Thermodynamics and Scale Relativity

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It is shown how the fractal paths of SR = scale relativity (following Nottale) can be introduced into a TD = thermodynamic context (following Asadov-Kechkin).

1 Preliminary remarks

The SR program of Nottale et al (cf. [1]) has produced a marvelous structure for describing quantum phenomena on the QM type paths of Hausdorff dimension two (see below). Due to a standard Hamiltonian TD dictionary (cf. [2]) an extension to TD phenomena seems plausible. However among the various extensive and intensive variables of TD it seems unclear which to embelish with fractality. We avoid this feature by going to [3] which describes the arrow of time in connection with QM and gravity. This introduces a complex time (**1A**) $\tau = t - (i\hbar/2)\beta$ where $\beta = 1/kT$ with $k = k_B$ the Bolzmann constant and a complex Hamiltonian (**1B**) $\mathfrak{H} = \mathfrak{E} - (i\hbar\Gamma/2)$ where $\mathfrak{E}$ is a standard energy term, e.g. (**1C**) $\mathfrak{E} \sim (1/2)mv^2 + \mathfrak{W}(x)$. One recalls that complex time has appeared frequently in mathematical physics. We will show how fractality can be introduced into the equations of [3] without resorting to several complex variables or quaternions.

Thus from [3] one has equations

$$\mathfrak{H} = \mathfrak{E} - \left(\frac{i\hbar}{2}\Gamma\right);\ \tau = t - \frac{i\hbar}{2}\beta;\ [\mathfrak{E}, \Gamma] = [\mathfrak{H}, \mathfrak{H}^\dagger] = 0; \quad (1.1)$$

$$\Psi = exp^{-\frac{i\mathfrak{H}\tau}{\hbar}}\psi; P_n = \frac{w_n}{Z}; w_n = \rho_n exp^{[-\mathfrak{E}_n\beta + \Gamma_n t]};$$

$$i\hbar\partial_\tau\Psi = \mathfrak{H}\Psi; \Psi = \sum C_n\psi_n;$$

$$\mathfrak{H}_n = \mathfrak{E}_n - \frac{i\hbar}{2}\Gamma_n; [\mathfrak{H}, \mathfrak{H}^\dagger] = 0;$$

$$\mathfrak{E}\psi_n = \mathfrak{E}_n\psi_n; \Gamma\psi_n = \Gamma_n\psi_n; (\psi_n, \psi_k) = \delta_{nk}.$$

One could introduce another complex variable here, say j with $j^2 = -1$, but this can be avoided.

Now go to the SR theory and recall the equations

$$\frac{\hat{d}}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right); \quad (1.2)$$

$$\mathcal{V} = \frac{\hat{d}x}{dt} = V - iU = \frac{1}{2}(v_+ + v_-) - \frac{i}{2}(v_+ - v_-);$$

$$\frac{\hat{d}}{dt} = \partial_t + \mathcal{V}\cdot\nabla - i\mathcal{D}\Delta;$$

$$\mathcal{H} = \frac{m}{2}\mathcal{V}^2 - im\mathcal{D}\nabla\cdot\mathcal{V} + \mathcal{W} = \frac{1}{2m}\mathcal{P}^2 - i\mathcal{D}\cdot\mathcal{P} + \mathcal{W}; \quad (1.3)$$

$$\mathcal{H} = \mathcal{V}\cdot\mathcal{P} - i\mathcal{D}\nabla\cdot\mathcal{P} - \mathcal{L};$$

$$\hat{\mathcal{V}} = \mathcal{V} - i\mathcal{D}\nabla; \quad (\partial_t + \hat{\mathcal{V}}\cdot\nabla)\mathcal{V} = -\frac{\nabla\mathcal{W}}{m}; \quad (1.4)$$

$$U = \mathcal{D}\nabla log(P); \quad P = |\psi|^2; \quad \psi = e^{i\mathfrak{S}/2m\mathcal{D}};$$

$$\mathfrak{Q} = -2m\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}}; \quad (1.5)$$

$$\mathcal{V} = -2i\mathcal{D}\nabla[log(\psi)]; \quad \mathfrak{S}_0 = 2m\mathcal{D};$$

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi - \frac{\mathcal{W}}{2m}\psi = 0; \quad (1.6)$$

$$\frac{dV}{dt} = \frac{F}{m} = U\cdot\nabla U + \mathcal{D}\Delta U.$$

This has been written for 3 space dimensions but we will restrict attention to a 1-D space based on x below.

We will combine the ideas in (1.1) and (1.2) in Section 2 below. Note here $\mathfrak{Q}$ is the QP = quantum potential (see e.g. [5–8] for background).

2 Combination and interaction

From (1.2)-(1.6) we see that the fractal paths in one space dimension have Hausdorff dimension 2 and we note that U in (1.2) is related to an osmotic velocity and completely determines the QP $\mathfrak{Q}$. Note that these equations (1.2)-(1.6) produce a macro-quantum mechanics (where $\mathcal{D} = \hbar/2m$ for QM). It is known that a QP represents a stabilizing organizational anti-diffusion force which suggests an important connection between the fractal picture above and biological processes involving life (cf. [1, 9–13]). We also refer to [14–16] for probabalistic aspects of quantum mechanics and entropy and recommend a number of papers of Agop et al (cf. [17]) which deal with fractality (usually involving Hausdorff dimension 2 or 3) in differential equations such as Ginzburg-Landau, Korteweg de-Vries, and Navier-Stokes; this work includes some formulations in Weyl-Dirac geometry (Feoli-Gregorash-Papini-Wood formulation) involving superconductivity in a gravitational context.

Now let us imagine that $\mathcal{W} \sim \mathfrak{W}$ and $V \sim v$ so that the energy terms in the real part of the SE arising from (1.2)-(1.6) will take the form

$$\mathfrak{E} \sim \frac{1}{2}mV^2 + \mathfrak{W} + \mathfrak{Q} \tag{2.1}$$

and we identify this with $\mathfrak{E}$ in the TD problem where

$$\mathfrak{Q} = -2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}; \quad P = |\Psi|^2. \tag{2.2}$$

One arrives at QM for $\mathcal{D} = \hbar/2m$ as mentioned above and one notes that the mean value $\bar{\mathfrak{E}}$ used in the analysis of [3] will now have the form

$$\bar{\mathfrak{E}} = \frac{1}{2}\int mV^2 Pdx + \int |\mathfrak{W}|^2 Pdx + \int \mathfrak{Q}Pdx \tag{2.3}$$

and the last term $\int \mathfrak{Q}Pdx$ has a special meaning in terms of Fisher information as developed in [5–7, 19–21]. In fact one has

$$\int \mathfrak{Q}Pdx = -2m\mathcal{D}^2 \int \frac{\partial_x^2 \sqrt{P}}{\sqrt{P}} Pdx = \tag{2.4}$$

$$= -\frac{\mathcal{D}^2}{2}\int \left[\frac{2P''}{P} - \left(\frac{P'}{P}\right)^2\right] Pdx = \frac{m\mathcal{D}^2}{2}\int \frac{(P')^2}{P}dx$$

In the quantum situation $\mathcal{D} = \hbar/2m$ leading to

$$\int \mathfrak{Q}Pdx = \frac{\hbar^2}{8m}\int \frac{(P')^2}{P}dx = \frac{\hbar^2}{8m}FI \tag{2.5}$$

where FI denotes Fisher information (cf. [7, 21]). And this term can be construed as a contribution from fractality.

One can now sketch very briefly the treatment of [3] based on (1.1). Thus one constructs a generalized QM (with arrow of time and connections to gravity for which we refer to [3]). The eigenvalues $\mathfrak{E}_n$, Γ_n, in (1.1) are exploited with

$$\rho_n = |C_n|^2; \quad P_n = \frac{w_n}{Z};$$

$$\Psi = \sum C_n \psi_n; \quad w_n = \rho_n e^{-\mathfrak{E}_n\beta + \Gamma_n t}. \tag{2.6}$$

One considers two special systems:

1. First let the eigenvectors Γ_n all be the same (decay free system) and then $w_n = \rho_n exp[-\mathfrak{E}_n\beta]$ which means that β is actually the inverse absolute temperature (multiplied by k_B) when $\mathfrak{E}_n$ is identified with the n-th energy level and the system is decay free.
2. Next let all the $\mathfrak{E}_n$ be the same so $w_n = \rho_n exp[-\Gamma_n t]$ and all the Γ_N have the sense of decay parameters if t is the conventional physical time.

Thus the solution space of the theory space can be decomposed into the direct sum of subspaces which have a given value of the energy or of the decay parameter. It is seen that for $\beta = constant$ the dynamical equation for the basis probabilities is

$$\frac{dP_n}{dt} = -(\Gamma_n - \bar{\Gamma})P_n; \quad \frac{d\bar{\Gamma}}{dt} = -D_\Gamma^2; \quad D_\Gamma^2 = \overline{(\Gamma - \bar{\Gamma})^2}. \tag{2.7}$$

From (2.7) one sees that $\bar{\Gamma}(t)$ is not increasing which means that the isothermal regime of evolution has an arrow of time, which is related to the average value of the decay operator. Thus P_n increases if $\bar{\Gamma} > \Gamma_n$ and decreases when $\bar{\Gamma} < \Gamma_n$. One can also show that in the general case of $\beta = \beta(t)$ the dynamical equations for the P_n have the form

$$\frac{dP_n}{dt} = -\left[\Gamma_n - \bar{\Gamma} + (\mathfrak{E}_n - \bar{\mathfrak{E}})\frac{d\beta}{dt}\right]P_n. \tag{2.8}$$

Here the specific function $d\beta/dt$ must be specified or extracted from a regime condition $f(t, \beta, \bar{\mathfrak{A}}(t, \beta)) = 0$ for some observable $\mathfrak{A}$ (e.g. $\bar{\mathfrak{E}} = constant$ is an adiabatic condition). In the adiabatic case for example when $\bar{\mathfrak{E}} = \sum_n \mathfrak{E}_n P_n = constant$ there results

$$\frac{d\beta}{dt} = -\frac{\overline{\mathfrak{E}T} - \bar{\mathfrak{E}}\bar{T}}{D_\mathfrak{E}^2} \tag{2.9}$$

where $D_\mathfrak{E}$ denotes a dispersion of the energy operator $\mathfrak{E}$. Using (2.8)-(2.9) one obtains

$$\frac{d\bar{\Gamma}}{dt} = -D_\Gamma^2\left[1 - \frac{(\overline{\mathfrak{E}T} - \bar{\mathfrak{E}}\bar{T})^2}{D_\mathfrak{E}^2 D_\Gamma^2}\right] \geq 0. \tag{2.10}$$

Subsequently classical dynamics is considered for $\hbar \to 0$ and connections to gravity are indicated with kinematically independent geometric and thermal times (cf. [3]).

Submitted on October 12, 2011 / Accepted on October 23, 2011

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