Trapping Regions for the Navier-Stokes Equations

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In 1999, J. C. Mattingly and Ya. G. Sinai used elementary methods to prove the existence and uniqueness of smooth solutions to the 2D Navier-Stokes equations with periodic boundary conditions. And they were almost successful in proving the existence and uniqueness of smooth solutions to the 3D Navier-Stokes equations using the same strategy. In this paper, we modify their technique to obtain a simpler proof of one of their results. We also argue that there is no logical reason why the 3D Navier-Stokes equations must always have solutions, even when the initial velocity vector field is smooth; if they do always have solutions, it is due to probability and not logic.

1 Introduction

In this paper, we examine the three-dimensional Navier-Stokes equations, which model the flow of incompressible fluids:

$$\left.\begin{aligned} &\frac{\partial u_i}{\partial t} + \sum_{j=1,2,3} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} \qquad i = 1,2,3 \\ &\sum_{i=1,2,3} \frac{\partial u_i}{\partial x_i} = 0 \end{aligned}\right\}, \qquad (1)$$

where $\nu > 0$ is viscosity, p is pressure, u is velocity, and $t > 0$ is time. We shall assume that both u and p are periodic in x. For simplicity, we take the period to be one. The first equation is Newton's Second Law, force equals mass times acceleration, and the second equation is the assumption that the fluid is incompressible.

Mattingly and Sinai [5] attempted to show that smooth solutions to 3D Navier Stokes equations exist for all initial conditions $u(x, 0) = u^0(x) \in C^\infty$ by dealing with an equivalent form of the Navier-Stokes equations for periodic boundary conditions:

$$\frac{\partial \omega_i}{\partial t} + \sum_{j=1,2,3} u_j \frac{\partial \omega_i}{\partial x_j} = \sum_{j=1,2,3} \omega_j \frac{\partial u_i}{\partial x_j} + \nu \Delta \omega_i \quad i = 1,2,3, \quad (2)$$

where the vorticity $\omega(x,t) = (\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}, \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1})$.

Their strategy was as follows: Represent the equations (2) as a Galerkin system in Fourier space with a basis $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}^3}$. A finite dimensional approximation of this Galerkin system can be associated to any finite subset $\mathcal{Z}$ of $\mathbb{Z}^3$ by setting $u^{(k)}(t) = \omega^{(k)}(t) = 0$ for all k outside of $\mathcal{Z}$. For each finite dimensional approximation of this Galerkin system, consider the system of coupled ODEs for the Fourier coefficients. Then construct a subset $\Omega(K)$ of the phase space (the set of possible configurations of the Fourier modes) so that all points in $\Omega(K)$ possess the desired decay properties. In addition, construct $\Omega(K)$ so that it contains the initial data. Then show that the dynamics never cause the sequence of Fourier modes to leave the subset $\Omega(K)$ by showing that the vector field on the boundary of $\Omega(K)$ points into the interior of $\Omega(K)$.

Unfortunately, their strategy only worked for the 3D Navier-Stokes equations when the Laplacian operator Δ in (2) was replaced by another similar linear operator. (Their strategy was in fact successful for the 2D Navier-Stokes equations.) In this paper, we attempt to apply their strategy to the original equations (1).

2 Navier-Stokes equations in Fourier space

Moving to Fourier space where

$$\left.\begin{aligned} u_i(x,t) &= \sum_{k\in\mathcal{Z}} u_i^{(k)}(t) e^{2\pi ikx} \\ p(x,t) &= \sum_{k\in\mathcal{Z}} p^{(k)}(t) e^{2\pi ikx} \\ |k| &= \sqrt{\sum_{j=1,2,3} k_j^2} \end{aligned}\right\}, \qquad (3)$$

let us consider the system of coupled ODEs for a finite-dimensional approximation to the Galerkin-system corresponding to (1),

$$\frac{du_i^{(k)}}{dt} = \Big(\sum_{\substack{q+r=k \\ q,r\in\mathcal{Z}}} \sum_{j=1,2,3} -2\pi i q_j u_i^{(q)} u_j^{(r)}\Big) - $$
$$- 4\pi^2 \nu |k|^2 u_i^{(k)} - 2\pi i k_i p^{(k)} \qquad i = 1,2,3, \qquad (4)$$

$$\sum_{i=1,2,3} k_i u_i^{(k)} = 0\,, \qquad (5)$$

where $\mathcal{Z}$ is a finite subset of $\mathbb{Z}^3$ in which $u^{(k)}(t) = p^{(k)}(t) = 0$ for each $k \in \mathbb{Z}^3$ outside of $\mathcal{Z}$. Like the Mattingly and Sinai paper, in this paper, we consider a generalization of this Galerkin-system:

$$\frac{du_i^{(k)}}{dt} = \Big(\sum_{\substack{q+r=k \\ q,r\in\mathcal{Z}}} \sum_{j=1,2,3} -2\pi i q_j u_i^{(q)} u_j^{(r)}\Big) - $$
$$- 4\pi^2 \nu |k|^\alpha u_i^{(k)} - 2\pi i k_i p^{(k)} \qquad i = 1,2,3, \qquad (6)$$

$$\sum_{i=1,2,3} k_i u_i^{(k)} = 0\,, \qquad (7)$$

where $\alpha \geqslant 2$. Multiplying each of the first three equations by k_i for $i = 1, 2, 3$ and adding the resulting equations together, we obtain

$$\sum_{\substack{q+r=k \\ q,r\in\mathcal{Z}}} \sum_{\substack{j=1,2,3 \\ l=1,2,3}} -2\pi i k_l q_j u_l^{(q)} u_j^{(r)} = 2\pi i |k|^2 p^{(k)}, \tag{8}$$

since $\sum_{i=1,2,3} k_i \frac{du_i^{(k)}}{dt} = 0$ (by equation (7)). Then substituting the above calculated expression for $p^{(k)}$ in terms of u into (6) we obtain

$$\frac{du_i^{(k)}}{dt} = \Big[\sum_{\substack{q+r=k \\ q,r\in\mathcal{Z}}} \sum_{\substack{j=1,2,3 \\ l=1,2,3}} 2\pi i \left(\frac{k_i k_l}{|k|^2} - \delta_{il} \right) q_j u_l^{(q)} u_j^{(r)} \Big] - - 4\pi^2 \nu |k|^\alpha u_i^{(k)} \qquad i = 1, 2, 3. \tag{9}$$

And since $\sum_{j=1,2,3} r_j u_j^{(r)} = 0$ and $q_j + r_j = k_j$, we can substitute k_j for q_j:

$$\frac{du_i^{(k)}}{dt} = \Big[\sum_{\substack{q+r=k \\ q,r\in\mathcal{Z}}} \sum_{\substack{j=1,2,3 \\ l=1,2,3}} 2\pi i \left(\frac{k_i k_l}{|k|^2} - \delta_{il} \right) k_j u_l^{(q)} u_j^{(r)} \Big] - - 4\pi^2 \nu |k|^\alpha u_i^{(k)} \qquad i = 1, 2, 3. \tag{10}$$

3 A new theorem

Now, we state and prove the following theorem:

Theorem: *Let $\{u^{(k)}(t)\}$ satisfy (10), where $\alpha > 2.5$. And let $1.5 < s < \alpha - 1$. Suppose there exists a constant $C_0 > 0$ such that $|u^{(k)}(0)| \leqslant C_0 |k|^{-s}$, for all $k \in \mathbb{Z}^3$. Then there exists a constant $C > C_0$ such that $|u^{(k)}(t)| \leqslant C|k|^{-s}$, for all $k \in \mathbb{Z}^3$ and all $t > 0$. (The constants, C_0 and C, are independent of the set $\mathcal{Z}$ defining the Galerkin approximation.)*

Proof: By the basic energy estimate (see [1,2,7]), there exists a constant $E \geqslant 0$ such that for each $t \geqslant 0$ and for any finite-dimensional Galerkin approximation defined by $\mathcal{Z} \subset \mathbb{Z}^3$, we have $\sum_{k\in\mathcal{Z}} \sum_{i=1,2,3} |u_i^{(k)}(t)|^2 \leqslant E$. Hence, for any $K > 0$, we can find a $C > C_0$ such that $|\Re(u^{(k)})| \leqslant C|k|^{-s}$ and $|\Im(u^{(k)})| \leqslant C|k|^{-s}$, for all $t \geqslant 0$ and $k \in \mathbb{Z}^3$ with $|k| \leqslant K$. Now let us consider the set,

$$\Omega(K) = \Big\{ \Big(\Re(u^{(k)}), \Im(u^{(k)}) \Big)_{k\in\mathbb{Z}^3} : |k| > K, \quad |\Re(u^{(k)})| \leqslant C|k|^{-s}, \quad |\Im(u^{(k)})| \leqslant C|k|^{-s} \Big\}. \tag{11}$$

We will show that if K is chosen large enough, any point starting in $\Omega(K)$ cannot leave $\Omega(K)$, because the vector field along the boundary $\partial\Omega(K)$ is pointing inward, i.e., $\Omega(K)$ is a trapping region. Since the initial data begins in $\Omega(K)$, proving this would prove the theorem.

We pick a point on $\partial\Omega(K)$ where $\Re(u_i^{(\bar{k})})$ or $\Im(u_i^{(\bar{k})}) = \pm C|\bar{k}|^{-s}$ for some $\bar{k} \in \mathcal{Z}$ such that $|\bar{k}| > K$ and some $i \in \{1, 2, 3\}$. (For definiteness, we shall assume that $\Re(u_i^{(\bar{k})}) = C|\bar{k}|^{-s}$, but the same line of argument which follows also applies to the other possibilities.) Then the following inequalities hold when K is chosen large enough:

$$\left| \sum_{\substack{q+r=\bar{k} \\ q,r\in\mathcal{Z}}} \sum_{\substack{j=1,2,3 \\ l=1,2,3}} 2\pi \left(\delta_{il} - \frac{\bar{k}_i \bar{k}_l}{|\bar{k}|^2} \right) \bar{k}_j \Im(u_l^{(q)} u_j^{(r)}) \right| \leqslant \sum_{\substack{q+r=\bar{k} \\ q,r\in\mathcal{Z}}} \sum_{\substack{j=1,2,3 \\ l=1,2,3}} 4\pi |\bar{k}_j| |u_l^{(q)}| |u_j^{(r)}| \leqslant \sum_{\substack{j=1,2,3 \\ l=1,2,3}} 4\pi |\bar{k}_j| \Big(\sum_{q\in\mathcal{Z}} |u_l^{(q)}|^2 \Big)^{1/2} \Big(\sum_{r\in\mathcal{Z}} |u_j^{(r)}|^2 \Big)^{1/2} \leqslant \sum_{\substack{j=1,2,3 \\ l=1,2,3}} 4\pi |\bar{k}_j| E < 4\pi^2 \nu |\bar{k}|^\alpha \frac{C}{|\bar{k}|^s} = 4\pi^2 \nu |\bar{k}|^\alpha |\Re(u_i^{(\bar{k})})|. \tag{12}$$

This establishes that the vector field points inward along the boundary of $\Omega(K)$ for all $t > 0$. So the trajectory never at any time leaves $\Omega(K)$. Then we have the desired estimate that $|u^{(k)}(t)| \leqslant C|k|^{-s}$ for all $t > 0$. ■

4 Discussion

Just as in the 1999 paper by Mattingly and Sinai [5], an existence and uniqueness theorem for solutions follows from our theorem by standard considerations (see [1, 2, 7]). The line of argument is as follows: By the Sobolev embedding theorem, the Galerkin approximations are trapped in a compact subset of L^2 of the 3-torus. This guarantees the existence of a limit point which can be shown to satisfy (10), where $\mathcal{Z} = \mathbb{Z}^3$. Using the regularity inherited from the Galerkin approximations, one then shows that there exists a unique solution to the generalized 3D Navier-Stokes equations where $\alpha > 2.5$.

The inequality (12) in the proof of our Theorem is not necessarily true when $\alpha = 2$. Because of this, there is nothing preventing the solutions to (10) from escaping the region $\Omega(K)$ when $\alpha = 2$. Hence, there is no logical reason why the standard 3D Navier-Stokes equations must always have solutions, even when the initial velocity vector field is smooth; if they do always have solutions, it is due to probability (see [6]) and not logic, just like the Collatz $3n + 1$ Conjecture and the Riemann Hypothesis (see [3, 4]). Of course, it is also possible that there is a counterexample to the famous unresolved conjecture that the Navier-Stokes equations always have solutions when the initial velocity vector field is smooth. But as far as the author knows, nobody has ever found such a counterexample.

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