Lorentzian Type Force on a Charge at Rest. Part II

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Some algebra and *seemingly* crystal clear arguments lead from the Coulomb force and the Lorentz transformation to the mathematical expression for the field of a moving charge. The field of a moving charge, applied to currents, has as consequences a magnetic force on a charge at rest, dubbed Lorentzian type force, and an electric field $\vec{E}$, the line integral of which, taken along a closed loop, is not equal to zero. Both consequences are falsified by experiment. Therefore we think that the arguments leading to the mathematical formulation of the field of a moving charge should be subject to a careful revision.

1　Citations

If someone asks me what time is, I do not know; if nobody asks me, I don't know either. [Rudolf Zelsacher]

2　Introduction

2.1　Miscellaneous

We will follow very closely the chain of thought taken by Edward Mills Purcell in [1]. We will use the Gaussian CGS units in order to underline the close relationship between electric field $\vec{E}$ and magnetic field $\vec{B}$.

Table 1: Definition of symbols

symbol	description
$j_x, \vec{J}$	current density
I	current
A, a	area
c	speed of light in vacuum
$v, \vec{v}$	speed, velocity
ϑ, α	angles
ω	anglular velocity
$N_e(x), n_e(x)$	current electron density, electron density
$\hat{R}$ etc.	unit vector in the direction of $\vec{R}$
$F(x, y, z, t),$ $F'(x', y', z', t')$	inertial systems in the usual sense as defined in e.g. [2]
β	$\frac{v}{c}$
$\vec{E}$	electric field
$\vec{B}$	magnetic field
q, Q, e, p	charge
h, a, r, R, s	distance
i, k, N, m	natural number variables
x, y, z	cartesian coordinates
t	time

2.2　The electric field $\vec{E}$ in F arising from a point charge q at rest in F′ and moving with $\vec{v}$ in F

The electric field $\vec{E}$ in F of a charge moving uniformly in F, at a given instant of time, is generally directed radially outward from its instantaneous position and given by [1]

$$\vec{E}(\vec{R}, \vartheta) = \frac{q(1 - \beta^2)}{R^2(1 - \beta^2 \sin^2 \vartheta)^{\frac{3}{2}}} \hat{R}. \tag{1}$$

R is the length of $\vec{R}$, the radius vector from the instantaneous position of the charge to the point of observation; ϑ is the angle between $\vec{v}\Delta t$, the direction of motion of charge q, and $\vec{R}$. Eq. 1, multiplied by Q, tells us the force on a charge Q at rest in F caused by a charge q moving in F (q is at rest in F').

3　Lorentzian type, i.e. magnetic like, force on a charge Q at rest

3.1　Boundary conditions that facilitate the estimation of the field characteristics

We have recently calculated the non-zero Lorentzian type force of a current in a wire on a stationary charge outside the wire by using conduction electrons all having the same speed [3]. We now expand the derivation given in [3] to systems with arbitrary conduction electron densities, i.e. to conduction electrons having a broader velocity range. Based on Eq. 1, describing the field of a moving charge, we derive geometric restrictions and velocity restrictions useful for our purposes. These boundary conditions allow the knowledge of important field characteristics, due to a non-uniform conduction electron density, at definite positions outside the wire.

3.1.1　The angular dependent characteristics of the field of a moving charge

For a given β, at one instant of time, the angle ϑ_c (theta change), between $\vec{R}$ and $\vec{v}\Delta t$, given by

$$\vartheta_c = \arcsin \frac{\left[1 - \left(1 - \beta^2\right)^{\frac{2}{3}}\right]^{\frac{1}{2}}}{\beta} \tag{2}$$

separates two regions: one where the absolute value of the field of the moving charge is less than $\frac{q}{R^2}$ and a second where the absolute value of the field of the moving charge is greater than $\frac{q}{R^2}$. For small velocities, e.g. $v = 2 \cdot 10^{-10}$ [cm/s], ϑ_c is $\approx \arcsin \sqrt{\frac{2}{3}}$ or about 54.7°. For $v = 2 \cdot 10^{10}$ [cm/s], ϑ_c is less than 60°. We will later need ϑ_c to estimate the effect of the field of conduction electrons at the position of a test charge Q. In Fig. 1 we have sketched in one quadrant the regions where the absolute value of the field of the moving charge is separated by ϑ_c. $2 \cdot 10^{10}$ cm/s or $2c/3$ is just an arbitrarily chosen and of course sufficiently high speed limit for conduction electrons to be used in our estimations.

$$\vartheta_c = \arcsin \frac{\left(1 - \left(1 - \beta^2\right)^{\frac{2}{3}}\right)^{\frac{1}{2}}}{\beta}$$

$$\text{for } v < c$$

$$\vartheta_c \cong \arcsin \left(\sqrt{\frac{2}{3} + \frac{\beta^2}{9} + \frac{4\beta^4}{81} \cdots} \right)$$

$$(\text{for } v = 2e^{10}[cms^{-1}] \quad \vartheta_c < 60°)$$

$$(\text{for } v \ll c$$

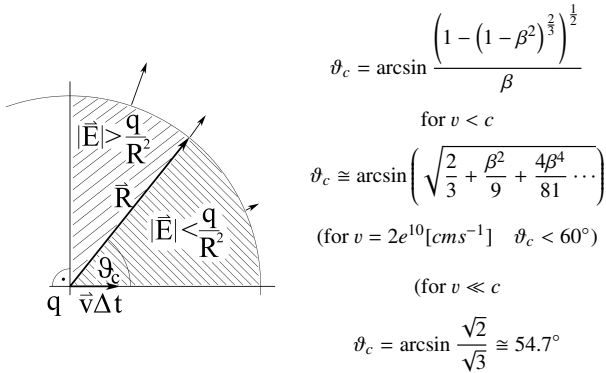$$\vartheta_c = \arcsin \frac{\sqrt{2}}{\sqrt{3}} \cong 54.7°$$

Fig. 1: The angle ϑ_c separates the region where the absolute value of the field of a moving charge is greater than $\frac{q}{R^2}$ from the region where the absolute value of the field of the moving charge is less than $\frac{q}{R^2}$.

3.1.2 The conduction electron density of a stationary current in a metal wire

We will use neutral wires and apply an electromotive force so that currents will flow in the wires. We also have in mind superconducting wires; at least we cool down the wires to near 0°[K] to reduce scattering. As in [1] we will restrict our investigation to a one dimensional current i.e. to velocities in one direction (v_x). A stationary current I, the number of electrons passing a point in a wire per unit of time, is then given by

$$I = \int \vec{j} d\vec{a} = A \, (-e) \, N_e \, (x) \, \bar{v}_x \, (x) \tag{3}$$

where A is the cross section of the wire, $\vec{j}$ or component j_x is the current density, $N_e(x)$ is the local conduction electron density and $\bar{v}_x(x)$ is the local mean velocity of the conduction electrons. For a stationary current $div \vec{j} = 0$. This indicates

that there can be no permanent pile up of charges anywhere in the wire. From our discussion with regard to ϑ_c in section 3.1.1 we know that for restricted velocities v_x of the conduction electrons and restricted angles ϑ the absolute value of the field of the conduction electron $\frac{e(1-\beta^2)}{r^2(1-\beta^2 \sin^2 \vartheta)^{\frac{3}{2}}}$, at the position of the test charge Q, is either greater than $\frac{e}{r^2}$ or less than $\frac{e}{r^2}$.

3.1.3 The line integral of the field of a moving charge

The field of a moving charge at an instant t_0 cannot be compensated by any stationary distribution of charges. The reason is that for the field of a moving charge in general

$$\oint \vec{E} d\vec{s} \neq 0. \tag{4}$$

We will use this property to estimate whether a variable electron density $n_e(x)$ along a wire can compensate the field due to the moving conduction electrons. In addition we will use this fact to show that currents in initially neutral wires produce electric fields whose line integral along a closed loop is non-zero.

3.2 The force of a pair of moving charges on a resting charge

In Fig. 2 we show two charges q_n and q_p moving in lab and a test charge Q at rest in lab. The indices n & p were chosen to emphasize that we will later use a negative elementary charge and a positive elementary charge, and calculate the effect of such pairs, one moving and the other stationary, on a test charge Q at rest in lab.
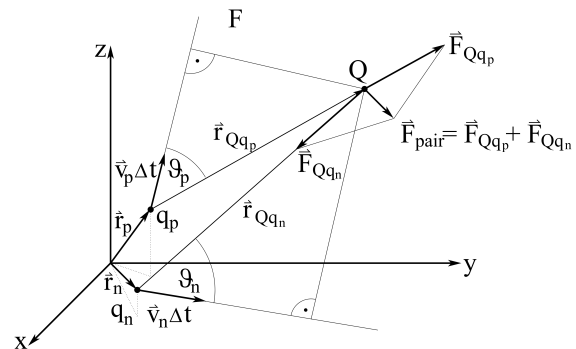

Fig. 2: The force $\vec{F}_{pair}$ on a resting charge Q caused by the two moving charges q_n and q_p. We assign the name $\vec{F}_{pair}$ to the result of the calculation of a force on a resting test charge Q, by at least two other charges having different velocities (including $\vec{v} = \vec{0}$).

The force $\vec{F}_{pair}$ exerted by this pair of charges, of q_n and

q_p, on the test charge Q is, according to Eq. 1, given by

$$
\vec{F}_{pair} = \vec{F}_{Qq_p} + \vec{F}_{Qq_n} =
$$

$$
= \frac{q_p Q \left(1 - \frac{v_p^2}{c^2}\right) \hat{r}_{Qq_p}}{r_{Qq_p}^2 \left(1 - \frac{v_p^2}{c^2} \sin^2 \vartheta_p\right)^{\frac{3}{2}}} + \frac{q_n Q \left(1 - \frac{v_n^2}{c^2}\right) \hat{r}_{Qq_n}}{r_{Qq_n}^2 \left(1 - \frac{v_n^2}{c^2} \sin^2 \vartheta_n\right)^{\frac{3}{2}}}. \quad (5)
$$

We are going to use such pairs of charges – specifically a conduction electron $(-e)$, and its partner, the nearest stationary proton (e) – in a current carrying wire and investigate the non vanishing field in lab produced by such pairs outside the wire. "Stationary" (or resting, or at rest) indicates that the "stationary charges" retain their mean position over time.

3.3 Lorentzian type force, part 1

We consider now two narrow wires isolated along their length, but connected at the ends, each having length 2a and lying in *lab* coaxial to the x-axis of F from $x = -a$ to $x = a$. In addition the system has a source of electromotive force applied so that a current I is flowing through the wires; in one of the wires I flows in the positive x direction and in the other wire I flows in the negative x direction. We also have in mind superconducting wires. On the z-axis of F fixed (stationary) at $(0, 0, h)$ a test charge Q is located. The system is sketched in Fig. 3. We will now calculate the Lorentzian type force $\vec{F}_{Lt}$ on the stationary test charge Q fixed at $(0, 0, h)$ exerted by the electrons of the current I and their nearest stationary protons at an instant t_0.
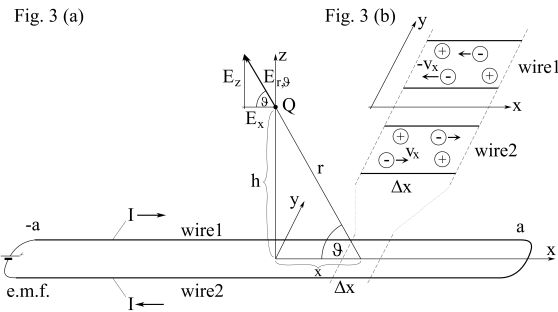

Fig. 3: (a) (b): We show in Fig. 3(a) the two wires carrying the current I extended along the x axis of F from $x = -a$ to $x = a$ and the charge Q at rest in F at $(0, 0, h)$. Additionally on the right-hand side a magnification of a small element Δx containing the two wires and labeled Fig. 3(b) can be seen. Fig. 3(b) shows some moving electrons and for each of these the nearest neighboring proton situated in the tiny element. We calculate the force on Q by precisely these pairs of charges.

The two wires are electrically neutral before the current is switched on. Therefore after the current is switched on we have an equal number of N electrons and N protons in the system - the same number N, as with the current switched off. We look at the system at one instant of *lab* time t_0, after

the current I is switched on and is constant. We consider the k electrons that make up the current I. For each of these k electrons e_i with $i = 1, 2, ..k$, having velocity $v_{x,i}$, we select the nearest neighboring stationary proton p_i with $i = 1, 2, ..k$. "Stationary" means that the charges labeled stationary retain their mean position over time. For each charge of the mobile electron-stationary proton pair, we use the same $\vec{r}_i$ as the vector from each of the two charges to Q. We use $\vartheta_i = \arcsin \frac{h}{r_i}$ as the angle between the x-axis and $\vec{r}_i$ for each pair of charges. As long as the velocity $v_{x,i}$ of a conduction electron is less than $2 \cdot 10^{10} [cm/s]$ and the angle $\vartheta_i = \arcsin \frac{h}{r_i}$, between the x-axis and the vector $\vec{r}_i$ from the current electron to test charge Q, is greater than $60°$(and less than $120°$), the contribution of the current electron to the absolute value of the field at $(0,0,h)$ is, according to our discussion in section 3.1.1, greater than $\frac{e}{r_i^2}$. The contribution of the nearest proton that completes the pair is $\frac{e}{r_i^2}$. If we restrict ϑ_i to between $60°$and $120°$, we will have an electric field $\vec{E} \neq \vec{0}$ at the position of Q pointing towards the wire. The Lorentzian type force $\vec{F}_{Lt}$ on the stationary test charge Q is then given by

$$
\vec{F}_{Lt} = Qe \sum_i \left\{ \left| \frac{\cos \vartheta_i}{r_i^2} \right| (-1)^{m_i} \left(1 - \frac{\left(1 - \frac{v_{x,i}^2}{c^2}\right)}{\left(1 - \frac{v_{x,i}^2}{c^2} \sin^2 \vartheta_i\right)^{\frac{3}{2}}} \right) \hat{x} + \right.
$$
$$
\left. + \frac{\sin \vartheta_i}{r_i^2} \left(1 - \frac{\left(1 - \frac{v_{x,i}^2}{c^2}\right)}{\left(1 - \frac{v_{x,i}^2}{c^2} \sin^2 \vartheta_i\right)^{\frac{3}{2}}} \right) \hat{z} \right\} = S_{Lt}\hat{S}. \quad (6)
$$

The m_i ($m_i = 0$ if $x_{e_i} - x_Q < 0, m_i = 1$ if $x_{e_i} - x_Q > 0$) ensures the correct sign for the x-component of the force. Eq. 6 shows that an equal Number N of positive and negative elementary charges (the charges of the wire loop) produces a force on a stationary charge, when a current is flowing. This force can be written as

$$
\vec{F}_{Lt} = F_{x,Lt}\hat{x} + F_{z,Lt}\hat{z} =
$$
$$
= \frac{\sqrt{F_{x,Lt}^2 + F_{z,Lt}^2}}{\sqrt{F_{x,Lt}^2 + F_{z,Lt}^2}} \left(F_{x,Lt}\hat{x} + F_{z,Lt}\hat{z} \right) = S_{Lt}\hat{S} \quad (7)
$$

with the unit vector $\vec{S}$ pointing from the position of the test charge $Q(0, 0, h)$ to a point $X(-a < X < a)$ on the x-axis. X will probably not be far from zero, but we leave this open as the resulting force vector $\vec{F}_{Lt} = S_{Lt}\vec{S}$ depends on the local current electron density in the wire. Note that $\frac{\left(1 - \frac{v_{x,i}^2}{c^2}\right)}{\left(1 - \frac{v_{x,i}^2}{c^2} \sin^2 \vartheta_i\right)^{\frac{3}{2}}}$ is greater than 1 as long as $v_{x,i} < 2 \cdot 10^{10}$ [cm/s] and $60°<$

$\vartheta_i < 120°$, as was shown in section 3.1.1 This means the field at (0, 0, h) points to the wire.

3.4 Lorentzian type force, part 2

Next we place the stationary charge Q at the position ($b > a, 0, h$), with $\vartheta_{max} = \arctan \frac{h}{b-a} < 54°$ (see Fig. 4).
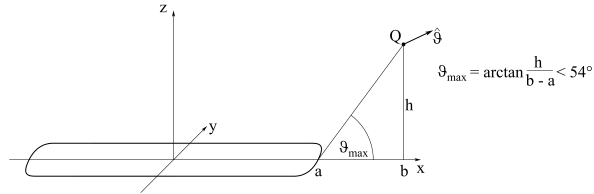

Fig. 4: If the test charge Q, is located at ($b, 0, h$) as shown here, with $\vartheta_{max} = \arctan \frac{h}{b-a} < 54°$, then the absolute value of the field of each of the conduction electrons at ($b, 0, h$) is less than that of a stationary charge for all velocities $0 < v_x < c$.

The force on the stationary test charge Q is given by Eq. 6. But now $\dfrac{\left(1 - \frac{v_{x,i}^2}{c^2}\right)}{\left(1 - \frac{v_{x,i}^2}{c^2}\sin^2\vartheta_i\right)^{\frac{3}{2}}}$ is less than 1 for $0 < v_{x,i} < 3 \cdot 10^{10}$ [cm/s] and $0° < \vartheta_i < 54°$ or $136° < \vartheta_i < 180°$ as was shown in section 3.1.1. This means the field at ($b, 0, h$) points away from the wire.

3.5 The line integral of the field of two parallel wires calculated at one instant t₀

We continue by estimating a specific line integral of the electric field outside the wire along the closed path shown in Fig. 5.
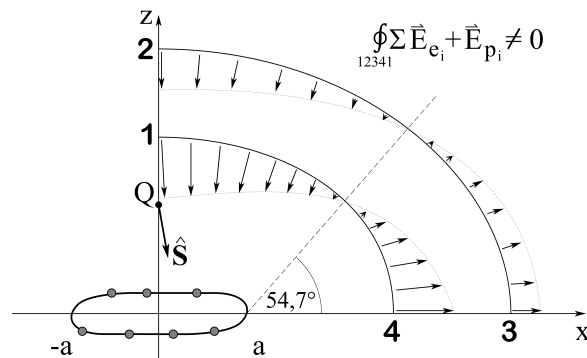

Fig. 5: Shows the electric field $\sum(\vec{E}_{e_i} + \vec{E}_{p_i})$ due to the moving conduction electrons and their partner protons of the system of Fig. 3. In addition the path 12341 is shown where the line integral of the electric field $\sum(\vec{E}_{e_i} + \vec{E}_{p_i})$ is estimated. $\vec{E}_s + \vec{E}_Q$, the field of the residual stationary charges of the system and the test charge Q, is not shown because the line integral of the field $\vec{E}_s + \vec{E}_Q$, along a closed path is zero.

The electric field of the system is a superposition of the field of the moving conduction electrons and their stationary

partner protons $\sum(\vec{E}_{e_i} + \vec{E}_{p_i})$, the field $\vec{E}_s$ of the residual stationary electrons and protons of the wire and the field $\vec{E}_Q$ of the resting test charge Q. The line Integral of $\vec{E}_s + \vec{E}_Q$ along every closed path is zero. The line integral of the electric field $\sum(\vec{E}_{e_i} + \vec{E}_{p_i})$ due to the moving conduction electrons and their partner protons is, according to our discussion in section 3.1.1 and the results given by Eq. 6 at positions like (0, 0, h) and (b, 0, h), less than zero from 1 to 2, zero from 2 to 3 (because here we have chosen a path perpendicular to the field), less than zero from 3 to 4 and zero from 4 to 1 (because here we have again chosen a path perpendicular to the field).

$$\oint_{12341} \vec{E} d\vec{s} = \oint_{12341} \left(\sum\left(\vec{E}_{e_i} + \vec{E}_{p_i}\right) + \vec{E}_s + \vec{E}_Q\right) d\vec{s} =$$

$$= \left[\oint_1^2 \left(\sum \vec{E}_{e_i} + \vec{E}_{p_i}\right) d\vec{s} + \oint_3^4 \left(\sum \vec{E}_{e_i} + \vec{E}_{p_i}\right) d\vec{s}\right] < 0. \quad (8)$$

A wire bent like the loop 12341 might be a good device for the experimental detection of $\vec{F}_{Lt}$. As we have mentioned in section 3.1.2 we do not expect pile-up effects of charges in the wire because from experiment we know the extreme precision to which Ohm's Law, is obeyed in metals. But we expect a variable electron density $n_e(x)$ (not to be confused with the variable conduction electron density $N_e(x)$) on the wires resulting from capacitive and shielding effects, together with the field component of the moving conduction electrons directed along the wire. The estimation of the line integral of the electric field of the system, resulting in Eq. 8, shows, by being non-zero, that no "stationary" static charge distribution on the wires is able to compensate the field due to the moving conduction electrons.

3.6 The force on a charge at rest due to a superconducting ring

We consider now a superconducting current carrying ring, with radius a, and assume that one of its conduction electrons e_i at t_0, at rest in its local inertial frame, has constant velocity $\vec{v}_i = \vec{\omega}_i \times \vec{r}_i$. Then, according to Eq. 5 and Fig. 6 the Lorentzian type force on a charge Q at rest at (0, 0, h) caused by this system is given by

$$\vec{F}_{Lt} = \sum_i \frac{Qe}{r_i^2 + h^2}\left[1 - \frac{1}{\left(1 - \beta_i^2\right)^{\frac{1}{2}}}\right]\cos\arctan\frac{a}{h}\hat{z} \quad (9a)$$

or if $v \ll c$

$$\vec{F}_{Lt} \approx \sum_i \frac{Qe}{r_i^2 + h^2}\left[1 - 1 - \frac{\beta_i^2}{2}\right]\cos\arctan\frac{a}{h}\hat{z} =$$

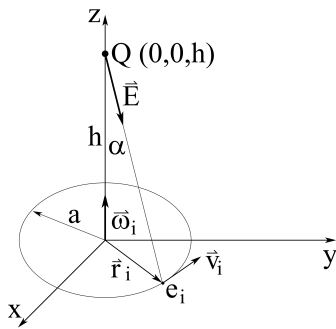$$= \sum_i -\frac{Qv_i}{c}\frac{ev_i}{2\left(r_i^2 + h^2\right)c}\cos\arctan\frac{a}{h}\hat{z}. \quad (9b)$$

Fig. 6: The electrical field, at the position of a charge Q at rest, caused by one of the charges e_i of the current in a superconducting wire.

As stated above we assume that the current carriers are at rest in a succession of individual local inertial frames when circling in the loop; i.e. the movement of the charges is well described by a polygon, with as many line segments as you like it. This view is supported by the experimental fact that currents flow for years in such loops without weakening, showing that the passage from one inertial frame to the next happens without much radiation.

3.7 The Field due to a constant electron density in the parallel wires connected at the ends

We now proceed to the case where the current electron density $N_e(x)$ is constant along the wires *by definition* to get an analytic expression for the force $\vec{F}_{Lt}$ on a stationary charge. This was calculated in [3] and here we just rewrite the result. The Lorentzian type force on a charge Q at rest due to a system like that shown in Fig. 2 is, by assuming a constant current electron density, given by

$$\vec{F}_{Lt} = -\frac{Q v_x}{c} \frac{2I \cos \vartheta_{min} \sin^2 \vartheta_{min}}{hc^2} \hat{z}. \qquad (10)$$

The force described by Eq. 10 is of the same order of magnitude as magnetic forces, as can be seen by comparing it to Eq. 11, the result of a similar derivation given in [1]

$$\vec{F} = \frac{q v_x}{c} \frac{2I}{rc^2} \hat{y}. \qquad (11)$$

4 Discussion

The one and only way to scientific truth is the comparison of theoretical conclusions with the experimental results. We have investigated the consequences of Eq. 1 - the elegant mathematical formulation of the field of a moving charge. By applying the field of a moving charge to currents in loops we derive a magnetic force on a charge at rest outside these loops. We have dubbed this force "Lorentzian type force"

and state that such a force has never been observed in experiments. In addition such current-carrying systems, when investigated by using the mathematical expression for the field of a moving charge, show an electric field whose line integral along a closed loop is non-zero. Also this prediction has never been observed by experimental means. We find the example of the Lorentzian type, i.e. magnetic, force on a charge at rest due to the superconducting ring (as given in 3.6), which also has been never observed, to be especially instructive because nothing disturbs the intrinsic symmetry. The overall conclusion from our investigation is that the arguments leading to the formula for the field of a moving charge should be subject to a careful revision.

Acknowledgements

References

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3. Zelsacher R. Lorentzian Type Force on a Charge at Rest. *Progress in Physics*, 2014, v. 10(1), 45–48.