Resonance and Fractals on the Real Numbers Set

Victor A. Panchelyuga, Maria S. Panchelyuga

Institute of Theoretical and Experimental Biophysics, Pushchino, Russia.
Research Institute of Hypercomplex Systems in Geometry and Physics, Friazino, Russia. E-mail: panvic333@yahoo.com

The paper shown that notions of resonance and roughness of real physical systems in applications to the real numbers set lead to existence of two complementary fractals on the sets of rational and irrational numbers accordingly. Also was shown that power of equivalence classes of rational numbers is connected with well known fact that resonance appear more easily for pairs of frequencies, which are small natural numbers.

1 Introduction

Well known that resonance is relation of two frequencies p and q, expressed by rational number $r \in \mathbb{Q}$:

$$r = \frac{p}{q}, \tag{1}$$

where $p, q \in \mathbb{N}$ and $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ is set of rational numbers. If r is irrational number, i.e. $r \in \mathbb{Q}^*$, where $\mathbb{Q}^*$ is set of irrational numbers, resonance is impossible.

Resonance definition as $r \in \mathbb{Q}$ leads to the next question. For real physical system p, q and, consequently, r cannot be a fixed number due to immanent fluctuations of the system. Consequently, condition $r \in \mathbb{Q}$ cannot be fulfilled all time because of irrational numbers, which fill densely neighborhood of any rational number. By these reasons, resonance condition $r \in \mathbb{Q}$ cannot be fulfilled and resonance must be impossible. But it is known that in reality resonance exists. The question is: in which way existence of resonance corresponds with it's definition as $r \in \mathbb{Q}$?

Also is known that resonance appear more easily for such $r \in \mathbb{Q}$ for which p and q are small numbers. As will be shown this experimental fact is closely connected with the question stated above.

2 Rational numbers distribution

The question stated above for the first time was considered by Kyril Dombrowski [1]. He suppose that despite the fact that rational numbers distributed densely along the number axis this distribution may be in some way non-uniform. In cited work K. Dombrowski used proposed by Khinchin [2] procedure of constructing of rational numbers set, based on the following continued fraction:

$$\left\{Q_i^{a_i}\right\} = \cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{\ddots\, a_i \pm \cfrac{1}{\dots}}}} \tag{2}$$

where $a_1, a_2, \dots, a_i = \overline{1, N}$, $i = \overline{1, N}$. Continued fraction (2) gives rational numbers, which belongs to interval $[0, 1]$. Is known that exists one-to-one correspondence between [0,1] and $[1, \infty)$ intervals. I.e., any regularities obtained from (2) on the interval $[0, 1]$ will be also true and for interval $[1, \infty)$.

In case $N \to \infty$ expression (2) leads to

$$\left\{Q_i^{a_i} \middle| N \to \infty\right\} \to \mathbb{Q}.$$

Apparently, in this case no distribution available, because rational numbers distributed along number axis densely.

For case of real physical system, condition $N \to \infty$ means that any parameters of the system must be defined with infinite accuracy. But in reality parameters values of the systems cannot be defined with such accuracy even if we have an ideal, infinite-accuracy measuring device. Such exact values simply don't exist because of quantum character of physical reality.

All this means that for considered physical phenomenon – resonance – we need to limit parameter i in (2) by some finite number N. Fig. 1 presents numerical simulation of (2) for the first two cases of finite N: $N = 1$, $N = 2$, and $N = 3$. In the case $N = 1$ (Fig. 1a) we have only one value $i = 1$, and from (2) we can obtain:

$$\left\{Q_1^{a_1}\right\} = \frac{1}{a_1}, \quad i = 1, \quad a_1 = \overline{1, \infty}. \tag{3}$$

In the case of $N = 2$, analogously:

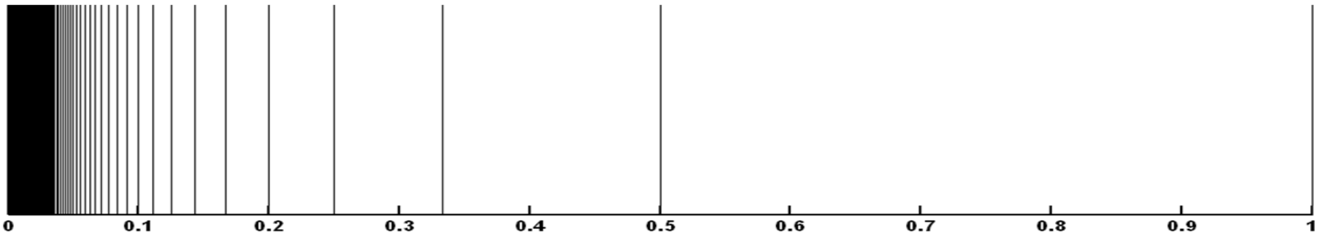
$$\left\{Q_i^{a_i}\right\} = \cfrac{1}{a_1 \pm \cfrac{1}{a_2}} = \frac{a_2}{a_1 a_2 \pm 1}, \quad i = 1, 2, \quad a_1, a_2 = \overline{1, \infty}. \tag{4}$$

For the case $N = 3$ we have

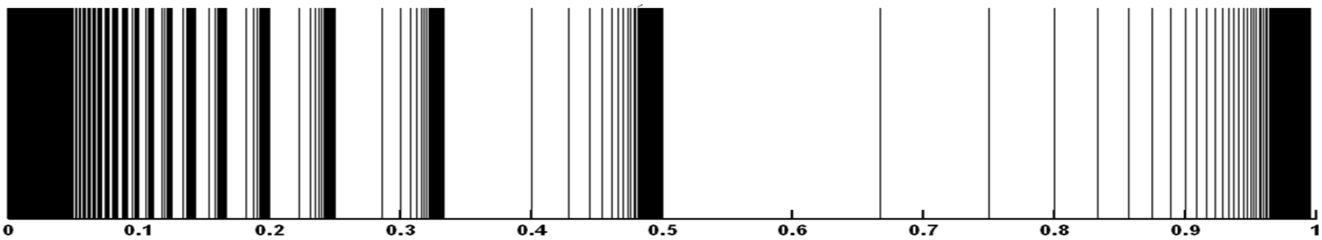
$$\left\{Q_i^{a_i}\right\} = \cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{a_3}}} = \frac{a_2 a_3 \pm 1}{a_1(a_2 a_3 \pm 1) \pm a_3}, \tag{5}$$

$$i = 1, 2, 3; \quad a_1, a_2, a_3 = \overline{1, \infty}.$$

It's easy to see that final set presented in Fig. 1c has a fractal character. Vicinity of every line in Fig. 1b is isomorphic to



a



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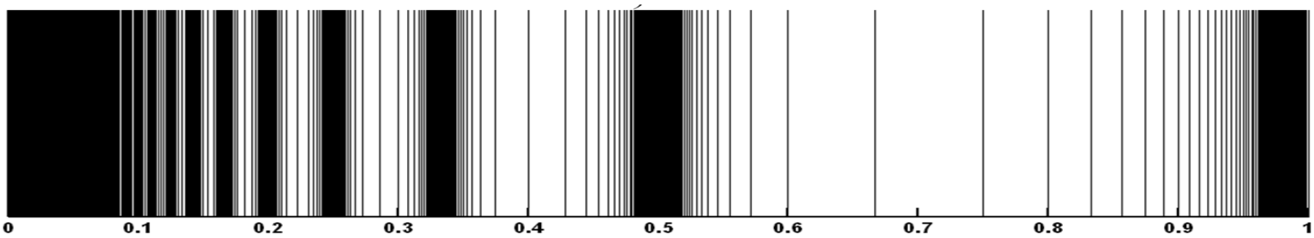

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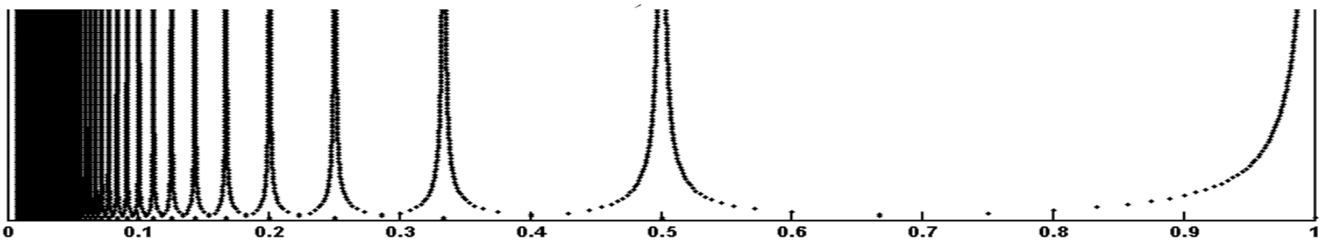

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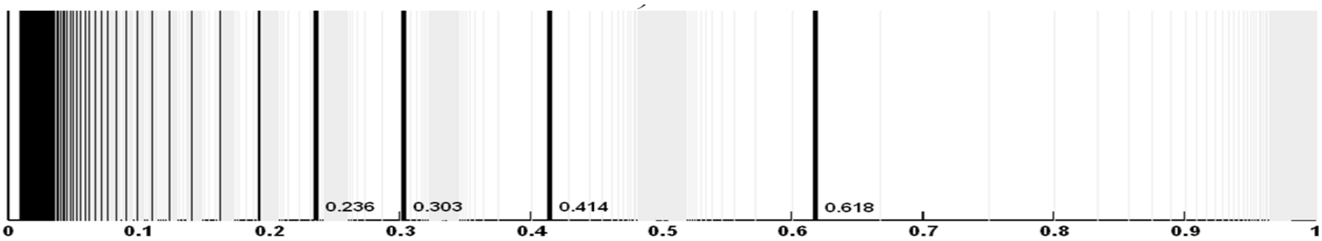

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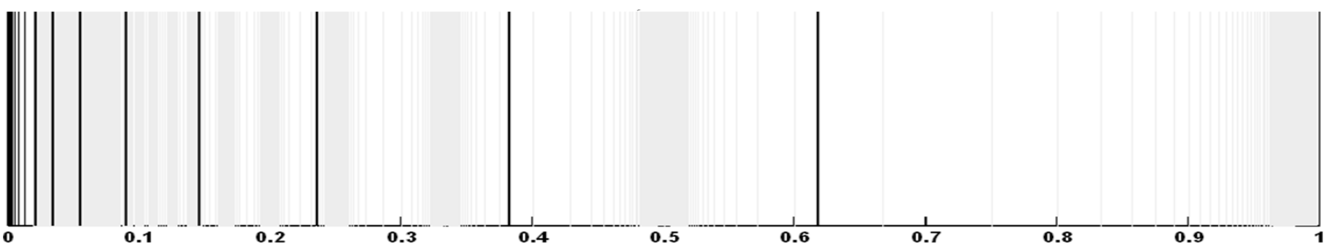


f

Fig. 1: Rational (a) – (d) and irrational (e) – (f) numbers distribution.

whole set in Fig. 1a. Consequently, vicinity of every line in Fig. 1c is isomorphic to whole set in Fig. 1b. Apparently that such regularity will be repeated on every next step of the algorithm and we can conclude that (2), in the case of $N \to \infty$, gives an example of mathematical fractal, which in the case of finite N gives an pre-fractal, which can be considered as physical fractal.

From Fig. 1c we can conclude that rational numbers for the case of finite N distributed along number axis inhomogeneously. This conclusion proves density distribution of rational numbers, constructed on the base of set presented in Fig. 1c, and given in Fig. 1d.

Summarizing, we can state that roughness of parameters of real physical system modeled by finite N in (2) leads to inhomogeous fractal distribution of rational numbers along number axis. As follows from Fig. 1d major maxima in the distribution defined by first steps of algorithm given in (3).

3 Equivalence classes of rational numbers and resonance

Expression (1) can be rewrite in terms of wavelength λ_p and λ_q, which corresponds to frequencies p and q:

$$r = \frac{p}{q} = \frac{\lambda_q}{\lambda_p}. \tag{6}$$

Suppose, that $\lambda_q > \lambda_p$. Then (6) means that wavelength λ_q is an integer part of λ_p. In this case resonance condition can be write in the form $\lambda_q \bmod \lambda_p = 0$, or in more general form:

$$n \bmod i = 0, \tag{7}$$

where $i, n \in \mathbb{N}$, $i, n = \overline{1, \infty}$. All i, which satisfy (7) gives integer divisors of natural number n. Fig. 2 gives graphical representation of numbers of integer divisors of n, obtained from (7).

Analogously to previous, roughness of physical system in the case of (7) can be modeled if instead of $n \to \infty$ will be used condition $n \to N$, where N is quite large, but finite natural number. In this case we can directly calculate power of equivalence classes of n, which belong to segment $[1, N]$. Result of the calculation for $N = 5000$ is given in Fig. 3.

As follows from Fig. 3a – b the power of equivalence classes is maximal only for first members of natural numbers axis.

From our point of view this result can explain the fact that resonance appears easier when p and q are small numbers. Really, for the larger power of equivalence classes exist the greater number of pairs p and q (different physical situations), which gives the same value of r, which finally make this resonance relation more easy to appear.

An interesting result, related to the power of equivalence classes, is presented in Fig. 4. This result for the first time was described, but not explained in [3]. In Fig. 4 are presented diagrams, obtained by means of the next procedure.

Number sequence, presented in Fig. 2, was divided onto equal Δn-points segments. In this way we obtain $\frac{N}{\Delta n}$ segments. The points in the segments was numerated from 1 to Δn. Finally all points with the same number in $\frac{N}{\Delta n}$ segments were summarized.

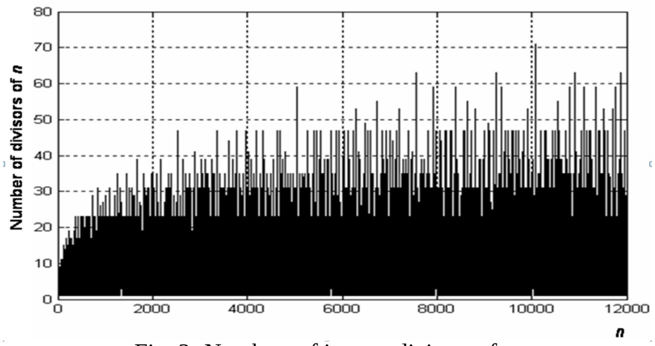
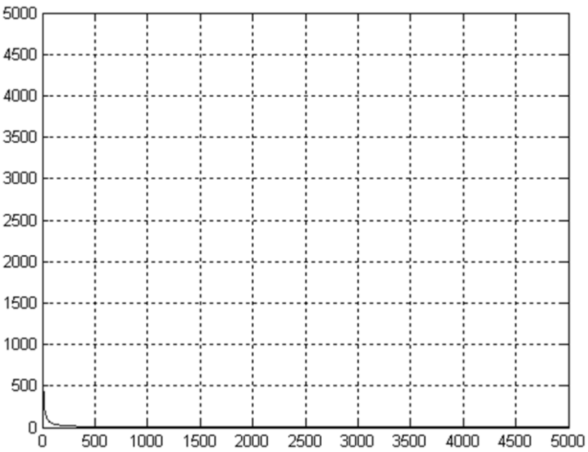
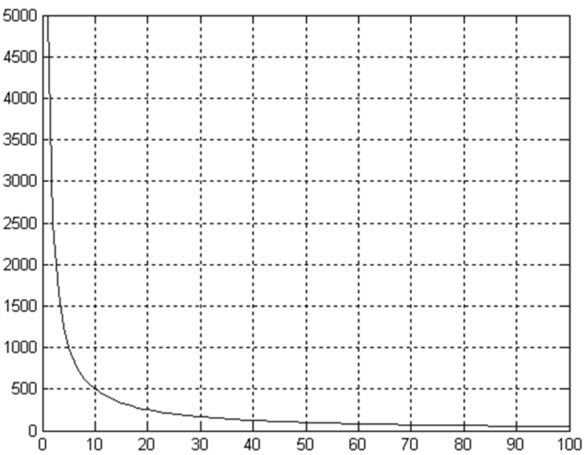


Fig. 2: Numbers of integer divisors of n.



(*a*)



(*b*)

Fig. 3: Power of equivalence classes for $N = 5000$, (a); magnified part of (a) for $N = 100$, (b). X-axis: value of N, Y-axis: power of equivalence classes.

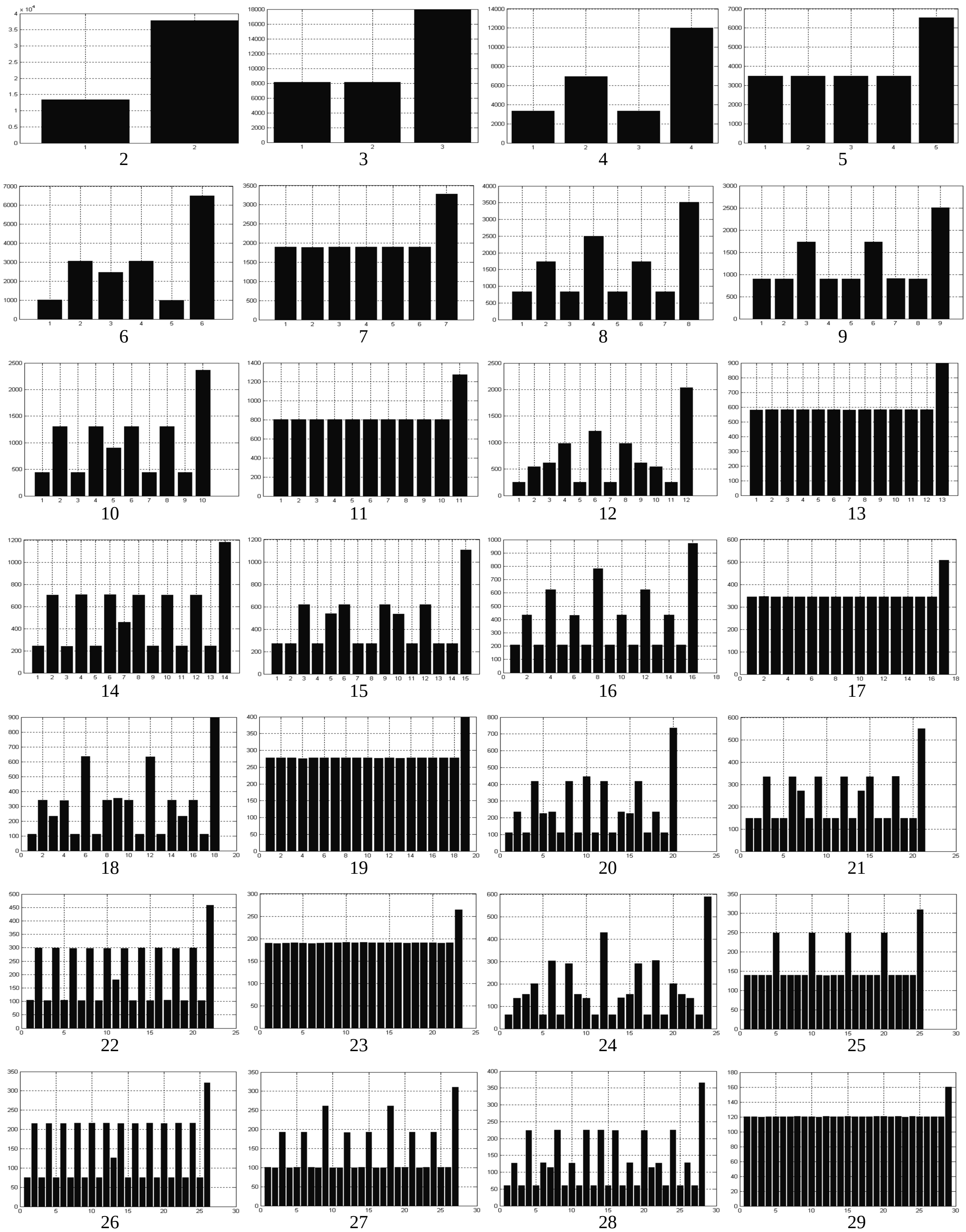


Fig. 4: Diagrams constructed on the base of sequence, presented in Fig. 2. The length of Δn-points segments pointed by number below the diagrams.

It can be seen from Fig. 4 that form of straight case when Δn is a prime number diagram always have a line. Otherwise presents some unique pattern. If we examine patterns, displayed in Fig. 4, we can find that in the role of buildings blocks, which define structure of the patterns with relatively big Δn, serve the patterns obtained for relatively small Δn. The patterns with small Δn based on numbers with greater power of equivalence classes and therefore manifests itself trough summarizing process in contradiction from relatively big values of Δn.

4 On irrational numbers distribution

Presented in Fig. 1c – d rational numbers distribution displays some rational maxima. Existence of such maxima means that in the case of rational relations, which correspond to the maxima, resonance will appear more easy and interaction between different parts of considered physical system will be more strong. If parameters of the system correspond to the maxima, such system becomes unstable, because of interaction, which is maximal for this case.

Analogously to rational maxima is interesting to consider existence of irrational maxima, which in opposition to rational one, must correspond to minimal interaction between parts of the system and to its maximal stability. Work [1] suppose that irrational maxima correspond to minima in rational numbers distribution. In the role of "the most irrational numbers" was proposed algebraic numbers, which are roots of equation

$$\alpha^2 + \alpha b + c = 0. \tag{8}$$

Assume that $c = -1$. Then

$$\alpha = \frac{1}{\alpha + b} = \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b + \cdots}}} = \frac{\sqrt{b^2+4} - b}{2}. \tag{9}$$

Infinite continued fraction gives the worst approximation for irrational number α the smaller is its $k+1$ component. So, the worst approximation will be in the case $b = 1$:

$$\alpha_1 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = \frac{\sqrt{5} - 1}{2} = 0.6180339. \tag{10}$$

The case $b = 1$ corresponds to co-called golden section. Farther calculations on the base of (9) give:

$$\alpha_2 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}} = \frac{\sqrt{8} - 2}{2} = 0.4142135,$$

$$\alpha_3 = 0.3027756,$$

$$\alpha_4 = 0.2360679,$$

$$\dots\dots\dots\dots\dots$$

Results of calculations are presented in Fig. 1e. Grey lines in Fig. 1e give rational numbers distribution, which is identical to Fig. 1c. Black lines give results of numerical calculation, based on (9) for $b = \overline{1, 100}$. Bold black lines point cases $\alpha_1, \dots, \alpha_4$.

As possible to see from Fig. 1e algebraic numbers with grows of b have tendency became closer to rational maxima. This result, indicate that such numbers, possibly, are not the best candidate for "the most irrational ones" [1].

In present work we don't state the task to find explicit form of irrational numbers fractal. It is clear, that first irrational maxima must be connected with golden section. The question is about the rest of the maxima. Fig. 1f gives another attempt to construct such maxima on the base of set, given by generalized golden proportion [4]. It is obvious from Fig. 1f that this case also is far away from desired result.

5 Summary

All results described in the paper are based on the notions of resonance and roughness of real physical system. This notions in applications to set of real numbers leads to existence of rational numbers distribution, which has fractal character. Maxima of the distribution (Fig. 1d) correspond to maximal sensitivity of the system to external influences, maximal interaction between parts of the system. Resonance phenomena are more stable and appear more easy if r (1) belong to rational maxima (Fig. 1d).

Obtained rational numbers distribution (Fig. 1c – d) contains also areas where density of rational numbers are minimal. It's logically to suppose that such minima correspond to maxima in irrational numbers distribution. We suppose that such distribution exists and is complementary to distribution of rational numbers. Maxima in such distribution correspond to high stability of the system, minimal interaction between parts of the system, minimal interaction with surrounding.

Both irrational and rational numbers distribution are related to the same physical system and must be consider together.

Question about explicit form of irrational numbers distribution remains open. At the moment we can only state that main maxima in this distribution must corresponds to co-called golden section (10).

Ideas about connection between resonance and rational numbers distribution can be useful in [4–8] where used the same mathematical apparatus, but initial postulates are based on the model of chain system.

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