A Wave Representation for Massless Neutrino Oscillations: The Weak Interaction Transmutes the Wave Function

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There are solutions of the Klein-Gordon equation for the massless neutrino that produce massless neutrino oscillation of flavor. These solutions serve as a counterexample to Pontecorvo, Maki, Nakagawa, and Sakata theory for neutrino oscillation of flavor, which implies neutrinos must have mass contrary to the standard model. We show that the wave function for the massless antineutrino for an inverse β decay (IBD) is a superposition of two independent solutions of the Klein-Gordon equation. One solution represents the latent incident wave upon an IBD. The other solution represents the latent reflected wave from the IBD. This superposition renders a compound modulated wave function with regard to amplitude and phase modulations. This compound modulation is shown to facilitate neutrino oscillation that may be massless and, therefore, consistent with the standard model. Extra to a massless counterexample, the weak interaction is shown to transmute the wave function during an IBD by changing the amounts of the latent incident and latent reflected wave functions that are allocated to the superposition.

1 Introduction

The Pontecorvo, Maki, Nakagawa, and Sakata (PMNS) theory for oscillation of neutrino (ν) flavor implies that the neutrino has a finite mass in contrast to the standard model [1]–[4]. PMNS theory, which was developed in the mid-twentieth century in the absence of a contending theory, soon became preeminent regarding neutrino oscillations including its implication that the neutrino must have a finite mass in order to oscillate. A counterexample to PMNS theory now exists: the quantum trajectory representation of quantum mechanics had predicted in 2017 that massless neutrino oscillation is an alternative possibility that is consistent with the standard model [5]. However, the quantum trajectory representation is presently arcane, for it is couched in a quantum Hamilton-Jacobi formulation [5]–[17]. As a result, PMNS theory has maintained its preeminence on neutrino oscillation. A way to overcome this preeminence is to describe **massless** neutrino oscillation in the more familiar wave function representation, which would be more accessible to a much broader audience. Our objective in this paper is to provide such.

A wave function representation that is a counterexample to PMNS theory is attainable. This theoretical counterexample renders massless neutrino oscillation while also showing that PMNS theory is not the exclusive explanation of neutrino oscillation. In this paper, we show that there are mathematical solutions of wave equations, which to the best of our knowledge have been used only a few times [18]–[23] to describe wave phenomena, and which invite further investigation. We study massless neutrino oscillation with these mathematical solutions of the Klein-Gordon equation for a massless antineutrino. This mathematical solution is synthesized by the superpositional principle from two independent solutions of the Klein-Gordon equation for an antineutrino before encountering a charged current interaction. The two solutions are the latent incident solution and the latent reflected solution. The "quantum action" of the Klein-Gordon equation is composed of both independent solutions of the Klein-Gordon equation [14] and can be seen as the order $\hbar^0$ term of the quantum action of QFT.

Extra to the initial goal of adducing a massless counterexample, the behavior of the synthesized solution also gives insight into the weak interaction (weak force). A byproduct of this investigation shows that the weak interaction without causing any exchange of energy can transmute the Klein-Gordon solution from a synthesized solution to a plane-wave solution.

The particular charged current interaction that we examine herein is the inverse beta decay (IBD) where [24]

$$\overline{\nu}_e + \mathrm{p} \xrightarrow{\mathrm{W}^+ \text{ boson exchange}} \mathrm{e}^+ + \mathrm{n}, \tag{1}$$

in which the antineutrino $\overline{\nu}$ participates as an electron antineutrino $\overline{\nu}_e$. The wave function for $\overline{\nu}$ is specified by $\overline{\psi}$. When $\overline{\nu}_e$ arrives at the point q_b ready for IBD absorbtion in (1), its $\overline{\psi}$ is assumed in this ab initio calculation to be then a traveling complex-exponential plane wave $\exp(ikq)$ with wave number k, in cartesian coordinate q, and tacitly with amplitude 1. While the ab initio calculation develops flavor oscillations for a massless $\overline{\nu}$, the conventional terminology "neutrino oscillation" is retained for referencing the oscillation phenomenon herein.

An outline of the rest of this paper follows. In §2 we develop a model by an ab initio computation for massless neutrino oscillation for an IBD. The wave function for the neutrino is synthesized from the latent solutions for the incident and reflected wave functions by the superpositional principle. The latent incident and latent reflected wave functions are traveling complex-exponential plane waves that are independent one-dimensional solutions of the Klein-Gordon equation. This synthesized solution is shown to be compoundly modulated with regard to amplitude and phase. This compound modulation induces periodic nonuniform propagation that in turn facilitates neutrino oscillation. The amplitude and phase modulations are individually analyzed. We apply the same modulation analyses to the wave function's spatial derivative. In this wave function representation for massless oscillation, the weak interaction changes the synthesized wave function to a traveling complex-exponential plane-wave solution, which is then ready for absorption by the IBD process. In §3, we examine selected didactic examples. The examples show that the individual contributions of phase modulation and amplitude modulation complement each other. Where one modulation is at a peak, the other is at a null. The examples also show that the compound modulations of the wave function and its derivative supplement each other. That is where the amplitude modulation increases dilation in one, it decreases it in the other. And where phase modulation rotates the phase of one clockwise, it rotates the other's phase counterclockwise. In §4 a brief discussion is presented. Together, the complementing and supplementing are shown to facilitate periodic nonuniform propagation that permits massless neutrino oscillation. Findings and conclusions are presented in §5.

2 Ab initio calculation

The one-dimensional stationary Klein-Gordon equation (SKGE) for an antineutrino with mass m and for the Cartesian dimension q is a second-order, linear, homogeneous ordinary differential equation given by [25]

$$-\hbar^2 c^2 \frac{\partial^2 \overline{\psi}(q)}{\partial q^2} + \left(m^2 c^4 - E^2\right) \overline{\psi}(q) = 0 \tag{2}$$

where $\hbar$ is Plank's constant, c is speed of light and E is energy. As such, the superpositional principle applies to the SKGE's solutions. The inertial reference frame for describing $\overline{\psi}$ of (2) is the frame for which the target proton of the IBD is at rest. This makes E dependent on the dynamics of the target proton. The threshold energy for executing an IBD is $E_{\text{threshold}} = 1.806$ MeV for ν_e and progressively greater for the analogous charged current interactions for ν_μ and ν_τ. Herein, it is always assumed the $\overline{\nu}$ has energy greater than the threshold energy. The notation $\overline{\psi}$ denotes that the wave function of the antineutrino is a solution of (2) but does not specify whether it is unispectral, $\overline{\psi} = \exp(ikq)$, or bispectral $\overline{\psi}_2$. Eq. (2) remains well posed should $m = 0$ in agreement with the standard model. Studying the case $m = 0$ is sufficient to render a massless counterexample to PMNS. For antineutrino energy E and nil mass, a set of independent solutions sufficient to solve (2) may be given by $\{\overline{\psi}, \check{\overline{\psi}}\} = \{\exp(+ikq), \exp(-ikq)\}$ where the wave number $k = E/(\hbar c)$.

The incident antineutrino is assumed to propagate in the $+q$ direction toward the target proton of an IBD, while any reflection from an IBD would propagate in the $-q$ direction. The solution $\overline{\psi} = \exp(ikq)$ is a unispectral wave function with one spectral component, $+k$ (the solution of the homogeneous SKGE is defined to within a constant in phase). Its derivative $\partial_q \overline{\psi} = ik\overline{\psi}$ is also unispectral and is displaced in phase from $\overline{\psi}_1$ by a constant $\pi/2$ radians. The amplitude of $\partial_q \overline{\psi}$ relative to that of $\overline{\psi}$ is multiplied by the factor k. Thus, the unispectral $\overline{\psi}(q)$ displays uniform rectilinear motion, which presents a constant relationship

$$\partial_q \overline{\psi} \,/\, \overline{\psi} \;=\; \partial_q \ln(\overline{\psi}) \;=\; ik \tag{3}$$

to any encountered current interactions. The constant character of (3) is expected, for $\overline{\psi}(q)$ is an exponential of the linear variable q. Uniform rectilinear propagation precludes flavor oscillations.

Let the incident antineutrino to an IBD have a bispectral wave function $\overline{\psi}_2$ with spectral components given by wave numbers $\{+k, -k\}$. We can synthesize a bispectral $\overline{\psi}_2$ by the superpositional principal from the set $\{\exp(+ikq), \exp(-ikq)\}$ of independent solutions for the SKGE. The incident bispectral $\overline{\psi}_2$ may be presented in a few representative forms as [5]

$$\overline{\psi}_2 = \overbrace{\underbrace{\alpha \exp(+ikq)}_{\text{latent incident wave}} + \underbrace{\beta \exp(-ikq)}_{\text{latent reflected wave}}}^{\text{bispectral solution of SKGE by superpositional principle}} \tag{4}$$

$$= \underbrace{(\alpha - \beta) \exp(ikq)}_{\text{traveling wave}} + \underbrace{2\beta \cos(kq)}_{\text{standing wave}}$$

$$= \underbrace{(\alpha + \beta) \cos(kq) + i(\alpha - \beta) \sin(kq)}_{\text{coherent standing waves}} \tag{5}$$

$$= \underbrace{A_{\overline{\psi}} \exp(i\, P_{\overline{\psi}}),}_{\text{compoundly modulated traveling wave}} \tag{6}$$

where all forms (4)–(6) are solutions of the SKGE. In (6), $\overline{\psi}_2$ is compoundly modulated for its amplitude $A_{\overline{\psi}}$ and phase $P_{\overline{\psi}}$ are modulated as given by

$$A_{\overline{\psi}} = \overbrace{[\alpha^2 + \beta^2 + 2\alpha\beta \cos(2kq)]^{1/2}}^{\text{amplitude modulation}}$$

and

$$P_{\overline{\psi}} = \overbrace{\arctan\left(\frac{\alpha - \beta}{\alpha + \beta} \tan(kq)\right)}^{\text{phase modulation [5]}}.$$

Eqs. (4)–(6) for the antineutrino's wave function are all representations of a wave function synthesized by the superpositional principle. As such, each individual equation of (4) through (6) represents a synthesized solution of the SKGE consistent with the orthodox interpretation of quantum mechanics. The coefficients α and β respectively specify the amplitudes for the latent incident and reflected waves associated with an IBD. Propagation of the latent incident wave in the $+q$ direction implies that $\alpha^2 > \beta^2$. The coefficients α and β are normalized by

$$\alpha^2 - \beta^2 = 1 \tag{7}$$

consistent with one $\overline{\nu}_e$ in (1) for an IBD (it is also the normalization used in the quantum trajectory representation). Knowing the value of one coefficient implies knowing the value of the other by normalization, (7). If the conditions $\alpha > 1$ and $0 < \beta^2 = \alpha^2 - 1$ exist, then bispectral propagation in the $+q$ direction follows. The bispectral propagation for $\overline{\nu}$ consistent with (4)–(6) is nonuniform, albeit still rectilinear, in the $+q$ direction. As such, $\overline{\psi}_2(q)$ may also be considered to be the wave function synthesized by the superposition of the latent incident wave and the the latent reflected wave upon each other. Note that herein the coefficients could have been expressed hyperbolically by $\alpha = \cosh(\gamma)$ and $\beta = \sinh(\gamma)$ consistent with (7).

For completeness, if the incident and reflected waves were neither latent nor superimposed, then the wave function representation would be in a two-dimensional space $\{q_{\text{incident}}, q_{\text{reflected}}\}$ given by

$$\overline{\psi}(q_{\text{incident}}, q_{\text{reflected}}) = \alpha \exp(+ikq_{\text{incident}}) + \beta \exp(-ikq_{\text{reflected}}),$$

which is not equivalent to $\overline{\psi}_2(q)$ of (4)–(6). Eqs. (4)–(6) individually show the superpositioning to describe $\overline{\psi}_{\text{superimposed}}$ in one-dimensional space by a single independent variable q. Also for completeness, a literature search for "reflected neutrinos" on the web has found nothing for reflected neutrinos from charged current interactions *per se* but did find an unpublished report regarding reflections of antique neutrinos from the big bang [26].

Let us examine the compoundly modulated traveling wave (6) in special situations for didactic reasons. Should $\beta = 0$, then the amplitude $A_{\overline{\psi}}$ and phase $P_{\overline{\psi}}$ would respectively become

$$A_{\overline{\psi}}\Big|_{\beta=0} = [\alpha^2 + \beta^2 + 2\alpha\beta\cos(2kq)]^{1/2}|_{\beta=0} = \alpha|_{\beta=0} = 1 \tag{8}$$

and

$$P_{\overline{\psi}}\Big|_{\beta=0} = \arctan\left(\frac{\alpha-\beta}{\alpha+\beta}\tan(kq)\right)\Big|_{\beta=0} = kq. \tag{9}$$

Then, (6) would represent unispectral propagation as expected. Next, we consider the case $(|\beta| = \alpha) \notin \{0 \le \beta^2 = \alpha^2 - 1\}$ and in violation of the normalization (7). Nevertheless, $|\beta| = \alpha$ is a limit point for $\beta \to \infty$. Should $\pm\beta = \infty$ (*i.e.* where a latent total reflection would preempt any IBD), then the amplitude would reduce to trigonometric identities with scaling factor 2α given by [27]

$$A_{\overline{\psi}}\Big|_{\beta=\alpha} = 2\alpha\left(\frac{1+\cos(2kq)}{2}\right)^{1/2} = 2\alpha\cos(kq) \tag{10}$$

and

$$A_{\overline{\psi}}\Big|_{-\beta=\alpha} = 2\alpha\left(\frac{1-\cos(2kq)}{2}\right)^{1/2} = 2\alpha\sin(kq) \tag{11}$$

consistent with (5). The corresponding phase would be

$$P_{\overline{\psi}}\Big|_{\beta=\alpha} = \arctan\left(\frac{\alpha-\beta}{\alpha+\beta}\tan(kq)\right)\Big|_{\beta=\alpha} = 0 \tag{12}$$

and

$$P_{\overline{\psi}}|_{-\beta=\alpha} = \arctan\left(\frac{\alpha-\beta}{\alpha+\beta}\tan(kq)\right)\Big|_{-\beta=\alpha} = \frac{\pi}{2} \tag{13}$$

also consistent with (5). Then, in either case and consistent with (4), (6) would represent a scaled standing cosine wave for $\beta = \alpha$ and a scaled standing sine wave for $-\beta = \alpha$. Standing waves, while mathematically permitted, would have relativistic issues in addition to the aforementioned total reflection issue. Thus, the representation for the wave function (6) covers all solutions of physical interest of (2) propagating in the $+q$ direction with normalization $\alpha^2 - \beta^2 = 1$ (7).

If the neutrino and antineutrino are considered to form a Majorana pair of particles (an unsettled question), then the wave functions for the neutrino and antineutrino would be complex conjugates of each other. Under the Majorana hypothesis, the latent reflected wave $\beta\exp(-ikq)$ in (4) would be the wave function for a neutrino with amplitude β. In this case, (6) would represent the superposition of the wave functions of the Majorana neutrino and antineutrino upon each other. This is consistent with Pontecorvo's proposal [28] that a mixed particle consisting of part antineutrino and part neutrino may exist. Furthermore, the set of independent solutions $\{\overline{\psi}, \check{\overline{\psi}}\} = \{\exp(+ikq), \exp(-ikq)\} = \{\overline{\psi}, \psi\}$ that solve the SKGE, form a pair of Majorana solutions that are sufficient to solve the SKGE. Any solution, e.g. (4)–(6), of the SKGE formed from this pair by the superpositional principle would itself have a Majorana partner that would also be its complex conjugate. While the wave functions given by (4)–(6) are Pontecorvo "mixed" solutions [28], they are still specified herein as $\overline{\psi}$s of the $\overline{\nu}$ as determined by the directional characteristic $(+q)$ of the latent incident wave.

Let us briefly discuss how this ab initio calculation describes the evolution of the bispectral $\overline{\psi}_2$ during consummation of an IBD. The weak interaction is not a "force" *per se*. It does not cause an energy exchange among its participants. Rather, for purposes of this paper, it enables beta decay where a neutron decays into a proton, electron, and neutrino, which is the inverse of an IBD (1). Let us consider that the weak interaction occurs in a black box over the short range of the weak interaction between q_a, where the antineutrino initially encounters the weak interaction, and q_b where the antineutrino is absorbed by the target proton. The short range of the weak interaction is given by $q_b - q_a \approx 10^{-18}$ m, a value much smaller than the radius of the proton. Within the black box $q_a < q < q_b$, the same set of independent solutions $\{\exp(+ikq), \exp(-ikq)\}$, which are sufficient to solve (2), are used to describe $\overline{\psi}_2$ while it is subject to the **forceless** weak interaction that precludes any energy exchange. In the absence of an energy exchange, the wave number k remains a constant in (4)–(6) during $\overline{\nu}_e$'s transit of the black box from q_a to q_b. But the coefficients $\{\alpha,\beta\}$ are changed! During the transit of $\overline{\nu}e$ from q_a to q_b in this ab initio calculation, the forceless weak interaction by W^+ exchange smoothly changes coefficients $\{\alpha,\beta\}|_{q_a} \to \{1,0\}|_{q_b}$ while continuously maintaining the normalization $\alpha^2 - \beta^2 = 1$ of (7). In other words, the coefficients while inside the black box boundaries become variables $\{\alpha(q),\beta(q)\}_{q_a \le q \le q_b}$ that are explicitly still subject to the normalization

$$\alpha^2(q) - \beta^2(q) = 1, \quad q_a \le q \le q_b,$$

which is consistent with (7). A smooth transition of the coefficients from $\{\alpha(q_a),\beta(q_a\}$ to $\{1,0\}|_{q_b}$ with C^1 continuity would be sufficient to maintain C^1 continuity of the $\overline{\nu}_e$'s wave function as it evolves, during its transit of the black box with constant E and wave number k, from a bispectral $\overline{\psi}_2(q_a)$ to a unispectral $\exp(ikq_b)$ ready to be absorbed. At q_b, the output transmitted wave function of the black box will have become a unispectral wave function as given by

$$\begin{aligned}\overline{\psi}_2(q_a) &= \alpha(q_a)\exp(ikq_a) + \beta(q_a)\exp(-ikq_a)\\ &= [1+\beta^2(q_a)]^{1/2}\exp(kq_a) + \beta(q_a)\exp(-ikq_a)\\ &\xrightarrow{q\to q_b,\ \therefore\ \beta(q)\to 0} \exp(ikq_b), \quad q_a \le q \le q_b\end{aligned} \tag{14}$$

under the influence of the exchange of the W^+ boson between the proton and antineutrino. In the extended black box, a provisional form for $\beta(q)$ with C^1 continuity during the transmutation of $\overline{\psi}$ from $\psi_2(q_a)$ to $\exp(ikq_b)$ in (14) is offered by

$$\beta(q) = \frac{\beta(q_a)}{2}\left[1+\cos\left(\frac{q-q_a}{q_b-q_a}\pi\right)\right], \quad q_a \le q \le q_b.$$

Again, no energy is exchanged between the proton and antineutrino by the W^+ boson exchange. (If the transmitted wave function at q_b had not been unispectral $\exp(ikq)$, then its initial values at q_a would have been flavor incompatible $\overline{\nu}(q_a) \neq \overline{\nu}_e(q_a)$, which would have preempted an IBD. Consummated IBDs are rare events.) The transmitted unispectral wave function $\exp(ikq)$ is the wave function for $\overline{\nu}_e$ in (1). The normalization $\alpha^2 - \beta^2 = 1$ (7) specifies that the value of the amplitude of the transmitted unispectral wave function is 1, consistent with the assumptions for $\overline{\nu}_e$'s wave function for (1). The transmitted unispectral $\overline{\nu}_e$ is compatible with being absorbed by the proton consistent with (1). The function of the black box in the IBD process (to change the input bispectral wave function to an output unispectral wave function of amplitude 1 in a forceless manner for $\overline{\nu}_e$'s E never changes) has been completed with the $\overline{\nu}_e$ positioned at q_b, ready to be absorbed with the target proton. The W^+ boson exchange has now been completed. The IBD carries on. The IBD completes consummation consistent with (1) where its parent particles, the proton and the unispectral antineutrino, are absorbed, and the IBD emits its daughter products, a positron and a neutron. The latent transmission coefficient T and reflective coefficient R of the black box for the weak interaction process are the expected

$$T = \frac{\alpha^2 - \beta^2}{\alpha^2} = \frac{1}{\alpha^2} \quad \text{and} \quad R = \frac{\beta^2}{\alpha^2}, \tag{15}$$

where the coefficients $\{\alpha,\beta\}$ are their pre-weak interaction values.

Flavor compatibility for an IBD is determined by the boundary conditions $\{\overline{\psi}, \partial_q\overline{\psi}\}$ at the black box's input barrier interface q_a. The black box in this ab initio calculation renders a transmitted unispectral wave function $\exp(ikq)$, if and only if $\overline{\psi}_2$ has proper IBD initial values for the black box, $\{\overline{\psi}, \partial_q\overline{\psi}\}_{q=q_a}$.

Future research may refine the aforementioned description of the evolution of the antineutrino's wave function in the black box. If so, the principle of superposition of the wave functions of the latent incident and the latent reflected waves could still describe a generalized (14). For example, future research may find that the transmitted wave function of energy E from the black box should have coefficients $\{(1+\beta_b^2)^{1/2}, \beta_b\}|_{q=q_b}$ with $\beta > 0$ for IBD absorption of the antineutrino. For a successful IBD, the black box model of the weak force would then transmute the incident wave function described by

$$\begin{aligned}&[1+\beta^2(q_a)]^{1/2}\exp(kq_a) + \beta(q_a)\exp(-ikq_a)\\ &\qquad\xrightarrow{q\to q_b,\ \therefore\ \beta(q)\to\beta_b}\\ &(1+\beta_b^2)^{1/2}\exp(ikq_b) + \beta_b\exp(-ikq_b)\end{aligned} \tag{16}$$

where $q_a < q \le q_b$. This generalizes (14) and would still describe a counterexample permitting massless neutrino oscillation. Eqs. (14) and (16) are analogous to the invariance of the Schwarzian derivative under a Möbius transformation in the quantum trajectory representation [14], [29].

Chirality and helicity are the same for massless leptons propagating with speed c. The quantum measure of helicity, normalized over a cycle of nonuniform propagation, for a massless antineutrino before encountering the black box, $q < q_a$, would by (4)–(6) be $\alpha^2 - \beta^2 = 1$, which is also the normalization (7). Upon completing the transit of the black box at q_b, the antineutrino, with $\overline{\psi} = \exp(ikq_b)$, would still have the helicity value of 1 conserving helicity (chirality). Thus, the interaction of the massless antineutrino with the black box would be reflectionless. This is consistent with (14) and (16). The concept of superimposing a latent reflected wave and the latent incident wave upon each other to achieve reflectionless transmission had initially been applied to an acoustical analogue [20].

The representation of $\overline{\psi}_2$ by (6) may be derived from the trigonometric form of (5) by using either Bohm's scheme for complex wave functions to render $\overline{\psi}_2$'s amplitude and phase [30] or by vector analysis. The amplitude $A_{\overline{\psi}} = [\alpha^2 + \beta^2 + 2\alpha\beta\cos(2kq)]^{1/2}$ is recognized as a re-expressed law of cosines where the exterior angle argument $2kq$ is the supplement of $\pi - 2kq$ or

$$A_{\overline{\psi}} = \underbrace{[\alpha^2 + \beta^2 - 2\alpha\beta\cos(\pi - 2kq)]^{1/2}}_{\text{law of cosines}} = \underbrace{[\alpha^2 + \beta^2 + 2\alpha\beta\cos(2kq)]^{1/2}}_{\text{law of cosines for exterior angles}}.$$

For completeness, the phase is established [30] by $P_{\overline{\psi}}(q) = \arctan\{\Im[\overline{\psi}(q)]/\Re[\overline{\psi}(q)]\}$, which by (5) renders

$$P_{\overline{\psi}} = \arctan\left(\frac{\alpha - \beta}{\alpha + \beta}\tan(kq)\right). \tag{17}$$

Also for completeness, the phase is related to the quantum Hamilton's characteristic function (quantum reduced action) $\mathcal{W}$ by $P_{\overline{\psi}} = \mathcal{W}/\hbar$ [7], [10], [14]. The $\mathcal{W}$ has been shown to change values monotonically [14] implying that $P_{\overline{\psi}}$ also behaves monotonically.

The bispectral $\overline{\psi}_2$ as represented by (6) exhibits the superposition of the latent incident and reflected wave functions upon each other that are described by functions of q (4). The superposition induces a compound modulation in $\overline{\psi}_2$, which in turn induces nonuniform rectilinear propagation for massless neutrinos as shown in §3. PMNS theory achieves nonuniform rectilinear propagation in one dimension by superimposing three different masseigenstates within the neutrino [1]–[4]. Application of Eq. (6)-like representations have been made to study step barriers [18] and tunneling [19].

Before an IBD, $q \leq q_a$, the nonuniform propagation of the compoundly modulated $\overline{\psi}_2(q)$ with q can be examined more closely by considering the phase and amplitude modulations separately. The phase modulation may be described by the phase displacement between the phase of the bispectral $\overline{\psi}_2$ given by (6) and the phase kq of the corresponding unispectral wave function $\exp(ikq)$, which propagates rectilinearly with uniform motion. This phase displacement is a rotational displacement in complex $\overline{\psi}$-space between $\overline{\psi}_2(q)$ and the unispectral $\exp(ikq)$. The phase displacement due to phase modulation $Pm_{\overline{\psi}}$ may be expressed in units of radians as a function of phase kq, also in units of radians, as given by

$$Pm_{\overline{\psi}} = \arctan\left(\frac{\alpha - \beta}{\alpha + \beta}\tan(kq)\right) - kq, \quad q \leq q_a \tag{18}$$

where kq, which is also the phase of unispectral $\exp(ikq)$, is not restricted to its principal value.

The derivative of phase with respect to q, for the bispectral wave function (6) is given by [5]

$$\begin{aligned}\frac{\partial \arctan\left(\frac{\alpha-\beta}{\alpha+\beta}\tan(kq)\right)}{\partial q} &= \frac{(\alpha^2 - \beta^2)k}{\alpha^2 + \beta^2 + 2\alpha\beta\cos(2kq)} \\ &= \frac{k}{\alpha^2 + \beta^2 + 2\alpha\beta\cos(2kq)}.\end{aligned} \tag{19}$$

Eq. (19) for the bispectral wave function exhibits nonuniform phase propagation that is periodic in q. The derivative of phase with respect to q remains positive definite for the denominator on the right side of (19) is always positive for all q by the Schwarzian inequality. Meanwhile, the corresponding derivative of phase for the unispectral wave function $\exp(ikq)$ is ik, which is constant and manifests uniform rectilinear propagation. For completeness in the quantum trajectory representation, the derivative of phase with regard to q renders the conjugate momentum $\partial_q\mathcal{W}$ divided by $\hbar$ [8]–[14].

The relative amplitude dilation $Am_{\overline{\psi}}$ due to amplitude modulation $A_{\overline{\psi}}$ of (6) or (8), relative to $(\alpha^2 + \beta^2)^{1/2}$, is defined to be a dimensionless variable that is a function of phase kq and given by

$$\begin{aligned}Am_{\overline{\psi}} &\equiv \frac{[\alpha^2 + \beta^2 + 2\alpha\beta\cos(2kq)]^{1/2} - (\alpha^2 + \beta^2)^{1/2}}{(\alpha^2 + \beta^2)^{1/2}} \\ &= \left[1 + \frac{2\alpha\beta\cos(2kq)}{\alpha^2 + \beta^2}\right]^{1/2} - 1, \quad q \leq q_a.\end{aligned} \tag{20}$$

Any finite $\beta = (\alpha^2 - 1)^{1/2}$ is sufficient to cause $\overline{\psi}_2$ to generate nonuniform rectilinear motion consistent with the compound modulation implied by (18) and (20).

As the wave function $\overline{\psi}_2$ for the antineutrino must be C^1 continuous until absorbed in an IBD, the behavior of its derivative $\partial_q\overline{\psi}_2$ must also be considered. If the dividend of $\partial_q\overline{\psi}_2 \,/\, \overline{\psi}_2$ were a constant or independent of q, then neutrino oscillation would not be supported as previously noted. From (4)–(6), the derivative of the bispectral wave function $\partial_q\overline{\psi}_2$ is given by

$$\begin{aligned}\partial_q\overline{\psi}_2 &= ik[\alpha\exp(ikq) - \beta\exp(-ikq)] \\ &= k[(\alpha - \beta)\cos(kq) - i(\alpha + \beta)\sin(kq)]\exp(i\pi/2) \\ &= k\underbrace{[\alpha^2 + \beta^2 - 2\alpha\beta\cos(2kq)]^{1/2}}_{\text{law of cosines}} \\ &\quad\times\exp\left[i\arctan\left(\frac{\alpha + \beta}{\alpha - \beta}\tan(kq)\right) + i\frac{\pi}{2}\right].\end{aligned} \tag{21}$$

A difference between (4)–(6) for $\overline{\psi}_2$ and (21) for $\partial_q\overline{\psi}_2$ is the change of the sign of β and the phase shift $\pi/2$. A finite β by (4) and (21) ensures that

$$\frac{\partial_q\overline{\psi}_2(q)}{\overline{\psi}_2(q)} = ik\left(\frac{\alpha\exp(ikq) - \beta\exp(-ikq)}{\alpha\exp(ikq) + \beta\exp(-ikq)}\right) \tag{22}$$

would be a variable of q in contrast to the unispectral case (3). The bispectral $\overline{\psi}_2(kq)$ propagates in a nonuniform manner that facilitates neutrino oscillation without the need for mass-eigenstates of PMNS theory.

There is an alternative expression for $\partial_q\overline{\psi}_2(kq)$ that conveniently shows its relation to $\overline{\psi}(kq-\pi/2)$. This relation is shown by (4) and (21) to be

$$\begin{aligned}\partial_q\overline{\psi}_2(kq) &= ik[\alpha\exp(ikq)-\beta\exp(-ikq)]\\ &= k\{\alpha\exp[i(kq+\pi/2)]+\beta\exp[-i(kq+\pi/2)]\}\\ &= k\,\overline{\psi}_2(kq+\pi/2).\end{aligned} \quad (23)$$

Eq. (23) can be generalized to

$$\begin{aligned}\partial_q\overline{\psi}_2(kq) &= k\overline{\psi}_2(kq+n_1\pi),\\ n_1 &= \pm 1/2,\ \pm 3/2,\ \pm 5/2,\ \cdots .\end{aligned} \quad (24)$$

where n_1 is bound by the antineutrino's creation point and the point q_a where an IBD commences. The bispectral derivative $\partial_q\overline{\psi}_2$ by (21)–(24), like $\partial_q\overline{\psi}_1$, is also a solution of the SKGE.

The derivative of the bispectral wave function is compoundly modulated. Its amplitude $A_{\overline{\psi}'}$ and phase $P_{\overline{\psi}'}$ are respectively given by

$$A_{\overline{\psi}'} = k[\alpha^2+\beta^2-2\alpha\beta\cos(2kq)]^{1/2}, \quad q\le q_a \quad (25)$$

and

$$P_{\overline{\psi}'} = \arctan\left(\frac{\alpha+\beta}{\alpha-\beta}\tan(kq)\right)+\frac{\pi}{2}, \quad q\le q_a. \quad (26)$$

Its relative amplitude dilation $Am_{\overline{\psi}'}$ due to amplitude modulation and its phase displacement (a rotation) due to phase modulation $Pm_{\overline{\psi}'}$ for $\partial_q\overline{\psi}_2(kq)$ are given respectively by

$$Am_{\overline{\psi}'} = k\left[1-\frac{2\alpha\beta\cos(2kq)}{\alpha^2+\beta^2}\right]^{1/2}-k, \quad q\le q_a \quad (27)$$

and

$$Pm_{\overline{\psi}'} = \arctan\left(\frac{\alpha+\beta}{\alpha-\beta}\tan(kq)\right)-kq, \quad q\le q_a. \quad (28)$$

The dilations and rotations of (27) and (28) for $\partial_q\overline{\psi}_2(kq)$ are analogous to those for $\overline{\psi}_2$, (20) and (18) respectively. While $\partial_q\overline{\psi}_2(kq)$ has compound modulation with the same period (oscillation cycle) as that of the associated $\overline{\psi}_2(kq)$, the of dilations and rotations differ by being out of phase, *cf.* (6) and (21)–(28). The relative amplitude dilation and phase rotation of $\partial_q\overline{\psi}_2(kq)$ are opposite to those of $\overline{\psi}_2(kq)$. This is desirable for flavor oscillation.

Let us now examine the measurement of momentum p for the bispectral antineutrino. The applicable quantum momentum operator herein is $\frac{\hbar}{i}\partial_q$. The orthodox measurement of momentum of the bispectral $\overline{\psi}_2$ with box normalization is over one repetitive cycle. This box length is π/k. The momentum of $\overline{\psi}_2$, using (4), (7) and (21), is given by

$$\begin{aligned}p &= \frac{\int_0^{\pi/k}\overline{\psi}_2^{\dagger}(q)\,\frac{\hbar}{i}\partial_q\overline{\psi}_2(q)\,dq}{\int_0^{\pi/k}\overline{\psi}_2^{\dagger}(q)\overline{\psi}_2(q)\,dq}\\ &= \hbar\frac{k\int_0^{\pi/k}[\alpha^2-\beta^2+2\alpha\beta\sin(2kq)]\,dq}{\int_0^{\pi/k}[\alpha^2+\beta^2+2\alpha\beta\cos(2kq)]\,dq}\\ &= \hbar\frac{(\alpha^2-\beta^2)\pi}{(\alpha^2+\beta^2)\pi/k} = \frac{\hbar k}{\alpha^2+\beta^2}.\end{aligned} \quad (29)$$

An orthodox measurement of momentum of the bispectral antineutrino (29) is a constant and positive definite, *i.e.* $p>0$, in the direction of latent incident wave (4). This is consistent with the quantum trajectory representation where the quantum reduced action $\mathcal{W}$ changes monotonically [14].

Let us extend our examination of p to find under what conditions $[\alpha^2-\beta^2+2\alpha\beta\sin(2kq)]$, the integrand in the numerator in (29), becomes negative over any portions of its repetitive cycle. The particular point of interest for investigation is $q=3\pi/(4k)$ where the integrand becomes

$$[\alpha^2-\beta^2+2\alpha\beta\sin(2kq)]_{q=3\pi/(4k)} = \overbrace{\alpha^2-\beta^2}^{=1}-2\alpha\beta. \quad (30)$$

For $|\beta|$ sufficiently small, (30) would be positive; sufficiently large, negative. The $|\beta|$ for which (30) is nil marks the upper bound where $[\alpha^2-\beta^2+2\alpha\beta\sin(2kq)]$, the integrand, is never negative. Because $-\beta^2$ is a negative quantity, the Schwarz inequality is not applicable to (30). The right side of (30) becomes nil for

$$2\alpha\beta = 1. \quad (31)$$

The particular values of α and $|\beta|$ that satisfy both Eqs. (7) and (31) are identified by $\alpha_{\text{threshold}}$ and $|\beta_{\text{threshold}}|$. The threshold coefficients separate $\alpha,|\beta|$-space into two domains: one where the integrand is always positive-definite; the other, not always positive consistent with the value of $\sin(2kq)$ in (29). Eq. (7) for normalization, $\alpha^2-\beta^2=1$, and (31) are sufficient to resolve $\alpha_{\text{threshold}}$ and $|\beta_{\text{threshold}}|$ by algebraic means. The solutions for the threshold coefficients are

$$\{\alpha_{\text{threshold}},\beta_{\text{threshold}}\} = \left\{\left(\tfrac{2^{1/2}+1}{2}\right)^{1/2},\left(\tfrac{2^{1/2}-1}{2}\right)^{1/2}\right\}. \quad (32)$$

The logic relationship

$$\alpha\ </>\ \alpha_{\text{threshold}} \iff |\beta|\ </>\ |\beta_{\text{threshold}}|$$

between α and β follows. If $|\beta|<|\beta_{\text{threshold}}|$, then the integrand $\overline{\psi}_2^{\dagger}(q)\,(\hbar/i)\partial_q\,\overline{\psi}_2(q)$ of (29) would always be positive (in the direction of the latent incident wave of (4)) for all q throughout the repetitive oscillation cycle. If $|\beta|>|\beta_{\text{threshold}}|$, then for some q, but not a preponderance of q of the repetitive oscillation cycle, the integrand $\overline{\psi}^{\dagger}\,(\hbar/i)\partial_q\,\overline{\psi}_2$ would be negative (in

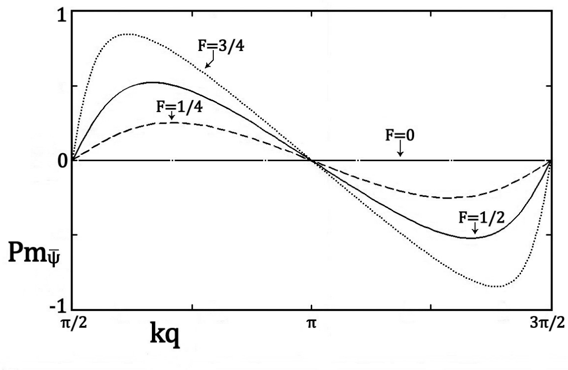

Fig. 1: The phase displacement due to phase modulation $Pm_{\overline{\psi}}$ as a function of kq over a Riemann sheet for selected values of F. Both $Pm_{\overline{\psi}}$ and kq are exhibited in units of radians.

the direction of the latent reflected wave of (4)). Nevertheless, even if $|\beta| > |\beta_{\text{threshold}}|$, the orthodox measure for momentum would still remain valid, for (29) yields positive momentum as $\alpha^2 - \beta^2 = 1 > 0$.

3 Examples

Let us now illustrate with didactic examples how a bispectral wave function facilitates massless flavor oscillation. We consider the contributions of phase and amplitude modulations separately. These contributions are examined for the selected cases given by

$$(\alpha,\beta) = (1,0),\ (4/15^{1/2}, 1/15^{1/2}),\ (2/3^{1/2}, 1/3^{1/2}),\ (4/7^{1/2}, 3/7^{1/2}). \tag{33}$$

These cases are compliant with normalization $\alpha^2 - \beta^2 = 1$ (7). The selected cases may be identified for convenience by the fraction $F \equiv \beta/\alpha = (\alpha^2-1)^{1/2}/\alpha = \beta/(1-\beta^2)^{1/2}$. Also, F is related to the reflection coefficient (15) for $F = R^{1/2}$. The fractions F for the selected cases with respect to (33) are given by

$$F = 0,\ 1/4,\ 1/2,\ 3/4. \tag{34}$$

Comparisons of the effects of either phase or amplitude modulations among the selected cases of F are developed as a function of phase kq measured in radians.

The value $F = 0$ represents a unispectral wave function, which precludes massless flavor oscillation. The unispectral $F = 0$ is still included for comparison to the bispectral Fs where $F = 1/4, 1/2, 3/4$. For comparison, the value $F_{\text{threshold}}$ for $2\alpha\beta = 1$ with normalization $\alpha^2 - \beta^2 = 1$, which establishes F's upper bound for no reversals of sign of the integrand $\overline{\psi}_2^{\dagger}\,(\hbar/i)\partial_q\,\overline{\psi}_2$ as a function of q (32) is given by

$$F_{\text{threshold}} = \frac{\beta_{\text{threshold}}}{\alpha_{\text{threshold}}} = \left(\frac{2^{1/2}-1}{2^{1/2}+1}\right)^{1/2} = 2^{1/2}-1 = \frac{1}{2^{1/2}+1} = 0.41421356\cdots.$$

We first consider phase modulation. The phase displacements $Pm_{\overline{\psi}}$ of (18) as a function of kq, where kq is also the phase of $\overline{\psi}$, are exhibited for the various values of F on Fig. 1 over the extended Riemann sheet $\pi/2 \le kq \le 3\pi/2$ of the arc tangent function on the right side of (6). The phase duration of the Riemann sheet is consistent with box normalization of $\overline{\psi}_2$. Each extended Riemann sheet specifies an oscillation cycle. Fig. 1 exhibits one cycle for phase modulation $Pm_{\overline{\psi}}$ over a Riemann sheet. The cycle of $Pm_{\overline{\psi}}$ for bispectral Fs has one concave segment and one convex segment. The cycle is repetitive over other Riemann sheets. As expected, a $Pm_{\overline{\psi}}$ for the unispectral F renders the horizontal straight line $Pm_{\overline{\psi}} = 0$. Thus, the unispectral case prohibits phase modulation, which does not facilitate flavor oscillation. The absolute value of $Pm_{\overline{\psi}}$ for $kq \ne \pi/2,\ \pi,\ 3\pi/2$ is shown on Fig. 1 to increase with increasing F. At $kq = \pi/2,\ \pi,\ 3\pi/2$, the phase difference $Pm_{\overline{\psi}} = 0$ for all F. These points $kq = \pi/2,\ \pi,\ 3\pi/2$ for $F \ne 0$, are inflection points of $Pm_{\overline{\psi}}$ with nil curvature, which are between $Pm_{\overline{\psi}}$'s alternating concave and convex segments. At these inflection points, $|Pm_{\overline{\psi}}(q)|$ attains its maximum slope (rate of change with kq). Had Fig. 1 included the standing-wave case where $F = 1$, then, consistent with (10) and (11), it would have generated a straight line from $Pm_{\overline{\psi}}(kq) = (\pi/2, \pi/2)$ to $(-\pi/2, 3/\pi/2)$ on an extended Fig. 1. Had the cases $F = -1/4,\ -1/2,\ -3/4$ been examined instead (*e.g.* the values of F for the analogous phase differences for $\partial_q\overline{\psi}_2$ would be negative), then Fig. 1 would have changed its exhibition of the antisymmetric phase modulation from the first-and-third (upper/left-and-lower/right) quadrants to the second-and-fourth of Fig. 1. The phase modulation $Pm_{\overline{\psi}}$ is antisymmetric within the Riemann sheet for

$$Pm_{\overline{\psi}}(\pi - kq) = -Pm_{\overline{\psi}}(\pi + kq), \quad 0 < q < \pi/2.$$

Each extended Riemann sheet contains one cycle of $Pm_{\overline{\psi}}$ for the bispectral $\overline{\psi}_2$.

For the amplitude modulation, $Am_{\overline{\psi}}$ is examined for $F = 0,\ 1/4,\ 1/2,\ 3/4$. Again, $F = 0$ represents the unispectral case, which does not support flavor oscillation. The amplitude modulations are exhibited on Fig. 2. Positive differences on Fig. 2 represent a dilation that is an expansion; negative differences, a contraction. The absolute values of $Am_{\overline{\psi}}$ for $kq \ne 3\pi/4, \pi/4$ are shown on Fig. 2 to increase with increasing F. In Fig. 2, $Am_{\overline{\psi}}$ for bispectral F is symmetric with its convex segments disjointed on the Riemann sheet. In comparing Figs. 1 and 2 for bispectral $F = 1/4,\ 1/2,\ 3/4$, either the

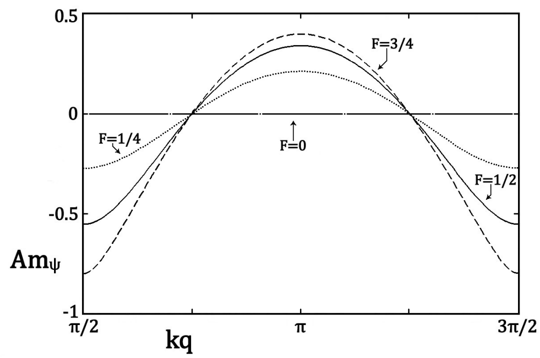

Fig. 2: The relative amplitude dilation $Am_{\overline{\psi}}$ as a function of kq over a Riemann sheet for selected values of F. $Am_{\overline{\psi}}$ is dimensionless, and kq is exhibited in units of radians.

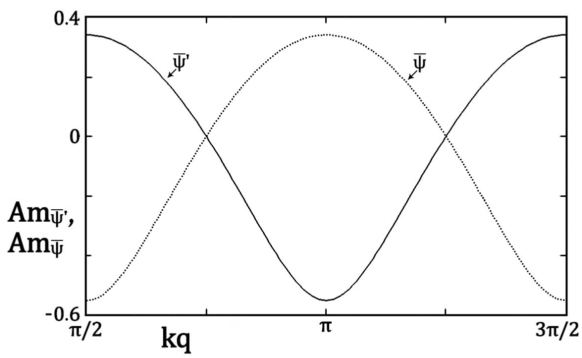

Fig. 3: The relative amplitude dilations due $Am_{\overline{\psi}}$ and $Am_{\overline{\psi}'}$ as functions of kq over a Riemann sheet for $F = 1/2$. For an unbiased $Am_{\overline{\psi}'}$, $k = 1$ to facilitate comparison to dimensionless $Am_{\overline{\psi}}$. The amplitude modulations are dimensionless, and kq is exhibited in units of radians.

$Pm_{\overline{\psi}}$ or the $Am_{\overline{\psi}}$ has an extremum where the other is nil. This ensures that at least one type of modulation of $\overline{\psi}_2$ is changing for all q on the extended Riemann sheet $\pi/2 \le kq \le 3\pi/2$. A local maximum rate of change of a modulation occurs at its zero-crossings where the modulation has inflection points between concave and convex segments as shown by Figs. 1 and 2. The greater (lesser) rate of change of modulation implies the greater (lesser) opportunity for flavor oscillation. The modulation extrema, where the rate of change of a particular modulation is nil, are isolated phase (kq) points where that particular modulation does not contribute to neutrino oscillation.

A comparison between the amplitude modulation $Am_{\overline{\psi}}$ of the bispectral $\overline{\psi}_2$ (6) and the amplitude modulation $Am_{\overline{\psi}'}$ of the associated bispectral $\partial_q\overline{\psi}_2$ (21) are presented in Fig. 3 for the particular values $F = 1/2$, and $k = 1$. As $Am_{\overline{\psi}'}$ by (25) has a linear factor k while $Am_{\overline{\psi}}$ does not, the choice $k = 1$ makes Fig. 3 unbiased. The amplitude modulations $Am_{\overline{\psi}}$ and $Am_{\overline{\psi}'}$ exhibit the same repetitive periodicity but are displaced in phase (kq) by the constant $\pi/2$ radians. This kq displacement increases the opportunity for neutrino oscillation for $Am_{\overline{\psi}}(kq)$ is positive (negative) where $Am_{\overline{\psi}'}(kq)$ is negative (positive). The ratio of amplitudes of $\partial_q\overline{\psi}_2(kq)$ relative to $\overline{\psi}_2(kq)$ by (6) and (21) is given as a function of phase (kq) in fractional form by

$$\left|\partial_q\overline{\psi}_2(kq)\right| : \left|\overline{\psi}_2(kq)\right| \rightsquigarrow \underbrace{\frac{\left|\partial_q\overline{\psi}_2(kq)\right|}{\left|\overline{\psi}_2(kq)\right|} = \frac{A_{\overline{\psi}'}(kq)}{A_{\overline{\psi}}(kq)}}_{\text{fractional form}}) \\ = \overbrace{k\left(\frac{\alpha^2+\beta^2-2\alpha\beta\cos(2kq)}{\alpha^2+\beta^2+2\alpha\beta\cos(2kq)}\right)^{1/2}} . \tag{35}$$

On the extended Riemann sheet $\pi/2 \le kq \le 3\pi/2$, the ratio $A_{\overline{\psi}'}(kq) : A_{\overline{\psi}}(kq)$ for $F = 1/2$ by (33)–(35) has maxima of $3k$ at $kq = \pi/2,\ 3\pi/2$; has a minimum of $k/3$ at $kq = \pi$; and equals k at $kq = 3\pi/4,\ 5\pi/4$ in accordance with (35). The values of the extrema of ratio in fractional form (35) may be generalized and are given on this extended Riemann sheet by

$$\left.\frac{A_{\overline{\psi}'}(kq)}{A_{\overline{\psi}}(kq)}\right|_{\text{maximum}} = k\frac{\alpha+\beta}{\alpha-\beta} \text{ at } kq = \frac{\pi}{2},\ \frac{3\pi}{2}$$

and

$$\left.\frac{A_{\overline{\psi}'}(kq)}{A_{\overline{\psi}}(kq)}\right|_{\text{minimum}} = k\frac{\alpha-\beta}{\alpha+\beta} \text{ at } kq = \pi.$$

The nature of (35) implies that its logarithmic presentation would exhibit for unbiased $k = 1$ a periodic antisymmetry within the extended Riemann sheet $\{\pi/2 \le kq \le 3\pi/2\}$ given by

$$\ln\left(\frac{A_{\overline{\psi}'}(kq)}{A_{\overline{\psi}}(kq)}\right) = -\ln\left(\frac{A_{\overline{\psi}'}(kq \pm \pi/2)}{A_{\overline{\psi}}(kq \pm \pi/2)}\right), \text{ for } k = 1.$$

The variation of the ratio (35) is one of the factors that facilitate flavor oscillation. On the other hand, the corresponding ratio for the unispectral case ($F = 0$) is the constant k for all q.

A comparison of (9) and (26) shows the relationship between $P_{\overline{\psi}}(kq)$ and $P_{\overline{\psi}'}(kq)$ is that the sign of β has changed (also the sign of the associated F would change). Therefore $P_{\overline{\psi}'}(kq) - \pi/2$ and $P_{\overline{\psi}}(kq)$ are a half-cycle out of phase. While the undulations of $P_{\overline{\psi}'}$ and $P_{\overline{\psi}}$ when summed are in opposition, their difference is reinforced. Their changing difference is another factor enabling flavor oscillation. The relative phase difference $\triangle P_{\overline{\psi}',\overline{\psi}}(kq)$ in radians between $Pm_{\overline{\psi}'}$ and $Pm_{\overline{\psi}}$ is reinforced for they are out of phase as shown by

$$\begin{aligned}\triangle P_{\overline{\psi}',\overline{\psi}}(kq) &= P_{\overline{\psi}'}(kq) - P_{\overline{\psi}}(kq) \\ &= P_{\overline{\psi}}(kq + \pi/2) + \pi/2 - P_{\overline{\psi}}(kq).\end{aligned} \tag{36}$$

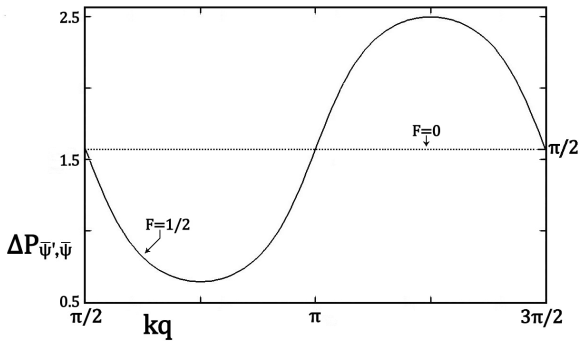

Fig. 4: The Phase difference $\triangle P_{\overline{\psi}',\overline{\psi}}(kq)$ as a function of kq over a Riemann sheet for $F = 0, 1/2$. Both $\triangle P_{\overline{\psi}',\overline{\psi}}(kq)$ and kq are exhibited in units of radians.

The relative phase difference $\triangle P_{\overline{\psi}',\overline{\psi}}(kq)$ is exhibited on Fig. 4 for $F = 1/2$ and $F = 0$ (the unispectral case). For the bispectral case, Fig. 4 also exhibits coherent reinforcement of the undulations of $P_{\overline{\psi}'}$ and $P_{\overline{\psi}}$ of $\triangle P_{\overline{\psi}',\overline{\psi}}(kq)$ consistent with (23). Larger undulations increase the opportunity for flavor oscillations.

The two factors, the ratio of amplitudes and the phase difference, describe the relative relationship between $\partial_q\overline{\psi}$ and $\overline{\psi}$ as a function of phase kq. The ratio of amplitudes (35) and the phase difference of Fig. 4 each complete one cycle on an extended Riemann sheet, *e.g.* $\pi/2 < kq < 3\pi/2$. However, their respective extrema are displaced by a quarter cycle $\pi/4$ from each other. The phase difference $\triangle P_{\overline{\psi}',\overline{\psi}}(kq)$ has extrema on the extended Riemann sheet at $kq = 3\pi/4,\ 5\pi/4$ while the ratio $A_{\overline{\psi}'} : A_{\overline{\psi}}(kq)$ has extrema at $kq = \pi/2,\ \pi,\ 3\pi/2$. Where one factor has an extremum at some particular kq, the other factor has an inflection point there. And where one factor has an inflection point, the other has an extremum. A local extremum for a factor implies that the factor has a local nil in facilitating flavor oscillation while the other factor having an inflection point implies a local peak in facilitating flavor oscillation. Furthermore, where one factor's support for flavor oscillation decreases, the other factor's support increases. Thus, the two factors complement each other to ensure that the bispectral antineutrino can facilitate possible flavor oscillation for some interaction throughout its repetitive cycle.

Both phase and amplitude modulations exhibit the same kq periodicity on Figs. 1–4. This may be shown by trigonometry for the general situation. Periodicity of phase modulation (19) is consistent with the extended Riemann sheet of the arc tangent,

$$(2n-1)\pi/2 \le kq \le (2n+1)\pi/2,\ n = 0, \pm1, \pm2, \cdots .$$

Hence, $Pm_{\overline{\psi}}(kq) = Pm_{\overline{\psi}}(kq+\pi)$. Periodicity of amplitude modulation (20) is consistent with the argument $2kq$ of the cosine term in the law of cosines completing its cycle 2π. Periodicity of $Am_{\overline{\psi}}$ is also given by

$$Am_{\overline{\psi}}(kq) = Am_{\overline{\psi}}(kq + n\pi),\ n = \pm1, \pm2, \pm3, \cdots .$$

For completeness, the quantum trajectory representation also has the same kq periodicity [5].

4 Discussion

Compound modulation makes $\partial_q\overline{\psi}_2 \,/\, \overline{\psi}_2$ a periodic variable in phase kq and spatially periodic for a given k. The phase and amplitude modulations complement each other for they are a quarter-cycle out of phase with each other as shown by Figs. 1 and 2. The modulations of $\overline{\psi}_2$ and $\partial_q\overline{\psi}_2$ supplement each other. The amplitude modulation induces continuous dilations with respect to phase kq of the $\partial_q\overline{\psi}_2(q)$ and $\overline{\psi}_2(q)$ differently by (25) and (8) respectively. The dilations of $\partial_q\overline{\psi}_2(q)$ and $\overline{\psi}_2(q)$ are opposed: where one is an expansion; the other is a contraction. These amplitude modulations being in opposition increase the amount of dilation (either expansion or contraction) of the ratio $\left|\partial_q\overline{\psi}_2(kq)\right| : \left|\overline{\psi}_2(kq)\right|$ with respect to phase kq as exhibited by (35) and Fig. 3. This increases the opportunity for neutrino oscillation. Meanwhile, phase modulation induces continuous rotations with respect to phase kq of $Pm_{\overline{\psi}}(q)$ (18) and $Pm_{\overline{\psi}'}(q)$ (28). These rotational displacements are opposed: where one rotation is clockwise; the other, counterclockwise. This opposition in rotations enlarges $\triangle P_{\overline{\psi}',\overline{\psi}}(kq)$ as exhibited by (36) and Fig. 4. This opposition between the behavior of $\overline{\psi}_2(q)$ and its derivative is typical of well behaved functions undergoing periodic motion. Note that either phase or amplitude modulation, by itself, could facilitate neutrino oscillation of the bispectral antineutrino. Together, they increase the opportunity for oscillation.

The transmutation of coefficients $\{\alpha,\beta\} \to \{1,0\}$ of (14) by the weak interaction nulls out the compound modulation of $\overline{\nu}_e$'s wave function without any exchange of energy. This is shown for phase modulation on Fig. 1 and for amplitude modulation on Fig. 2 where modulation effects decrease with decreasing absolute values of $|F|$ and are completely nulled at $|F| = 0$.

The periodic, nonuniform propagation by a massless antineutrino results in flavor oscillations where the antineutrino in a particular phase (kq) segment within an oscillation cycle may execute a flavor-compatible current interaction with C^1 continuity of its wave function. Future work may show that these segments for various flavors $\{\overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau\}$ may be disjointed, and the segments for the flavors may not densely fill the oscillation cycle.

Should the segments for the active flavors $\{\overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau\}$ not densely fill the oscillation cycle, then the voids of the oscillation cycle would be locations where the antineutrino is inactive and would behave as the elusive sterile antineutrino $\overline{\nu}_s$ [31], [32]. By precept, the sterile antineutrino was hypothesized to be subject only to gravity and explicitly not to the weak interaction. The MiniBooNE Collaboration has recently inferred its existence from experiment [31], but such existence has not yet been independently confirmed by other ongoing experiments [32]. As the hypothetical sterile antineutrino would not partake in charged current interactions, the voids in the oscillation cycle could manifest the existence of this hypothetical sterile antineutrino. This hypothetical sterile antineutrino, by (2)–(6), could be massless and have a bispectral wave function. As this hypothetical bispectral sterile antineutrino could propagate nonuniformly, it would oscillate in flavor to become an active antineutrino $\{\overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau\}$. Flavor oscillation of the sterile antineutrino would imply that it would have the same right handedness of the active antineutrinos. Again, this support for the existence of the sterile antineutrino is predicated on the existence of voids in the oscillation cycle.

The orthodox measurement of the momentum operator $\frac{\hbar}{i}\partial_q$ acting on a bispectral antineutrino over a box length, which is consistent with an oscillation cycle, has been shown by (29) to give a finite positive momentum in the direction of the latent incident wave (4). An IBD event is a good way to observe antineutrinos for the antineutrino reacts only to gravity and the weak interaction. Observed momentum, in principle, need not be averaged over a box length. Should future work find that box normalization is too coarse, then restricting the absolute value of β to $|\beta| \le |\beta_{\text{threshold}}| = [(2^{1/2}-1)/2]^{1/2}$ (32) would maintain positive momentum for the bispectral antineutrino throughout the oscillation cycle, *i.e.*

$$\overline{\psi}_2^\dagger(q)\,\frac{\hbar}{i}\partial_q\overline{\psi}_2(q) > 0$$

by (30)–(32) for all q within the box normalization.

Future work may also show that the different charged or neutral current interactions may scramble the flavors. In other words, the antineutrino flavors may be interaction dependent where the values of $\overline{\psi}$ and $\partial_q\overline{\psi}$ for some given E at a point q_0 may specify an antineutrino of a particular flavor for an interaction while concurrently at q_0 also specifying a different flavor associated with another different interaction. This would cause the segments for the various flavors of the oscillation cycle to overlap.

Future work may also yield a better understanding of IBDs and the weak force. Nevertheless, the concept of a bispectral wave function representation should be robust enough to adjust assumptions and still facilitate flavor oscillation by a massless antineutrino.

5 Findings and conclusions

The principal finding is the existence of a wave function representation for massless neutrino oscillation of flavor, which is a counterexample to PMNS theory's finding that $m > 0$. The wave function representation for $m = 0$ is compatible with an orthodox interpretation of the bispectral wave function, $\overline{\psi}_2$. One spectral component represents the embedded latent incident wave function for an IBD; the other, the embedded latent reflected wave function. Such a bispectral wave function is capable of flavor oscillations without any need for mass-eigenstates, which confirms that PMNS theory is not the exclusive theory for neutrino oscillation. Once created, a bispectral, massless antineutrino, with super-threshold energy ($E > 1.806$ MeV), has the possibility by flavor oscillation to initiate an IBD.

The co-principal finding, which is extra to the massless oscillation finding, is that the forceless weak interaction for this oscillation model transmutes the wave function of the antineutrino from bispectral to unispectral. There is no energy exchange during the transmutation for the weak interaction is forceless. In general, the weak interaction can transmute the wave function to a different superposition of its set of independent solutions without any exchange of energy.

The first secondary finding is that flavor oscillations are compatible with classifying neutrinos to be Majorana leptons.

The second secondary finding is that the elusive sterile neutrino may be just where the antineutrino is in a location, q, in the oscillation cycle where its values $\{\overline{\psi}_2, \partial_q\overline{\psi}_2\}|_q$ are incompatible initial values for initiating a current interaction of any flavor there (sterile is not a flavor). This finding is predicated upon the existence of such a location in the oscillation cycle.

The third secondary finding establishes a relationship between the amplitude β of the latent embedded reflected wave and the opportunity to observe negative momentum, *i.e.*, $\overline{\psi}_2^\dagger(q)\,\frac{\hbar}{i}\partial_q\overline{\psi}_2(q) < 0$. There exists a $\beta_{\text{threshold}}$ for which, if $|\beta| < |\beta_{\text{threshold}}|$, then $\overline{\psi}_2^\dagger(q)\,\frac{\hbar}{i}\partial_q\overline{\psi}_2(q) > 0$ for all q before an IBD. For cases of super-threshold $|\beta|$, the orthodox quantum measurement of momentum over one repetitive box length would still yield positive momentum (29).

The fourth secondary finding confirms the similar prediction for massless neutrino oscillation by the less familiar quantum trajectory representation of quantum mechanics [5]. This finding also substantiates that wave mechanics and quantum trajectories are equivalent for free particles [7], [33]. In addition, incisive insights rendered by the wave function representation complement those of the trajectory representation to substantiate massless neutrino oscillation.

A tertiary finding supports Pontecorvo's suggestion [28] that a neutrino may be composed of a mixture of neutrino and antineutrino components.

In conclusion, massless neutrino oscillation implies the validity of the standard model to consider neutrinos to be massless.

A co-conclusion is that the forceless weak interaction prepares the antineutrino for interaction with other particles by transmuting the antineutrino's wave function. The transmutation changes the wave function in this ab initio calculation from a bispectral wave function to a unispectral wave function $\exp(ikq)$ without an exchange of energy. Conversely, the wave function of the antineutrino manifests the effects of the forceless weak interaction by a change in the superposition of its independent solutions for a given energy.

A secondary conclusion is the confirmation of the similar prediction of the validity of the standard model by the quantum trajectory representation, which substantiates that such a prediction is not an anomaly of the quantum trajectory representation.

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