A New Finslerian Unified Field Theory of Physical Interactions

Indranu Suhendro
E-mail: spherical_symmetry@yahoo.com

In this work, we shall present the foundational structure of a new unified field theory of physical interactions in a geometric world-space endowed with a new kind of Finslerian metric. The intrinsic non-metricity in the structure of our world-geometry may have direct, genuine connection with quantum mechanics, which is yet to be fully explored at present. Building upon some of the previous works of the Author, our ultimate aim here is yet another quantum theory of gravity (in just four space-time dimensions). Our resulting new theory appears to present us with a novel Eulerian (intrinsically motion-dependent) world-geometry in which the physical fields originate.

1 Introduction

This work is a complementary exposition to our several previous attempts at the geometrization of matter and physical fields, while each of them can be seen as an independent, self-contained, coherent unified field theory.

Our primary aim is to develop a new foundational world-geometry based on the intuitive notion of a novel, fully naturalized kind of Finsler geometry, which extensively mimics the Eulerian description of the mechanics of continuous media with special emphasis on the world-velocity field, in the sense that the whole space-time continuum itself is taken to be globally dynamic on both microscopic and macroscopic scales. In other words, the world-manifold itself, as a whole, is not merely an ambient four-dimensional geometric background, but an open (self-closed, yet unbounded), co-moving, self-organizing, self-projective entity, together with the individual particles (objects) encompassed by its structure.

2 Elementary construction of the new world-geometry

Without initial recourse to the common structure of Finsler geometry, whose exposition can easily be found in the literature, we shall build the essential geometric world-space of our new theory somewhat from scratch.

We shall simply start with an intuitive vision of intrinsically motion-dependent objects, whose fuzzy Eulerian behavior, on the microscopic scale, is generated by the structure of the world-geometry in the first place, and whose very presence, on the macroscopic scale, affects the entire structure of the world-geometry. In this sense, the space-time continuum itself has a dynamic, non-metric character at heart, such that nothing whatsoever is intrinsically "fixed", including the defining metric tensor itself, which evolves, as a structural entity of global coverage, in a self-closed (self-inclusive) yet unbounded (open) manner.

In the present theory, the Universe is indeed an evolving, holographic (self-projective) four-dimensional space-time continuum U_4 with local curvilinear coordinates x^α and an intrinsically fuzzy (quantum-like), possibly degenerate, non-metric field ψ. As such, U_4 may encompass all possible metric-compatible (sub-)universes, especially those of the General Theory of Relativity. In this sense, U_4 may be viewed as a Meta-Universe, possibly without admitting any apparent boundary between its microscopic (interior) and macroscopic (exterior) mechanisms, as we shall see.

If we represent the metric-compatible part of the geometric basis of U_4 as $g_\alpha(x)$, then, following our unification scenario, the total geometric basis of our generally non-metric manifold shall be given by

$$g_\alpha(x,u) = g_\alpha(x) + \psi_\alpha u$$
$$g^\alpha(x,u) = (g_\alpha(x,u))^{-1}$$
$$\langle g_\alpha(x,u), g^\beta(x,u)\rangle = \delta^\beta_\alpha$$

where $u = \frac{dx^\alpha}{ds}\, g_\alpha(x,u)$ is the world-velocity field along the world-line

$$s(x,u) = \int \sqrt{g_{\alpha\beta}(x,u)\, dx^\alpha dx^\beta}$$

(with $g_{\alpha\beta}(x,u)$ being the components of the generalized metric tensor to be subsequently given below), and where δ^β_α are the components of the Kronecker delta. (Needless to say, the Einstein summation convention is applied throughout this work as usual.) Here the inner product is indicated by $\langle \ldots, \ldots \rangle$. We then have

$$\frac{\partial}{\partial x^\beta}\, g_\alpha(x,u) = \frac{\partial}{\partial x^\beta}\, g_\alpha(x) + u\,\frac{\partial \psi_\alpha}{\partial x^\beta} + \psi_\alpha \nabla_\beta\, u\,,$$

where ∇ denotes the gradient, that is, the covariant derivative.

The components of the symmetric, bilinear metric tensor $g(x,u)$ for the given geometric basis are readily given by

$$g_{\alpha\beta}(x,u) = \langle g_\alpha(x,u), g_\beta(x,u)\rangle$$
$$g_{\alpha\lambda}(x,u)\, g^{\beta\lambda}(x,u) = \delta^\beta_\alpha\,.$$

As such, we obtain

$$g_{\alpha\beta}(x,u) = g_{\alpha\beta}(x) + 2\,\hat{u}_{(\alpha}\psi_{\beta)} + \phi^2(x,u)\;\psi_\alpha\,\psi_\beta\,.$$

As usual, round brackets enclosing indices indicate symmetrization; subsequently, anti-symmetrization shall be indicated by square brackets. In the above relation, $\hat{u}_\alpha = \langle u, g_\alpha (x)\rangle$ and

$$\phi^2 (x, u) = g_{\alpha\beta} (x, u)\, u^\alpha u^\beta$$

is the squared length of the world-velocity vector, which varies from point to point in our world-geometry. As we know, this squared length is equal to unity in metric-compatible Riemannian geometry.

The connection form of our world-geometry is obtained through the inner product

$$\Gamma^\lambda_{\alpha\beta} (x, u) = \left\langle g^\lambda (x, u) , \frac{\partial}{\partial x^\beta} g_\alpha (x, u) \right\rangle .$$

In an explicit manner, we see that

$$\Gamma^\lambda_{\alpha\beta} (x, u) = \Gamma^\lambda_{\alpha\beta} (x) + \left(\frac{\partial \psi_\alpha}{\partial x^\beta} \right) u^\lambda + \psi_\alpha \nabla_\beta u^\lambda .$$

In accordance with our previous unified field theories (see, for instance, [1–5]), the above expression must generally be asymmetric, with the torsion being given by the anti-symmetric form

$$\Gamma^\lambda_{[\alpha\beta]} (x, u) = \Gamma^\lambda_{[\alpha\beta]} (x) + \frac{1}{2} \left(\frac{\partial \psi_\alpha}{\partial x^\beta} - \frac{\partial \psi_\beta}{\partial x^\alpha} \right) u^\lambda + + \frac{1}{2} \left(\psi_\alpha \nabla_\beta u^\lambda - \psi_\beta \nabla_\alpha u^\lambda \right) .$$

In contrast to the case of a Riemannian manifold (without background embedding), we have the following unique case:

$$\nabla_\beta g_\alpha (x, u) \equiv \frac{\partial}{\partial x^\beta} g_\alpha (x, u) - \Gamma^\lambda_{\alpha\beta} (x, u)\; g_\lambda (x, u) = = \frac{1}{2} \psi_\alpha \psi_\lambda \left(\nabla_\beta u^\lambda \right) \psi$$

for which, additionally, $\Gamma^\lambda_{\alpha\beta} (x)\; \psi_\lambda = 0$. Consequently, the covariant derivative of the world-metric tensor fails to vanish in the present theory, as we obtain the following non-metric expression:

$$\nabla_\lambda g_{\alpha\beta} (x, u) = \psi_\alpha \psi_\beta \psi_\sigma \nabla_\lambda u^\sigma .$$

At this point, in order to correspond with Finsler geometry in a manifest way, we shall write

$$\nabla_\lambda g_{\alpha\beta} (x, u) = \Phi_{\alpha\beta\sigma} \nabla_\lambda u^\sigma$$

and

$$g_{\alpha\beta} (x, u) = \frac{1}{2} \frac{\partial^2}{\partial u^\alpha \partial u^\beta} \phi^2 (x, u)$$

in such a way that the following conditions are satisfied:

$$\Phi_{\alpha\beta\lambda} = \psi_\alpha \psi_\beta \psi_\lambda ,$$

$$\frac{1}{2} \Phi_{\alpha\beta\lambda} = \frac{1}{2} \Phi_{(\alpha\beta\lambda)} = \frac{1}{2} \frac{\partial}{\partial u^\lambda} g_{\alpha\beta} (x, u) = = \frac{1}{4} \frac{\partial^3}{\partial u^\alpha \partial u^\beta \partial u^\lambda} \phi^2 (x, u) ,$$

$$\Phi_{\alpha\beta\lambda} u^\lambda = 0 ,$$

$$\psi_\alpha u^\alpha = 0 .$$

Once the velocity field is known, the Hessian form of the metric tensor enables us to write, in the momentum representation for a geometric object with mass m (initially at rest, locally),

$$g_{\alpha\beta} (x, u) = \frac{1}{2} m^2 \frac{\partial^2}{\partial p^\alpha \partial p^\beta} \phi^2 (x, u) ,$$

$$p^\alpha = m\, u^\alpha$$

such that, with $\phi^2 (x, u)$ being expressed in parametric form, physical geometry, that is, the existence of a geometric object in space-time, is essentially always related to mass and its energy content.

Taking into account the projective angular tensor given by

$$\Omega_{\alpha\beta} (x, u) = g_{\alpha\beta} (x, u) - \frac{1}{\phi^2 (x, u)} u_\alpha u_\beta ,$$

$$\Omega_{\alpha\lambda} (x, u)\; \Omega^{\beta\lambda} (x, u) = \delta^\beta_\alpha - \frac{1}{\phi^2 (x, u)} u_\alpha u^\beta ,$$

$$\Omega_{\alpha\beta} (x, u)\; u^\beta = 0 ,$$

where n is the number of dimensions of the geometric space (in our case, of course, $n = 4$), in the customary Finslerian way, it can easily be shown that

$$\Phi_{\alpha\beta\lambda} = \frac{1}{n} \Big(\Omega_{\alpha\beta} (x, u)\; \Phi_\lambda + \Omega_{\beta\lambda} (x, u)\; \Phi_\alpha + + \Omega_{\lambda\alpha} (x, u)\; \Phi_\beta - \frac{1}{\Phi_\sigma \Phi^\sigma} \Phi_\alpha \Phi_\beta \Phi_\lambda \Big) ,$$

$$\Phi_\alpha = g^{\sigma\rho} (x, u)\; \Phi_{\sigma\rho\alpha} = 2 \frac{\partial}{\partial u^\alpha} \ln \sqrt{\det (g (x, u))} ,$$

$$\frac{\partial}{\partial u^\alpha} \ln \sqrt{\det (g (x, u))} = \frac{1}{2} g^{\rho\sigma} (x, u) \frac{\partial}{\partial u^\alpha} g_{\rho\sigma} (x, u)$$

for which, in our specific theory, we have, with $\psi^2 = = g_{\alpha\beta} (x, u)\; \psi^\alpha \psi^\beta$,

$$\Phi_{\alpha\beta\lambda} = \frac{\psi^2}{n} \Big(\Omega_{\alpha\beta} (x, u)\; \psi_\lambda + \Omega_{\beta\lambda} (x, u)\; \psi_\alpha + + \Omega_{\lambda\alpha} (x, u)\; \psi_\beta - \frac{1}{\psi^2} \psi_\alpha \psi_\beta \psi_\lambda \Big) .$$

We may note that, along the world-line, for the intrinsic geodesic motion of a particle given by the parallelism

$$\frac{Du^\alpha}{Ds} = (\nabla_\beta u^\alpha)\, u^\beta = 0 ,$$

the Finslerian condition

$$\frac{D}{Ds}\, g_{\alpha\beta}(x,u) = 0$$

is always satisfied, along with the supplementary condition

$$\frac{D}{Ds}\, \phi^2(x,u) = 0\,.$$

Consequently, we shall also have

$$\frac{D}{Ds}\, \Omega_{\alpha\beta}(x,u) = 0\,.$$

It is essential to note that, unlike in Weyl geometry, we shall not expect to arrive at the much simpler gauge condition $\nabla_\lambda\, g_{\alpha\beta}(x,u) = g_{\alpha\beta}(x,u)\; A_\lambda(\psi)$. Instead, we shall always employ the following alternative general form:

$$\nabla_\lambda\, g_{\alpha\beta}(x,u) = \frac{1}{\phi^2(x,u)}\left(\delta_u\, g_{\alpha\beta} - 2\,\hat{u}_{(\alpha}\,\psi_{\beta)}\right)\,\psi_\sigma\,\nabla_\lambda\, u^\sigma$$

where, as we can easily see, the diffeomorphic structure of the metric tensor for the condition of non-metricity of our world-geometry is manifestly given by

$$\delta_u\, g_{\alpha\beta} \equiv g_{\alpha\beta}(x,u) - g_{\alpha\beta}(x) = \\ = 2\,\hat{u}_{(\alpha}\,\psi_{\beta)} + \phi^2(x,u)\;\psi_\alpha\,\psi_\beta$$

3 Explicit physical (Eulerian) structure of the connection form

Having recognized the structural non-metric character of our new world-geometry in the preceding section, we shall now seek to outline the explicit physical structure of the connection form for the purpose of building a unified field theory.

We first note that the non-metric connection form of our theory can always be given by the general expression

$$\Gamma^\lambda_{\alpha\beta}(x,u) = \frac{1}{2}\, g^{\lambda\sigma}(x,u)\left(\frac{\partial}{\partial x^\beta}\, g_{\sigma\alpha}(x,u) - \right.$$
$$\left. - \frac{\partial}{\partial x^\sigma}\, g_{\alpha\beta}(x,u) + \frac{\partial}{\partial x^\alpha}\, g_{\beta\sigma}(x,u)\right) +$$
$$+ \Gamma^\lambda_{[\alpha\beta]}(x,u) - g^{\lambda\sigma}(x,u)\left(g_{\alpha\rho}(x,u)\,\Gamma^\rho_{[\sigma\beta]}(x,u) + \right.$$
$$\left. + g_{\beta\rho}(x,u)\,\Gamma^\rho_{[\sigma\alpha]}(x,u)\right) +$$
$$+ \frac{1}{2}\, g^{\lambda\sigma}(x,u)\left(\nabla_\beta\, g_{\sigma\alpha}(x,u) - \right.$$
$$\left. - \nabla_\sigma\, g_{\alpha\beta}(x,u) + \nabla_\alpha\, g_{\beta\sigma}(x,u)\right).$$

Then, using the results given in the previous section, in direct relation to our previous metric-compatible unification theory of gravity, electromagnetism, material spin, and the nuclear interaction [4], where the electromagnetic field and material spin are generated by the torsion field, we readily obtain

$$\Gamma^\lambda_{\alpha\beta}(x,u) = \frac{1}{2}\, g^{\lambda\sigma}(x,u)\left(\frac{\partial}{\partial x^\beta}\, g_{\sigma\alpha}(x,u) - \right.$$
$$\left. - \frac{\partial}{\partial x^\sigma}\, g_{\alpha\beta}(x,u) + \frac{\partial}{\partial x^\alpha}\, g_{\beta\sigma}(x,u)\right) +$$
$$+ \frac{e}{2\,m\,c^2}\,\phi^2(x,u)\left(F_{\alpha\beta}\,u^\lambda - F^\lambda{}_\alpha\,u_\beta - F^\lambda{}_\beta\,u_\alpha\right) +$$
$$+ S^\lambda_{\alpha\beta} - g^{\lambda\sigma}(x,u)\left(g_{\alpha\rho}(x,u)\; S^\rho_{\sigma\beta} + g_{\beta\rho}(x,u)\; S^\rho_{\sigma\alpha}\right) +$$
$$+ \frac{1}{2}\, g^{\lambda\sigma}(x,u)\,\psi_\rho(\psi_\sigma\,\psi_\alpha\,\nabla_\beta\, u^\rho - \psi_\alpha\,\psi_\beta\,\nabla_\sigma\, u^\rho +$$
$$+ \psi_\beta\,\psi_\sigma\,\nabla_\alpha\, u^\rho)\,.$$

Here it is interesting to note that even when $\psi = 0$, which gives a metric-compatible ("classical") case, our connection form already explicitly depends on the world-velocity (in addition to position), hence the unified field theory of physical interactions outlined in [4] can somehow already be considered as being a Finslerian one despite the fact that it is metric-compatible.

We recall, still from [4], that the electromagnetic field F and the material spin field S have a common geometric origin, which is the structural torsion of the space-time manifold, and are essentially given by the following expressions:

$$F_{\alpha\beta} = 2\,\frac{m\,c^2}{e}\,\Gamma^\lambda_{[\alpha\beta]}\,u_\lambda\,,$$

$$S^\lambda_{\alpha\beta} = S^\lambda{}_\alpha\,u_\beta - S^\lambda{}_\beta\,u_\alpha\,,$$

$$S^{\alpha\beta}\,u_\beta = 0,\; S^{\alpha\beta} = S^{[\alpha\beta]}\,,$$

$$\Gamma^\lambda_{[\alpha\beta]} = \frac{e}{2\,m\,c^2}\,F_{\alpha\beta}\,u^\lambda + S^\lambda_{\alpha\beta}\,,$$

where m is the (rest) mass, e is the electric charge, and c is the speed of light in vacuum, such that the physical fields are intrinsic to the space-time geometry itself, as manifest in generalized geodesic equation of motion $\frac{Du^\alpha}{Ds} = 0$, which naturally yields the general relativistic equation of motion of a charged, massive particle in the gravitational field

$$m\,c^2\left(\frac{du^\alpha}{ds} + \Delta^\alpha_{\beta\lambda}\,u^\beta\,u^\lambda\right) = e\,F^\alpha{}_\beta\,u^\beta\,,$$

$$\Delta^\alpha_{\beta\lambda} = \frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial\, g_{\sigma\beta}}{\partial x^\lambda} - \frac{\partial\, g_{\beta\lambda}}{\partial x^\sigma} + \frac{\partial\, g_{\lambda\sigma}}{\partial x^\beta}\right).$$

In other words, the physical fields other than gravity (chiefly, the electromagnetic field) can also be represented as part of the internal structure of the free-fall of a particle. Just like gravity, being fully geometrized in our theory, these non-holonomic (vortical) fields are no longer external entities merely added into the world-picture in order to interact with gravity and the structure of space-time itself, thereby essentially fulfilling the geometrization program of physics as stated, for example, in [6].

Correspondingly, the nuclear (Yang-Mills) interaction is essentially given in our theory as an internal electromagnetic interaction by

$$F^i_{\alpha\beta} = 2\,\omega^i_\lambda\,\Gamma^\lambda_{[\alpha\beta]}\,,$$

$$F_{\alpha\beta} = \frac{m\,c^2}{e}\,F^i_{\alpha\beta}\,u_i\,(i=1,2,3)\,,$$

where ω^i_α are the components of the tetrad (projective) field relating the global space-time to the internal three-dimensional space of the nuclear interaction.

In this direction, we may also define the extended electromagnetic field, which explicitly depends on the world-velocity, through

$$\tilde{F}_{\alpha\beta}\,(x,u) = \phi^2\,(x,u)\,F_{\alpha\beta} = 2\,\phi^2\,(x,u)\,\frac{mc^2}{e}\,\Gamma^\lambda_{[\alpha\beta]}\,u_\lambda\,.$$

4 Substantial structure of covariant differentiation in U_4

Given an arbitrary world-tensor $T\,(x,u)$ at any point in our Finslerian world-geometry, we have the following elementary substantial derivatives:

$$\frac{d}{d\tau}\,T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) = = \frac{\partial}{\partial x^\eta}\left(T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{dx^\eta}{d\tau} + \frac{\partial}{\partial u^\eta}\left(T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\eta}{\partial\tau}\,,$$

$$\frac{d}{dx^\eta}T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) = = \frac{\partial}{\partial x^\eta}T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) + \frac{\partial}{\partial u^\delta}\left(T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\delta}{\partial x^\eta}\,,$$

where τ is a global parameter.

In this way, the substantial structure of covariant differentiation in U_4 shall be given by

$$\begin{aligned}
\tilde{\nabla}_\eta\,T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) &= \\
&= \frac{\partial}{\partial x^\eta}\,T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) + \frac{\partial}{\partial u^\varsigma}\left(T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\varsigma}{\partial x^\eta} + \\
&+ \Gamma^\alpha_{\delta\eta}\,(x,u)\,T^{\delta\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) + \Gamma^\beta_{\delta\eta}\,(x,u)\,T^{\alpha\delta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) + \ldots + \\
&+ \Gamma^\gamma_{\delta\eta}\,(x,u)\,T^{\alpha\beta\ldots\delta}_{\rho\sigma\ldots\lambda}\,(x,u) - \Gamma^\delta_{\rho\eta}\,(x,u)\,T^{\alpha\beta\ldots\gamma}_{\delta\sigma\ldots\lambda}\,(x,u) - \\
&- \Gamma^\delta_{\sigma\eta}\,(x,u)\,T^{\alpha\beta\ldots\gamma}_{\rho\delta\ldots\lambda}\,(x,u) - \ldots - \Gamma^\delta_{\lambda\eta}\,(x,u)\,T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\delta}\,(x,u)
\end{aligned}$$

along with the more regular (point-oriented) form

$$\begin{aligned}
\nabla_\eta\,T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) &= \frac{\partial}{\partial x^\eta}\,T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) + \\
&+ \Gamma^\alpha_{\delta\eta}\,(x,u)\,T^{\delta\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) + \Gamma^\beta_{\delta\eta}\,(x,u)\,T^{\alpha\delta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) + \cdots + \\
&+ \Gamma^\gamma_{\delta\eta}\,(x,u)\,T^{\alpha\beta\ldots\delta}_{\rho\sigma\ldots\lambda}\,(x,u) - \Gamma^\delta_{\rho\eta}\,(x,u)\,T^{\alpha\beta\ldots\gamma}_{\delta\sigma\ldots\lambda}\,(x,u) - \\
&- \Gamma^\delta_{\sigma\eta}\,(x,u)\,T^{\alpha\beta\ldots\gamma}_{\rho\delta\ldots\lambda}\,(x,u) - \cdots - \Gamma^\delta_{\lambda\eta}\,(x,u)\,T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\delta}\,(x,u)\,.
\end{aligned}$$

Turning our attention to the world-metric tensor, we see that the expression

$$\tilde{\nabla}_\lambda\,g_{\alpha\beta}\,(x,u) = \frac{\partial}{\partial x^\lambda}\,g_{\alpha\beta}\,(x,u) + \frac{\partial}{\partial u^\sigma}\,(g_{\alpha\beta}\,(x,u))\,\frac{\partial u^\sigma}{\partial x^\lambda} - - \Gamma^\rho_{\alpha\lambda}\,(x,u)\,g_{\rho\beta}\,(x,u) - \Gamma^\rho_{\beta\lambda}\,(x,u)\,g_{\alpha\rho}\,(x,u)$$

may enable us to establish a rather indirect metricity-like condition. This can be done by invoking the condition

$$\Phi_{\alpha\beta\sigma}\Gamma^\sigma_{\rho\lambda}\,(x,u)\,u^\lambda = 0$$

and by setting

$$\tilde{\nabla}_\lambda\,g_{\alpha\beta}\,(x,u) = 0\,.$$

Now, with the help of the already familiar relations

$$\frac{\partial}{\partial u^\lambda}\,g_{\alpha\beta}\,(x,u) = \Phi_{\alpha\beta\lambda}\,,$$

$$g^{\alpha\beta}\,(x,u)\,\frac{\partial}{\partial u^\lambda}\,g_{\alpha\beta}\,(x,u) = 2\frac{\partial}{\partial u^\lambda}\ln\,\sqrt{\det\left(g\,(x,u)\right)}$$

we shall again have

$$\nabla_\lambda\,g_{\alpha\beta}\,(x,u) = \Phi_{\alpha\beta\sigma}\,\nabla_\lambda\,u^\sigma\,.$$

5 Generalized curvature forms

We are now equipped enough with the basic structural relations to investigate curvature forms in our theory. In doing so, we shall derive a set of generalized Bianchi identities corresponding to a peculiar class of field equations, including some possible conservation laws (in rather special circumstances).

In a direct customary manner, we have the extended expression

$$\begin{aligned}
&\left(\tilde{\nabla}_\nu\,\tilde{\nabla}_\mu - \tilde{\nabla}_\mu\,\tilde{\nabla}_\nu\right)T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) = \\
&= \left(\nabla_\nu\,\nabla_\mu - \nabla_\mu\,\nabla_\nu\right)T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u) + \\
&+ \frac{\partial}{\partial u^\eta}\left(\nabla_\mu T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\eta}{\partial x^\nu} - \\
&- \frac{\partial}{\partial u^\eta}\left(\nabla_\nu T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\eta}{\partial u^\mu} + \\
&+ \nabla_\nu\left(\frac{\partial}{\partial u^\eta}\left(T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\eta}{\partial x^\mu}\right) - \\
&- \nabla_\mu\left(\frac{\partial}{\partial u^\eta}\left(T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\eta}{\partial x^\nu}\right) + \\
&+ \frac{\partial}{\partial u^\delta}\left(\frac{\partial}{\partial u^\eta}\left(T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\eta}{\partial x^\mu}\right)\frac{\partial u^\delta}{\partial x^\nu} - \\
&- \frac{\partial}{\partial u^\delta}\left(\frac{\partial}{\partial u^\eta}\left(T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}\,(x,u)\right)\frac{\partial u^\eta}{\partial x^\nu}\right)\frac{\partial u^\delta}{\partial x^\mu}
\end{aligned}$$

for which the essential part is

$$(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu)\, T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}(x,u) =$$
$$= R^\eta{}_{\rho\mu\nu}(x,u)\, T^{\alpha\beta\ldots\gamma}_{\eta\sigma\ldots\lambda}(x,u) +$$
$$+ R^\eta{}_{\sigma\mu\nu}(x,u)\, T^{\alpha\beta\ldots\gamma}_{\rho\eta\ldots\lambda}(x,u) + \ldots +$$
$$+ R^\eta{}_{\lambda\mu\nu}(x,u)\, T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\eta}(x,u) -$$
$$- R^\alpha{}_{\eta\mu\nu}(x,u)\, T^{\eta\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}(x,u) -$$
$$- R^\beta{}_{\eta\mu\nu}(x,u)\, T^{\alpha\eta\ldots\gamma}_{\rho\sigma\ldots\lambda}(x,u) -$$
$$- \ldots - R^\gamma{}_{\eta\mu\nu}(x,u)\, T^{\alpha\beta\ldots\eta}_{\rho\sigma\ldots\lambda}(x,u) -$$
$$- 2\,\Gamma^\eta_{[\mu\nu]}(x,u)\, \nabla_\eta T^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\lambda}(x,u)\,.$$

Here the world-curvature tensor, that is, the generalized, Eulerian Riemann tensor, is given by

$$R^\alpha{}_{\beta\rho\sigma}(x,u) = \frac{\partial}{\partial x^\rho}\Gamma^\alpha_{\beta\sigma}(x,u) - \frac{\partial}{\partial x^\sigma}\Gamma^\alpha_{\beta\rho}(x,u) +$$
$$+ \Gamma^\lambda_{\beta\sigma}(x,u)\,\Gamma^\alpha_{\lambda\rho}(x,u) - \Gamma^\lambda_{\beta\rho}(x,u)\,\Gamma^\alpha_{\lambda\sigma}(x,u)$$

for which the corresponding curvature form of mobility may simply be given by

$$\tilde{R}^\alpha{}_{\beta\rho\sigma}(x,u) = \frac{\partial}{\partial x^\rho}\Gamma^\alpha_{\beta\sigma}(x,u) + \frac{\partial}{\partial u^\lambda}\left(\Gamma^\alpha_{\beta\sigma}(x,u)\right)\frac{\partial u^\lambda}{\partial x^\rho} -$$
$$- \frac{\partial}{\partial x^\sigma}\Gamma^\alpha_{\beta\rho}(x,u) - \frac{\partial}{\partial u^\lambda}\left(\Gamma^\alpha_{\beta\rho}(x,u)\right)\frac{\partial u^\lambda}{\partial x^\sigma} +$$
$$+ \Gamma^\lambda_{\beta\sigma}(x,u)\,\Gamma^\alpha_{\lambda\rho}(x,u) - \Gamma^\lambda_{\beta\rho}(x,u)\,\Gamma^\alpha_{\lambda\sigma}(x,u)\,.$$

We can now write the following fundamental decomposition:

$$R^\alpha{}_{\beta\rho\sigma}(x,u) = B^\alpha{}_{\beta\rho\sigma}(x,u) + M^\alpha{}_{\beta\rho\sigma}(x,u) +$$
$$+ N^\alpha{}_{\beta\rho\sigma}(x,u) + U^\alpha{}_{\beta\rho\sigma}(x,u)\,,$$

$$B^\alpha{}_{\beta\rho\sigma}(x,u) = \frac{\partial}{\partial x^\rho}\Delta^\alpha_{\beta\sigma}(x,u) - \frac{\partial}{\partial x^\sigma}\Delta^\alpha_{\beta\rho}(x,u) +$$
$$+ \Delta^\lambda_{\beta\sigma}(x,u)\,\Delta^\alpha_{\lambda\rho}(x,u) - \Delta^\lambda_{\beta\rho}(x,u)\,\Delta^\alpha_{\lambda\sigma}(x,u)\,,$$

$$M^\alpha{}_{\beta\rho\sigma}(x,u) = \breve{\nabla}_\rho K^\alpha_{\beta\sigma}(x,u) - \breve{\nabla}_\sigma K^\alpha_{\beta\rho}(x,u) +$$
$$+ K^\lambda_{\beta\sigma}(x,u)\,K^\alpha_{\lambda\rho}(x,u) - K^\lambda_{\beta\rho}(x,u)\,K^\alpha_{\lambda\sigma}(x,u)\,,$$

$$N^\alpha{}_{\beta\rho\sigma}(x,u) = \breve{\nabla}_\rho Q^\alpha_{\beta\sigma}(x,u) - \breve{\nabla}_\sigma Q^\alpha_{\beta\rho}(x,u) +$$
$$+ Q^\lambda_{\beta\sigma}(x,u)\,Q^\alpha_{\lambda\rho}(x,u) - Q^\lambda_{\beta\rho}(x,u)\,Q^\alpha_{\lambda\sigma}(x,u)\,,$$

$$U^\alpha{}_{\beta\rho\sigma}(x,u) = K^\lambda_{\beta\sigma}(x,u)\,Q^\alpha_{\lambda\rho}(x,u) - K^\lambda_{\beta\rho}(x,u)\,Q^\alpha_{\lambda\sigma}(x,u) +$$
$$+ Q^\lambda_{\beta\sigma}(x,u)\,K^\alpha_{\lambda\rho}(x,u) - Q^\lambda_{\beta\rho}(x,u)\,K^\alpha_{\lambda\sigma}(x,u)\,,$$

where the Eulerian Levi-Civita connection, the Eulerian contorsion tensor, and the connection of non-metricity are respectively given by

$$\Delta^\lambda_{\alpha\beta}(x,u) = \frac{1}{2}\,g^{\lambda\sigma}(x,u)\left(\frac{\partial}{\partial x^\beta}g_{\sigma\alpha}(x,u) - \frac{\partial}{\partial x^\sigma}g_{\alpha\beta}(x,u) + \right.$$
$$\left. + \frac{\partial}{\partial x^\alpha}g_{\beta\sigma}(x,u)\right),$$
$$K^\lambda_{\alpha\beta}(x,u) = \Gamma^\lambda_{[\alpha\beta]}(x,u) -$$
$$- g^{\lambda\sigma}(x,u)\left(g_{\alpha\rho}(x,u)\,\Gamma^\rho_{[\sigma\beta]}(x,u) + g_{\beta\rho}(x,u)\,\Gamma^\rho_{[\sigma\alpha]}(x,u)\right),$$
$$Q^\lambda_{\alpha\beta}(x,u) = \frac{1}{2}\,g^{\lambda\sigma}(x,u)\left(\nabla_\beta g_{\sigma\alpha}(x,u) - \nabla_\sigma g_{\alpha\beta}(x,u) + \right.$$
$$\left. + \nabla_\alpha g_{\beta\sigma}(x,u)\right),$$

such that $\breve{\nabla}$ represents covariant differentiation with respect to the symmetric connection $\Delta(x,u)$ alone. The curvature tensor given by $B(x,u)$ is, of course, the Eulerian Riemann-Christoffel tensor, generalizing the one of the General Theory of Relativity which depends on position alone.

Of special interest, for the world-metric tensor, we note that

$$(\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma)\, g_{\alpha\beta}(x,u) = R_{\alpha\beta\rho\sigma}(x,u) + R_{\beta\alpha\rho\sigma}(x,u) -$$
$$- 2\,\Gamma^\lambda_{[\rho\sigma]}(x,u)\,\nabla_\lambda g_{\alpha\beta}(x,u)$$

where, with the usual notation, $R_{\alpha\beta\rho\sigma}(x,u) = g_{\alpha\lambda}(x,u)\,R^\lambda{}_{\beta\rho\sigma}(x,u)$. That is, more specifically, while keeping in mind that

$$\Phi_{\alpha\beta\lambda} = \frac{\partial}{\partial u^\lambda}g_{\alpha\beta}(x,u) = \psi_\alpha\,\psi_\beta\,\psi_\lambda\,,$$

we have

$$(\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma)\, g_{\alpha\beta}(x,u) = R_{\alpha\beta\rho\sigma}(x,u) +$$
$$+ R_{\beta\alpha\rho\sigma}(x,u) - 2\,\Gamma^\lambda_{[\rho\sigma]}(x,u)\,\Phi_{\alpha\beta\gamma}\nabla_\lambda u^\gamma.$$

As such, we have a genuine homothetic curvature given by

$$H_{\alpha\beta}(x,u) = R^\lambda{}_{\lambda\alpha\beta}(x,u) =$$
$$= \breve{\nabla}_\alpha Q_\beta(x,u) - \breve{\nabla}_\beta Q_\alpha(x,u) =$$
$$= \frac{\partial}{\partial x^\alpha}Q_\beta(x,u) - \frac{\partial}{\partial x^\beta}Q_\alpha(x,u)\,,$$
$$Q_\alpha(x,u) = Q^\lambda_{\lambda\alpha}(x,u) = \frac{1}{2}\,g^{\lambda\beta}(x,u)\,\nabla_\alpha g_{\lambda\beta}(x,u) =$$
$$= \psi^2\,\psi_\beta\,\nabla_\alpha u^\beta\,.$$

Upon setting

$$\theta_\alpha(x,u) = \frac{1}{2}\,\psi_\beta\,\nabla_\alpha u^\beta,$$

we have

$$H_{\alpha\beta}(x,u) = \psi^2\left(\frac{\partial}{\partial x^\alpha}\theta_\beta(x,u) - \frac{\partial}{\partial x^\beta}\theta_\alpha(x,u) - \right.$$
$$\left. - 2\left(\theta_\alpha(x,u)\,\frac{\partial \ln\psi}{\partial x^\beta} - \theta_\beta(x,u)\,\frac{\partial \ln\psi}{\partial x^\alpha}\right)\right).$$

At this point, the generalized, Eulerian Ricci tensor is given in the form

$$R_{\alpha\beta}(x,u) = R^{\lambda}{}_{\alpha\lambda\beta}(x,u) = Z_{\alpha\beta}(\Delta(x,u), K(x,u)) + $$
$$+ N_{\alpha\beta}(Q(x,u)) + X_{\alpha\beta}(K(x,u), Q(x,u)),$$
$$Z_{\alpha\beta}(\Delta(x,u), K(x,u)) = B^{\lambda}{}_{\alpha\lambda\beta}(x,u) + M^{\lambda}{}_{\alpha\lambda\beta}(x,u),$$
$$N_{\alpha\beta}(Q(x,u)) = N^{\lambda}_{\alpha\lambda\beta}(x,u),$$
$$X_{\alpha\beta}(K(x,u), Q(x,u)) = U^{\lambda}{}_{\alpha\lambda\beta}(x,u),$$

which admits the peculiar anti-symmetric part

$$R_{[\alpha\beta]}(x,u) = \frac{1}{2}\left(\frac{\partial}{\partial x^{\alpha}} K^{\lambda}_{\beta\lambda}(x,u) - \frac{\partial}{\partial x^{\beta}} K^{\lambda}_{\alpha\lambda}(x,u)\right) + $$
$$+ \frac{1}{2}\left(\frac{\partial}{\partial x^{\alpha}} Q_{\beta}(x,u) - \frac{\partial}{\partial x^{\beta}} Q_{\alpha}(x,u)\right) + $$
$$+ \breve{\nabla}_{\lambda}\Gamma^{\lambda}_{[\alpha\beta]}(x,u) + $$
$$+ \Gamma^{\lambda}_{[\alpha\beta]}(x,u) K^{\sigma}_{\lambda\sigma}(x,u) + \Gamma^{\lambda}_{[\alpha\beta]}(x,u) Q_{\lambda}(x,u) + $$
$$+ \Gamma^{\lambda}_{[\beta\sigma]}(x,u) Q^{\sigma}_{\alpha\lambda}(x,u) - \Gamma^{\lambda}_{[\alpha\sigma]}(x,u) Q^{\sigma}_{\beta\lambda}(x,u) + $$
$$+ \frac{1}{2}\left(K^{\lambda}_{\alpha\sigma}(x,u) K^{\sigma}_{\lambda\beta}(x,u) - K^{\lambda}_{\beta\sigma}(x,u) K^{\sigma}_{\lambda\alpha}(x,u)\right),$$

where we have made use of the fact that $K^{\lambda}_{[\alpha\beta]}(x,u) = \Gamma^{\lambda}_{[\alpha\beta]}(x,u)$. Let us also keep in mind that the explicit physical structure of the connection form forming our various curvature expressions, as it relates to gravity, electromagnetism, material spin, and the nuclear interaction, is given in Section 3 of this work, naturally following [4].

We can now obtain the complete Eulerian generalization of the first Bianchi identity as follows:

$$R_{\alpha\beta\rho\sigma}(x,u) + R_{\alpha\rho\sigma\beta}(x,u) + R_{\alpha\sigma\beta\rho}(x,u) = $$
$$= -2\, g_{\alpha\lambda}(x,u)\left(\frac{\partial}{\partial x^{\sigma}}\Gamma^{\lambda}_{[\beta\rho]}(x,u) + \frac{\partial}{\partial x^{\beta}}\Gamma^{\lambda}_{[\rho\sigma]}(x,u) + \right.$$
$$\left. + \frac{\partial}{\partial x^{\rho}}\Gamma^{\lambda}_{[\sigma\beta]}(x,u)\right) - $$
$$- 2\, g_{\alpha\lambda}(x,u)\left(\Gamma^{\lambda}_{\gamma\beta}(x,u)\Gamma^{\gamma}_{[\rho\sigma]}(x,u) + \right.$$
$$\left. + \Gamma^{\lambda}_{\gamma\rho}(x,u)\Gamma^{\gamma}_{[\sigma\beta]}(x,u) + \Gamma^{\lambda}_{\gamma\sigma}(x,u)\Gamma^{\gamma}_{[\beta\rho]}(x,u)\right) + $$
$$+ 2\,\Phi_{\alpha\lambda\gamma}\left(\Gamma^{\lambda}_{[\rho\sigma]}(x,u)\nabla_{\beta}u^{\gamma} + \Gamma^{\lambda}_{[\sigma\beta]}(x,u)\nabla_{\rho}u^{\gamma} + \right.$$
$$\left. + \Gamma^{\lambda}_{[\beta\rho]}(x,u)\nabla_{\sigma}u^{\gamma}\right).$$

Similarly, after a somewhat lengthy calculation, we obtain, for the generalization of the second Bianchi identity,

$$\nabla_{\lambda} R_{\alpha\beta\rho\sigma}(x,u) + \nabla_{\rho} R_{\alpha\beta\sigma\lambda}(x,u) + \nabla_{\sigma} R_{\alpha\beta\lambda\rho}(x,u) = $$
$$= 2\left(\Gamma^{\gamma}_{[\rho\sigma]}(x,u) R_{\alpha\beta\gamma\lambda}(x,u) + \Gamma^{\gamma}_{[\sigma\lambda]}(x,u) R_{\alpha\beta\gamma\rho}(x,u) + \right.$$
$$\left. + \Gamma^{\gamma}_{[\lambda\rho]}(x,u) R_{\alpha\beta\gamma\sigma}(x,u)\right) + $$
$$+ \Gamma^{\gamma}_{\beta\rho}(x,u)\left((\nabla_{\lambda}\Phi_{\alpha\gamma\eta})\nabla_{\sigma}u^{\eta} - (\nabla_{\sigma}\Phi_{\alpha\gamma\eta})\nabla_{\lambda}u^{\eta}\right) + $$
$$+ \Gamma^{\gamma}_{\beta\sigma}(x,u)\left((\nabla_{\rho}\Phi_{\alpha\gamma\eta})\nabla_{\lambda}u^{\eta} - (\nabla_{\lambda}\Phi_{\alpha\gamma\eta})\nabla_{\rho}u^{\eta}\right) + $$
$$+ \Gamma^{\gamma}_{\beta\lambda}(x,u)\left((\nabla_{\sigma}\Phi_{\alpha\gamma\eta})\nabla_{\rho}u^{\eta} - (\nabla_{\rho}\Phi_{\alpha\gamma\eta})\nabla_{\sigma}u^{\eta}\right) - $$
$$- \Gamma^{\gamma}_{\beta\rho}(x,u)\Phi_{\alpha\gamma\eta}\left(R^{\eta}{}_{\mu\sigma\lambda}(x,u)u^{\mu} + 2\Gamma^{\mu}_{[\sigma\lambda]}(x,u)\nabla_{\mu}u^{\eta}\right) - $$
$$- \Gamma^{\gamma}_{\beta\sigma}(x,u)\Phi_{\alpha\gamma\eta}\left(R^{\eta}{}_{\mu\lambda\rho}(x,u)u^{\mu} + 2\Gamma^{\mu}_{[\lambda\rho]}(x,u)\nabla_{\mu}u^{\eta}\right) - $$
$$- \Gamma^{\gamma}_{\beta\lambda}(x,u)\Phi_{\alpha\gamma\eta}\left(R^{\eta}{}_{\mu\rho\sigma}(x,u)u^{\mu} + 2\Gamma^{\mu}_{[\rho\sigma]}(x,u)\nabla_{\mu}u^{\eta}\right) + $$
$$+ \Phi_{\alpha\gamma\eta}(\nabla_{\rho}u^{\eta})\left(\nabla_{\sigma}\Gamma^{\gamma}_{\beta\lambda}(x,u) - \nabla_{\lambda}\Gamma^{\gamma}_{\beta\sigma}(x,u)\right) + $$
$$+ \Phi_{\alpha\gamma\eta}(\nabla_{\sigma}u^{\eta})\left(\nabla_{\lambda}\Gamma^{\gamma}_{\beta\rho}(x,u) - \nabla_{\rho}\Gamma^{\gamma}_{\beta\lambda}(x,u)\right) + $$
$$+ \Phi_{\alpha\gamma\eta}(\nabla_{\lambda}u^{\eta})\left(\nabla_{\rho}\Gamma^{\gamma}_{\beta\sigma}(x,u) - \nabla_{\sigma}\Gamma^{\gamma}_{\beta\rho}(x,u)\right),$$

where

$$\nabla_{\sigma}\Gamma^{\lambda}_{\alpha\beta}(x,u) - \nabla_{\beta}\Gamma^{\lambda}_{\alpha\sigma}(x,u) = -R^{\lambda}{}_{\alpha\beta\sigma}(x,u) + $$
$$+ \Gamma^{\rho}_{\alpha\beta}(x,u)\Gamma^{\lambda}_{\rho\sigma}(x,u) - \Gamma^{\rho}_{\alpha\sigma}(x,u)\Gamma^{\lambda}_{\rho\beta}(x,u) - $$
$$- 2\Gamma^{\rho}_{[\beta\sigma]}(x,u)\Gamma^{\lambda}_{\alpha\rho}(x,u).$$

By contraction, we may extract a physical density field as follows:

$$J_{\alpha}(x,u) = $$
$$= -\nabla_{\beta}\left(\frac{1}{2}\left(R^{\beta}{}_{\alpha}(x,u) + {}^{*}R^{\beta}{}_{\alpha}(x,u)\right) - \frac{1}{2}\delta^{\beta}_{\alpha} R(x,u)\right),$$

where ${}^{*}R^{\alpha}{}_{\beta}(x,u) = R^{\alpha\lambda}_{\beta\lambda}(x,u)$ are the components of the generalized Ricci tensor of the second kind and $R(x,u) = R^{\lambda}{}_{\lambda}(x,u) = {}^{*}R^{\lambda}{}_{\lambda}(x,u)$ is the generalized Ricci scalar. As we know, the Ricci tensor of the first kind and the Ricci tensor of the second kind coincide only when the connection form is metric-compatible. The asymmetric, generally non-conservative world-entity given by

$$G^{\alpha}{}_{\beta}(x,u) = \frac{1}{2}\left(R^{\alpha}{}_{\beta}(x,u) + {}^{*}R^{\alpha}{}_{\beta}(x,u)\right) - \frac{1}{2}\delta^{\alpha}_{\beta} R(x,u)$$

will therefore represent the generalized Einstein tensor, such that we may have a corresponding geometric object given by

$$C^{\alpha}(x,u) \equiv -g^{\alpha\beta}(x,u) J_{\beta}(x,u) = $$
$$= \nabla_{\beta} G^{\beta\alpha}(x,u) - G^{\beta}{}_{\lambda}(x,u)\nabla_{\beta} g^{\lambda\alpha}(x,u).$$

6 Quantum gravity from the physical vacuum of U_4

We are now in a position to derive a quantum mechanical wave equation from the underlying structure of our present theory. So far, our field equations appear too complicated to handle for this particular purpose. It is quite enough that we know the structural content of the connection form, which encompasses the geometrization of the known classical fields. However, if we deal with a particular case, namely, that of physical vacuum, we shall immediately be able to speak of one type of emergent quantum gravity.

Assuming now that the world-geometry U_4 is devoid of "ultimate physical substance" (that is, intrinsic material confinement on the most fundamental scale) other than, perhaps, primordial radiation, the field equation shall be given by

$$R_{\alpha\beta}(x,u)=0$$

for which, in general, $R^{\alpha}{}_{\beta\mu\nu}(x,u)=W^{\alpha}_{\beta\mu\nu}(x,u)\neq 0$, where $W(x,u)$ is the generalized Weyl conformal tensor. In this way, all physical fields, including matter, are mere appearances in our geometric world-structure. Consequently, from $R_{(\alpha\beta)}(x,u)=0$, the emergent picture of gravity is readily given by the symmetric Eulerian Ricci tensor for the composite structure of gravity, that is, explicitly,

$$B_{\alpha\beta}(\Delta(x,u))=-\Big(M_{\alpha\beta}\big(K(x,u)\big)+N_{\alpha\beta}\big(Q(\psi)\big)+ +U_{\alpha\beta}\big(K(x,u),Q(\psi)\big)\Big),$$

where we have written $Q(x,u)=Q(\psi)$, such that, in this special consideration, gravity can essentially be thought of as exterior electromagnetism as well as arising from the quantum fuzziness of the background non-metricity of the world-geometry. In addition, from $R_{[\alpha\beta]}(x,u)=0$, we also have the following anti-symmetric counterpart:

$$R_{[\alpha\beta]}(\Delta(x,u),K(x,u))=\frac{\partial}{\partial x^{\beta}}Q_{\alpha}(\psi)-\frac{\partial}{\partial x^{\alpha}}Q_{\beta}(\psi)- -\Gamma^{\lambda}_{[\alpha\beta]}(x,u)\,Q_{\lambda}(\psi)+ +\Gamma^{\lambda}_{[\alpha\sigma]}(x,u)\,Q^{\sigma}_{\beta\lambda}(\psi)-\Gamma^{\lambda}_{[\beta\sigma]}(x,u)\,Q^{\sigma}_{\alpha\lambda}(\psi)\,,$$

$$Q_{\alpha}(\psi)=\frac{1}{2}\,\psi^{2}\,\psi_{\beta}\,\nabla_{\alpha}\,u^{\beta}\,.$$

Correspondingly, we shall set, for the "quantum potential",

$$Q_{\alpha}(\psi)=\frac{\partial}{\partial x^{\alpha}}\ln\bar{\psi}$$

such that the free, geodesic motion of a particle along the fuzzy world-path $s(x,u)=\tau\left(\psi\left(\bar{\psi}\right)\right)$ in the empty U_4 can simultaneously be described by the pair of dynamical equations

$$\frac{Du^{\alpha}}{Ds}=0\,,$$

$$\frac{D\bar{\psi}}{Ds}=0\,,$$

since, as we have previously seen, $Q_{\alpha}\left(\psi\left(\bar{\psi}\right)\right)u^{\alpha}=0$.

Immediately, we obtain the geometrically non-linear wave equation

$$\frac{1}{\sqrt{\det(g(x,u))}}\frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta}(x,u)\sqrt{\det(g(x,u))}\frac{\partial\bar{\psi}}{\partial x^{\beta}}\right)= =\left(R(\Delta(x,u),K(x,u))+\Lambda(Q(\psi))\right)\bar{\psi}$$

that is,

$$\left(\Delta^{2}_{B}-\widehat{R}(x,u)\right)\bar{\psi}=0\,,$$

where

$$\Delta^{2}_{B}=\frac{1}{\sqrt{\det(g(x,u))}}\frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta}(x,u)\sqrt{\det(g(x,u))}\frac{\partial}{\partial x^{\beta}}\right)$$

is the covariant four-dimensional Beltrami wave operator and, with the explicit dependence of ψ on $\bar{\psi}$,

$$\widehat{R}(x,u)=R(\Delta(x,u),K(x,u))+\Lambda\left(Q\left(\psi\left(\bar{\psi}\right)\right)\right)$$

is the emergent curvature scalar of our quantum field, for which

$$\Lambda(Q(\psi))=\widehat{N}\left(Q\left(\psi\left(\bar{\psi}\right)\right)\right)-\frac{1}{\bar{\psi}^{2}}\,g^{\alpha\beta}(x,u)\,\frac{\partial\bar{\psi}}{\partial x^{\alpha}}\frac{\partial\bar{\psi}}{\partial x^{\beta}}\,,$$

$$\widehat{N}\left(Q\left(\psi\left(\bar{\psi}\right)\right)\right)=N\left(Q\left(\psi\left(\bar{\psi}\right)\right)\right)+ +U\left(K(x,u),Q\left(\psi\left(\bar{\psi}\right)\right)\right)-g^{\alpha\beta}(x,u)\,\breve{\nabla}_{\beta}Q_{\alpha}\left(\psi\left(\bar{\psi}\right)\right).$$

In terms of the Eulerian Ricci scalar, which is now quantized by the wave equation, we have a quantum gravitational wave equation with two quantized intrinsic sources, namely, the torsional source $M(x,u)$, which combines the electromagnetic and material sources, and the quantum mechanical source $\Lambda\left(Q\left(\psi\left(\bar{\psi}\right)\right)\right)=\Lambda(Q(x,u))$,

$$\left(\Delta^{2}_{B}-B(x,u)\right)\bar{\psi}=M(x,u)\,\bar{\psi}+\Lambda\left(Q\left(\psi\left(\bar{\psi}\right)\right)\right)\bar{\psi}$$

thereby completing the quantum gravitational picture at an elementary stage.

7 Special analytic form of geodesic paths

Here we are interested in the derivation of the generalized geodesic equation of motion such that our geodesic paths correspond to the formal solution of the quantum gravitational wave equation in the preceding section. Indeed, owing to the wave function $\bar{\psi}=\bar{\psi}(x,u)$, these geodesic paths shall be conformal ones.

For our purpose, let $\Psi(x)=const.$ represent a family of hypersurfaces in U_4 such that with respect to a mobile hypersurface Σ, for $\frac{\partial}{\partial x^{\alpha}}(\Psi(x))\,\delta x^{\alpha}=0$, there exists a genuine unit normal velocity vector, given by $n^{\alpha}=\frac{dx^{\alpha}}{d\tau}$, at some point whose extended path can be parametrized by $\tau=\tau(s)$, that is

$$n_{\alpha}=\zeta\left(x,\frac{\partial}{\partial x}\Psi(x)\right)\frac{\partial}{\partial x^{\alpha}}\Psi(x)$$

$$g_{\alpha\beta}(x,u)\;n^{\alpha}\,\delta\,n^{\beta}=0\,.$$

The essential partial differential equation representing any quantum gravitational hypersurface Σ_{ψ} can then simply be represented by the arbitrary parametric form $\zeta\left(x, \frac{\partial}{\partial x}\Psi(x)\right) = \zeta\left(\bar{\psi}\right) = const$ such that

$$\int_a^b \left(\phi(x,u) - \zeta\left(\bar{\psi}\right)\frac{d}{d\tau}\Psi(x)\right)d\tau \geqslant 0$$

where a and b are two points in Σ_ψ.

Keeping in mind once again that $\psi_\alpha\, u^\alpha = 0$ and that

$$u_\alpha = \frac{1}{2}\frac{\partial}{\partial u^\alpha}\phi^2(x,u)$$

$$\frac{\partial}{\partial x^\lambda} g_{\alpha\beta}(x,u) = $$
$$= \Gamma_{\alpha\beta\lambda}(x,u) + \Gamma_{\beta\alpha\lambda}(x,u) + \psi_\alpha\,\psi_\beta\,\psi_\sigma\,\nabla_\lambda u^\sigma$$

the generalized Euler-Lagrange equation corresponding to our situation shall then be given by

$$\frac{d}{ds}\left(\frac{\partial}{\partial u^\alpha}\phi^2(x,u)\right) - \frac{\partial}{\partial x^\alpha}\phi^2(x,u) +$$
$$+\frac{\partial}{\partial u^\beta}\left(\phi^2(x,u)\right)\frac{\partial u^\beta}{\partial x^\alpha} + b_\alpha(x,u) = 0\,,$$

where the "external" term is given by

$$b_\alpha(x,u) = 4\,\Gamma^\lambda_{[\alpha\beta]}(x,u)\,u_\lambda\,u^\beta.$$

As a matter of straightforward verification, we have

$$\frac{du_\alpha}{ds} - \Gamma_{\beta\alpha\lambda}(x,u)\,u^\beta\,u^\lambda = 0$$

A unique general solution to the above equation corresponding to the quantum displacement field $\psi = \psi\left(\bar{\psi}\right)$, which, in our theory, generates the non-metric nature of the world-manifold U_4, can now be obtained as

$$s(x,u) = s\left(\psi\left(\bar{\psi}\right)\right) = C_1 + C_2\int\exp\left(\int H\left(\psi\left(\bar{\psi}\right)ds\right)\right)ds$$

where C_1 and C_2 are integration constants. This is such that, at arbitrary world-points a and b, we have the conformal relation (for $C = C_2$)

$$ds_b = \exp\left(C\int H\left(\psi\left(\bar{\psi}\right)\right)ds\right)\,ds_a\,,$$

which sublimely corresponds to the case of our previous quantum theory of gravity [3].

8 Geometric structure of the electromagnetic potential

As another special consideration, let us now attempt to extensively describe the geometric structure of the electromagnetic potential in our theory.

Due to the degree of complicatedness of the detailed general coordinate transformations in U_4, let us, for the sake of tangibility, refer a smoothly extensive coordinate patch $P(x)$ to the four-dimensional tangent hyperplane $M_4(y)$, whose metric tensor η is Minkowskian, such that an ensemble of Minkowskian tangent hyperplanes, that is,

$$\sum\nolimits_{a\,=\,1,2,\ldots,N} M_4^{(a)}(y)$$

cannot globally cover the curved manifold U_4 without breaking analytic continuity (smoothness), at least up to the third order. Denoting the "invariant derivative" by $\nabla_A = E^\alpha_A(x,u)\frac{\partial}{\partial x^\alpha}$, this situation can then basically be described by

$$g_{\alpha\beta}(x,u) = E^A_\alpha(x,u)\,E^B_\beta(x,u)\,\eta_{AB}\,,$$

$$E^A_\alpha(x,u) = \frac{\partial y^A}{\partial x^\alpha}, E^\alpha_A(x,u) = \left(E^A_\alpha(x,u)\right)^{-1}\,,$$

$$y^A = y^A(x,u)\,, x^\alpha = x^\alpha(y)\,,$$

$$E^A_\alpha(x,u)\,E^\beta_A(x,u) = \delta^\beta_\alpha, E^\alpha_A(x,u)\,E^B_\alpha(x,u) = \delta^B_A\,,$$

$$\Gamma^\lambda_{\alpha\beta}(x,u) = E^\lambda_A(x,u)\frac{\partial}{\partial x^\beta}E^A_\alpha(x,u) = $$
$$= E^\lambda_A(x,u)\,E^B_\beta(x,u)\,\nabla_B E^A_\alpha(x,u)\,.$$

Of fundamental importance in our unified field theory are, of course, the torsion tensor given by

$$\Gamma^\lambda_{[\alpha\beta]}(x,u) = \frac{1}{2}E^\lambda_A(x,u)\left(\frac{\partial}{\partial x^\beta}E^A_\alpha(x,u) - \frac{\partial}{\partial x^\alpha}E^A_\beta(x,u)\right)$$

and the curvature tensor given by

$$R^\lambda{}_{\sigma\alpha\beta}(x,u) = $$
$$= -E^\lambda_A(x,u)\left(\left(\frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\right)E^A_\sigma(x,u)\right) = $$
$$= E^A_\sigma(x,u)\left(\left(\frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\right)E^\lambda_A(x,u)\right)\,.$$

Additionally, we can also see that

$$R_{\rho\sigma\alpha\beta}(x,u) = $$
$$= E^A_\sigma(x,u)\left(\left(\frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\right)E_{A\rho}(x,u)\right) +$$
$$+\left(\frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\right)g_{\rho\sigma}(x,u)\,.$$

Immediately, we obtain

$$R^\lambda{}_{\sigma\alpha\beta}(x,u) = E^\lambda_A(x,u)\,E^B_\alpha(x,u)\,E^C_\beta(x,u)\,\times$$
$$\times\left(\left(\nabla_B\nabla_C - \nabla_C\nabla_B\right)E^A_\sigma(x,u)\right) - 2\,\Gamma^\lambda_{\sigma\rho}(x,u)\,\Gamma^\rho_{[\alpha\beta]}(x,u)\,.$$

Introducing a corresponding internal ("isotopic") curvature form through

$$\bar{R}^\alpha{}_{\beta AB}(x,u) = E^\alpha_C(x,u)\left(\left(\nabla_A\nabla_B - \nabla_B\nabla_A\right)E^C_\beta(x,u)\right),$$

we can write

$$R^{\lambda}{}_{\sigma\alpha\beta}(x,u) = E^{A}_{\alpha}(x,u)\,E^{B}_{\beta}(x,u)\,\bar{R}^{\lambda}{}_{\sigma AB}(x,u) - $$
$$ - 2\,\Gamma^{\lambda}_{\sigma\rho}(x,u)\,\Gamma^{\rho}_{[\alpha\beta]}(x,u)\,.$$

In physical terms, we therefore see that

$$R^{\lambda}{}_{\sigma\alpha\beta}(x,u) = E^{A}_{\alpha}(x,u)\,E^{B}_{\beta}(x,u)\,\bar{R}^{\lambda}{}_{\sigma AB}(x,u) - $$
$$ - 2\,\Gamma^{\lambda}_{\sigma\rho}(x,u)\,S^{\rho}_{\alpha\beta} - \frac{e}{mc^2}\,\phi^2(x,u)\,\Gamma^{\lambda}_{\sigma\rho}(x,u)\,F_{\alpha\beta}\,u^{\rho}\,,$$

where the electromagnetic field tensor can now be expressed by the extended form (given in Section 3)

$$\tilde{F}_{\alpha\beta}(x,u) = 2\,\frac{mc^2}{e}\,\phi^2(x,u)\;\Gamma^{\lambda}_{[\alpha\beta]}(x,u)\,u_{\lambda}\,,$$

that is,

$$\tilde{F}_{\alpha\beta}(x,u) = \frac{mc^2}{e}\,\phi^2(x,u)\left(\frac{\partial u_{\alpha}}{\partial x^{\beta}} - \frac{\partial u_{\beta}}{\partial x^{\alpha}} - \right.$$
$$\left. - \,E^{A}_{\alpha}(x,u)\,E^{B}_{\beta}(x,u)\left(\nabla_B u_A - \nabla_B u_A\right)\right).$$

An essential feature of the electromagnetic field in our unified field theory therefore manifests as a field of vorticity, somewhat reminiscent of the case of fluid dynamics, that is,

$$\tilde{F}_{\alpha\beta}(x,u) = $$
$$ = 2\,\frac{mc^2}{e}\,\phi^2(x,u)\left(\omega_{\alpha\beta} - E^{A}_{\alpha}(x,u)\,E^{B}_{\beta}(x,u)\,\Theta_{AB}\right)\,,$$

where the vorticity field is given in two referential forms by

$$\omega_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_{\alpha}}{\partial x^{\beta}} - \frac{\partial u_{\beta}}{\partial x^{\alpha}}\right),$$
$$\Theta_{AB} = \frac{1}{2}\left(\nabla_B u_A - \nabla_A u_B\right).$$

For our regular Eulerian electromagnetic field, we simply have

$$F_{\alpha\beta} = F_{\alpha\beta}(x,u) = 2\,\frac{mc^2}{e}\left(\omega_{\alpha\beta} - E^{A}_{\alpha}(x,u)\,E^{B}_{\beta}(x,u)\,\Theta_{AB}\right).$$

After some algebraic (structural) factorization, a profound physical solution to our most general Eulerian expression for the electromagnetic field can be obtained in integral form as

$$\varphi_{\alpha}(x,u) = \frac{mc^2}{e}\oint_C \phi^2(x,u)\left(\frac{\partial}{\partial x^{\beta}}\,E^{A}_{\alpha}(x,u)\right)u_A\,dx^{\beta}$$

such that $\tilde{F}_{\alpha\beta}(x,u) = \frac{\partial}{\partial x^{\beta}}\,\varphi_{\alpha}(x,u) - \frac{\partial}{\partial x^{\alpha}}\,\varphi_{\beta}(x,u)$, that is, in order to preserve the customary gauge invariance, our electromagnetic field shall manifestly be a "pure curl". This structural form is, of course, given in the domain of a vortical path C covered by a quasi-regular surface spanned in two directions and essentially given by the form $d\sigma^{AB} = d_1\,y^A(x,u)\,d_2\,y^B(x,u) - d_1\,y^B(x,u)\,d_2\,y^A(x,u)$. Upon using Gauss theorem, we therefore see that.

$$\varphi_{\alpha}(x,u) = \frac{1}{2}\,\frac{mc^2}{e}\,\times$$
$$\times\iint_{\sigma}\phi^2(x,u)\left(\left(\nabla_B\nabla_A - \nabla_A\nabla_B\right)E^{C}_{\alpha}(x,u)\right)u_C\,d\sigma^{AB}.$$

In other words, we have

$$\varphi_{\alpha}(x,u) = -\,\frac{1}{2}\,\frac{mc^2}{e}\int\!\!\int_{\sigma}\phi^2(x,u)\,\bar{R}^{\lambda}{}_{\alpha AB}(x,u)\,u_{\lambda}\,d\sigma^{AB}$$

or, with $d\sigma^{\alpha\beta} = E^{\alpha}_{A}(x,u)\,E^{\beta}_{B}(x,u)\,d\sigma^{AB}$,

$$\varphi_{\alpha}(x,u) = -\,\frac{1}{2}\,\frac{mc^2}{e}\int\!\!\int_{\sigma}\phi^2(x,u)\,\times$$
$$\times\left(R^{\lambda}{}_{\alpha\beta\sigma}(x,u) + 2\,\Gamma^{\lambda}_{\alpha\rho}(x,u)\,\Gamma^{\rho}_{[\beta\sigma]}(x,u)\right)u_{\lambda}\,d\sigma^{\beta\sigma},$$

which means that

$$\varphi_{\alpha}(x,u) = -\,\frac{1}{2}\,\frac{mc^2}{e}\int\!\!\int_{\sigma}\phi^2(x,u)\,\times$$
$$\times\left(R^{\lambda}{}_{\alpha\beta\sigma}(x,u) + 2\,\Gamma^{\lambda}_{\alpha\rho}(x,u)\,S^{\rho}_{\beta\sigma}(x,u)\right)u_{\lambda}\,d\sigma^{\beta\sigma} - $$
$$ - \,\frac{1}{2}\int\!\!\int_{\sigma}\Gamma^{\lambda}_{\alpha\rho}(x,u)\,F_{\beta\sigma}(x,u)\,u^{\rho}\,u_{\lambda}\,d\sigma^{\beta\sigma}.$$

Combining the above expression with the geodesic equation of motion given by $\frac{du_{\alpha}}{ds} = \Gamma^{\lambda}_{\alpha\beta}(x,u)\,u_{\lambda}\,u^{\beta}$, we finally obtain the integral equation of motion

$$\varphi_{\alpha}(x,u) = -\,\frac{1}{2}\,\frac{mc^2}{e}\int\!\!\int_{\sigma}\phi^2(x,u)\,\times$$
$$\times\left(R^{\lambda}{}_{\alpha\beta\sigma}(x,u) + 2\,\Gamma^{\lambda}_{\alpha\rho}(x,u)\,S^{\rho}_{\beta\sigma}(x,u)\right)u_{\lambda}\,d\sigma^{\beta\sigma} - $$
$$ - \,\frac{1}{2}\int\!\!\int_{\sigma}\left(\frac{du_{\alpha}}{ds}\right)F_{\beta\sigma}(x,u)\,d\sigma^{\beta\sigma},$$

which shows, for the first time, the explicit dependence of the electromagnetic potential on world-velocity (as well as local acceleration), global curvature, and the material spin field.

9 Closing remarks

In the foregoing presentation, we have created a new kind of Finsler space, from which we have built the foundation of a unified field theory endowed with propagating torsion and curvature. Previously [1, 5], we have done it without the "luxury" of killing the metricity condition of Riemannian geometry; at present, the asymmetric connection form of our world-geometry, in addition to the metric and curvature, is a function of both position and world-velocity. Therefore, looking back on our previous works, we may conclude that, in particular, the theories outlined in [3,4], as a whole, appear to be a natural bridge between generalized Riemannian and Finslerian structures.

A very general presentation of my own version of the theory of non-linear connection has also been given in [3], where, in immediate relation to [4], the enveloping evolutive world-structure can be seen as some kind of conformal Finsler space with torsion. The union between [3] and [4] has indeed already given us the essence of a fully geometric quantum theory of gravity, with electromagnetism and the Yang-Mills gauge field included. The present work mainly serves to complement and enrich this purely geometric union.

Submitted on June 27, 2009 / Accepted on August 06, 2009

References

1. Suhendro I. A four-dimensional continuum theory of space-time and the classical physical fields. *Progress in Physics*, 2007, v. 4, 34–46.
2. Suhendro I. A new semi-symmetric unified field theory of the classical fields of gravity and electromagnetism. *Progress in Physics*, 2007, v. 4, 47–62.
3. Suhendro I. A new conformal theory of semi-classical quantum general relativity. *Progress in Physics*, 2007, v. 4, 96–103.
4. Suhendro I. A unified field theory of gravity, electromagnetism, and the Yang-Mills gauge field. *Progress in Physics*, 2008, v. 1, 31–37.
5. Suhendro I. Spin-curvature and the unification of fields in a twisted space. Svenska fysikarkivet, Stockholm, 2008.
6. Borissova L. and Rabounski D. Fields, vacuum, and the mirror Universe. 2nd edition, Svenska fysikarkivet, Stockholm, 2009, p. 26–29.