Formation of Singlet Fermion Pairs in the Dilute Gas of Boson-Fermion Mixture

Vahan Minasyan and Valentin Samoilov

Scientific Center of Applied Research, JINR, Dubna, 141980, Russia

E-mails: mvahan@scar.jinr.ru; scar@off-serv.jinr.ru

We argue the formation of a free neutron spinless pairs in a liquid helium -dilute neutron gas mixture. We show that the term, of the interaction between the excitations of the Bose gas and the density modes of the neutron, meditate an attractive interaction via the neutron modes, which in turn leads to a bound state on a spinless neutron pair. Due to presented theoretical approach, we prove that the electron pairs in superconductivity could be discovered by Frölich earlier then it was made by the Cooper.

1 Introduction

In 1938, the connection between the ideal Bose gas and superfluidity in helium was first made by London [1]. The ideal Bose gas undergoes a phase transition at sufficiently low temperatures to a condition in which the zero-momentum quantum state is occupied by a finite fraction of the atoms. This momentum-condensed phase was postulated by London to represent the superfluid component of liquid ^{4}He. With this hypothesis, the beginnings of a two- fluid hydrodynamic model of superfluids was developed by Landau [2] where he predicted the notation of a collective excitations so- called phonons and rotons.

The microscopic theory most widely- adopted was first described by Bogoliubov [3], who considered a model of a non-ideal Bose-gas at the absolute zero of temperature. In 1974, Bishop [4] examined the one-particle excitation spectrum at the condensation temperature T_c.

The dispersion curve of superfluid helium excitations has been measured accurately as a function of momentum [5]. At the lambda transition, these experiments show a sharp peak inelastic whose neutron scattering intensity is defined by the energy of the single particle excitations, and there is appearing a broad component in the inelastic neutron scattering intensity, at higher momenta. To explain the appearance of a broad component in the inelastic neutron scattering intensity, the authors of papers [6–7] proposed the presence of collective modes in superfluid liquid ^{4}He, represented a density excitations. Thus the collective modes are represent as density quasiparticles [8]. Such density excitations and density quasiparticles appear because of the remaining density operator term that describes atoms above the condensate, a term which was neglected by Bogoliubov [3].

Previously, the authors of ref [9] discovered that, at the lambda transition, there was scattering between atoms of the superfluid liquid helium, which is confirmed by the calculation of the dependence of the critical temperature on the interaction parameter, here the scattering length. On other hand, as we have noted, there are two types of excitation in superfluid helium at lambda transition point [5]. This means it is necessary to revise the conditions that determine the Bose-Einstein condensation in the superfluid liquid helium. Obviously, the peak inelastic neutron scattering intensity is connected with the registration of neutron modes in a neutron-spectrometer which, in turn, defines the nature of the excitations. So we may conclude that the registration of single neutron modes or neutron pair modes occurs at the lambda transition, from the neutron-spectrometer.

In this letter, we proposed new model for Bose-gas by extending the concept of a broken Bose-symmetry law for bosons in the condensate within applying the Penrose-Onsager definition of the Bose condensation [10]. After, we show that the interaction term between Boson modes and Fermion density modes is meditated by an effective attractive interaction between the Fermion modes, which in turn determines a bound state of singlet Fermion pair in a superfluid Bose liquid- Fermion gas mixture.

We investigate the problem of superconductivity presented by Frölich [11]. Hence, we also remark the theory of superconductivity, presented by Bardeen, Cooper and Schrieffer [12], and by Bogoliubov [13] (BCSB). They asserted that the Frölich effective attractive potential between electrons leads to shaping of two electrons with opposite spins around Fermi level into the Cooper pairs [14]. However, we demonstrate the term of the interaction between electrons and ions of lattice meditates the existence of the Frölich singlet electron pairs.

2 New model of a superfluid liquid helium

First, we present new model of a dilute Bose gas with strongly interactions between the atoms, to describe the superfluid liquid helium. This model considers a system of N identical interacting atoms via S-wave scattering. These atoms, as spinless Bose-particles, have a mass m and are confined to a box of volume V. The main part of the Hamiltonian of such system is expressed in the second quantization form as:

$$\hat{H}_a = \sum_{\vec{p}\neq 0} \frac{p^2}{2m} \hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} + \frac{1}{2V} \sum_{\vec{p}\neq 0} U_{\vec{p}} \hat{\varrho}_{\vec{p}} \hat{\varrho}^+_{\vec{p}} . \quad (1)$$

Here $\hat{a}^+_{\vec{p}}$ and $\hat{a}_{\vec{p}}$ are, respectively, the “creation” and “annihilation” operators of a free atoms with momentum $\vec{p}$; $U_{\vec{p}}$ is the Fourier transform of a S-wave pseudopotential in the momentum space:

$$U_{\vec{p}} = \frac{4\pi d\hbar^2}{m}\,, \tag{2}$$

where d is the scattering amplitude; and the Fourier component of the density operator presents as

$$\hat{\varrho}_{\vec{p}} = \sum_{\vec{p}_1} \hat{a}^+_{\vec{p}_1-\vec{p}}\,\hat{a}_{\vec{p}_1}\,. \tag{3}$$

According to the Bogoliubov theory [3], it is necessary to separate the atoms in the condensate from those atoms filling states above the condensate. In this respect, the operators $\hat{a}_0$ and $\hat{a}^+_0$ are replaced by c-numbers $\hat{a}_0 = \hat{a}^+_0 = \sqrt{N_0}$ within the approximation of the presence of a macroscopic number of condensate atoms $N_0 \gg 1$. This assumption leads to a broken Bose-symmetry law for atoms in the condensate state. To extend the concept of a broken Bose-symmetry law for bosons in the condensate, we apply the Penrose-Onsager definition of Bose condensation [10]:

$$\lim_{N_0,N\to\infty} \frac{N_0}{N} = const. \tag{4}$$

This reasoning is a very important factor in the microscopic investigation of the model non-ideal Bose gas because the presence of a macroscopic number of atoms in the condensate means new excitations in the model Bose-gas for superfluid liquid helium:

$$\frac{N_{\vec{p}\neq 0}}{N_0} = \alpha \ll 1\,,$$

where $N_{\vec{p}\neq 0}$ is the occupation number of atoms in the quantum levels above the condensate; α is the small number. Obviously, conservation of the total number of atoms suggests that the number of the Bose-condensed atoms N_0 essentially deviates from the total number N:

$$N_0 + \sum_{\vec{p}\neq 0} N_{\vec{p}\neq 0} = N\,,$$

which is satisfied for the present model. In this context,

$$\alpha = \frac{N - N_0}{N_0 \sum_{\vec{p}\neq 0} 1} \to 0\,,$$

where $\sum_{\vec{p}\neq 0} 1 \to \infty$.

For futher calculations, we replace the initial assumptions of our model by the approximation

$$\lim_{N_0\to\infty} \frac{N_{\vec{p}}}{N_0} \approx \delta_{\vec{p},0} \tag{5}$$

The next step is to find the property of operators $\frac{\hat{a}^+_{\vec{p}_1-\vec{p}}}{\sqrt{N_0}}$, $\frac{\hat{a}_{\vec{p}_1-\vec{p}}}{\sqrt{N_0}}$ by applying (5). Obviously,

$$\lim_{N_0\to\infty} \frac{\hat{a}^+_{\vec{p}_1-\vec{p}}}{\sqrt{N_0}} = \delta_{\vec{p}_1,\vec{p}} \tag{6}$$

and

$$\lim_{N_0\to\infty} \frac{\hat{a}_{\vec{p}_1-\vec{p}}}{\sqrt{N_0}} = \delta_{\vec{p}_1,\vec{p}}\,. \tag{7}$$

Excluding the term $\vec{p}_1 = 0$, the density operators of bosons $\hat{\varrho}_{\vec{p}}$ and $\hat{\varrho}^+_{\vec{p}}$ take the following forms:

$$\hat{\varrho}_{\vec{p}} = \sqrt{N_0}\,\left(\hat{a}^+_{-\vec{p}} + \sqrt{2}\,\hat{c}_{\vec{p}}\right) \tag{8}$$

and

$$\hat{\varrho}^+_{\vec{p}} = \sqrt{N_0}\,\left(\hat{a}_{-\vec{p}} + \sqrt{2}\,\hat{c}^+_{\vec{p}}\right) \tag{9}$$

where $\hat{c}_{\vec{p}}$ and $\hat{c}^+_{\vec{p}}$ are, respectively, the Bose-operators of density-quasiparticles presented in reference [8], which in turn are the Bose-operators of bosons used in expressions (6) and (7):

$$\hat{c}_{\vec{p}} = \frac{1}{\sqrt{2N_0}} \sum_{\vec{p}_1\neq 0} \hat{a}^+_{\vec{p}_1-\vec{p}}\,\hat{a}_{\vec{p}_1} = \frac{1}{\sqrt{2}} \sum_{\vec{p}_1\neq 0} \delta_{\vec{p}_1,\vec{p}}\,\hat{a}_{\vec{p}_1} = \frac{\hat{a}_{\vec{p}}}{\sqrt{2}} \tag{10}$$

and

$$\hat{c}^+_{\vec{p}} = \frac{1}{\sqrt{2N_0}} \sum_{\vec{p}_1\neq 0} \hat{a}^+_{\vec{p}_1}\,\hat{a}_{\vec{p}_1-\vec{p}} = \frac{1}{\sqrt{2}} \sum_{\vec{p}_1\neq 0} \delta_{\vec{p}_1,\vec{p}}\,\hat{a}^+_{\vec{p}_1} = \frac{\hat{a}^+_{\vec{p}}}{\sqrt{2}}\,. \tag{11}$$

Thus, we reach to the density operators of atoms $\hat{\varrho}_{\vec{p}}$ and $\hat{\varrho}^+_{\vec{p}}$, presented by Bogoliubov [3], at approximation $\frac{N_0}{N}$=*const*, which describes the gas of atoms ^{4}He with strongly interaction via S-wave scattering:

$$\hat{\varrho}_{\vec{p}} = \sqrt{N_0}\,\left(\hat{a}^+_{-\vec{p}} + \hat{a}_{\vec{p}}\right) \tag{12}$$

and

$$\hat{\varrho}^+_{\vec{p}} = \sqrt{N_0}\,\left(\hat{a}_{-\vec{p}} + \hat{a}^+_{\vec{p}}\right) \tag{13}$$

which shows that the density quasiparticles are absent.

The identical picture is observed in the case of the density excitations, as predicted by Glyde, Griffin and Stirling [5–7] proposing $\hat{\varrho}_{\vec{p}}$ in the following form:

$$\hat{\varrho}_{\vec{p}} = \sqrt{N_0}\,\left(\hat{a}^+_{-\vec{p}} + \hat{a}_{\vec{p}} + \tilde{\varrho}_{\vec{p}}\right) \tag{14}$$

where terms involving $\vec{p}_1 \neq 0$ and , $\vec{p}_1 \neq \vec{p}$ are written separately; and the operator $\tilde{\varrho}_{\vec{p}}$ describes the density-excitations:

$$\tilde{\varrho}_{\vec{p}} = \frac{1}{\sqrt{N_0}} \sum_{\vec{p}_1\neq 0,\vec{p}_1\neq\vec{p}} \hat{a}^+_{\vec{p}_1-\vec{p}}\,\hat{a}_{\vec{p}_1}\,. \tag{15}$$

After inserting (6) and (7) into (15), the term, representing the density-excitations vanishes because $\tilde{\varrho}_{\vec{p}} = 0$.

Consequently, the Hamiltonian of system, presented in (1) with also (12) and (13), represents an extension of the Bogoliubov Hamiltonian, with the approximation $\frac{N_0}{N}$ = *const*, which in turn does not depend on the actual amplitude of interaction. In the case of strongly interacting atoms, the Hamiltonian takes the following form:

$$\hat{H}_a = \sum_{\vec{p}\neq 0} \left(\frac{p^2}{2m} + mv^2\right) \hat{a}^+_{\vec{p}}\,\hat{a}_{\vec{p}} + \frac{mv^2}{2} \sum_{\vec{p}\neq 0} \left(\hat{a}^+_{-\vec{p}}\,\hat{a}^+_{\vec{p}} + \hat{a}_{\vec{p}}\,\hat{a}_{-\vec{p}}\right), \tag{16}$$

where $v = \sqrt{\frac{U_{\vec{p}} N_0}{mV}} = \sqrt{\frac{4\pi d\hbar^2 N_0}{m^2 V}}$ is the velocity of sound in the Bose gas, and which depends on the density atoms in the condensate $\frac{N_0}{V}$.

For the evolution of the energy level, it is a necessary to diagonalize the Hamiltonian $\hat{H}_a$ which is accomplished by introduction of the Bose-operators $\hat{b}^+_{\vec{p}}$ and $\hat{b}_{\vec{p}}$ by using of the Bogoliubov linear transformation [3]:

$$\hat{a}_{\vec{p}} = \frac{\hat{b}_{\vec{p}} + L_{\vec{p}}\,\hat{b}^+_{-\vec{p}}}{\sqrt{1 - L^2_{\vec{p}}}}\,, \tag{17}$$

where $L_{\vec{p}}$ is the unknown real symmetrical function of a momentum $\vec{p}$.

Substitution of (17) into (16) leads to

$$\hat{H}_a = \sum_{\vec{p}} \varepsilon_{\vec{p}}\,\hat{b}^+_{\vec{p}}\,\hat{b}_{\vec{p}} \tag{18}$$

hence we infer that $\hat{b}^+_{\vec{p}}$ and $\hat{b}_{\vec{p}}$ are the "creation" and "annihilation" operators of a Bogoliubov quasiparticles with energy:

$$\varepsilon_{\vec{p}} = \left[\left(\frac{p^2}{2m}\right)^2 + p^2 v^2\right]^{1/2}. \tag{19}$$

In this context, the real symmetrical function $L_{\vec{p}}$ of a momentum $\vec{p}$ is found

$$L^2_{\vec{p}} = \frac{\frac{p^2}{2m} + mv^2 - \varepsilon_{\vec{p}}}{\frac{p^2}{2m} + mv^2 + \varepsilon_{\vec{p}}}\,. \tag{20}$$

As is well known, the strong interaction between the helium atoms is very important and reduces the condensate fraction to 10 percent or $\frac{N_0}{N} = 0.1$ [5], at absolute zero. However, as we suggest, our model of dilute Bose gas may be valuable in describing thermodynamic properties of superfluid liquid helium, because the S-wave scattering between two atoms, with coordinates $\vec{r}_1$ and $\vec{r}_2$ in coordinate space, is represented by the repulsive potential delta-function $U_{\vec{r}} = \frac{4\pi d\hbar^2 \delta_{\vec{r}}}{m}$ from $\vec{r} = \vec{r}_1 - \vec{r}_2$. The model presented works on the condensed fraction $\frac{N_0}{N} \ll 1$ and differs from the Bogoliubov model where $\frac{N_0}{N} \approx 1$.

3 Formation singlet spinless neutron pairs

We now attempt to describe the thermodynamic property of a helium liquid-neutron gas mixture. In this context, we consider a neutron gas as an ideal Fermi gas consisting of n free neutrons with mass m_n which interact with N interacting atoms of a superfluid liquid helium. The helium-neutron mixture is confined in a box of volume V. The Hamiltonian of a considering system $\hat{H}_{a,n}$ consists of the term of the Hamiltonian of Bogoliubov excitations $\hat{H}_a$ in (18) and the term of the Hamiltonian of an ideal Fermi neutron gas as well as the term of interaction between the density of the Bogoliubov excitations and the density of the neutron modes:

$$\hat{H}_{a,n} = \sum_{\vec{p},\sigma} \frac{p^2}{2m_n}\,\hat{a}^+_{\vec{p},\sigma}\,\hat{a}_{\vec{p},\sigma} + \sum_{\vec{p}} \varepsilon_{\vec{p}}\,\hat{b}^+_{\vec{p}}\,\hat{b}_{\vec{p}} + \frac{1}{2V}\sum_{\vec{p}\neq 0} U_0\,\hat{\varrho}_{\vec{p}}\,\hat{\varrho}_{-\vec{p},n}\,, \tag{21}$$

where $\hat{a}^+_{\vec{p},\sigma}$ and $\hat{a}_{\vec{p},\sigma}$ are, respectively, the operators of creation and annihilation for free neutron with momentum $\vec{p}$, by the value of its spin z-component $\sigma =^+_- \frac{1}{2}$; U_0 is the Fourier transform of the repulsive interaction between the density of the Bogoliubov excitations and the density modes of the neutrons:

$$U_0 = \frac{4\pi d_0 \hbar^2}{\mu}\,, \tag{22}$$

where d_0 is the scattering amplitude between a helium atoms and neutrons; $\mu = \frac{m \cdot m_n}{m + m_n}$ is the relative mass.

Hence, we note that the Fermi operators $\hat{a}^+_{\vec{p},\sigma}$ and $\hat{a}_{\vec{p},\sigma}$ satisfy to the Fermi commutation relations $[\cdots]_+$ as:

$$\left[\hat{a}_{\vec{p},\sigma},\ \hat{a}^+_{\vec{p}',\sigma'}\right]_+ = \delta_{\vec{p},\vec{p}'}\ \delta_{\sigma,\sigma'}\,, \tag{23}$$

$$[\hat{a}_{\vec{p},\sigma}, \hat{a}_{\vec{p}',\sigma'}]_+ = 0\,, \tag{24}$$

$$[\hat{a}^+_{\vec{p},\sigma}, \hat{a}^+_{\vec{p}',\sigma'}]_+ = 0\,. \tag{25}$$

The density operator of neutrons with spin σ in momentum $\vec{p}$ is defined as

$$\hat{\varrho}_{\vec{p},n} = \sum_{\vec{p}_1,\sigma} \hat{a}^+_{\vec{p}_1-\vec{p},\sigma}\,\hat{a}_{\vec{p}_1,\sigma}\,, \tag{26}$$

where $\hat{\varrho}^+_{\vec{p},n} = \hat{\varrho}_{-\vec{p},n}$.

The operator of total number of neutrons is

$$\sum_{\vec{p},\sigma} \hat{a}^+_{\vec{p},\sigma}\,\hat{a}_{\vec{p},\sigma} = \hat{n}; \tag{27}$$

on other hand, the density operator, in the term of the Bogoliubov quasiparticles $\hat{\varrho}_{\vec{p}}$ included in (21), is expressed by following form, to application (17) into (12):

$$\hat{\varrho}_{\vec{p}} = \sqrt{N_0}\,\sqrt{\frac{1 + L_{\vec{p}}}{1 - L_{\vec{p}}}}\,\left(\hat{b}^+_{-\vec{p}} + \hat{b}_{\vec{p}}\right). \tag{28}$$

Hence, we note that the Bose- operator $\hat{b}_{\vec{p}}$ commutates with the Fermi operator $\hat{a}_{\vec{p},\sigma}$ because the Bogoliubov excitations and neutrons are an independent.

Now, inserting of a value of operator $\hat{\varrho}_{\vec{p}}$ from (28) into (21), which in turn leads to reducing the Hamiltonian of system $\hat{H}_{a,n}$:

$$\hat{H}_{a,n} = \sum_{\vec{p},\sigma} \frac{p^2}{2m_n}\,\hat{a}^+_{\vec{p},\sigma}\,\hat{a}_{\vec{p},\sigma} + \sum_{\vec{p}} \varepsilon_{\vec{p}}\,\hat{b}^+_{\vec{p}}\,\hat{b}_{\vec{p}} + \frac{U_0\sqrt{N_0}}{2V}\sum_{\vec{p}} \sqrt{\frac{1 + L_{\vec{p}}}{1 - L_{\vec{p}}}}\,\left(\hat{b}^+_{-\vec{p}} + \hat{b}_{\vec{p}}\right)\hat{\varrho}_{-\vec{p},n}\,. \tag{29}$$

Hence, we note that the Hamiltonian of system $\hat{H}_{a,n}$ in (29) is a similar to the Hamiltonian of system an electron gas-phonon gas mixture which was proposed by Frölich at solving of the problem superconductivity (please, see the Equation (16) in Frölich, *Proc. Roy. Soc. A*, 1952, v.215, 291–291 in the reference [11]), contains a subtle error in the term of the interaction between the density of phonon modes and the density of electron modes which represents a third term in right side of Equation (16) in [11] because the later is described by two sums, one from which goes by the wave vector $\vec{w}$ but other sum goes by the wave vector $\vec{k}$. This fact contradicts to the definition of the density operator of the electron modes $\hat{\varrho}_{\vec{w}}$ (please, see the Equation (12) in [11]) which in turn already contains the sum by the wave vector $\vec{k}$, and therefore, it is not a necessary to take into account so-called twice summations from $\vec{k}$ and $\vec{w}$ for describing of the term of the interaction between the density of phonon modes and the density of electron modes Thus, in the case of the Frölich, the sum must be taken only by wave vector w, due to definition of the density operator of electron modes with the momentum of phonon $\vec{w}$.

To allocate anomalous term in the Hamiltonian of system $\hat{H}_{a,n}$, which denotes by third term in right side in (29), we apply the Frölich approach [11] which allows to do a canonical transformation for the operator $\hat{H}_{a,n}$ within introducing an operator $\tilde{H}$:

$$\tilde{H} = \exp\left(\hat{S}^{+}\right)\, \hat{H}_{a,n} \exp\left(\hat{S}\right), \tag{30}$$

which is decayed by following terms:

$$\begin{aligned}\tilde{H} &= \exp\left(\hat{S}^{+}\right)\, \hat{H}_{a,n} \exp\left(\hat{S}\right) = \\ &= \hat{H}_{a,n} - \left[\hat{S}, \hat{H}_{a,n}\right] + \frac{1}{2}\left[\hat{S}, \left[\hat{S}, \hat{H}_{a,n}\right]\right] - \cdots,\end{aligned} \tag{31}$$

where the operators represent as:

$$\hat{S}^{+} = \sum_{\vec{p}} \hat{S}^{+}_{\vec{p}} \tag{32}$$

and

$$\hat{S} = \sum_{\vec{p}} \hat{S}_{\vec{p}} \tag{33}$$

and satisfy to a condition $\hat{S}^{+} = -\hat{S}$.

In this respect, we assume that

$$\hat{S}_{\vec{p}} = A_{\vec{p}}\left(\hat{\varrho}_{\vec{p},n}\hat{b}_{\vec{p}} - \hat{\varrho}^{+}_{\vec{p},n}\hat{b}^{+}_{\vec{p}}\right), \tag{34}$$

where $A_{\vec{p}}$ is the unknown real symmetrical function from a momentum $\vec{p}$. In this context, at application $\hat{S}_{\vec{p}}$ from (34) to (33) with taking into account $\hat{\varrho}^{+}_{-\vec{p},n} = \hat{\varrho}_{\vec{p},n}$, then we obtain

$$\hat{S} = \sum_{\vec{p}} \hat{S}_{\vec{p}} = \sum_{\vec{p}} A_{\vec{p}}\, \hat{\varrho}_{\vec{p},n}\left(\hat{b}_{=\vec{p}} - \hat{b}^{+}_{\vec{p}}\right). \tag{35}$$

In analogy manner, at $\hat{\varrho}^{+}_{-\vec{p},n} = \hat{\varrho}_{\vec{p},n}$, we have

$$\begin{aligned}\hat{S}^{+} &= \textstyle\sum_{\vec{p}} \hat{S}^{+}_{\vec{p}} = \sum_{\vec{p}} A_{\vec{p}}\, \hat{\varrho}^{+}_{\vec{p},n}\left(\hat{b}^{+}_{\vec{p}} - \hat{b}_{-\vec{p}}\right) = \\ &= -\sum_{\vec{p}} A_{\vec{p}}\, \hat{\varrho}_{\vec{p},n}\left(\hat{b}_{-\vec{p}} - \hat{b}^{+}_{\vec{p}}\right).\end{aligned} \tag{36}$$

To find $A_{\vec{p}}$, we substitute (29), (35) and (36) into (31). Then,

$$\begin{aligned}\left[\hat{S}, \hat{H}_{a,n}\right] &= \tfrac{1}{V}\textstyle\sum_{\vec{p}} A_{\vec{p}}\, U_0\sqrt{N_0}\sqrt{\tfrac{1+L_{\vec{p}}}{1-L_{\vec{p}}}}\, \hat{\varrho}_{\vec{p},n}\, \hat{\varrho}_{-\vec{p},n} + \\ &+ \sum_{\vec{p}} A_{\vec{p}}\, \varepsilon_{\vec{p}}\left(\hat{b}^{+}_{\vec{p}} + \hat{b}_{-\vec{p}}\right)\hat{\varrho}_{-\vec{p},n},\end{aligned} \tag{37}$$

$$\frac{1}{2}\left[\hat{S}, \left[\hat{S}, \hat{H}_{a,n}\right]\right] = \sum_{\vec{p}} A^2_{\vec{p}}\, \varepsilon_{\vec{p}}\, \varrho_{\vec{p},n}\, \hat{\varrho}_{-\vec{p},n} \tag{38}$$

and $[\hat{S}, [\hat{S}, [\hat{S}, \hat{H}_{a,n}]]] = 0$ within application a Bose commutation relations as $[\varrho_{\vec{p}_1,n}, \hat{\varrho}_{\vec{p}_2,n}] = 0$ and $[\hat{a}^{+}_{\vec{p}_1,\sigma}\hat{a}_{\vec{p}_1,\sigma}, \hat{\varrho}_{\vec{p}_2,n}] = 0$.

Thus, the form of new operator $\tilde{H}$ in (31) takes a following form:

$$\begin{aligned}\tilde{H} &= \textstyle\sum_{\vec{p}} \varepsilon_{\vec{p}}\, \hat{b}^{+}_{\vec{p}}\, \hat{b}_{\vec{p}} + \sum_{\vec{p},\sigma} \frac{p^2}{2m_n}\, \hat{a}^{+}_{\vec{p},\sigma}\, \hat{a}_{\vec{p},\sigma} + \\ &+ \frac{1}{2V}\sum_{\vec{p}} U_0\sqrt{N_0}\sqrt{\frac{1+L_{\vec{p}}}{1-L_{\vec{p}}}}\left(\hat{b}^{+}_{-\vec{p}} + \hat{b}_{\vec{p}}\right)\hat{\varrho}_{-\vec{p},n} - \\ &- \sum_{\vec{p}} A_{\vec{p}}\, \varepsilon_{\vec{p}}\left(\hat{b}^{+}_{-\vec{p}} + \hat{b}_{\vec{p}}\right)\hat{\varrho}_{-\vec{p},n} + \sum_{\vec{p}} A^2_{\vec{p}}\, \varepsilon_{\vec{p}}\, \hat{\varrho}_{\vec{p},n}\, \hat{\varrho}_{-\vec{p},n} - \\ &- \frac{1}{V}\sum_{\vec{p}} A_{\vec{p}} U_0\sqrt{N_0}\sqrt{\frac{1+L_{\vec{p}}}{1-L_{\vec{p}}}}\, \hat{\varrho}_{\vec{p},n}\, \hat{\varrho}_{-\vec{p},n}.\end{aligned} \tag{39}$$

The transformation of the term of the interaction between the density of the Bogoliubov modes and the density neutron modes is made by removing of a second and fifth terms in right side of (39) which leads to obtaining of a quantity for $A_{\vec{p}}$:

$$A_{\vec{p}} = \frac{U_0\sqrt{N_0}}{2\varepsilon_{\vec{p}} V}\sqrt{\frac{1+L_{\vec{p}}}{1-L_{\vec{p}}}}. \tag{40}$$

In this respect, we reach to reducing of the new Hamiltonian of system (39):

$$\begin{aligned}\tilde{H} &= \textstyle\sum_{\vec{p}} \varepsilon_{\vec{p}}\, \hat{b}^{+}_{\vec{p}}\, \hat{b}_{\vec{p}} + \sum_{\vec{p},\sigma} \frac{p^2}{2m_n}\, \hat{a}^{+}_{\vec{p},\sigma}\hat{a}_{\vec{p},\sigma} - \\ &- \frac{1}{V}\sum_{\vec{p}} A_{\vec{p}} U_0\sqrt{N_0}\sqrt{\frac{1+L_{\vec{p}}}{1-L_{\vec{p}}}}\, \hat{\varrho}_{\vec{p},n}\, \hat{\varrho}_{-\vec{p},n} + \\ &+ \sum_{\vec{p}} A^2_{\vec{p}}\, \varepsilon_{\vec{p}}\, \hat{\varrho}_{\vec{p},n}\, \hat{\varrho}_{-\vec{p},n}.\end{aligned} \tag{41}$$

As result, the new form of Hamiltonian system takes a following form:

$$\tilde{H} = \sum_{\vec{p}} \varepsilon_{\vec{p}}\, \hat{b}^{+}_{\vec{p}}\, \hat{b}_{\vec{p}} + \hat{H}_n, \tag{42}$$

where $\hat{H}_n$ is the effective Hamiltonian of a neutron gas which contains an effective interaction between neutron modes:

$$\hat{H}_n = \sum_{\vec{p},\sigma} \frac{p^2}{2m_n} \hat{a}^+_{\vec{p},\sigma} \hat{a}_{\vec{p},\sigma} + \frac{1}{2V} \sum_{\vec{p}} V_{\vec{p}} \hat{\varrho}_{\vec{p},n} \hat{\varrho}_{-\vec{p},n} , \tag{43}$$

where $V_{\vec{p}}$ is the effective potential of the interaction between neutron modes which takes a following form at substituting a value of $A_{\vec{p}}$ from (40) into (41):

$$V_{\vec{p}} = -2A_{\vec{p}} U_0 \sqrt{N_0} \sqrt{\frac{1+L_{\vec{p}}}{1-L_{\vec{p}}}} + 2A^2_{\vec{p}} \varepsilon_{\vec{p}} V = = -\frac{U_0^2 N_0 \left(1+L_{\vec{p}}\right)}{V\varepsilon_{\vec{p}}\left(1-L_{\vec{p}}\right)} . \tag{44}$$

In this letter, we consider following cases:

1. At low momenta atoms of a helium $p \ll 2mv$, the Bogoliunov's quasiparticles in (19) represent as the phonons with energy $\varepsilon_{\vec{p}} \approx pv$ which in turn defines a value $L^2_{\vec{p}} \approx \frac{1-\frac{p}{mv}}{1+\frac{p}{mv}} \approx \left(1-\frac{p}{mv}\right)^2$ in (20) or $L_{\vec{p}} \approx 1-\frac{p}{mv}$. In this context, the effective potential between neutron modes takes a following form:

$$V_{\vec{p}} \approx -\frac{2mU_0^2 N_0}{Vp^2} = -\frac{4\pi\hbar^2 e_1^2}{p^2} . \tag{45}$$

The value e_1 is the effective charge, at a small momenta of atoms:

$$e_1 = \frac{U_0}{\hbar}\sqrt{\frac{mN_0}{2V\pi}} .$$

2. At high momenta atoms of a helium $p \gg 2mv$, we obtain $\varepsilon_{\vec{p}} \approx \frac{p^2}{2m} + mv^2$ in (19) which in turn defines $L_{\vec{p}} \approx 0$ in (20). Then, the effective potential between neutron modes presents as:

$$V_{\vec{p}} \approx -\frac{mU_0^2 N_0}{Vp^2} = -\frac{4\pi\hbar^2 e_2^2}{p^2} , \tag{46}$$

where e_2 is the effective charge, at high momenta of atoms:

$$e_2 = \frac{U_0}{2\hbar}\sqrt{\frac{mN_0}{V\pi}} .$$

Consequently, in both cases, the effective scattering between two neutrons is presented in the coordinate space by a following form:

$$V(\vec{r}) = \frac{1}{V}\sum_{\vec{p}} V_{\vec{p}}\, e^{i\frac{\vec{p}\vec{r}}{\hbar}} = -\frac{e_*^2}{r} , \tag{47}$$

where $e_* = e_1$, at small momenta of atoms; and $e_* = e_2$, at high momenta.

The term of the interaction between two neutrons $V(\vec{r})$ in the coordinate space mediates the attractive Coulomb interaction between two charged particles with mass of neutron m_n, having the opposite effective charges e_* and $-e_*$, which together create a neutral system. Indeed, the effective Hamiltonian of a neutron gas in (43) is rewrite down in the space of coordinate by following form:

$$\hat{H}_n = \sum_{i=1}^{\frac{n}{2}} \hat{H}_i = -\frac{\hbar^2}{2m_n}\sum_{i=1}^{n}\Delta_i - \sum_{i<j}\frac{e_*^2}{\mid \vec{r}_i - \vec{r}_j \mid} , \tag{48}$$

where $\hat{H}_i$ is the Hamiltonian of system consisting two neutron with opposite spin which have a coordinates $\vec{r}_i$ and $\vec{r}_j$:

$$\hat{H}_i = -\frac{\hbar^2}{2m_n}\Delta_i - \frac{\hbar^2}{2m_n}\Delta_j - \frac{e_*^2}{\mid \vec{r}_i - \vec{r}_j \mid} . \tag{49}$$

The transformation of considering coordinate system to the relative coordinate $\vec{r} = \vec{r}_i - \vec{r}_j$ and the coordinate of center mass $\vec{R} = \frac{\vec{r}_i + \vec{r}_j}{2}$, we have

$$\hat{H}_i = -\frac{\hbar^2}{4m_n}\Delta_R - \frac{\hbar^2}{m_n}\Delta_r - \frac{e_*^2}{r} . \tag{50}$$

In analogy of the problem Hydrogen atom, two neutrons with opposite spins is bound as a spinless neutron pair with binding energy:

$$E_n = -\frac{m_n e_*^4}{4\hbar^2 n^2} = -\frac{const}{n^2}\left(\frac{N_0}{V}\right)^2 , \tag{51}$$

where n is the main quantum number which determines a bound state on a neutron pair, at $const > 0$.

Thus, a spinless neutron pair with mass $m_0 = 2m_n$ is created in a helium liquid-dilute neutron gas mixture.

4 Formation of the Frölich electron pairs in superconductivity

We now attempt to describe the thermodynamic property of the model a phonon-electron gas mixture confined in a box of volume V. In this context, we consider an electron gas consisting of n free electrons with mass m_e which interact with phonon modes of lattice by constancy interaction [11]. The Frölich Hamiltonian has a following form:

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_2 \tag{52}$$

with

$$\hat{H}_0 = \sum_{\vec{k},\sigma} \varepsilon_{\vec{k}}\, \hat{d}^+_{\vec{k},\sigma} \hat{d}_{\vec{k},\sigma} , \tag{53}$$

$$\hat{H}_1 = \sum_{\vec{w}} \hbar w s\, \hat{b}^+_{\vec{w}} \hat{b}_{\vec{w}} , \tag{54}$$

$$\hat{H}_2 = i\sum_{\vec{w}} D_w \left(\hat{b}_{\vec{w}} \hat{\varrho}^+_{\vec{w}} - \hat{b}^+_{\vec{w}} \hat{\varrho}_{\vec{w}}\right) , \tag{55}$$

where $\hat{d}^+_{\vec{k},\sigma}$ and $\hat{d}_{\vec{k},\sigma}$ are, respectively, the Fermi operators of creation and annihilation for free electron with wave-vector

$\vec{k}$ and energy $\varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m_e}$, by the value of its spin z-component $\sigma =^{+}_{-} \frac{1}{2}$; s is the velocity of phonon; $\hat{b}^{+}_{\vec{w},\sigma}$ and $\hat{b}_{\vec{w},\sigma}$ are, respectively, the Bose operators of creation and annihilation for free phonon with wave-vector $\vec{w}$ and energy$\hbar ws$; D_w is the constant of the interaction between the density of the phonon excitations and the density modes of the electrons which equals to $D_w = \sqrt{\frac{\alpha\hbar ws}{V}}$ (where $\alpha = \frac{C^{"2}}{2Ms^2 \frac{n}{V}}$ is the constant characterizing of the metal; $C^{"}$ is the constant of the interaction; M is the mass of ion); $\hat{\varrho}_{\vec{w}}$ is the density operator of the electron modes with wave vector $\vec{w}$ which is defined as:

$$\hat{\varrho}_{\vec{w}} = \sum_{\vec{k},\sigma} \hat{d}^{+}_{\vec{k}-\vec{w},\sigma} \hat{d}_{\vec{k},\sigma} \tag{56}$$

and

$$\hat{\varrho}^{+}_{\vec{w}} = \sum_{\vec{k},\sigma} \hat{d}^{+}_{\vec{k},\sigma} \hat{d}_{\vec{k}-\vec{w},\sigma}\,, \tag{57}$$

where $\hat{\varrho}^{+}_{\vec{w}} = \hat{\varrho}_{-\vec{w}}$.

Hence, we note that the Fermi operators $\hat{d}^{+}_{\vec{k},\sigma}$ and $\hat{d}_{\vec{k},\sigma}$ satisfy to the Fermi commutation relations $[\cdots]_{+}$ presented in above for neutrons (23–25).

Obviously, the Bose- operator $\hat{b}_{\vec{w}}$ commutates with the Fermi operator $\hat{d}_{\vec{k},\sigma}$ because phonon excitations and electron modes are an independent.

Now, we introduce new transformation of the Bose-operators of phonon modes $\hat{b}^{+}_{\vec{w}}$ and $\hat{b}_{\vec{w}}$ by the new Bose - operators of phonon excitations $\hat{c}^{+}_{\vec{w}}$ and $\hat{c}_{\vec{w}}$ which help us to remove an anomalous term:

$$\hat{b}_{\vec{w}} = -i\hat{c}_{\vec{w}} \tag{58}$$

and

$$\hat{b}^{+}_{\vec{w}} = i\hat{c}^{+}_{\vec{w}}\,. \tag{59}$$

Then, $\hat{H}_1$ in (56) and $\hat{H}_2$ in (57) take following forms:

$$\hat{H}_1 = \sum_{\vec{w}} \hbar ws\, \hat{c}^{+}_{\vec{w}} \hat{c}_{\vec{w}}\,, \tag{60}$$

$$\hat{H}_2 = \sum_{\vec{w}} D_w \left(\hat{c}_{\vec{w}}\, \hat{\varrho}^{+}_{\vec{w}} + \hat{c}^{+}_{\vec{w}}\, \hat{\varrho}_{\vec{w}}\right) = \sum_{\vec{w}} D_w\, \hat{\varrho}_{\vec{w}} \left(\hat{c}_{\vec{w}} + \hat{c}^{+}_{\vec{w}}\right). \tag{61}$$

To allocate anomalous term in the Hamiltonian of system $\hat{H}$ in (54), presented by the term in (63), we use of the canonical transformation for the operator $\hat{H}$ presented by formulae (30). Due to this approach, we obtain new form for operator Hamiltonian $\tilde{H}$:

$$\tilde{H} = \sum_{\vec{w}} \hbar ws\, \hat{b}^{+}_{\vec{w}}\, \hat{b}_{\vec{w}} + \hat{H}_e\,, \tag{62}$$

where

$$\hat{H}_e = \sum_{\vec{k},\sigma} \varepsilon_{\vec{k}}\, \hat{d}^{+}_{\vec{k},\sigma} \hat{d}_{\vec{k},\sigma} + \frac{1}{2V} \sum_{\vec{w}} V_{\vec{w}}\, \hat{\varrho}_{\vec{w}}\, \hat{\varrho}_{-\vec{w}}\,, \tag{63}$$

hence $V_{\vec{w}}$ is the effective potential of the interaction between electron modes, which at taking into account $D_w = \sqrt{\frac{\alpha\hbar ws}{V}}$, has the form:

$$V_{\vec{w}} = -\frac{2D_w^2 V}{\hbar ws} = -2\alpha\,. \tag{64}$$

Consequently, the effective scattering between two electrons in the coordinate space takes a following form:

$$V(\vec{r}) = \frac{1}{V} \sum_{\vec{w}} V_{\vec{w}}\, e^{i\vec{w}\vec{r}} = -2\alpha\delta(\vec{r}) \tag{65}$$

at using of $\frac{1}{V}\sum_{\vec{w}} e^{i\vec{w}\vec{r}} = \delta(\vec{r})$.

Using of the relative coordinate $\vec{r} = \vec{r}_i - \vec{r}_j$ and the coordinate of center mass $\vec{R} = \frac{\vec{r}_i + \vec{r}_j}{2}$, we reach to the Hamiltonian of system consisting two electron with opposite spins:

$$\hat{H}_i = -\frac{\hbar^2}{4m_e}\,\Delta_R - \frac{\hbar^2}{m_e}\,\Delta_r + V(\vec{r})\,. \tag{66}$$

To find the binding energy $E < 0$ of electron pair, we search the solution of the Schrödinger equation with introduction of wave function $\psi(\vec{r})$:

$$\hat{H}_i\, \psi_s(\vec{r}) = E\psi_s(\vec{r})\,.$$

In this respect, we have a following equation

$$-\frac{\hbar^2}{m_e}\,\Delta_r \psi_s(\vec{r}) + V(\vec{r})\,\psi_s(\vec{r}) = E\psi(\vec{r}) \tag{67}$$

which may determine the binding energy $E < 0$ of electron pair, if we claim that the condition $\frac{p_f d}{\hbar} \ll 1$ always is fulfilled. This reasoning implies that the effective scattering between two electrons is presented by the coordinate space:

$$V(\vec{r}) = \frac{1}{V} \sum_{\vec{w}} V_{\vec{w}}\, e^{i\vec{w}\vec{r}} = 4\pi \int_0^{w_f} V_{\vec{w}}\, w^2\, \frac{\sin(wr)}{wr}\, dw\,, \tag{68}$$

where we introduce a following approximation as $\frac{sin(wr)}{wr} \approx 1 - \frac{w^2 r^2}{6}$ at conditions $w \le w_f$ and $w_f d \ll 1$ ($w_f = \left(\frac{3\pi^2 n}{V}\right)^{\frac{1}{3}}$ is the Fermi wave number). The later condition defines a state for distance r between two neighboring electrons which is a very small $r \ll \frac{1}{w_f} = \left(\frac{V}{3\pi^2 n}\right)^{\frac{1}{3}}$ where $\frac{4\pi w_f^3}{3} = \frac{n}{2V}$. Then,

$$V(\vec{r}) \approx -\frac{\alpha n}{V} + \alpha \left(\frac{n}{V}\right)^{\frac{5}{3}} r^2. \tag{69}$$

Thus, the effective interaction between electron modes $V(\vec{r}) = -2\alpha\delta(\vec{r})$, presented in (65) is replaced by a screening effective scattering presented by (69). This approximation means that there is an appearance of a screening character in the effective scattering because one depends on the density electron modes. Now, denoting $E = E_s$, and then, we arrive to an important equation for finding a binding energy E_s of singlet electron pair:

$$\left[-\frac{\hbar^2}{m_e}\Delta_r - \frac{n\alpha}{V} + \alpha\left(\frac{n}{V}\right)^{\frac{5}{3}} r^2\right]\psi_s(r) = E_s\,\psi_s(r)\,, \qquad (70)$$

which we may rewrite down as:

$$\frac{d^2\psi_s(r)}{dr^2} + \left(\lambda - \theta^2 r^2\right)\psi_s(r) = 0\,, \qquad (71)$$

where we take $\theta = -\sqrt{\frac{m_e\alpha}{\hbar^2}\left(\frac{n}{V}\right)^{\frac{5}{3}}}$, and $\lambda = \frac{m_e E_s}{\hbar^2} - \frac{\alpha m_e n}{\hbar^2 V}$.

Now, introducing the wave function $\psi_s(r)$ via the Chebishev-Hermit function $H_s(it)$ from an imaginary number as argument it [15] (where i is the imaginary one; t is the real number; $s = 0; 1; 2; \ldots$), the equation (71) has a following solution:

$$\psi_s(\vec{r}) = e^{-\theta \cdot r^2} H_s\left(\sqrt{\theta}\, r\right),$$

where

$$H_s(it) = i^s e^{-t^2}\frac{d^s e^{t^2}}{dt^s}$$

at $\theta < 0$, where

$$\lambda = \theta\left(s + \frac{1}{2}\right).$$

Consequently, the quantity of the binding energy E_s of electron pair with mass $m_0 = 2m_e$ takes a following form:

$$E_s = -\sqrt{\frac{\alpha\hbar^2}{m_e}\left(\frac{n}{V}\right)^{\frac{5}{3}}}\left(s + \frac{1}{2}\right) + \frac{\alpha n}{V} < 0 \qquad (72)$$

at $s = 0; 1; 2; \ldots$.

The normal state of electron pair corresponds to quantity $s = 0$ which defines maximal binding energy of electron pair:

$$E_0 = -\sqrt{\frac{\alpha\hbar^2}{m_e}\left(\frac{n}{V}\right)^{\frac{5}{3}}} + \frac{\alpha n}{V} < 0\,. \qquad (73)$$

This fact implies that the formation of the superconducting phase in superconductor is appeared by condition for density of metal $\frac{n}{V}$:

$$\frac{n}{V} > \left(\frac{C^2 m_e}{2Ms^2\hbar^2}\right)^{\frac{3}{2}}.$$

At choosing $C \approx 10$ eV [11]; $M \approx 5{\times}10^{-26}$ kg; $s \approx 3{\times}10^3$ m, we may estimate density of electron $\frac{n}{V} > 10^{27}$ m^{-3} which may represent as superconductor.

Acknowledgements

We thank Professor Marshall Stoneham for help with the English.

Submitted on April 12, 2010 / Accepted on April 19, 2010

References

1. London F. *Nature*, 1938, v. 141, 643.
2. Landau L. *Journal of Physics (USSR)*, 1941, v. 5, 77.
3. Bogoliubov N.N. On the theory of superfludity. *Journal of Physics (USSR)*, 1947, v. 11, 23.
4. Bishop R.F. *J. Low Temp. Physics*, 1974, v. 15, 601.
5. Blagoveshchenskii N.N. et al. *Physical Review B*, 1994, v. 50, 16550.
6. Glyde H.R., Griffin A. *Physical Review Letters*, 1990, v. 65, 1454.
7. Stirling W.G.,Griffin A., Glyde H.R. *Physical Review B*, 1990, v. 41, 4224.
8. Minasyan V.N. et al. *Physical Review Letters*, 2003, v. 90, 235301.
9. Morawetz K. et al. *Physical Review B*, 2007, v. 76, 075116.
10. Penrose O., Onsager L. *Physical Review*, 1956, v. 104, 576.
11. Frölich H. *Proc. Roy. Soc.*, 1952, v. A215, 576.
12. Bardeen J., Cooper L.N., and Schrieffer J.R. *Physical Review*, 1957, v. 108, 1175.
13. Bogoliubov N.N. *Nuovo Cimento*, 1958, v. 7, 794.
14. Cooper L.N. *Physical Review*, 1956, v. 104, 1189.
15. Lavrentiev N.A. and Shabat B.V. Methods of the theory of a function of a complex variable. Nauka, Moscow, 1972.