{01}\,,$$

$$W_{ii} = E_{11} + x_i^2E_{22}\,,\quad W_{ij} = W_{ji} = x_ix_jE_{22}\,,$$

$$(i,j = 1,2,3;\ i\neq j)\,,$$

where

$$E_{00} = \frac{1}{8\pi}\rho^2 f^2 E\,, \quad E_{01} = \frac{1}{8\pi}\rho^2 f f_1 E\,,$$
$$E_{11} = \frac{1}{8\pi}\rho^2 \ell_1^2 E\,, \quad E_{22} = \frac{1}{8\pi}(-\ell_1^2 - \ell^2 + \rho^2 f_1^2)E$$

with

$$E = \frac{q^2}{f^2\ell^2} = \frac{\varepsilon^2}{\rho^2 g^4}\,.$$

Regarding the Ricci tensor $R_{\alpha\beta}$, we already know [4] that it is a symmetric $\Theta(4)$-invariant tensor defined by the functions

$$Q_{00} = Q_{00}(t,\rho)\,, \quad Q_{01} = Q_{01}(t,\rho)\,,$$
$$Q_{11} = Q_{11}(t,\rho)\,, \quad Q_{22} = Q_{22}(t,\rho)$$

as follows

$$R_{00} = Q_{00}\,, \;\; R_{0i} = R_{i0} = Q_{01}x_i\,, \;\; R_{ii} = Q_{11} + x_i^2 Q_{22}\,,$$
$$R_{ij} = R_{ji} = x_i x_j Q_{22}\,, \quad (i,j = 1,2,3;\; i \neq j)\,.$$

So, assuming that the cosmological constant vanishes, we have to do from the outset with four simple equations of gravitation, namely

$$Q_{00} - \frac{R}{2} f^2 + \frac{8\pi k}{c^4} E_{00} = 0\,,$$
$$Q_{01} - \frac{R}{2} f f_1 + \frac{8\pi k}{c^4} E_{01} = 0\,,$$
$$Q_{11} + \frac{R}{2} \ell_1^2 + \frac{8\pi k}{c^4} E_{11} = 0\,,$$
$$Q_{11} + \rho^2 Q_{22} - \frac{R}{2}(\rho^2 f_1^2 - \ell^2) + \frac{8\pi k}{c^4}(E_{11} + \rho^2 E_{22}) = 0\,.$$

An additional simplification results from the fact that the mixed components of the electromagnetic energy-momentum tensor satisfy the condition $\Sigma W_\alpha^\alpha = 0$, and then the equations of gravitation imply (by contraction) that the scalar curvature R vanishes. Moreover, introducing as usual the functions $h = \rho f_1$, $g = \rho\ell_1$, and taking into account that $q = \dfrac{\varepsilon f \ell}{\rho^3 \ell_1^2}$, we obtain

$$E = \frac{\varepsilon^2}{\rho^2 g^4}, \qquad E_{00} = \frac{\varepsilon^2}{8\pi}\frac{f^2}{g^4}\,, \qquad E_{01} = \frac{\varepsilon^2}{8\pi}\frac{f f_1}{g^4}\,,$$
$$E_{11} = \frac{\varepsilon^2}{8\pi}\frac{\ell_1^2}{g^4}\,, \quad E_{11} + \rho^2 E_{22} = \frac{\varepsilon^2}{8\pi}\frac{(-\ell^2 + h^2)}{g^4}\,,$$

so that by setting

$$\nu^2 = \frac{k}{c^4}\varepsilon^2,$$

we get the definitive form of the equations of gravitation

$$Q_{00} + \frac{\nu^2}{g^4} f^2 = 0\,, \tag{3.1}$$

$$Q_{01} + \frac{\nu^2}{g^4} f f_1 = 0\,, \tag{3.2}$$

$$Q_{11} + \frac{\nu^2}{g^4}\ell_1^2 = 0\,, \tag{3.3}$$

$$Q_{11} + \rho^2 Q_{22} + \frac{\nu^2}{g^4}(-\ell^2 + h^2) = 0\,. \tag{3.4}$$

4 Stationary solutions outside the charged spherical source

In the case of a stationary field, the functions Q_{00}, Q_{01}, Q_{11}, Q_{22} depend only on ρ and their expressions are already known [3, 4]

$$Q_{00} = f\left(-\frac{f''}{\ell^2} + \frac{f'\ell'}{\ell^3} - \frac{2f'g'}{\ell^2 g}\right), \tag{4.1}$$

$$Q_{01} = \frac{h}{\rho f} Q_{00}\,, \tag{4.2}$$

$$Q_{11} = \frac{1}{\rho^2}\left(-1 + \frac{g'^2}{\ell^2} + \frac{g g''}{\ell^2} - \frac{\ell' g g'}{\ell^3} + \frac{f' g g'}{f\ell^2}\right), \tag{4.3}$$

$$Q_{11} + \rho^2 Q_{22} = \frac{f''}{f} + \frac{2g''}{g} - \frac{f'\ell'}{f\ell} - \frac{2\ell' g'}{\ell g} + \frac{h^2}{f^2} Q_{00}\,. \tag{4.4}$$

On account of (4.2), the equation (3.2) is written as

$$\left(Q_{00} + \frac{\nu^2}{g^4} f^2\right) h = 0$$

so that it is verified because of (3.1).

Consequently it only remains to take into account the equations (3.1), (3.3), (3.4).

From (3.1) we obtain

$$\frac{\nu^2}{g^4} = -\frac{Q_{00}}{f^2}$$

and inserting this expression into (3.4) we obtain the relation

$$f^2(Q_{11} + \rho^2 Q_{22}) - (-\ell^2 + h^2)Q_{00} = 0$$

which, on account of (4.1) and (4.4), reduces, after cancelations, to the simple equation

$$\frac{g''}{g'} = \frac{f'}{f} + \frac{\ell'}{\ell}$$

which does not contain the unknown function h and implies

$$f\ell = cg', \quad (c = \text{const}). \tag{4.5}$$

Next, from (3.1) and (3.3) we deduce the equation

$$Q_{11} - \frac{Q_{00}}{f^2}\ell_1^2 = 0 \tag{4.6}$$

which does not contain the function h either.

Now, from (4.5) we find

$$f = \frac{cg'}{\ell}$$

and inserting this expression of f into (4.6), we obtain an equation which can be written as

$$\frac{d}{d\rho}\left(\frac{F'}{2g'}\right) = 0$$

with

$$F = g^2 - \frac{g^2 g'^2}{\ell^2}\,.$$

It follows that

$$F = 2A_1 g - A_2\,, \quad (A_1 = \text{const},\ A_2 = \text{const}),$$

and

$$g'^2 = \ell^2\left(1 - \frac{2A_1}{g} + \frac{A_2}{g^2}\right). \tag{4.7}$$

On account of (4.5), the derivative g' does not vanish. In fact $g' = 0$ implies either $f = 0$ or $\ell = 0$, which gives rise to a degenerate spacetime metric, namely a spacetime metric meaningless physically. Then, in particular, it follows from (4.7) that

$$1 - \frac{2A_1}{g} + \frac{A_2}{g^2} > 0\,.$$

The constant A_1, obtained by means of the Newtonian approximation, is already known:

$$A_1 = \frac{km}{c^2} = \mu\,.$$

In order to get A_2, we insert first

$$\frac{f'}{f} = \frac{g''}{g'} - \frac{\ell'}{\ell}$$

into (4.3) thus obtaining

$$\rho^2 Q_{11} = -1 + \frac{g'^2}{\ell^2} + \frac{2gg''}{\ell^2} - \frac{2\ell' g g'}{\ell^3}\,. \tag{4.8}$$

Next by setting

$$Q(g) = 1 - \frac{2A_1}{g} + \frac{A_2}{g^2}$$

we have

$$g' = \ell\sqrt{Q(g)}\,,$$

$$g'' = \ell'\sqrt{Q(g)} + \ell^2\left(\frac{A_1}{g^2} - \frac{A_2}{g^3}\right)$$

and inserting these expressions of g' and g'' into (4.8), we find

$$\rho^2 Q_{11} = -\frac{A_2}{g^2}\,.$$

The equation (3.3) gives finally the value of the constant A_2:

$$A_2 = \nu^2 = \frac{k\varepsilon^2}{c^4}\,.$$

It follows that the general stationary solution outside the charged spherical source is defined by two equations, namely

$$f\ell = c\,\frac{dg}{d\rho}\,, \tag{4.9}$$

$$\frac{dg}{d\rho} = \ell\sqrt{1 - \frac{2\mu}{g} + \frac{\nu^2}{g^2}}\,, \tag{4.10}$$

$$\left(\mu = \frac{km}{c^2}, \quad \nu = \frac{\sqrt{k}}{c^2}\,|\varepsilon|, \quad 1 - \frac{2\mu}{g} + \frac{\nu^2}{g^2} > 0\right).$$

The interdependence of the two fields, gravitational and electric, in now obvious: The electric charge ε, which defines the electric field, is also involved in the definition of the gravitational field by means of the term

$$\frac{\nu^2}{g^2} = \frac{k}{c^4}\left(\frac{\varepsilon}{g}\right)^2.$$

On the other hand, since

$$q = \frac{\varepsilon f \ell}{\rho^3 \ell_1^2} = \frac{c\varepsilon}{\rho g^2}\frac{dg}{d\rho}\,,$$

the components of the electric field:

$$V_{01} = -V_{10} = qx_1 = \frac{c\varepsilon}{g^2}\frac{dg}{d\rho}\frac{x_1}{\rho} = \\ = -\,c\varepsilon\,\frac{\partial}{\partial x_1}\left(\frac{1}{g}\right) = -c\frac{\partial}{\partial x_1}\left(\frac{\varepsilon}{g}\right),$$

$$V_{02} = -V_{20} = qx_2 = -c\,\frac{\partial}{\partial x_2}\left(\frac{\varepsilon}{g}\right),$$

$$V_{03} = -V_{30} = qx_3 = -\,c\,\frac{\partial}{\partial x_3}\left(\frac{\varepsilon}{g}\right)$$

result from the electric potential:

$$\frac{\varepsilon}{g} = \frac{\varepsilon}{g(\rho)}$$

which is thus defined by means of the curvature radius $g(\rho)$, namely by the fundamental function involved in the definition of the gravitational field.

Note that, among the functions occurring in the spacetime metric, only the function $h = \rho f_1$ does not appear in the equations (4.9) and (4.10). The problem does not require a uniquely defined h. Every differentiable function h satisfying the condition $|h| \leqslant \ell$ is allowable. And every allowable h gives rise to a possible conception of the time coordinate. Contrary to the Special Relativity, we have to do, in General Relativity, with an infinity of possible definitions of the time coordinate. In order to elucidate this assertion in the present situation, let us denote by ρ_1 the radius of the spherical stationary source, and consider a photon emitted radially from the sphere $\|x\| = \rho_1$ at an instant τ. The equation of motion of this photon, namely

$$f(\rho)dt + h(\rho)d\rho = \ell(\rho)d\rho$$

implies

$$\frac{dt}{d\rho} = \frac{-h(\rho) + \ell(\rho)}{f(\rho)}$$

whence $\tau = t - \psi(\rho)$ with

$$\psi(\rho) = \int_{\rho_1}^{\rho} \frac{-h(u) + \ell(u)}{f(u)}\, du\,.$$

For every value of $\rho \geqslant \rho_1$, $\pi(t,\rho) = t - \psi(\rho)$ is the instant of radial emission of a photon reaching the sphere $\|x\| = \rho$ at the instant t. The function $\pi(t,\rho)$ will be called *propagation function*, and we see that to each allowable h there corresponds a uniquely defined propagation function. Moreover each propagation function characterizes uniquely a conception of the notion of time. Regarding the radial velocity of propagation of light, namely

$$\frac{d\rho}{dt} = \frac{f(\rho)}{-h(\rho) + \ell(\rho)}\,,$$

it is not bounded by a barrier as in Special Relativity. In the limit case where the allowable h equals ℓ, this velocity becomes infinite.

This being said, we return to the equations (4.9) and (4.10) which contain the remaining unknown functions f, ℓ, g. Their investigation necessitates a rather lengthy discussion which will be carried out in another paper. At present we confine ourselves to note two significant conclusions of this discussion:

(a) Pointwise sources do not exist, so that the spherical source cannot be reduced to a point. In particular the notion of black hole is inconceivable;

(b) Among the solutions defined by (4.9) and (4.10), particularly significant are those obtained by introducing the radial geodesic distance

$$\delta = \int_0^{\rho} \ell(u)du\,.$$

Then we have to define the curvature radius $G(\delta) = g(\rho(\delta))$ by means of the equation

$$\frac{dG}{d\delta} = \sqrt{1 - \frac{2\mu}{G} + \frac{\nu^2}{G^2}}$$

the solutions of which need specific discussion according as $\nu^2 - \mu^2 > 0$ or $\nu^2 - \mu^2 = 0$ or $\nu^2 - \mu^2 < 0$. The first approach to this problem appeared in the paper [1].

We note finally that the derivation of the Reissner-Nordström metric contains topological errors and moreover identifies erroneously the fundamental function $g(\rho)$ with a radial coordinate. This is why the Reissner-Nordström metric is devoid of geometrical and physical meaning.

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