Optics of the Event Horizon Telescope

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It is suggested in this article that part of the signal in the 1.3 mm range from Sagittarius A* originates inside the central collapsar, rather than coming entirely from its accretion disc. The suggestion has its origin in the discovery that the classic article of Oppenheimer and Snyder contains a basic error in its assertion that the light, from a collapsing object lying entirely within its own photonsphere, is progressively cut off as the object shrinks towards its gravitational radius, where a large part of the Oppenheimer-Snyder collapsar's material is concentrated. The signal from the collapsar has certain features which may make it possible to distinguish its image from that of the accretion disc.

1 Introduction

At the centre of our galaxy, 8 kp distant from us, there is an object named Sagittarius A* whose mass is 4.1 megasuns. It is popularly classified as a black hole, with a spherical* region of radius 1.2×10^7 km around it bounded by an "event horizon"; according to black-hole theory no light from Sagittarius A* can cross this horizon.

In two recent articles [1, 2] it was shown that there is a solution of the field equations of General Relativity for such a supermassive object, which has no singularity at $r = 0$, and which allows light signals to cross the horizon. The latter property of the solution was demonstrated for the case of rays which are normal to the event horizon, and the present article demonstrates that it may be extended to all orientations. In addition we consider the range of angles for which light originating at the surface of such a collapsar crosses the photonsphere, at 1.5 times the gravitational radius, and consequently may reach a terrestrial telescope. There is currently a project called the Event-Horizon Telescope [3] (EHT) designed to look at the signal from the neighbourhood of Sagittarius A* in the 1.3 mm range.

Central to the widespread belief in the validity of black-hole theory is the article of Oppenheimer and Snyder (OS) [4]. This reported, without giving details, an investigation of the light signal from a supermassive object, arriving at the following conclusion

> All energy from the surface of the star will be reduced very much in escaping ... by the gravitational deflection of light which will prevent the escape of radiation except through a cone about the outward normal of progressively shrinking aperture as the star contracts. The star thus tends to close itself off from any communication with a distant observer.

The property of the OS metric claimed by Penrose, which he needed as a prerequisite for his singularity theorem [5], was the stronger one known as the *trapped surface*. The publications cited above show that neither of these properties in fact holds for the OS metric.

In the following two sections we shall use precisely the OS metric to show that the progressively shrinking aperture of the emission cone has no effect on the size of the image of the collapsing object, and only a marginal effect on its total luminosity. This result leads us to suggest that the signal from Sagittarius A* comes partly from the surface of the collapsar itself, and not entirely from the accretion disc, as is assumed in most current analyses. The accretion disc may well have a substantially higher temperature than the collapsar, but that is probably offset by the vastly greater area of the latter. Note also that the millimetre range of wavelength investigated by the EHT corresponds to the maximum of a Planck spectrum of just a few degrees Kelvin; to support our analysis, the collapsar must retain only the merest relic of its thermal energy.

The OS article reached another conclusion, stated in their Abstract, namely

> ... an external observer sees the star asymptotically shrinking to its gravitational radius.

This result contradicts directly Penrose's description of the OS results and was verified by me in [1]. The point is that OS showed that there is a common system of coordinates applicable to both the exterior and interior of the collapsar. My article [1] demonstrated that the density distribution of the OS "dust cloud" becomes concentrated near the surface as it shrinks to the gravitational radius; no exotic process like the modern black-hole one of "spaghettification" [6] occurs when a notional spaceship crosses the event horizon. OS should be considered responsible for the notion that further shrinkage occurs within the gravitational radius only in so far as they gave their article the misleading title "On continued gravitational contraction".

It should be noted that in the exterior, and hence in what should now be recognized as the universal, time frame the collapsar's shrinkage to the gravitational radius takes an infinite lapse of time. We shall show in the following section that in the limit there is an underlying infinite red shift, which

*For the purposes of this article we ignore its spin.

causes not only the surface itself, but also all light signals approaching it, to be infinitely slowed down. This is the real significance of the event horizon, but it is my contention that a real collapsar, with an internal pressure resulting from the intervention of forces other than gravitational, stops shrinking before it reaches the gravitational radius. For example, we have investigated [7] a collapsar whose equation of state is an idealized form of neutron fluid*, and for which, above a certain mass, its maximum density lies between the event horizon and the photonsphere.

2 The exterior light orbits

Darwin [9, 10] described the null geodesics of the Schwarzschild metric

$$ds^2 = \frac{r-2m}{r}\,dt^2 - \frac{r}{r-2m}\,dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2, \quad (1)$$

where $2m$ is the gravitational radius. He extended the standard theory of light deflection, his method being equivalent to minimising the action integral for a light ray with small impact parameter starting from infinity; for orbits in the plane $\theta = \pi/2$,

$$\delta \int L d\phi = 0, \quad (2)$$

with the lagrangian

$$L = \left[\frac{r-2m}{r}\,t'^2 - \frac{r}{r-2m}\,r'^2 - r^2\right]^{1/2}, \quad (3)$$

where a prime denotes differentiation with respect to ϕ. The Lagrange equation for the cyclic coordinate t is

$$\left[\frac{d}{d\phi} - \frac{L'}{L}\right]\frac{r-2m}{r}\,t' = 0. \quad (4)$$

The corresponding conservation integral for ϕ enables us to put $L'/L = 2r'/r$, so we obtain

$$t' = \frac{r^3}{p(r-2m)}, \quad (5)$$

the constant p being the impact parameter

$$p = \lim_{r\to\infty} r^2 \frac{d\phi}{dt}. \quad (6)$$

The ray orbit is then obtained by substituting for t' and then putting $L = 0$, that is

$$r'^2 = \frac{r^4}{p^2} - r^2 + 2mr. \quad (7)$$

Darwin deduced that a ray with impact parameter p greater than $3m\sqrt{3}$ returns to $r = \infty$; the deflection angle may be many multiples of 2π as p approaches $3m\sqrt{3}$, and in the limiting case $p = 3m\sqrt{3}$ the ray circles indefinitely at $r = 3m$, which is nowadays called the *photonsphere*. For p less than this, the ray is captured, and it goes to what Darwin termed the "barrier", nowadays called the *event horizon*, at $r = 2m$. He also repeated the point previously made by OS, that the journey from $r = 3m$ to $r = 2m$ takes an infinite time. When the collapse is incomplete, the surface being at $r = r_1 > 2m$, a ray arrives there making an angle with the normal of

$$\xi = \tan^{-1}\left[\left(\frac{r_1^2}{p^2} - 1 + \frac{2m}{r_1}\right)^{-1/2}\right], \quad (8)$$

and in the limiting case $r_1 = 2m$ this becomes

$$\xi = \tan^{-1}\left(\frac{p}{2m}\right). \quad (9)$$

We may deduce directly the orbits of rays exiting from the barrier; those falling within a cone of semiangle $\tan^{-1}(3\sqrt{3}/2) = 68.9$ degrees go to our telescope at "infinity", forming an image of parallax $6m\sqrt{3}$. Any collapsar with $2m < r_1 < 3m$ has this same parallax, but at $3m$ the cone has opened up fully to 90 degrees. A collapsar bigger than $3m$ has a parallax bigger than $6m\sqrt{3}$, while for much larger collapsars, like white dwarfs of solar mass, light deflection is negligible, and the parallax is then simply twice the surface radius. In Figure 1 a number of rays have been plotted, leaving various points in the surface, when $r_1 = 2.2m$, and going towards our telescope; we note that the rays going to the edge of the image come from points on the "invisible face" of the collapsar.

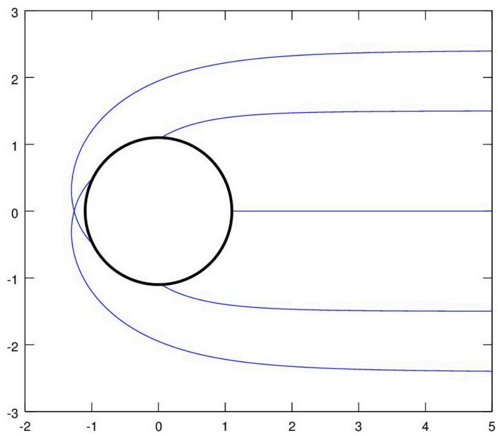

Fig. 1: The light rays issuing from the surface of a collapsar at 1.1 times the gravitational radius, collimated towards a distant telescope. The outer rays are close to the edge of the image, which has a diameter of 5.2 times the gravitational radius; these rays have their sources on what, in the absence of gravitational lensing, would be the invisible part of the surface. The unit of distance is the gravitational radius.

*This model is simply that of Oppenheimer and Volkoff [8] with a different boundary condition at the origin.

For Sagittarius A* the minimum parallax, according to the above analysis, and with the distance of EHT from the galactic centre equal to 2.4×10^{17} km, is 52 arc microseconds, which exceeds the best available current value [3] by about 50 percent. The image profile, that is its intensity $C(p)$ as p goes from zero to $3m\sqrt{3}$, is given by

$$C(p) = \left| \frac{r_1 \sin\phi_0 \cos\xi}{p} \frac{d\phi_0}{dp} \right| , \tag{10}$$

where ξ is given by (8), that is

$$\cos\xi = \sqrt{\frac{r_1^3 - p^2 r_1 + 2p^2 m}{r_1^3 + 2p^2 m}} , \tag{11}$$

and ϕ_0 is the angle between the outward normal at the surface and the ray's final direction, that is

$$\phi_0 = \int_{r_1}^{\infty} \frac{p\,dr}{\sqrt{r^4 - p^2 r^2 + 2mp^2 r}} , \tag{12}$$

leading to

$$\frac{d\phi_0}{dp} = \int_{r_1}^{\infty} \frac{r^4 dr}{\left(r^4 - p^2 r^2 + 2mp^2 r\right)^{3/2}} . \tag{13}$$

Note that, for $r_1 \gg 2m$, $\xi = \phi_0$, $p = r_1 \sin\phi_0$, and $C(p) = 1/r_1 =$ const. with ϕ_0 going from $-\pi/2$ to $\pi/2$, giving a uniform circular image of radius r_1; for our case ϕ_0 takes all real values. In Figure 2 the image profile $C(p)$ is plotted. The edge of the image is at $p = 3m\sqrt{3} = 2.598\, r_0$, where r_0 is the gravitational radius. Note that, though $C(p)$ drops to zero at $p = 2.388 r_0$, there is a bright fringe between that value and $p = 2.588\, r_0$; though not shown in the Figure, there is a series of narrower fringes between the latter value and the edge of the image at $p = 2.598\, r_0$. The fringes result from light rays circling close to the photonsphere before finally escaping to reach the telescope, their minima occurring at p-values for which ϕ_0 are integer multiples of π.

A ray which leaves the surface in a direction falling outside the limiting cone, that is with an orbit described by $p > 3m\sqrt{3}$, turns round before reaching the photonsphere, and returns to the barrier after an infinite time.

None of this accords with the OS description, in which the cone closes down to zero at $r = 2m$.

3 The interior light orbits

According to the OS [4] model, the surface of the collapsar completely contracts to the barrier only at $t = \infty$; in the words of that article

> ...an external observer sees the star asymptotically shrinking to its gravitational radius.

Specifically $r_1(t)$ is given by

$$t = -\frac{2}{3}\sqrt{\frac{r_1^3}{2m}} - 2\sqrt{2mr_1} + 2m \ln \frac{\sqrt{r_1} + \sqrt{2m}}{\sqrt{r_1} - \sqrt{2m}} . \tag{14}$$

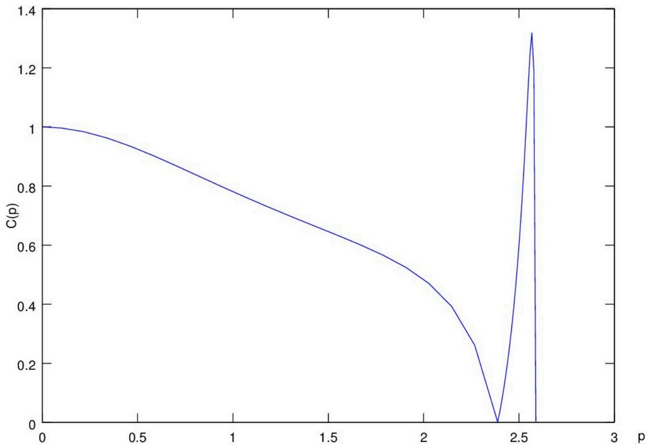

Fig. 2: The image profile $C(p)$ formed by the rays in Figure 1. Again the gravitational radius is the distance unit on the horizontal axis p, and $C(p)$ is normalized to $C(0) = 1$.

For $r < r_1$ the OS metric is

$$ds^2 = \frac{r^3}{2mR^3}\left(\frac{dr}{r} - \frac{dR}{R}\right)^2 - \frac{r^2}{R^2}\,dR^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 , \tag{15}$$

where the coordinate R lies between 0 and 1, and is related to t in a manner to be determined by matching conditions imposed at the surface.

The interior null geodesics in the plane $\theta = \pi/2$ are constructed from the Lagrangian

$$L = \left[\frac{r^3}{2mR^3}\left(\frac{r'}{r} - \frac{R'}{R}\right)^2 - \frac{r^2}{R^2} R'^2 - r^2 \right]^{1/2} . \tag{16}$$

The Lagrange equations for r and R are, putting $L'/L = 2r'/r$ as in the exterior case,

$$\frac{2rr''}{R^3} - \frac{2r^2 R''}{R^4} - \frac{3r'^2}{R^3} - \frac{2rr'R'}{R^4} + \frac{5r^2 R'^2}{R^5} + \frac{4mrR'^2}{R^2} + 4mr = 0 \tag{17}$$

and

$$\frac{2r^2 r''}{R^4} - \frac{2r^3 R''}{R^5} - \frac{3rr'^2}{R^4} - \frac{2r^2 r'R'}{R^5} + \frac{5r^3 R'^2}{R^6} + \frac{4mr^2 R''}{R^2} - \frac{4mr^2 R'^2}{R^3} = 0 . \tag{18}$$

Combining these to eliminate r'', we obtain

$$R'' = R + \frac{2R'^2}{R} , \tag{19}$$

for which a sufficiently general solution, for $0 < R < 1$, is

$$R = \sin\phi_0 \csc\phi \quad (\phi_0 < \phi < \pi - \phi_0) . \tag{20}$$

Then, as in the exterior case, we obtain a first order equation for r by substituting this in L and putting $L = 0$, namely

$$r' = \sqrt{\frac{2mr\sin^3\phi_0}{\sin^5\phi}} - r\cot\phi\,, \tag{21}$$

with the solution

$$r = \frac{m\sin\phi_0}{2\sin\phi}\left(A - \sin\phi_0\cot\phi\right)^2. \tag{22}$$

A ray which arrives at $R = 1$, that is $\phi = \phi_0$, with $r = r_1$ has $A = 2\sqrt{r_1/(2m)} + \cos\phi_0$; the special case $\phi_0 = 0$ was given in eq (14) of [1]. At this point the ray has gradient

$$r' = \csc\phi_0\sqrt{2mr_1} - r_1\cot\phi_0\,. \tag{23}$$

4 Matching at the surface

OS [4] matched their metric with the exterior (1) by defining the *cotime* $y(r, R)$ related to t by

$$\frac{t}{2m} = -\frac{2}{3}\,y^{3/2} - 2\sqrt{y} + \ln\frac{\sqrt{y}+1}{\sqrt{y}-1}\,; \tag{24}$$

this they required to satisfy $y(r_1, 1) = r_1/(2m)$. To match the two metric tensors at $R = 1$ they then put

$$y = \frac{r}{2Rm} + \frac{R^2-1}{2}\,. \tag{25}$$

With the metrics matched at the surface, that means the refractive indices are also matched, so the corresponding light rays should join smoothly there. Eq (23) for the interior ray gives, at $r = r_1$,

$$r'^2 + r^2 - 2mr = \left(r_1\csc\phi_0 - \sqrt{2mr_1}\cot\phi_0\right)^2, \tag{26}$$

so the value

$$p = \frac{r_1^2\sin\phi_0}{r_1 - \sqrt{2mr_1}\cos\phi_0} \tag{27}$$

gives a smooth connection between the interior (23) and exterior (7) rays at $r = r_1$. Differentiating (24) and (25), we then find that the values of t' also match at r_1, which confirms that the light speed r'/t' is continuous there.

It may now be seen that, as r_1 approaches $2m$, the speed of light at the surface goes to zero, which generalizes the particular case treated in [1], where the light ray was normal to the surface. Such behaviour may be understood as resulting from the infinite "dust" density there (see below). This behaviour will be modified by the intervention of nongravitational forces; in particular we have studied the effect of the Fermi degeneracy pressure in a neutron star [7], for which the density has a finite maximum well separated from both the surface and $r = 0$. Thus, for a collapsar made of real stellar matter, it makes sense to consider a state of equilibrium whose radius exceeds the gravitational, and for which light leaves the surface with a finite speed; this was the situation depicted in Figure 1.

I add that the matching relation (25) is not unique, though OS stated that it was. In my previous articles [1, 2] the alternative

$$y = \frac{r}{2Rm} - \frac{(1-R)(5-R)}{4} \tag{28}$$

was given. This is part of a wider family of matching relations, and, for this particular choice, has certain advantages in respect of causality.

The infinite surface density of the OS final state may be seen in their calculation of the scalar density ρ, namely

$$\rho = \frac{3R^3}{8\pi r^3}\,. \tag{29}$$

Multiplying this by their three-volume element, we obtain

$$\rho\sqrt{-g}\,dR\,d\theta\,d\phi = \frac{3R^2\sin\theta}{8\pi}\,dR\,d\theta\,d\phi\,, \tag{30}$$

which, in terms of r, gives the density

$$\rho\sqrt{-g}\,dr\,d\theta\,d\phi = \frac{3R^2\sin\theta}{8\pi}\left(\frac{\partial R}{\partial r}\right)_t dr\,d\theta\,d\phi\,. \tag{31}$$

The partial derivative is given, at cotime $y = 1$, by

$$\left(\frac{\partial R}{\partial r}\right)_t = \left(\frac{\partial R}{\partial r}\right)_y = \frac{1}{3m(1-R^2)}\,, \tag{32}$$

giving infinite density at $R = 1$.

Actually we have found that the density in the shell just inside r_1 is very much reduced for a supermassive object like Sagittarius A*, and I propose that the material there is an electron gas with a nearly stationary nucleonic background* which should have broadly similar optical properties to both the OS dust cloud and the neutron star. In these cases the light speed will still be considerably reduced near the surface, but will remain finite.

5 Discussion

The suggestion about the origin of the EHT image of Sagittarius A*, namely that part of the light we receive comes from the collapsar itself, has implications for the direction future observations with the telescope should take. A central problem is to explain the present-day value of the parallax, which is 37μas as opposed to the 52μas we obtained in Section 2. We note that the size of this image is not at all well defined, because of the need to separate the signal from the background noise of nearby objects; this is reflected by the wide error bar in the above parallax. It should be noted also that the image of the accretion disc has almost the same diameter as the one

This entails classifying Sagittarius A as a *supermassive white giant*.

described in Section 2 if the disc is inside the photonsphere, and that its image is larger if it lies outside the photonsphere.

The fringes of the image, described in Section 2, do not seem to have been noticed previously, though they are surely present also in the image of the accretion disc. To distinguish between the two images, arising, as they do, from two superimposed sources of almost the same diameter, will require further analysis along the lines of Section 2; the principal difference is the three-dimensional form of the collapsar, as opposed to the flat, effectively two-dimensional form of the disc. Some progress, both in image enhancement and in theoretical modelling, would help to clarify matters.

The classic article of Oppenheimer and Snyder [4], based in turn on the equally classic one of Tolman [11], was essential for the construction of the matched orbits. In particular these articles (see also [12]) enable us to identify the comoving coordinate R used in Section 3. But the step required to describe fully the orbits of particles of "dust", that is the stellar material, and of light rays near the surface, is the identification of the time coordinate $t(r, R)$ made in our earlier article [1] and in Section 4 of this one.

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