On the Fluid Model of the Spherically Symmetric Gravitational Field

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The radial flow within the frame of analogue hydrodynamic approach to gravitational field with spherical symmetry is reviewed. Such alternative models of gravity, for example the river model of black holes and the analogue gravity, do not satisfy the continuity equation for the radial fluid flow. The presented model considers a case of incompressible fluid with non-zero source-sink field that can reconcile the continuity equation with the analogue gravity. Based on modelling of a fluid parcel's evolution with time, three cases are reviewed resulting in the Schwarzschild, the Schwarzschild-de Sitter (SdS) and the Schwarzschild-Anti de Sitter(AdS) metrics. The parameters of the model are exactly determined. The model can support a view on the de Sitter cosmology and can serve as its alternative interpretation via such hydrodynamic approach.

1 Introduction

General Relativity (GR) is a widely accepted theory of gravitation. However, in spite of its mathematical beauty and concordance with experiments, as it is well known, it also has a few difficulties: first of all, it is still problematic to merge GR with quantum mechanics; secondly, GR is not fully sufficient in explaining few observable effects in the cosmology (such as rotation curves of the galaxies); and lastly it is not a singularity-free theory. In this article an alternative approach to gravitation based on the fluid/aether model is reviewed.

Such interpretations (not dismissing GR) always existed in parallel, starting from Lenz and Sommerfeld who reported his ideas in Lectures on Theoretical Physics [12] in 1944. In the 1960s, a number of authors discussed this topic following Lenz's idea, see [10, 11]. The approach uses Special Relativity (SR) only to derive the same results as GR [3–5,7,9]. Even if this model still captures the interest of the researchers, it is not widely accepted, and usually is considered through the prism of a "heuristic" approach as it was reviewed in [13].

Such four-vector model of gravity describes a spherically symmetric gravitational field via the Lorenz invariant four-potential which are the same as the components of four-vector "aether" velocity

$$v^\alpha = \left(\frac{\phi}{c}, v^r, v^\varphi, v^\theta\right) \quad (1)$$

where ϕ is the scalar gravitational potential *, and

$$v = \sqrt{\frac{2Gm}{r}} \quad (2)$$

is the radial velocity as measured by co-moving observer given for the case of a static, non-rotating mass m without charge and $v^\varphi = v^\theta = 0$. The velocity in case of the Kerr-Newman metric is obtained in [6], and in case of the de Sitter metric is reviewed in [3]. According to such approach the curvature of spacetime is the consequence of movement of some medium (or even space itself [2]). The concept implies that *something moves and therefore space curves*, [4–6]. Due to this motion the special relativistic length contraction leads to spatial curvature in gravity and the special relativistic time dilation causes time dilation in gravitational field respectively.

The Schwarzschild metric written in the (− + + +) sign convention generated by radial flow is given by

$$ds^2 = -c^2\left(1 - \frac{v^2}{c^2}\right)dt^2 + \left(1 - \frac{v^2}{c^2}\right)^{-1} dr^2 + r^2\, d\Omega^2 \quad (3)$$

where $d\Omega^2 = \sin^2\theta\, d\phi^2 + d\theta^2$ and the coordinate velocity is given by (2). Even if such model fully suffices to describe all effects of GR, it has two drawbacks: first, it is based on the abstract concept of moving space and does not hypothesize about the nature of what moves. It should be *something* that moves instead of nothing. Secondly, it is applicable to spherically symmetrical fields only. The second point is not as solid as the first one, because most of the objects in the universe demonstrate spherical symmetry, especially in the physics of elementary particles where the phenomena of gravitation originates.

2 The analogue gravity and its problem with the hydrodynamic continuity equation

Though, even if the ideas for a fluid theory of the gravitation were reported before [16], recently, as a continuation and generalization of such approach, the analogue gravity model was proposed [1, 14, 15]. It is based explicitly on *fluid hydrodynamics*, and it uses the acoustic metric for a moving fluid in general form (not only for spherically symmetric case) as

$$g^{\mu\nu} = \frac{\rho}{c}\begin{bmatrix} -\left(c^2 - v^2\right) & \vdots & -v^j \\ \cdots\cdots & . & \cdots\cdots \\ -v^i & \vdots & \delta^{ij} \end{bmatrix}. \quad (4)$$

*For example, the reader may check that such effective potential given by $(v_0 c)$ (its second term of the Taylor series) leads to the correction of Newtonian potential and to the same result for the anomalous perihelion precession of Mercury as GR.

In spherically symmetric case it suggests that density of the fluid should change as $r^{-3/2}$ and therefore the conformal factor appears as in the acoustic metric as

$$ds^2 \propto r^{-3/2}\left[-c^2\left(1-\frac{v^2}{c^2}\right)dt^2+ +\left(1-\frac{v^2}{c^2}\right)^{-1}dr^2+r^2d\Omega^2\right]. \quad (5)$$

Then it creates an issue for the metric itself. The suggested workaround [1] is to represent the fluid density as perturbation $\rho = \rho_0 + \rho'$ i.e. as linearized fluctuations around the background value. This is good to model the metric in approximation but again the first term does not satisfy the continuity equation.

It should be noted that such value for the velocity (2) in the frame of the fluid analogue model of gravity is not derived from any hydrodynamic equation. Moreover the inflow through the sphere of radius r as $4\pi r^2 b = r^{3/2}$ is clearly incompatible with the continuity equation. The presented approach suggests to resolve the conformal factor problem in the analogue gravity by conjecturing the fluid's *constant density* and sink-source term in the continuity equation which represents an evolution of fluid parcel's volume with time in the Lagrangian frame .

3 The continuity equation for the model

Let's consider an ideal inviscid isentropic fluid. In Lagrangian co-moving frame of reference the use of relativistic equation of the continuity is not *required* and also because, as discussed in [4], the metric in the co-moving frame is flat. In case of presence of sink-source term the equation of continuity in Lagrangian frame is

$$\frac{\partial\rho}{\partial t} + \rho\nabla\cdot(\vec{v}) = \sigma \quad (6)$$

where σ is the sink-source term. In case of constant density ρ_0 it reduces to

$$\nabla\cdot(\vec{v}) = \frac{\sigma}{\rho_0} = \frac{\partial\dot{V}}{\partial V} \quad (7)$$

where the rate of volume production per time within a control volume was denoted as $\dot{V}$. Let's now consider the spherically symmetric case and take some volume with radius r. Using the Gauss-Ostrogradsky theorem then

$$4\pi r^2 v(r) = \dot{V}(r) = \frac{1}{\rho_0}\int_0^r \sigma(r)4\pi r^2 dr \quad (8)$$

where $\dot{V}$ represents the total volume integral of sink-sources σ within a sphere of radius r. So the radial velocity can be obtained from (8) as

$$v(r) = \frac{\dot{V}}{4\pi r^2}\,. \quad (9)$$

In (9) the rate of volume production is a function of time in Lagrangian frame $\dot{V}(t)$, or in Eulerian frame is a function of only radial distance $\dot{V}(r)$ respectively, and the flow is stationary.

It is important to make note on a sign of the velocity (2). The approach is valid for both – for positive and negative values of the velocity (2) because it comes to the metric (3) as squared value. Many authors treat the river model of gravity with radial flow going in inward direction to the center of gravity. However, in the present model it is considered opposite – the outward flow of the fluid and the positive sign for velocity (placing coordinate center at the point mass) which means that the flow is decelerated going from the point mass center and has also negative acceleration.

4 The linear model, the Schwarzschild metric

Let's now consider the point mass m and the spherical coordinate center is placed in m. The point mass m emits the volume parcels V_n of the fluid at some constant rate ω_m with initial position $r = 0$ and time $t = 0$. The parameter ω_m is denoted in such way because of an assumption that it depends on the property of point mass itself or even may be linearly proportional to the value of point mass m. So every time interval

$$\Delta t = 1/\omega_m\,, \quad (10)$$

one n^{th} parcel of the fluid V_n appears near the point m and no initial velocity is considered. Following the above, let's assume that every parcel V_n further grows linearly with time in its respective Lagrangian frame as *

$$V_n = \omega V_0 t \quad (11)$$

where $V_0 = m_0\rho_0$ and ω are some external constants which do not depend on the property of point mass, and ω is in the same way linearly proportional to a parameter m_0. Then the total number of produced parcels during time t is

$$n = \omega_m t\,. \quad (12)$$

So, the volume of n^{th} parcel in row is given by

$$V_n = \frac{\omega}{\omega_m}V_0 n\,. \quad (13)$$

Importantly, time in Lagrangian frame (local co-moving frame of every fluid's parcel) is synchronized with time of the observer resting at infinity (see [5] for more details on this). So, the time interval given by (10) is the same in the co-moving frame of parcel as well as in the reference frame of point mass.

In order to find $\dot{V}$ within a sphere of some fixed radius r, first a total volume produced by sum of all such parcels has

*For simplicity one can imagine the emitted volume parcels V_n as growing spherical bubbles, though fluid parcels have no actual form.

to be defined. Summation of (13) yields

$$V(t) = \sum_1^n V_n = \frac{\omega}{\omega_m} V_0 \frac{n^2}{2} = \frac{1}{2}\,\omega_m \omega V_0 t^2 \tag{14}$$

where an approximation that $n \approx n + 1$ for a relatively big number of parcels was used. Taking time derivative and substituting into (9) leads to

$$\upsilon = \frac{dr}{dt} = \frac{\omega_m \omega V_0 t}{4\pi r^2}\,. \tag{15}$$

Solving this differential equation for $r(t)$ one can find the equation of motion for the fluid as

$$r(t) = \left(\frac{3\omega_m \omega V_0 t^2}{8\pi} + c_1\right)^{1/3} \tag{16}$$

where c_1 is an arbitrary constant and represents initial position of parcel at time $t = 0$ which has to be zero, so c_1 is zeroed. Expressing $t(r)$ from (16) and substituting this into the original equation (15) results in the fluid velocity $\upsilon(r)$ in Lagrangian frame as

$$\upsilon = \frac{dr}{dt} = \left(\frac{1}{6\pi}\frac{\omega_m \omega V_0}{r}\right)^{1/2}. \tag{17}$$

So as a result, the radial velocity is inversely proportional to the square root of the radial distance as (2), which reproduces the Schwarzschild metric. But still, the unknown parameters in the expression are to be determined.

The fluid acceleration is

$$\frac{d\upsilon}{dt} = \frac{\partial \upsilon}{\partial t} + (\upsilon\nabla)\,\upsilon\,. \tag{18}$$

For a stationary radial flow the acceleration is given only by the convective term, therefore

$$a = \nabla\left(\frac{\upsilon^2}{2}\right) = -\frac{1}{12\pi}\frac{\omega_m \omega V_0}{r^2}\,. \tag{19}$$

This acceleration is negative for the positive value of the velocity (17), and as the coordinate center was placed in the center of mass m, it means that the flow is decelerated in outward direction. However, as it was noted above, the corresponding metric (3) remains the same regardless of the velocity sign.

5 The volume conversion relation and the uncertainty principle

Let's introduce the volume V_m such as

$$V_m = \frac{m}{\rho_0} \tag{20}$$

where m is the mass of the point source. And let's assume that ω_m represents de Broglie wave frequency of the mass m, and m_0 is given by the uncertainty principle with rigorous factor of two (where it originates because of the non-commutativity of the quantum operators [8]) as

$$m_0 c^2 = \rho_0 V_0 = \frac{1}{2}\,\hbar\omega\,. \tag{21}$$

This means that the fluid parcel's mass m_0 is not observable during the time ω^{-1}. Then

$$V_m \omega = 2\omega_m V_0\,. \tag{22}$$

Further this expression will be referred as the volume conversion relation with the exact factor of two. Therefore (17) becomes

$$\upsilon = \left(\frac{\omega^2}{12\pi\rho_0}\frac{m}{r}\right)^{1/2}. \tag{23}$$

Regarding the mass-energy conservation, the point mass m does not act as actual source studied in classical fluid dynamics, because at time $t = 0$ an outgoing parcel has zero volume $V_n = 0$ and zero mass accordingly, therefore there is no actual mass flow from the point mass m. The linear mass growth of a parcel is also governed by the uncertainty principle and it is not observable during the time ω^{-1}.

6 The hyperbolic model, the SdS metric

Presumably the linear dependency of $V_n(t)$ in the model above can be just an approximation of some unknown odd function and the linear function of t in (11) represents just a first term of its Taylor series. Choosing to test the hyperbolic sine one may assume that V_n changes with time in its respective Lagrangian frame as

$$V_n = V_0 \sinh(\omega t)\,. \tag{24}$$

Considering that time in co-moving frame of parcel now is not synchronized with time running at the clock of the observer at rest at infinity, but the time coordinate transform is given by

$$t' = \frac{1}{\omega}\sinh(\omega t) \tag{25}$$

where t' is proper time in co-moving parcel's frame.

Following the same procedure, as in the previous model, the total number of produced fluid parcels during time t is given by (12). And the volume of n^{th} parcel in row is given by

$$V_n = V_0 \sinh\left(\frac{\omega}{\omega_m} n\right). \tag{26}$$

The sum of all such parcels provides the total volume produced by time t as

$$V(t) = \sum_1^n V_n = V_0 \frac{\sinh^2\left(\frac{n}{2}\frac{\omega}{\omega_m}\right)}{\sinh\left(\frac{1}{2}\frac{\omega}{\omega_m}\right)} \tag{27}$$

where $n \approx n+1$ for a relatively big number of parcels. The value of $\sinh\left(\frac{1}{2}\frac{\omega}{\omega_m}\right)$ is very small and can be easily approximated without a loss of precision as $\frac{1}{2}\frac{\omega}{\omega_m}$ *. Then, using trigonometric identity and t instead of n let's rewrite (27) in simpler form as

$$V(t) = \frac{\omega_m V_0}{\omega}\left(\cosh(\omega t)+1\right) \tag{28}$$

where factor 1/2 disappears because of the trigonometric conversion. Taking time derivative and using the volume conversion relation (22) it becomes

$$\dot{V} = \frac{1}{2}\,\omega V_m \sinh(\omega t)\,. \tag{29}$$

With the use of (9) the differential equation is

$$v = \frac{dr(t)}{dt} = \frac{\omega V_m \sinh(\omega t)}{8\pi r(t)^2}\,. \tag{30}$$

Solution for $r(t)$ provides the equation of motion as

$$r(t) = \left({r_0}^3 + \frac{3V_m\cosh(\omega t)}{8\pi}\right)^{1/3}. \tag{31}$$

Applying boundary condition as $r = 0$ when $t = 0$ the equation of motion becomes simply

$$r(t) = \left(\frac{3V_m}{8\pi}\right)^{1/3}\left(\cosh(\omega t)-1\right)^{1/3}. \tag{32}$$

Expressing the hyperbolic sine from this and then substituting it into (30) leads to

$$v(r) = \left(\frac{V_m\omega^2}{12\pi r} + \frac{\omega^2 r^2}{9}\right)^{1/2} \tag{33}$$

or with use of the definition of V_m (20) the resulting radial velocity is

$$v(r) = \left(\frac{\omega^2}{12\pi\rho_0}\frac{m}{r} + \frac{\omega^2 r^2}{9}\right)^{1/2}. \tag{34}$$

So the hyperbolic model leads to the same radial velocity as in the previous model (23), but with the additional term. Using (18) the fluid acceleration is

$$a = -\frac{\omega^2}{24\pi\rho_0}\frac{m}{r^2} + \frac{\omega^2 r}{9}\,. \tag{35}$$

7 Determination of the model parameters

The association of the first term in (35) with Newtonian gravitational acceleration allows expressing the value for fluid density via ω as

$$\rho_0 = \frac{\omega^2}{24\pi G}\,. \tag{36}$$

*For example for the proton mass such approximation would give an error of order less than 10^{-40}.

Then substituting ω from this into the second term of (35) gives the repulsive acceleration as

$$a_{rep} = \frac{8\pi}{3}\,\rho_0 G r\,. \tag{37}$$

This term can be also treated as the Newtonian gravitational force from uniformly distributed mass that has the equation of state $p = -\rho c^2$ and satisfies stress-energy equivalent

$$\rho_0 + \sum_i \frac{p_i}{c^2} = -2\rho_0 \tag{38}$$

as given in [13, see the expressions (45–46)]. Assuming the constant density ρ_0 (36) is equal to the critical density, the value for ω can be defined via the Hubble constant as

$$\omega = 3H\,. \tag{39}$$

And the repulsive acceleration as given by (35) is

$$a_{rep} = H^2 r = \frac{c^2\Lambda}{3}\,r\,. \tag{40}$$

The radial velocity of the fluid (34) based on (3) and using (39) leads to

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Gm}{c^2 r}-\frac{H^2r^2}{c^2}\right)c^2dt^2 + \\ &+\left(1-\frac{2Gm}{c^2 r}-\frac{H^2r^2}{c^2}\right)^{-1}dr^2 + r^2d\Omega^2 \end{aligned} \tag{41}$$

that corresponds to the Schwarzschild-de Sitter metric for the hyperbolic model.

8 The harmonic model, the Schwarzschild-AdS metric

Using the sine function in (25) which could be treated as a simple harmonic oscillation of a fluid parcel volume $V_n(t)$. Following the same procedure (substituting sinh() with sin() instead) it is easy to see that the result would be the same as it was in previous model (34) but with a difference in sign of the second term

$$v(r) = \left(\frac{\omega^2}{12\pi\rho_0}\frac{m}{r} - \frac{\omega^2 r^2}{9}\right)^{1/2} \tag{42}$$

which with the use of (39) and (3) obviously leads to the Schwarzschild-Anti de Sitter metric.

9 Conclusions

The model results in full accordance with known metrics with exact accuracy by the coefficients based on assumptions of the volume conversion equation (22) and of the equality of the fluid density to the critical density value. The forces, the Newtonian gravitational and the repulsive cosmological, both appear natively in the hyperbolic model. Therefore the model may support a view on applicability of the static de Sitter metric for cosmology. In presented approach the de Sitter Universe is also empty in the sense that the mass of the matter is attributed to the medium with constant density ρ_0. While the matter objects may reside statically at the fixed coordinates of the metrics (41), the space-time curvature (resulting in both attractive gravitation and repulsion) originates in a motion of the medium. The equation of state and the stress-energy of such fluid were suggested (38). However, one should be cautious to apply GR for further analysis of the solutions, because only Special Relativity is considered in the frame of the present approach.

The fluid parcels can be treated as virtual particles emitted by an elementary particle with the constant rate given by the de Broglie frequency, and on the other hand they can be considered as "growing bubbles of space". An individual parcel is not observable during the cosmological time, and its mass and volume are constrained by the uncertainty principle as shown.

The evolution of parcel's volume with time was modelled by odd functions. The odd functions have property of being asymmetric under time-reversal transformation. The requirement for such time asymmetry to generate velocities applicable to describe different metrics for gravitational field could be a topic for future study. Further analysis is required on finite boundary conditions (when a fluid parcel originates at time $t = 0$ at finite radius) and on corresponding event horizons. The temporal coordinate transform (25) as a base of the hyperbolic model, a possible correspondence of the cosmological scale factor to the proposed volume increase require further analysis.

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