A Classical Model of Gravitation

Pieter Wagener

Department of Physics, NMMU South Campus, Port Elizabeth, South Africa

E-mail: Pieter.Wagener@nmmu.ac.za

A classical model of gravitation is proposed with time as an independent coordinate. The dynamics of the model is determined by a proposed Lagrangian. Applying the canonical equations of motion to its associated Hamiltonian gives conservation equations of energy, total angular momentum and the z component of the angular momentum. These lead to a Keplerian orbit in three dimensions, which gives the observed values of perihelion precession and bending of light by a massive object. An expression for gravitational redshift is derived by accepting the local validity of special relativity at all points in space. Exact expressions for the GEM relations, as well as their associated Lorentz-type force, are derived. An expression for Mach's Principle is also derived.

1 Introduction

The proposed theory is based on two postulates that respectively establish the dynamics and kinematics of a system of particles subject to a gravitational force. The result is a closed particle model that satisfies the basic experimental observations of the force.

The details of applications and all derivations are included in the doctoral thesis of the author [1].

2 Postulates

The model is based on two postulates:

Postulate 1: The **dynamics** of a system of particles subject to gravitational forces is determined by the Lagrangian,

$$L = -m_0(c^2 + v^2)\exp\frac{R}{r}\,, \tag{1}$$

where m_0 is *gravitational rest mass* of a test body moving at velocity $\mathbf{v}$ in the vicinity of a massive, central body of mass M, $\gamma = 1/\sqrt{1 - v^2/c^2}$, $R = 2GM/c^2$ is the Schwarzschild radius of the central body.

Postulate 2: Special Relativity (SR) is valid instantan-eously and locally at all points in the reference system of the central massive body. This gives the **kinematics** of the system.

3 Conservation equations

Applying the canonical equations of motion to the Hamiltonian, derived from the Lagrangian, leads to three conservation equations:

$$E = m_0 c^2\,\frac{e^{R/r}}{\gamma^2} = \text{total energy} = \text{constant}\,, \tag{2}$$

$$\mathbf{L} = e^{R/r}\,\mathbf{M}\,, \tag{3}$$
$$= \text{total angular momentum} = \text{constant}\,,$$

$$L_z = e^{R/r}\,m_0 r^2 \sin^2\theta\,\dot{\phi}\,, \tag{4}$$
$$= z \text{ component of } \mathbf{L} = \text{constant}\,,$$

where $\mathbf{M} = (\mathbf{r}\times m_0\mathbf{v})$. Equations (2), (3) and (4) give the quadrature of motion:

$$\frac{d\Psi}{du} = \pm\left[\frac{e^{2Ru}}{L^2} - u^2 - \frac{Ee^{Ru}}{L^2}\right]^{-1/2}, \tag{5}$$

where $u = 1/r$, $L = |\mathbf{L}|$ and Ψ is defined by

$$|\mathbf{M}| = m_0 r^2\,\frac{d\Psi}{dt}\,. \tag{6}$$

Expanding the exponential terms to second degree yields a differential equation of generalized Keplerian form,

$$\frac{d\Psi}{du} = (au^2 + bu + c)^{-1/2}, \tag{7}$$

where

$$\left.\begin{aligned} u &= \frac{1}{r} \\ a &= \frac{R^2(4-E)}{2L^2} - 1 \\ b &= \frac{R(2-E)}{L^2} \\ c &= \frac{1-E}{L^2} \end{aligned}\right\}, \tag{8}$$

and the convention $m_0 = c = 1$ was used.

Integrating (7) gives the orbit of a test particle as a generalized conic,

$$u = K(1 + \epsilon\cos k\Psi)\,, \tag{9}$$

where the angles are measured from $\Psi = 0$, and

$$k = (-a)^{\frac{1}{2}}\,, \tag{10}$$

$$K = -\frac{b}{2a}\,, \tag{11}$$

$$\epsilon = \left(1 - \frac{4ac}{b}\right)^{\frac{1}{2}}. \tag{12}$$

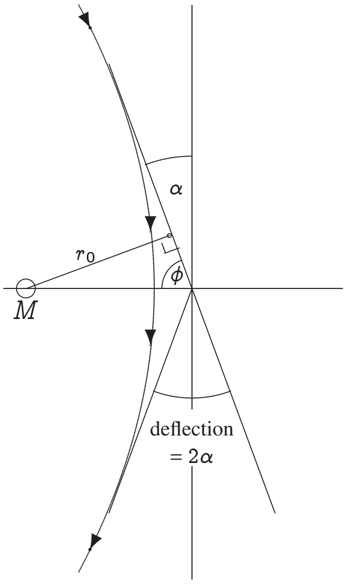

Fig. 1: Deflection of light.

4 Gravitational redshift

Assuming the validity of $\gamma d\tau = dt$ of SR at each point in space and taking frequencies as the inverses of time, (2) yields

$$\nu = \nu_0 \, e^{-R/2r} \qquad (\nu_0 = \text{constant}), \tag{13}$$

which, to first approximation in $\exp(-R/2r)$, gives the observed gravitational redshift.

5 Perihelion precession

In the case of an ellipse ($\epsilon < 1$), the presence of the coefficient k causes the ellipse not to be completed after a cycle of $\theta = 2\pi$ radians, i.e. the perihelion is shifted through a certain angle. This shift, or precession, can be calculated as (see Appendix A.1):

$$\Delta\phi = \frac{3\pi R}{\bar{a}\,(1-\epsilon^2)}\,, \tag{14}$$

where $\bar{a}$ is the semi-major axis of the ellipse. This expression gives the observed perihelion precession of Mercury.

6 Deflection of light

We define a photon as a particle for which $v = c$. From (2) it follows that $E = 0$ and the eccentricity of the conic section is found to be (see Appendix A.2)

$$\epsilon = \frac{r_0}{R}\,, \tag{15}$$

where r_0 is the impact parameter. Approximating r_0 by the radius of the sun, it follows that $\epsilon > 1$. From Fig. 1 we see that the trajectory is a hyperbola with total deflection equal to $2R/r_0$. This is in agreement with observation.

7 Lorentz-type force equation

The corresponding force equation is found from the associated Euler-Lagrange equations:

$$\dot{\mathbf{p}} = \mathbf{E}m + m_0\mathbf{v}\times\mathbf{H}\,, \tag{16}$$

where

$$\mathbf{p} \;=\; m_0\,\dot{\mathbf{r}} = m_0\,\mathbf{v}\,, \tag{17}$$

$$m \;=\; \frac{m_0}{\gamma^2}\,, \tag{18}$$

$$\mathbf{E} \;=\; -\,\hat{\mathbf{r}}\,\frac{GM}{r^2}\,, \tag{19}$$

$$\mathbf{H} \;=\; \frac{GM\,(\mathbf{v}\times\mathbf{r})}{c^2 r^3}\,. \tag{20}$$

The force equation shows the deviation from Newton's law of gravitation. The above equations are analogous to the gravitoelectromagnetic (GEM) equations derived by Mashhoon [2] as a lowest order approximation to Einstein's field equations for $v \ll c$ and $r \gg R$.

8 Mach's Principle

An *ad hoc* formulation for Mach's Principle has been presented as [3, 4]

$$G \cong \frac{Lc^2}{M}\,, \tag{21}$$

where: L = radius of the universe,
M = mass of the universe ≅ mass of the distant stars.

This relation can be found by applying the energy relation of (2) to the system of Fig. 2.

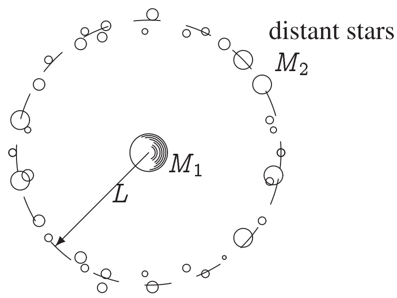

Fig. 2: Mutual gravitational interaction between a central mass M_1 and the distant stars of total mass M_2.

The potential at M_2 due to M_1 is $\Phi_1{=}GM_1/L{=}R_1c^2/2L$ and the potential of the shell at M_1 is $\Phi_2{=}GM_2/L{=}R_2c^2/2L$. Furthermore, since M_1 and M_2 are in relative motion, the value of γ will be the same for both of them. Applying (2) to the mutual gravitational interaction between the shell of distant stars and the central body then gives

$$E = M_1c^2\exp\frac{R_2}{L} = M_2c^2\exp\frac{R_1}{L}\,.$$

Since $L > R_2 \gg R_1$ we can realistically approximate the exponential to first order in R_2/L. After some algebra we get $R_2 \approx L$, which gives the Mach relation,

$$\frac{2GM_2}{Lc^2} \approx 1\,.$$

9 Comparison with General Relativity

The equations of motion of General Relativity (GR) are approximations to those of the proposed Lagrangian. This can be seen as follows.

The conservation equations of (2), (3) and (4) can also be derived from a generalized metric,

$$ds^2 = e^{-R/r}\,dt^2 - e^{R/r}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right). \quad (22)$$

Comparing this metric with that of GR,

$$ds^2 = \left(1 - \frac{R}{r}\right) dt^2 - \frac{1}{1-\frac{R}{r}}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2, \quad (23)$$

we note that (23) is a first order approximation to the time and radial coefficients, and a zeroth order approximation to the angular coefficients of (22). It implies that all predictions of GR will be accommodated by the Lagrangian of (1) within the orders of approximation.

Comparing (5) with the corresponding quadrature of GR,

$$\frac{d\theta}{du} = \pm\left[\frac{1-E}{J^2} + \frac{uRE}{J^2} - u^2 + Ru^3\right]^{-1/2}, \quad (24)$$

we note that it differs from the Newtonian limit, or the Keplerian form of (7), by the presence of the Ru^3 term. The form of this quadrature does not allow the conventional Keplerian orbit of (9).

A Appendix

A.1 Precession of the perihelion

After one revolution of 2π radians, the perihelion of an ellipse given by the conic of (9) shifts through an angle $\Delta\phi = \frac{2\pi}{k} - 2\pi$ or, from (10), as

$$\Delta\phi = 2\pi\left[(-a)^{-1/2} - 1\right], \quad (25)$$

where a is given by (8). The constants of motion E and L are found from the boundary conditions of the system, i.e. $du/d\theta = 0$ at $u = 1/r_-$ and $1/r_+$, where r_+ and r_- are the maximum and minimum radii respectively of the ellipse. We find [1]

$$\left.\begin{aligned} E &\approx 1 + \frac{R}{2\bar{a}} \\ \frac{R^2}{L^2} &\approx \frac{2R}{\bar{a}\,(1-\epsilon^2)} \end{aligned}\right\}, \quad (26)$$

where $\bar{a} = (r_+ + r_-)/2$ is the semi-major axis of the approximate ellipse. Substituting these values in (8) gives

$$a = \frac{3R}{\bar{a}(1-\epsilon^2)} - 1\,. \quad (27)$$

Substituting this value in (25) gives (14).

A.2 Deflection of light

We first have to calculate the eccentricity ϵ of the conic for this case,

$$\epsilon = \left(1 - \frac{4ac}{b^2}\right)^{1/2}.$$

For a photon, setting $v = c$ in (8) gives

$$\epsilon^2 = \left[-1 + \frac{L^2}{R^2}\right]. \quad (28)$$

At the distance of closest approach, $r = r_0 = 1/u_0$, we have $d\theta/du = 0$, so that from (5):

$$L^2 = \frac{e^{2Ru_0}}{u_0^2} = r_0^2\, e^{2R/r_0}. \quad (29)$$

From (28) and (29), and ignoring terms of first and higher order in R/r_0, we find

$$\epsilon \approx \frac{r_0}{R}\,. \quad (30)$$

For a hyperbola $\cos\phi = 1/\epsilon$, so that (see Fig. 1):

$$\begin{aligned} \sin\alpha &= 1/\epsilon \\ \Rightarrow \alpha &\approx 1/\epsilon \\ \Rightarrow 2\alpha &\approx 2R/r_0 = \text{total deflection.} \end{aligned}$$

Submitted on April 07, 2008
Accepted on April 10, 2008

References

1. Wagener P. C. Principles of a theory of general interaction. PhD thesis, University of South Africa, 1987. An updated version is to be published as a book during 2008.
2. Mashhoon. Gravitoelectromagnetism: a brief review. In L. Iorio, editor, *Measuring Gravitomagnetism: A Challenging Enterprise*, pages 29–39. Nova Publishers, 2007. arXiv: gr-qc/0311030.
3. Chiu H.-Y. and Hoffmann W. P. (eds.). *Gravitation and Relativity*, Ch. 7., W. A. Benjamin Inc., New York, 1964.
4. Sciama D. W. The physical foundations of General Relativity. Doubleday, 1969.