Vacuum Background Field in General Relativity

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We assume here a slightly varying cosmological term which readily induces a permanent background field filling the physical vacuum. A precise form of the variable cosmological term is introduced containing an infinitesimal Killing vector which accounts for the space-time variation of this term. As a result the term can be added to the Einstein Lagrangian without affecting the varied action δS. As a result, we showed in an earlier publications that the permanent background field filling the vaccum is excited in the vicinity of matter which precisely corresponds to its gravitational field classically described by a pseudo-tensor. With this preparation, the global energy-momentum tensor of matter and gravity field is no longer a pseudo-tensor and is formally conserved like the Einstein tensor. In the excited state, this antisymmetric tensor can be conveniently symmetrized by applying the Belinfante procedure which automatically self excludes far from matter since the background field tensor is naturally symmetric.

Introduction

The substance of this study is inspired by the following considerations. In the framework of the Theory of General Relativity (GR), the Einstein tensor exhibits a *conceptually* conserved property, while any corresponding stress-energy tensor does not, which leaves the theory with a major inconsistency. When pure matter is the source, a so-called "pseudo-tensor" describing its gravitational field is introduced so that the four-momentum of both matter and its gravity field is conserved [1]. Unfortunately in this approach, the gravitational field maybe transformed away at any point and by essence, its pseudo-tensor cannot appear in the Einstein's field equations, as it should be.

We will tackle the problem in another way : Restricting our study to neutral massive flow, we proceed as follows. We introduce a space-time variable term that supersedes the so-called cosmological term Λg_{ab} in the Einstein's field equations [2]. Under this latter assumption, we formally show that the gravity field of a massive source is no longer described by a vanishing *pseudo-tensor*, but it is represented by a *true* tensor which can explicitly appear with the bare matter tensor together with another specific field, on the right hand side of the Einstein's field equations. Inspection also shows that this *global* stress-energy tensor now complies with the intrinsic conservation property of the Einstein tensor as it should be. As a result, the physical vacuum is here filled with a *homogeneous vacuum background field* which is always present in the so-called Einstein's "source free" equations and whose tensor exhibits a conserved property. Our theory leads to admit that matter causes the surrounding background field to produce its gravitational field which decreases asymptotically to the level of this vacuum field. Naturally, since we will deal with energy-momentum canonical field tensors which are not symmetric, the total angular momentum of the isolated system is not conserved. In this case, it is always possible to apply the symmetrizing procedure to these tensors according to J. Belinfante [3]. In the absence of matter, the inferred Belinfante tensor reduces to the symmetric background field tensor as it should be.

Notations

Space-time Latin indices run from $a = b$: 0, 1, 2, 3, while spatial Greek indices run from $\alpha = \beta$: 1, 2, 3. The space-time signature is -2. In the present text, $\varkappa$ is Einstein's constant $4\pi G/c^4$, where G is Newton's gravitational constant.

1 The field equations in General Relativity

1.1 The problem of the conserved gravity tensor

The General Theory of Relativity requires a 4-dimensional pseudo-Riemannian manifold. A Riemannian manifold is characterized by the line element $ds^2 = g_{ab}\,dx^a dx^b$. It is well known that by varying the action $\mathcal{S} = \mathfrak{L}_E d^4x$ with respect to the metric tensor g_{ab} with the Lagrangian density given by

$$\mathfrak{L}_E = \sqrt{-g}\, g_{ab}\left[\left\{{}^{e}{}_{ab}\right\}\left\{{}^{d}{}_{de}\right\} - \left\{{}^{d}{}_{ae}\right\}\left\{{}^{e}{}_{bd}\right\}\right], \tag{1.1}$$

$$g = \det \| g_{ab} \|. \tag{1.2}$$

Also one infers the symmetric Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R\,, \tag{1.3}$$

where

$$R_{bc} = \partial_a \left\{{}^{a}{}_{bc}\right\} - \partial_c \left\{{}^{a}{}_{ba}\right\} - \left\{{}^{d}{}_{bc}\right\}\left\{{}^{a}{}_{da}\right\} - \left\{{}^{d}{}_{ba}\right\}\left\{{}^{a}{}_{dc}\right\} \tag{1.4}$$

is the Ricci tensor with its contraction R, the curvature scalar (the $\left\{{}^{e}{}_{ab}\right\}$ denote the Christoffel symbols of the second kind). The 10 source free field equations are

$$G_{ab} = 0\,. \tag{1.5}$$

The second rank Einstein tensor G_{ab} is symmetric and is only function of the metric tensor components g_{ab} and their first and second order derivatives. The relations

$$\nabla_a G^a_b = 0 \tag{1.6}$$

are the conservation identities provided that the tensor G_{ab} has the form [4]

$$G_{ab} = k\left[R_{ab} - \frac{1}{2}\, g_{ab}\,(R + 2\Lambda)\right], \tag{1.7}$$

where k is a constant, which is here assumed to be 1, while Λ is usually named the *cosmological constant*.

Einstein's field equations for a source free field are

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R - \Lambda g_{ab} = 0\,. \tag{1.8}$$

In the case where the field source is present, the field equations become

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R - \Lambda g_{ab} = \varkappa\, T_{ab}\,, \tag{1.8 bis}$$

where T_{ab} is the energy-momentum tensor of the source.

However, unlike the Einstein tensor G_{ab} which is *conceptually conserved*, the conditions

$$\nabla_a T^a_b = 0 \tag{1.9}$$

are never satisfied in a general coordinates system [5]. Therefore, the Einstein tensor G_{ab} which *intrinsically* obeys a conservation condition inferred from the Bianchi's identities, is generally related with a tensor T_{ab} which obviously *fails to satisfy the same requirement*.

Hence, we are faced here with a major inconsistency in GR which can be removed in the case of a neutral massive source upon a small constraint.

1.2 The tensor density representation

We first set

$$\mathfrak{g}^{ab} = \sqrt{-g}\, g^{ab} \tag{1.10}$$

thus the Einstein tensor density is

$$\mathfrak{G}^{ab} = \sqrt{-g}\, G^{ab}\,, \tag{1.10 bis}$$

$$\mathfrak{G}^c_a = \sqrt{-g}\, G^c_a\,, \tag{1.10 ter}$$

$$\mathfrak{R}^{ab} = \sqrt{-g}\, R^{ab}\,. \tag{1.11}$$

In the density notations, the field equations with a source (1.8) will read

$$\mathfrak{G}^{ab} = \mathfrak{R}^{ab} - \frac{1}{2}\, g^{ab}\,\mathfrak{R} - \sqrt{-g}\, g^{ab}\,\Lambda = \varkappa\,\mathfrak{T}^{ab}\,, \tag{1.12}$$

where $\mathfrak{T}^{ab} = \sqrt{-g}\, T^{ab}$.

2 The new approach on gravity

2.1 The canonical gravity pseudo-tensor

Let us consider the energy momentum tensor for neutral matter density ρ

$$T_{ab} = \rho\, c^2 u_a u_b \tag{2.1}$$

as the right hand side of the standard field equations

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R = \varkappa\, T_{ab}\,. \tag{2.2}$$

The conservation condition for this tensor are written

$$\nabla_a T^a_b = \sqrt{-g}\,\partial_a T^a_b - \frac{1}{2}\, T_{ac}\partial_b\, g_{ac} = 0 \tag{2.3}$$

with the tensor density

$$\mathfrak{T}^a_b = \sqrt{-g}\, T^a_b\,. \tag{2.4}$$

However, across a given hypersurface dS_b, the integral

$$P^a = \frac{1}{c}\int T^{ab}\sqrt{-g}\, dS_b \tag{2.5}$$

is conserved only if [6]

$$\partial_a\,\mathfrak{T}^a_b = 0\,. \tag{2.6}$$

This problem can be cured only if the metric admits a Killing vector field [7]. If this is not so, we write (2.3) for the *bare* matter tensor density

$$\partial_a(\mathfrak{T}^a_b)_{\text{matter}} = \frac{1}{2}\,(\mathfrak{T}^{cd})_{\mathfrak{matter}}\,\partial_b\, g_{cd}\,. \tag{2.7}$$

Inspection then shows that

$$R_{il}\, d\mathfrak{g}^{il} = \sqrt{-g}\left[-R^{ie} + \frac{1}{2}\, g^{ie}R\right] dg_{ie} = \\ = -\varkappa\,(\mathfrak{T}^{ie})_{\text{matter}}\, dg_{ie}\,. \tag{2.8}$$

Taking now into account the Lagrangian formulation for R_{il} which is

$$R_{il} = \frac{d\mathfrak{L}_E}{\mathfrak{g}^{il}} = \partial_k\left[\frac{\partial\mathfrak{L}_E}{\partial(\partial_k\,\mathfrak{g}^{il})}\right] - \frac{d\mathfrak{L}_E}{\partial\,\mathfrak{g}^{il}}\,, \tag{2.9}$$

we obtain

$$-\varkappa\,(\mathfrak{T}^{il})_{\text{matter}}\, dg_{il} = \left\{\partial_k\left[\frac{\partial\mathfrak{L}_E}{\partial\,(\partial_k\,\mathfrak{g}^{il})}\right] - \frac{\partial\mathfrak{L}_E}{\partial\mathfrak{g}^{il}}\right\} d\mathfrak{g}^{il} = \\ = \partial_k\left[\frac{\partial\mathfrak{L}_E\, d\mathfrak{g}^{il}}{\partial\,(\partial_k\,\mathfrak{g}^{il})}\right] - d\mathfrak{L}_E$$

or

$$-\varkappa\,(\mathfrak{T}^{il})_{\mathfrak{matter}}\,\partial_m\, g_{il} = \partial_k\left[\frac{\partial\mathfrak{L}_E\,\partial_m(\partial\mathfrak{g}^{il})}{\partial(\partial_k\,\mathfrak{g}^{il})} - \delta^k_m\,\mathfrak{L}_E\right] = \\ = 2\,\varkappa\,\partial_k(\mathfrak{t}^k_m)_{\text{field}}\,, \tag{2.10}$$

where $(\mathfrak{t}_m^k)_{\text{field}}$ denotes the field tensor density extracted from

$$2\,\varkappa\,(\mathfrak{t}_m^k)_{\text{field}} = \frac{\partial\mathfrak{L}_E\,\partial_m(\partial\mathfrak{g}^{il})}{\partial(\partial_k\,\mathfrak{g}^{il})} - \delta_m^k\,\mathfrak{L}_E \tag{2.11}$$

so that we have the explicit canonical form

$$(\mathfrak{t}_m^k)_{\text{field}} = \frac{1}{2\varkappa}\left\{\frac{\partial\mathfrak{L}_E\,\partial_m(\partial\mathfrak{g}^{il})}{\partial(\partial_k\,\mathfrak{g}^{il})} - \delta_m^k\,\mathfrak{L}_E\right\} \tag{2.12}$$

where

$$\partial_k(\mathfrak{T}_i^k)_{\text{matter}} = \frac{1}{2}\,(\mathfrak{T}^{ek})_{\text{matter}}\,\partial_k\,g_{ei} = -\partial_k(\mathfrak{t}_i^k)_{\text{field}}$$

that is, the required conservation relation is

$$\partial_k\left[(\mathfrak{T}_i^k)_{\text{matter}} + (\mathfrak{t}_i^k)_{\text{field}}\right] = 0\,. \tag{2.13}$$

Looking back of the deduction, (2.12) defines the *canonical gravity pseudo-tensor density of matter*

$$(\mathfrak{t}_m^k)_{\text{pseudogravity}} = \frac{1}{2\varkappa}\left\{\frac{\partial\mathfrak{L}_E\,\partial_m(\partial\mathfrak{g}^{il})}{\partial(\partial_k\,\mathfrak{g}^{il})} - \delta_m^k\,\mathfrak{L}_E\right\}. \tag{2.14}$$

Expressed with the explicit form of the Lagrangian density $\mathfrak{L}_E$ (1.1), (2.14) can be written in the form

$$(\mathfrak{t}_m^k)_{\text{pseudogravity}} = \\ = \frac{1}{2\varkappa}\left(\left\{{}^{k}_{\;il}\right\}\partial_m\,\mathfrak{g}^{il} - \left\{{}^{i}_{\;il}\right\}\partial_m\,\mathfrak{g}^{lk} - \delta_m^k\,\mathfrak{L}_E\right). \tag{2.15}$$

This is the *mixed Einstein-Dirac pseudo-tensor density* [8] which is not symmetric on k and m, and therefore is not suitable for basing a definition of angular momentum on.

Thus, our aim is to look for:

- A *true* tensor;
- A *symmetric* tensor.

2.2 The new canonical tensor

In the density notations, the field equations with a massive source (1.8 bis) can be re-written as

$$\mathfrak{G}^{ab} = \mathfrak{R}^{ab} - \frac{1}{2}\,\mathfrak{g}^{ab}\,\mathfrak{R} - \mathfrak{g}^{ab}\,\zeta = \varkappa\,(\mathfrak{T}^{ab})_{\text{matter}}\,, \tag{2.16}$$

where in place of the constant cosmological term $\Lambda\,\sqrt{-g}$, we have introduced a *scalar density* denoted as

$$\zeta = \Xi\,\sqrt{-g}\,. \tag{2.17}$$

Unlike Λ, the scalar Ξ is slightly space-time variable and can be regarded as a Lagrangian characterizing a *specific vacuum background field.*

We will choose the variation of Ξ as follows

$$\Xi = \nabla_a\,\kappa^a, \tag{2.17 bis}$$

where κ^a is a Killing vector. Hence

$$\zeta = \sqrt{-g}\;\nabla_a\,\kappa^a. \tag{2.17 ter}$$

We will first write the field equations with a massive source together with its gravity tensor density

$$\mathfrak{G}^{ab} = \mathfrak{R}^{ab} - \frac{1}{2}\,g^{ab}\,\mathfrak{R} = \varkappa\left[(\mathfrak{T}^{ab})_{\text{matter}} + (\mathfrak{t}^{ab})_{\text{gravity}}\right] \tag{2.18}$$

where $(\mathfrak{t}^{ab})_{\text{gravity}}$ is related to ζ as

$$\mathfrak{G}^{ab} = \mathfrak{R}^{ab} - \frac{1}{2}\,g^{ab}\,\mathfrak{R} = \varkappa\left[(\mathfrak{T}^{ab})_{\text{matter}} + \frac{g^{ab}\zeta}{2\varkappa}\right]. \tag{2.19}$$

Re-instating the term ζ accordingly, the gravitational field tensor density now reads

$$(\mathfrak{t}_m^k)_{\text{gravity}} = \frac{1}{2\varkappa}\left\{\frac{\partial\mathfrak{L}_E\,\partial_m(\partial\mathfrak{g}^{il})}{\partial(\partial_k\,\mathfrak{g}^{il})} - \delta_m^k\,(\mathfrak{L}_E - \zeta)\right\}. \tag{2.20}$$

A first inspection shows that ζ represents the Lagrangian density of the background field, therefore the modified field equations (2.19) should be derived from an Einstein Lagrangian density different from $\mathfrak{L}_E$ (1.1) and which includes ζ.

By choosing the form (2.17 ter), we check that

$$\zeta = \sqrt{-g}\;\nabla_a\,\kappa^a = \partial_a\left(\sqrt{-g}\;\kappa^a\right).$$

Now, if we write the new action as

$$\mathcal{S}_M = \int \mathfrak{L}_M\,d^4x = \int \mathfrak{L}_E\,d^4x + \int \partial_a\left(\sqrt{-g}\;\kappa^a\right)d^4x$$

due to Gauss' theorem we see that the last integral can be transformed in an integral extended to an hyperfurface which does not contribute in the variation of $\mathcal{S}_M$ and

$$\delta\int \mathfrak{L}_M\,d^4x = \delta\int \mathfrak{L}_E\,d^4x\,.$$

Therefore, it is legitimate to maintain $(\mathfrak{t}_m^k)_{\text{gravity}}$ as per (2.20).

The presence of the scalar density ζ characterizing the background field is here of central importance, as it means that $(\mathfrak{t}_m^k)_{\text{gravity}}$ can never be zero in contrast to the classical theory where the gravitational field is only described by an awkward *pseudo-tensor*.

The quantity $(\mathfrak{t}_m^k)_{\text{gravity}}$ constitutes thus a *true* tensor density describing the gravity field attached to the neighbouring matter.

It is then easy to show that we have the conserved quantity

$$\partial_a\left[(\mathfrak{T}_a^b)_{\text{matter}} + (\mathfrak{t}_a^b)_{\text{gravity}}\right] = 0\,. \tag{2.21}$$

In this picture and examining (2.20), we clearly see that the gravitational field of matter appears as an *excited state* of the homogeneous background energy field which permanently fills the physical vacuum.

Far from its matter source, the field sharply decreases down to the level of the background field described by the tensor density $(\mathrm{t}^{ab})_{\text{background field}}$. Therefore the "source free" field equations should always retain a non-zero right hand side according to

$$\mathfrak{G}^{ab} = \mathfrak{R}^{ab} - \frac{1}{2}\, g^{ab}\, \mathfrak{R} = \varkappa\, (\mathrm{t}^{ab})_{\text{background field}} \tag{2.22}$$

which are the equivalent of (1.8)

$$\mathfrak{G}^{ab} = \mathfrak{R}^{ab} - \frac{1}{2}\, g^{ab}\, \mathfrak{R} = \varkappa\, \frac{g^{ab}\zeta}{2\varkappa}\,. \tag{2.23}$$

In this case, the conservation law applied to the right hand side of the tensor density field equations is straightforward

$$\partial_a (\mathrm{t}^b_a)_{\text{background field}} = \partial_a \left(\frac{\zeta}{2\varkappa}\, \delta^b_a \right) = 0\,. \tag{2.24}$$

2.3 Symmetrization of the gravity tensor

Let us consider the new gravity tensor expressed with the explicit form of the Lagrangian density $\mathfrak{L}_E$ (1.1):

$$(\mathrm{t}^k_m)_{\text{gravity}} = \\ = \frac{1}{2\varkappa} \left[\left\{ {}^{k}{}_{il} \right\} \partial_m\, \mathfrak{g}^{il} - \left\{ {}^{i}{}_{il} \right\} \partial_m\, \mathfrak{g}^{lk} - \delta^k_m \left(\mathfrak{L}_E - \zeta \right) \right]. \tag{2.25}$$

Like we mentioned, this tensor includes the Einstein-Dirac pseudo-tensor which is not symmetric. We can however follow the *Belinfante procedure* used to symmetrize the canonical tensor $(\Theta^k_m)_{\text{gravity}}$ that extracted from $(\mathrm{t}^k_m)_{\text{gravity}} = \sqrt{-g}\, (\Theta^k_m)_{\text{gravity}}$.

The *total angular momentum* is known to be the sum

$$M^{cba} = x^b (\Theta^{ca})_{\text{gravity}} - x^a (\Theta^{cb})_{\text{gravity}} + S^{cab}, \tag{2.26}$$

where S^{cab} is the contribution of the *intrinsic angular momentum*. By definition,

$$S^{cab} = -S^{cba}.$$

Local conservation of the total angular momentum, i.e. $\nabla_c M^{cab} = 0$, requires that

$$\nabla_c S^{cab} = (\Theta^{ab})_{\text{gravity}} - (\Theta^{ba})_{\text{gravity}}\,. \tag{2.27}$$

We now add a tensor Υ^{bca} which is antisymmetric with respect to the first two indices b, c:

$$(\mathrm{t}^{ca})_{\text{gravity}} = (\Theta^{ca})_{\text{gravity}} + \nabla_b\, \Upsilon^{bca}, \tag{2.28}$$

where

$$\Upsilon^{cba} = \frac{1}{2} \left(S^{cba} + S^{bab} - S^{acb} \right). \tag{2.29}$$

The $(\mathrm{t}^{ab})_{\text{gravity}}$ should be identified to the *Belinfante-Rosenfeld tensor* [9] which is found to be symmetric.

In addition, the antisymmetry of Υ^{cba} guarantees that the conservation law remains unchanged

$$\nabla_a (\Theta^a_b)_{\text{gravity}} = \nabla_a (\mathrm{t}^a_b)_{\mathfrak{gravity}} = 0\,. \tag{2.30}$$

Staying far distant from matter (unexcited state), we have

$$(\Theta^{ab})_{\text{gravity}} \longrightarrow (\mathrm{t}^{ab})_{\text{background field}}\,, \qquad \Upsilon^{cba} = 0\,.$$

By essence, $(\mathrm{t}^{ab})_{\text{background field}}$ is thus symmetric.

Conclusions and outlook

Like we mentioned in an earlier publication, from the beginning of General Relativity, the cosmological constant Λ has played an unsavory rôle Einstein included this constant in his theory, because he wanted to have a cosmological model of the Universe which he wrongly thought static. Shortly after the works published by De Sitter and Lemaître, he decided to reject it.

But to-day, despite its smallness, a term like Λ seems to be badly needed to explain some astronomical observations, all related with the basic dynamical expanding model (Robertson-Walker et al.), even though its occurrence was never clearly explained.

In the classical General Relativity, the space-time is either filled with ponderomotive energy or devoid of source, which is accepted as a physical vacuum. However, numerous experiments predict that quantum vacuum is not "empty" but permanently subjected to virtual particles exchanges of energy.

Heisenberg's Uncertainty Principle, which allows for this process to take a place, has not been used in our demonstration, but it certainly plays a role in the variable property of the cosmological background field which our study relied on.

To sum up all that above, we have eventually reached the following important results:

- The gravitational energy can be represented by a true tensor;
- Its nonlocalizability doesnot hold anymore;
- The existence of a vacuum field is inferred from GR, which confirms the quantum predictions.

This last conclusion is noteworthy since our theory shows that General Relativity and Quantum Physics have convergent results.

Submitted on July 8, 2016 / Accepted on July 14, 2016

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