From the Geometry of the FLRW to the Gravitational Dynamics

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The approach when the scale factor that describes the expansion of space, being its pure geometrical property, is derived from the dynamical (the Friedman) equations is questioned. The opposite path when the geometry determines the dynamics is more consistent, but not vice versa. Starting from the FLRW, the equivalent form of the metric in static coordinates is proposed. Based of few models for $a(t)$ the corresponding static metrics are derived. Further dynamics and the analogue of the Friedman equations can be obtained as consequence. The embedding of the FLRW geometry into the higher-dimensional Minkowski space as the hypersurface can be considered as the base for the model. The deceleration parameter for the Schwarzschild-de Sitter (SdS) case is reviewed based on such approach.

1 Introduction

In recent author's work [5] the hydrodynamic model of spherically symmetric gravitational field was reviewed. As it was shown the gravitational metrics can be modelled by expanding parcels of the fluid based on the respective functions of the volume change with time in co-moving frame. As it has explicit similarity with the space expansion, the present attempt is to use the geometrical approach to describe spherically symmetric gravitational filed starting from the FLRW metric.

2 The FLRW metric

Let's consider the static pseudo-Minkowski coordinates with the observer M at rest in the center. The static spherical coordinates are to be denoted as t, r, θ, ϕ, where r is coordinate distance to the observer P who is at rest, but is attached to the point of expanding space. The co-moving coordinates are given as T, R, θ, ϕ, where R is co-moving distance (from M to P). Respectively, time T is measured by the observer P. If space expands, the point P, attached to it, moves in the static coordinate system, so as observed by M, the motion of P represents the function of coordinate distance $r(t)$. The correspondence between the static coordinate and the co-moving distance is given by

$$r = Ra \tag{1}$$

where $a(T)$ is the scale factor. Then the proper velocity measured by the observed P,*

$$v = \frac{dr}{dT} = \frac{dR}{dT}\,a + R\,\frac{da}{dT} \tag{2}$$

and point P is at rest in its reference frame, so the first term is identically zero therefore

$$v = R\dot{a}\,. \tag{3}$$

*This is not coordinate velocity. This velocity is ratio of coordinate distance change to time measured in co-moving observer's clock.

Using (1) then

$$v = \frac{dr}{dT} = r(T)\,\frac{\dot{a}}{a}\,. \tag{4}$$

This expression provides the velocity of the space motion due to its expansion or the velocity of the reference frame attached to point P in the static coordinate system where r is the coordinate distance.

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric in the spatially flat case ($k = 0$) is given by

$$ds^2 = -c^2 dT^2 + a(T)^2\left(dR^2 + R^2 d\Omega^2\right) \tag{5}$$

where $d\Omega^2 = \sin^2\theta d\phi^2 + d\theta^2$ and $a(T)$ is the scale factor. The metric is written explicitly in comoving coordinates, attached to the point of expanding space. Using (1) we may write

$$dr = \dot{a}RdT + adR$$

from which

$$dR = \frac{dr}{a} - v\,\frac{dT}{a}\,.$$

Substituting it into the FLRW metric (5) leads to

$$ds^2 = -c^2\left(1 - \frac{v^2}{c^2}\right)dT^2 - 2v\,dT\,dr + dr^2 + r^2 d\Omega^2 \tag{6}$$

which is the Gullstrand-Painlevé form of the metric which is spatially flat and describes co-moving observer in its free float with velocity v. The transformation of time coordinate T from co-moving to fixed frame of reference t is given by

$$dT = dt - \frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1} dr\,. \tag{7}$$

The substitution of this expression into (6) leads to the respective static metric

$$ds^2 = -c^2\left(1 - \frac{v^2}{c^2}\right)dt^2 + \left(1 - \frac{v^2}{c^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{8}$$

where velocity v is to be determined from (4). The velocity v is called the river velocity in [2, 4] and the shift in ADM formalism. Importantly, the metric (8) is equivalent to the FLRW, but written in the static coordinate systems of the observer M. Such form of the metric is known, starting from Lenz and Sommerfeld [11] and used in the river model of black holes and similar analogous models [3, 4] for the spherically symmetric gravitational field.

The proposed approach starts from a certain function for the scale factor $a(T)$, and then the solution of the equation (4) provides the velocity $v(r)$ resulting in the corresponding metric in static coordinate system based on (8).

As it was stressed in the author's previous work [5], the sign of the velocity v does not play a role, as it comes to the metric as squared value. In the present approach the value of the velocity as given in (4) is obviously positive ($\dot{a} > 0$) and as coordinate center is placed in the center point of M the velocity is radial and directed outwards.

3 The case of the de Sitter metric

The easiest case to demonstrate the proposed approach is the de Sitter metric. The starting point is $a(T) = e^{H_0 T}$, or equivalently, the constancy of the Hubble constant with time

$$H_0 = \frac{\dot{a}}{a} \,. \tag{9}$$

Then using (4) gives

$$v = rH_0 \,. \tag{10}$$

And substitution into (8) leads to

$$ds^2 = -\left(1 - \frac{H_0^2 r^2}{c^2}\right) c^2 dt^2 + \left(1 - \frac{H_0^2 r^2}{c^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{11}$$

which is the de Sitter metric as expected.

4 The case of the Schwarzschild metric

Let's now assume that

$$a(T) \propto T^{2/3} \,. \tag{12}$$

Then, using (4), it follows that

$$v(r) = \frac{c_1}{r^{1/2}} \tag{13}$$

where c_1 - is an integration constant. Then the substitution into (8) leads to the form the Schwarzschild metric with precision by constant c_1. In order to determine the meaning of the integration constant, it is required to normalize (12), for example in such way that $a(0) = 1$

$$a = (\omega T + 1)^{2/3} \tag{14}$$

where ω is some constant. Then it would imply

$$r = r_0\,(\omega T + 1)^{2/3} \tag{15}$$

where r_0 is the initial coordinate distance at time $T = 0$. Then the velocity

$$v = \frac{2}{3}\left(\frac{\omega^2 r_0^3}{r}\right)^{1/2} \,. \tag{16}$$

The equation shows that the integration constant in (13) should have correspondence to the initial volume and therefore to the central mass, if one introduces a density in the equation. Proposed boundary conditions allow to put the scale factor function in direct relation with the particle mass and to remove the initial singularity.

Interestingly from (13) and (1) the scale factor in terms of the coordinate distance is simply $r = r_0 a$. From that, using (1), the co-moving distance is $R = r_0$. As expected, the scale factor a changes with time instead of the co-moving distance R which remains constant and always equals to its initial value in the static coordinates.

5 The Schwarzschild-de Sitter (SdS) metric

As suggested in [10] the scale factor that describes current Universe expansion within the frame of standard model of the cosmology has following form

$$a(T) = \sinh\left(\frac{3}{2} H_0 T\right)^{2/3} \,. \tag{17}$$

This corresponds (differing by factor of 2) to proposed in the hyperbolic model [5]*

$$a\,(T) = (\cosh(3H_0 T) - 1)^{1/3} \,. \tag{18}$$

Then using (4)

$$v = \frac{dr}{dT} = r_0\, \frac{H_0 \sinh(3H_0 T)}{(\cosh(3H_0 T) - 1)^{2/3}} \tag{19}$$

from which

$$r(T) = r_0\,(\cosh(3H_0 T) - 1)^{1/3} \tag{20}$$

where r_0 is integration constant with dimension of length. Expressing hyperbolic sine from this and substitution into (19) leads to

$$v = \left(H_0^2 r^2 + \frac{2 r_0^3 H_0^2}{r}\right)^{1/2} \,. \tag{21}$$

Exact determination of the constant r_0 for the volume can be found in [5]. It was suggested that such volume can be associated with the mass via the fluid density. The substitution into (8) leads to the SdS metric

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Gm}{c^2 r} - \frac{H_0^2 r^2}{c^2}\right) c^2 dt^2 + \\ &+\left(1 - \frac{2Gm}{c^2 r} - \frac{H_0^2 r^2}{c^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \,. \end{aligned} \tag{22}$$

*Obviously the presented approach has direct correspondence to the cited author's fluid model via $\dot{V} \propto a^2 \dot{a}$ and $V(t) \propto a^3$.

6 The embedding the FLRW geometry

The embedding of the de Sitter geometry in the pseudo-Euclidian five-dimensional space is well known and was obtained by Robertson [7, 8]. This corresponds to embedding of the spatially flat FLRW metric with $a(t) = e^{H_0 t}$. However, as demonstrated in [9] and reviewed in [1] the generalization of the FLRW metric ($k = 0$) embedding is possible via reconstruction of the respective curve and the Minkowski five-dimensional metric is

$$t' = \frac{1}{2}\int \frac{\dot{a}^2 - 1}{\dot{a}}\, dT\,, \quad r' = \frac{1}{2}\int \frac{\dot{a}^2 + 1}{\dot{a}}\, dT\,, \qquad \text{and } x' = x \quad y' = y \quad x' = z\,. \tag{23}$$

The embedding of the FLRW metric with the hyperbolic function as (17) was reviewed in [6], however it was concluded that the integral has no analytical expression.

7 On the deceleration parameter for the SdS metric

Presented approach provides a simple way to determine the deceleration parameter

$$q_0 = -\frac{\ddot{a}a}{\dot{a}^2}\,. \tag{24}$$

And as

$$\alpha = \ddot{a}R \quad \upsilon = \dot{a}R \quad r = Ra \tag{25}$$

then for the SdS metric using the deceleration parameter can be expressed via coordinate distance as

$$q_0 = \frac{Gm - H_0^2 r^3}{2Gm + H_0^2 r^3}\,. \tag{26}$$

In case of mass m is uniformly distributed within a sphere and if the density is expressed in terms of $\Omega_M = \rho/\rho_{crit}$ then the deceleration parameter is

$$q_0 = \frac{1}{2}\frac{\Omega_M - 2}{\Omega_M + 1}\,. \tag{27}$$

In case of $\Omega_M = 0.27$ it gives the deceleration parameter $q_0 = -0.68$ which is close to the observed value. As example the equation results in $q_0 = -1$ for empty the de Sitter Universe, and $q_0 = -0.4$ in case of $\Omega_M = 1$.

8 The Friedman equations

In the frame of present approach the dynamical Friedman equations appear as a result of the original scale factor function. In general case, as the resulting metric provides us with the values for acceleration $\alpha(r)$ and the velocity $\upsilon(r)$ and with use of (25) the Friedman equations are derived. In case of the SdS metric, the first Friedman equation can be directly obtained from the result (21)

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[1 + \frac{2}{a^3}\right]. \tag{28}$$

In the reverse way it obviously would reproduce (17). In case of uniformly distributed matter it has following form

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\,(1 + 2\Omega_M)\,. \tag{29}$$

The second Friedman equation is from (21)

$$\frac{\ddot{a}}{a} = H_0^2\left[1 - \frac{1}{a^3}\right] \tag{30}$$

or for uniformly distributed matter in terms of Ω_M

$$\frac{\ddot{a}}{a} = H_0^2\left[-\frac{1}{2}\Omega_M + 1\right]. \tag{31}$$

Another types of functions $a(t)$ can be proposed and in that way would originate different dynamical equations that could be analysed for its compliance with the cosmological observations.

9 Conclusions

The spatial expansion phenomena is considered as the space flow. The curvature of space-time in the static four-dimensional coordinate systems emerges as the consequence of such motion. Then the dynamics and the physical forces are derived from the resulting metric. The scale factor being the primary property of space should have the fundamental significance (instead of being secondary consequence of the dynamical equations). Because of the reviewed boundary conditions the scale factor may originate on the elementary particle level and can be a key for understanding the origin of gravity. The function for $a(t)$ that results in the SdS metric was reviewed, the deceleration parameter is determined (27) and the result is close to the observed value.

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References

1. Akbar M. M. Embedding FLRW Geometries in Pseudo-Euclidian abd Anti-de Sitter Spaces. arXiv: gr-qc/1702.00987v2.
2. Braeck S., Gron O. A river model of space. arXiv: gr-qc/1204.0419.
3. Czerniawski J. The possibility of a simple derivation of the Schwarzschild metric. arXiv: gr-qc/0611104.
4. Hamilton A. J. S., Lisle J. P. The river model of black holes. *American Journal of Physics*, 2008, v. 76, 519–532. arXiv: gr-qc/0411060.
5. Kritov A. On the Fluid Model of the Spherically Symmetric Gravitational Field. *Progress in Physics*, 2019, v. 15 (2), 101–105.
6. Lachiéze-Rey M. The Friedman-Lemaître models in perspective. *Astronomy ans Astrophysics*, 2000, v. 364, 894–900.
7. Robertson H. P. On Relativistic Cosmology. *Phylosophy Magazin*, 1928, v. 5, 835–848.
8. Robertson H. P. Relativistic Cosmology. *Review of Modern Physics*, 1933, v. 5, 62–90.
9. Rosen J. Embedding of Various Relativistic Riemannian Spaces in Pseudo-Euclidean Spaces. *Review of Modern Physics*, 1965, v. 37, 204–214.
10. Sazhin M. V., Sazhina O. S., Chadayammuri U. The Scale Factor in the Universe with Dark Energy. arXiv: astro-ph/1109.2258v1.
11. Sommerfeld A. Electrodynamics. Lectures on Theoretical Physics, Vol. III. Academic Press, New York, 1952.