Relativistically Correct Electromagnetic Energy Flow

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Detailed study of the energy and momentum carried by the electromagnetic field can be a source of clues to possible new physics underlying the Maxwell equations. But such study has been impeded by expressions for the parameters of the electromagnetic energy flow that are inconsistent with the transformation rules of special relativity. This paper begins by correcting a basic parameter, the local velocity of electromagnetic energy flow. This correction is derived by the direct application of the transformation rules of special relativity. After this correction, the electromagnetic energy-momentum tensor can then be expressed in a reference system comoving with the energy flow. This tensor can be made diagonal in the comoving system, and brought into a canonical form depending only on the energy density and one other parameter. The corrected energy flow and its energy-momentum tensor are illustrated by a simple example using static electric and magnetic fields. The proposal that electromagnetic momentum results from the motion of a relativistic mass contained in the fields is examined in the context of the corrected flow velocity. It is found that electromagnetic field momentum, though real, cannot be explained as due only to the motion of relativistic mass. The paper concludes that introducing the requirement of consistency with special relativity opens the study of electromagnetic energy and momentum to new possibilities.

1 Introduction

The Feynman example of a rotating disk with a magnet at its center and charged spheres on its perimeter provides a convincing argument that, to preserve the principle of angular momentum conservation, the field momentum of even a static electromagnetic field must be considered physically real.*

Since the energy density and momentum density of the electromagnetic field are real, it is important to investigate the details of the energy flow that they represent. Since special relativity is the symmetry theory of electrodynamics, it is essential that such investigations respect the transformation laws of special relativity.

In Section 2 a previously proposed candidate expression for a basic parameter, the velocity of energy flow at a given point in the electromagnetic field, is shown to be inconsistent with the transformation rules of special relativity and therefore incorrect. A corrected velocity expression is derived by explicit use of these rules.

Section 3 derives the electromagnetic energy-momentum tensor in a reference system comoving with this corrected velocity, and shows that it can be made diagonal and reduced to a canonical form that depends on two parameters derived from the values of the electric and magnetic fields.

Section 4 illustrates the results of the previous sections with an example using static electric and magnetic fields.

Section 5 considers the question of a relativistic mass density derived from the energy density of the electromagnetic field. It is found that this mass density does not correctly relate the momentum density to the flow velocity. Electromagnetic field momentum, although real, is not due only to the motion of relativistic mass.

Section 6 concludes that introducing the requirement of special relativistic covariance into the study of the flow of energy in electromagnetic fields opens up new possibilities for investigation of such flows.

Electromagnetic formulas in this paper are taken from Griffiths [6] and Jackson [7] with translation to Heaviside-Lorentz units. I denote four-vectors as $\mathsf{A} = A^0\mathsf{e}_0 + \mathbf{A}$ where e_0 is the time unit vector and the three-vector part is understood to be $\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3$. In the Einstein summation convention, Greek indices range from 0 to 3, Roman indices from 1 to 3. The Minkowski metric tensor is $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$. Three-vectors are written with bold type $\mathbf{A}$, and their magnitudes as A. Thus $|\mathbf{A}| = A$.

2 Velocity of energy flow

We begin with a basic parameter of the electromagnetic field. The flow velocity of the energy contained in the field at a given event can be defined as the velocity of a comoving observer who measures a zero energy flux there. Expressed in the precise language of Lorentz boosts:†

> The laboratory system coordinate velocity of the flow of electromagnetic field energy at a given event is the velocity $\mathbf{V}$ of a Lorentz boost that transforms the laboratory reference system into a reference system in which the Poynting energy

*Feynman et al [4], Section 17-4, Section 27-6, and Figure 17-5. Quantitative matches of field to mechanical angular momentum are found, for example, in Romer [12] and Boos [2].

†The Lorentz boost formalism used here is defined in Appendix A.

> flux vector is null at that event. An observer at that event and at rest in this system, which we call the comoving system and denote by primes, therefore measures a zero energy flux. The zero flux measurement indicates that this observer is comoving with the flow of energy. Such an observer has coordinate velocity **V** relative to the laboratory*, and therefore **V** is the laboratory system coordinate velocity of the energy flow at the given event.

A previously proposed† candidate for the laboratory system coordinate velocity of the electromagnetic energy flow is the momentum density divided by the relativistic mass (defined as energy density divided by the square of the speed of light). Denoting this velocity as $\mathbf{u}_e$ gives

$$\frac{\mathbf{u}_e}{c} = \frac{\mathbf{G}}{(\mathcal{E}/c)} = \frac{2\,(\mathbf{E}\times\mathbf{B})}{(E^2+B^2)} \tag{1}$$

where $\mathbf{G} = \mathbf{S}/c^2 = (\mathbf{E}\times\mathbf{B})\,/c$ is the linear momentum density of an electromagnetic field with electric and magnetic field vectors **E** and **B** and Poynting energy flux vector **S**. The $\mathcal{E} = \left(E^2+B^2\right)/2$ is the electromagnetic energy density, and c is the vacuum velocity of light.

If $\mathcal{E}/c$ and **G** were the time and space parts of a four-vector, then a Lorentz boost from the laboratory system using boost velocity $\mathbf{V} = \mathbf{u}_e$ would produce a comoving reference system (denoted by primes) in which the space part of that four-vector, that is $\mathbf{G}'$ and hence the Poynting vector $\mathbf{S}' = c^2\mathbf{G}'$, would vanish, indicating a system comoving with the energy flow.‡ Thus $\mathbf{u}_e$ would be the comoving velocity of the energy flow.

However, $\mathcal{E}/c$ and **G** are *not* components of a four-vector. There is no four-vector momentum density of the form $\mathsf{G} = (\mathcal{E}/c)\,\mathsf{e}_0 + \mathbf{G}$.

Rather, $\mathcal{E}$ and $c\,\mathbf{G}$ are the T^{00} and T^{0i} components of the second-rank electromagnetic energy-momentum tensor

$$T^{\alpha\beta} = \begin{pmatrix} \mathcal{E} & c\,G_1 & c\,G_2 & c\,G_3 \\ c\,G_1 & M_{11} & M_{12} & M_{13} \\ c\,G_2 & M_{21} & M_{22} & M_{23} \\ c\,G_3 & M_{31} & M_{32} & M_{33} \end{pmatrix} \tag{2}$$

where $M_{ij} = -\left(E_iE_j + B_iB_j\right) + \frac{1}{2}\,\delta_{ij}\,\left(E^2+B^2\right)$.

*See Appendix A for a demonstration that any point at rest in the primed system moves with laboratory system coordinate velocity **V**.

†In a discussion of the Poynting theorem in material media, but with no special attention to Lorentz covariance, Born and Wolf [3] Section 14.2, Eq. (8) identify $\mathbf{u}_e$ as the *velocity of energy transport* or *ray velocity.* Section B.2 of Smith [15] echoes Born and Wolf but provides no new derivation. (The first edition of Born and Wolf's text appeared in 1959.) Geppert [5] writes a non-covariant equation with the same identification. More recently, Sebens [13,14] relies on these and other sources to identify $\mathbf{u}_e$ as the electromagnetic mass flow velocity. (Following Sebens, expand $(E-B)^2 \geq 0$ and use the definitions of $\mathcal{E}$ and **G** to prove that $|\mathbf{u}_e| \leq c$.)

‡See Appendix A for a demonstration that $\mathbf{G}'$ would be zero.

A related point is made by Rohrlich [11], using the so-called von Laue theorem to argue that *integrals* of $\mathcal{E}/c$ and **G** over hyperplanes may in some cases transform as four-vectors. But we are treating these quantities locally, at a particular event. Von Laue's theorem does not imply that the local field functions $\mathcal{E}/c$ and **G** themselves transform as components of a four-vector. They do not. Rather than attempting to derive a four-vector from $\mathcal{E}/c$ and **G**, we show how to use them in a relativistically correct manner as they are. See also Section 6.3 of [10].

Since $\mathcal{E}$ and $c\,\mathbf{G}$ are components of $T^{\alpha\beta}$, contributions to the boost transformation from the other components of $T^{\alpha\beta}$ would produce a comoving system in which $\mathbf{G}'$ and the Poynting vector would not vanish. *The electromagnetic energy flow velocity is not* $\mathbf{u}_e$.

The failure of $\mathbf{u}_e$ to be the correct flow velocity can be contrasted with the well-understood theory of the flow of electric charge. The charge density ρ and the current density vector **J** are shown by the divergence of the Maxwell field tensor to form a four-vector of the form $\mathsf{J} = c\rho\,\mathsf{e}_0 + \mathbf{J}$. In general, J can be timelike, spacelike, or null. If spacelike, there is no velocity $\mathbf{v}_q$ less than the speed of light with $\mathbf{J} = \rho\,\mathbf{v}_q$. But if we consider, for example, a system in which all the moving charges have the same sign, it can be shown that J is timelike and hence the definition $\mathbf{u}_q = \mathbf{J}/\rho$ does produce a vector of magnitude less than the speed of light. Then a Lorentz boost with boost velocity $\mathbf{V} = \mathbf{u}_q$ indeed leads to a comoving primed reference system in which the current flux density $\mathbf{J}'$ vanishes§, and $\mathbf{u}_q$ is therefore the correct flow velocity of the moving charge.

But the fact that J transforms as a four-vector is crucial to this argument. If it were not a four-vector transforming as in Appendix A, the system reached by boost $\mathbf{V} = \mathbf{u}_q$ would have a residual current flow $\mathbf{J}' \neq 0$, and $\mathbf{u}_q$ would therefore not be the correct flow velocity. The equation $\mathbf{J} = \rho\,\mathbf{u}_q$ would still follow from the definition of $\mathbf{u}_q$, but that formula would not imply that $\mathbf{u}_q$ is the correct velocity of the flowing charge.

The failure of $\mathbf{u}_e$ as a candidate for the flow velocity of electromagnetic energy is precisely because, unlike J, the expression written here in four-vector form $\mathsf{G} = (\mathcal{E}/c)\,\mathsf{e}_0 + \mathbf{G}$ actually does *not* transform as a four-vector. The equation $\mathbf{G} = \left(\mathcal{E}/c^2\right)\mathbf{u}_e$ (or equivalently $\mathbf{S} = \mathcal{E}\,\mathbf{u}_e$) still follows from the definition of $\mathbf{u}_e$, but that formula does not imply that $\mathbf{u}_e$ is the correct velocity of the flowing energy.

However, the correct boost velocity **V** can be found by starting from $\mathbf{u}_e$ and applying a scalar correction factor. The corrected velocity **V** will have the same *direction* as $\mathbf{u}_e$ but not the same *magnitude.* To find this corrected velocity **V** it is best to turn to a direct method, using the transformation rules for the fields **E** and **B**.

§Substitute $c\rho = J^0$ and **J** for G^0 and **G** in Appendix A to see that $\mathbf{J}'$ vanishes.

The rules for transformation of electric and magnetic fields by a boost with velocity $\mathbf{V}$ can be written in a special relativistically correct but not manifestly covariant form as*

$$\begin{aligned}\mathbf{E}' &\doteq \gamma\left(\mathbf{E}+\frac{\mathbf{V}}{c}\times\mathbf{B}\right)+(1-\gamma)\,\frac{\mathbf{V}\,(\mathbf{V}\cdot\mathbf{E})}{V^2}\\ \mathbf{B}' &\doteq \gamma\left(\mathbf{B}-\frac{\mathbf{V}}{c}\times\mathbf{E}\right)+(1-\gamma)\,\frac{\mathbf{V}\,(\mathbf{V}\cdot\mathbf{B})}{V^2}\end{aligned} \qquad (3)$$

where the Lorentz factor is $\gamma = \left(1 - V^2/c^2\right)^{-1/2}$. The $\doteq$ symbol means that the components of the three-vector on the left side of this symbol expressed in the primed coordinate system are numerically equal to the components of the three-vector on the right side of this symbol expressed in the original unprimed system. If $\mathbf{a}' \doteq \mathbf{c}$ and $\mathbf{b}' \doteq \mathbf{d}$, it is easily proved that: (a) $\mathbf{a}'\times\mathbf{b}' \doteq \mathbf{c}\times\mathbf{d}$ and (b) $(\mathbf{a}'\cdot\mathbf{b}') = (\mathbf{c}\cdot\mathbf{d})$.

Define the boost velocity $\mathbf{V}$ to be an unknown but rotationally scalar quantity λ times $\mathbf{u}_e$

$$\mathbf{V} = \lambda\,\mathbf{u}_e\,. \qquad (4)$$

Since $\mathbf{u}_e$ and $\mathbf{V}$ are perpendicular to both the electric and magnetic fields, it follows that $(\mathbf{V}\cdot\mathbf{E}) = (\mathbf{V}\cdot\mathbf{B}) = 0$. Thus, (3) reduces to †

$$\begin{aligned}\mathbf{E}' &\doteq \gamma\left(\mathbf{E}+\frac{\mathbf{V}}{c}\times\mathbf{B}\right)\\ \mathbf{B}' &\doteq \gamma\left(\mathbf{B}-\frac{\mathbf{V}}{c}\times\mathbf{E}\right).\end{aligned} \qquad (5)$$

Insert (5) into the definition $\mathbf{S}' = c\mathbf{E}'\times\mathbf{B}'$. Using (4) and then (1) leads to‡

$$\mathbf{S}' = c\mathbf{E}'\times\mathbf{B}' \doteq \gamma^2 c\,(\mathbf{E}\times\mathbf{B})\left((u_e^2/c^2)\,\lambda^2 - 2\lambda + 1\right). \qquad (6)$$

Notice that (6) verifies the statement above that $\mathbf{u}_e$ is not the comoving velocity of the energy flow. Setting $\lambda = 1$ in (4) makes $\mathbf{V} = \mathbf{u}_e$. But setting $\lambda = 1$ in (6) makes

$$\mathbf{S}' \doteq \gamma^2 c\,(\mathbf{E}\times\mathbf{B})\left(u_e^2/c^2 - 1\right) \quad \text{when} \quad \lambda = 1 \qquad (7)$$

which is not zero, except in the unphysical limit $u_e = V = c$.

For a second and more important use of (6), choose λ to solve the quadratic equation $\left((u_e^2/c^2)\lambda^2 - 2\lambda + 1\right) = 0$. Then (6) makes $\mathbf{S}' = 0$. The solution is

$$\lambda = \frac{1}{(u_e/c)^2}\left\{1 - \sqrt{1-(u_e/c)^2}\right\}. \qquad (8)$$

From (4), the correct velocity of the energy flow is therefore

$$\mathbf{V} = \lambda\,\mathbf{u}_e = \frac{1}{(u_e/c)^2}\left\{1 - \sqrt{1-(u_e/c)^2}\right\}\mathbf{u}_e \qquad (9)$$

where $\mathbf{u}_e$ is defined in (1).

This $\mathbf{V}$ is the relativistically correct boost velocity from the laboratory frame to the comoving reference frame in which $\mathbf{S}' = 0$, and is therefore the laboratory system coordinate velocity of the electromagnetic energy flow§.

Since both $\mathbf{V}$ and $\mathbf{u}_e$ are parallel to the energy flux vector $\mathbf{S}$, the energy flow velocity can also be written as $\mathbf{V} = V\,(\mathbf{S}/S)$ where the magnitude V is given by¶

$$(V/c) = \frac{1}{(u_e/c)}\left\{1 - \sqrt{1-(u_e/c)^2}\right\}. \qquad (10)$$

This equation can be inverted to give

$$(u_e/c) = \frac{2\,(V/c)}{1+(V/c)^2} \qquad (11)$$

which can be used to write the correction factor λ in (8) as a function of the corrected velocity

$$\lambda = \frac{1+(V/c)^2}{2}\,. \qquad (12)$$

3 Comoving energy-momentum tensor

The derivation of a reference system comoving with the flow of energy allows the electromagnetic energy-momentum tensor to be examined in more detail. The energy-momentum tensor in (2) can be transformed into the comoving (primed) coordinate system that was produced by the Lorentz boost $\mathbf{V}$. In this system, the electromagnetic energy-momentum tensor is represented by the tensor components $T'^{\alpha\beta}$ in which the cG'_i elements are zero.

$$T'^{\alpha\beta} = \begin{pmatrix} \mathcal{E}' & 0 & 0 & 0\\ 0 & M'_{11} & M'_{12} & M'_{13}\\ 0 & M'_{21} & M'_{22} & M'_{23}\\ 0 & M'_{31} & M'_{32} & M'_{33}\end{pmatrix} \qquad (13)$$

where

$$\begin{aligned}\mathcal{E}' &= \frac{1}{2}\left(E'^2 + B'^2\right) = \mathcal{E}\,\frac{1-(V/c)^2}{1+(V/c)^2}\\ \text{and}\quad M'_{ij} &= -\left(E'_iE'_j + B'_iB'_j\right) + \delta_{ij}\mathcal{E}'\,.\end{aligned} \qquad (14)$$

We can now make another Lorentz transformation, an orthogonal spatial rotation at fixed time, to diagonalize the real, symmetric sub-matrix M_{ij} in (13).

The required spatial rotation can be defined as the product of two proper rotations. First rotate the coordinate system to bring the $\mathbf{e}'_3$ axis into the $\mathbf{V}' \doteq \mathbf{V}$ direction. Denote

*See Section 11.10 of Jackson [7], eqn (11.149).

†Note that $\mathbf{V}'\doteq\mathbf{V}$ as defined in Appendix A, together with (5) and property (b) of the symbol $\doteq$ noted above, imply that $(\mathbf{V}'\cdot\mathbf{E}') = \mathbf{V}\cdot\gamma\,[\mathbf{E}+(\mathbf{V}/c)\times\mathbf{B}] = \gamma(\mathbf{V}\cdot\mathbf{E}) = 0$. Similarly, $(\mathbf{V}'\cdot\mathbf{B}') = 0$.

‡See a detailed derivation of (6) in Appendix B.

§Appendix C gives details of the comoving system for possible values of $(\mathbf{E}\cdot\mathbf{B})$ at a given event.

¶Footnote † on page 4 proves that $0 \le u_e \le c$. As u_e/c increases from 0 to 1, (10) shows that V/c increases monotonically from 0 to 1, with $V \le u_e$ at every point. It follows that $0 \le V \le c$ also.

this rotated system by tildes. Rotations do not change three-vectors, which are invariant objects under rotations. However, rotations do change the *components* of three-vectors. Thus $\tilde{\mathbf{V}} = \mathbf{V}'$, $\tilde{\mathbf{E}} = \mathbf{E}'$, and $\tilde{\mathbf{B}} = \mathbf{B}'$, but in the tilde system $\tilde{\mathbf{V}}$ now has components $\tilde{V}_1 = \tilde{V}_2 = 0$ and $\tilde{V}_3 = V$. Then using Footnote † on page 5, we have $0 = (\mathbf{E}' \cdot \mathbf{V}') = (\tilde{\mathbf{E}} \cdot \tilde{\mathbf{V}}) = V\tilde{E}_3$. Since the magnitude $V \neq 0$, it follows that $\tilde{E}_3 = 0$. A similar argument proves that $\tilde{B}_3 = 0$. Thus the {33} component of the energy-momentum tensor when expressed in the tilde system is $\tilde{T}^{33} = -\left(\tilde{E}_3^2 + \tilde{B}_3^2\right) + \tilde{\mathcal{E}} = \tilde{\mathcal{E}}$. The tensor from (13), when expressed in the tilde system, becomes

$$\tilde{T}^{\alpha\beta} = \begin{pmatrix} \tilde{\mathcal{E}} & 0 & 0 & 0 \\ 0 & \tilde{M}_{11} & \tilde{M}_{12} & 0 \\ 0 & \tilde{M}_{21} & \tilde{M}_{22} & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{E}} \end{pmatrix} \tag{15}$$

where $\tilde{\mathcal{E}} = \mathcal{E}'$.

Since the invariant trace of the electrodynamic energy-momentum tensor vanishes*, it follows from (15) that

$$0 = \eta_{\alpha\beta}\tilde{T}^{\alpha\beta} = -\tilde{\mathcal{E}} + \tilde{M}_{11} + \tilde{M}_{22} + \tilde{\mathcal{E}} \tag{16}$$

and hence $\tilde{M}_{11} = -\tilde{M}_{22}$. Also, the symmetry of the energy-momentum tensor makes $\tilde{M}_{21} = \tilde{M}_{12}$. Thus

$$\tilde{T}^{\alpha\beta} = \begin{pmatrix} \tilde{\mathcal{E}} & 0 & 0 & 0 \\ 0 & -\tilde{\psi} & \tilde{\xi} & 0 \\ 0 & \tilde{\xi} & \tilde{\psi} & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{E}} \end{pmatrix} \tag{17}$$

where $\tilde{\psi} = \tilde{M}_{22}$ and $\tilde{\xi} = \tilde{M}_{12}$.

A second proper rotation, this time about the $\tilde{\mathbf{e}}_3$ axis, produces the final coordinate system, denoted with double primes. After this rotation, $E''_3 = \tilde{E}_3 = 0$, $B''_3 = \tilde{B}_3 = 0$, and $\mathbf{V}'' = \tilde{\mathbf{V}}$ has components $V''_1 = V''_2 = 0$ and $V''_3 = V$. The only effect of this second rotation is to diagonalize the matrix $\begin{pmatrix} -\tilde{\psi} & \tilde{\xi} \\ \tilde{\xi} & \tilde{\psi} \end{pmatrix}$. The energy momentum tensor then has its final, diagonal form in the double-prime system

$$T''^{\alpha\beta} = \begin{pmatrix} \mathcal{E}'' & 0 & 0 & 0 \\ 0 & -a'' & 0 & 0 \\ 0 & 0 & a'' & 0 \\ 0 & 0 & 0 & \mathcal{E}'' \end{pmatrix} \tag{18}$$

where $\mathcal{E}'' = \tilde{\mathcal{E}} = \mathcal{E}'$. The parameter a'' has absolute value $|a''| = \left\{\tilde{\psi}^2 + \tilde{\xi}^2\right\}^{1/2}$ where $\pm\left\{\tilde{\psi}^2 + \tilde{\xi}^2\right\}^{1/2}$ are the two eigenvalues of the matrix $\begin{pmatrix} -\tilde{\psi} & \tilde{\xi} \\ \tilde{\xi} & \tilde{\psi} \end{pmatrix}$ that were calculated during the diagonalization process. The sign of a'' depends on the directions and relative magnitudes of the electric and magnetic fields.

*See Section 7.8 of Rindler [9].

The rotation that takes the system from the primed to the double-primed system is then the product of the first and second rotations. The various representations of the boost velocity are related by $\mathbf{V}'' = V\mathbf{e}''_3 = \tilde{\mathbf{V}} = V\tilde{\mathbf{e}}_3 = \mathbf{V}' \triangleq \mathbf{V}$ where all of these vectors have the same original magnitude V.

The energy-momentum tensor in the double-prime system is diagonal and in a canonical form that depends only on the energy density $\mathcal{E}''$ in the comoving system and one other parameter a''.

Section 4 shows that there are realistic electromagnetic cases in which $a'' \neq 0$ and hence the diagonal elements M''_{ii} for $i = 1, 2, 3$ are not all equal, unlike the analogous elements in the energy-momentum tensor of a perfect fluid†, all of which are equal by definition, a fact of relevance for future studies that might attempt a fluid-dynamic model of electrodynamic energy flow.

4 Example: crossed static fields

Consider an example with static, perpendicular electric and magnetic fields.‡ Choose the Cartesian axes of the laboratory system so that $\mathbf{E} = E\,\hat{\mathbf{x}}$ and $\mathbf{B} = B\,\hat{\mathbf{y}}$. Then (1) becomes

$$\frac{\mathbf{u}_\mathrm{e}}{c} = \left(\frac{2E\,B}{E^2 + B^2}\right)\hat{\mathbf{z}}\,. \tag{19}$$

The energy flow velocity is thus $\mathbf{V} = V\,\hat{\mathbf{z}}$ where $V = \lambda\,u_\mathrm{e}$ with λ from (8). Inserting this $\mathbf{V}$ into (5) with the above values of the electric and magnetic fields gives

$$\begin{aligned} \mathbf{E}'' &\triangleq \gamma\left(E - \frac{VB}{c}\right)\hat{\mathbf{x}} \\ \mathbf{B}'' &\triangleq \gamma\left(B - \frac{VE}{c}\right)\hat{\mathbf{y}}\,. \end{aligned} \tag{20}$$

Thus the definitions in (2) when applied in the double-prime system give $M''_{ij} = 0$ for $i \neq j$ and

$$\begin{aligned} M''_{11} &= -E''^2_1 + \mathcal{E}'' = -\frac{1}{2}\left(E''^2_1 - B''^2_2\right) \\ M''_{22} &= -B''^2_2 + \mathcal{E}'' = \frac{1}{2}\left(E''^2_1 - B''^2_2\right) \\ M''_{33} &= \mathcal{E}'' = \frac{1}{2}\left(E''^2_1 + B''^2_2\right)\,. \end{aligned} \tag{21}$$

where E''_1 and B''_2 are the components of $\mathbf{E}''$ and $\mathbf{B}''$, respectively, in (20).

The step of rotating from primed to double-primed reference systems that was necessary to move from (13) to (18) above was not necessary here due to a propitious choice of original laboratory reference system. The Lorentz boost with velocity $\mathbf{V} = V\,\hat{\mathbf{z}}$ produces an already diagonal energy momentum tensor with $M''_{ij} = 0$ for $i \neq j$.

†See Part I, Chapter 2, Section 10 of Weinberg [16].

‡The center of a parallel plate capacitor at the center of a long solenoid, for example.

Consider the case $E \neq B$. From (19), this inequality implies that $u_e < c$ and hence, from (10), that $V < c$, a physically possible value. Also $E \neq B$ implies, either from the invariance of $\left(E^2 - B^2\right)$ noted in Appendix C or directly from (20), that $2M''_{22} = \left(E_1''^2 - B_2''^2\right) = \left(E^2 - B^2\right) \neq 0$. Thus $E \neq B$ implies that $M''_{11} = -M''_{22} \neq 0$ and hence that $M''_{11} \neq M''_{22}$.

Comparing (21) to (14) and (18) shows that for the crossed-field example with $E{\neq}B$, the energy-momentum tensor in the double-prime system is (18) with

$$a'' = \left(E^2 - B^2\right)/2 \neq 0$$
$$\text{and} \quad \mathcal{E}'' = \mathcal{E}\left(1 - (V/c)^2\right) / \left(1 + (V/c)^2\right) \tag{22}$$

where $\mathcal{E} = \left(E^2 + B^2\right)/2$.

As asserted at the end of Section 3, the inequality $E \neq B$ in the crossed-field example shows a physically reasonable case for which $a'' \neq 0$ and the M''_{ii} for $i = 1, 2, 3$ are not all equal.*

There are questions about the interpretation of this example globally†. In our use of this example, however, we need not consider the question of so-called *hidden momentum* required to balance the total field momentum‡. Here, the only relevant use of this example is to illustrate the correct local definition of the energy flow velocity and comoving energy-momentum tensor in a vacuum region where the fields are well known — at the center of the parallel plate capacitor far from the edges.

5 Relativistic mass density

The energy density $\mathcal{E}$ of either static or time-varying vacuum electromagnetic fields can be used to define a *relativistic* mass density§

$$\mathcal{M}_{\text{rel}} = \mathcal{E}/c^2 \,. \tag{23}$$

The adjective *relativistic* indicates that this mass density is analogous to a single-particle relativistic mass $m_{\text{rel}} = \gamma m = e/c^2$ where e is the particle relativistic energy, m is the invariant or rest mass of the particle, and $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ is the Lorentz factor of its velocity $\mathbf{v}$.

It follows from (23) that the flow velocity of the energy $\mathcal{E}$, the velocity $\mathbf{V}$ derived in Section 2 and summarized in (9), must also be the flow velocity of the relativistic mass $\mathcal{M}_{\text{rel}}$.

In the single-particle case, the same m_{rel} can be used to relate the momentum of the particle to its velocity, $\mathbf{p} = \gamma m \mathbf{v} = m_{\text{rel}}\mathbf{v}$. However in the case of electromagnetic fields, the same mass density $\mathcal{M}_{\text{rel}}$ cannot be used for both purposes.

Due to the correction of the flow velocity in Section 2, which was necessitated by adherence to the transformation rules of special relativity, the relation between momentum density $\mathbf{G}$ and corrected flow velocity $\mathbf{V}$ is

$$\mathbf{G} = \left(\frac{\mathcal{M}_{\text{rel}}\,\mathbf{V}}{\lambda}\right) \neq \mathcal{M}_{\text{rel}}\mathbf{V} \tag{24}$$

where λ is the correction factor in (12).

The inequality in (24) shows that the electromagnetic momentum density at an event is not equal to the electromagnetic mass density at that event times the relativistically correct mass flow velocity there.

The explicit expression for the correction factor λ from (12) quantifies the extent of the inequality. The effective mass for momentum calculation is the larger value $\mathcal{M}_{\text{rel}}/\lambda$ rather than $\mathcal{M}_{\text{rel}}$¶.

The failure of $\mathcal{M}_{\text{rel}}\mathbf{V}$ to equal the momentum density $\mathbf{G}$ in (24) suggests that vacuum electromagnetic field momentum cannot be explained only by the motion of relativistic mass. There must be another source of real electromagnetic field momentum.

6 Conclusion

The electromagnetic field contains energy and momentum. Calculation of the energy flow velocity and energy-momentum tensor in a relativistically correct manner opens the subject to new insights into that energy and momentum. For example, the energy-momentum tensor measured by an observer comoving with the flow velocity is obtained in diagonal, canonical form suggestive of possible fluid dynamical models. And the momentum density of the electromagnetic field is shown to require some source other than the flow of relativistic mass.

Appendix A: Lorentz boosts

Consider a Lorentz transformation from an *unprimed* coordinate system (which we also refer to as the *laboratory* system) with coordinates $x = (x^0, x^1, x^2, x^3)$ to a *primed* coordinate system with coordinates $x' = (x'^0, x'^1, x'^2, x'^3)$ where $x^0 = ct$ and $x'^0 = ct'$. The most general proper, homogeneous Lorentz transformation from the unprimed to the primed systems can be written as a Lorentz boost times a rotation.‖

Definition of Lorentz boost

A Lorentz boost transformation is parameterized by a boost velocity vector $\mathbf{V}$ with components (V_1, V_2, V_3) and magnitude $V = \left(V_1^2 + V_2^2 + V_3^2\right)^{1/2}$. Using the Einstein summation

*Our use of this example is based on $E \neq B$. The case $E = B \neq 0$ would have to be approached as a limit, as discussed in Appendix C (c). With $E \neq 0$ and $B = E(1 + \delta)$, retaining leading order in the small quantity δ gives $(u_e/c) \approx (1 - \delta^2/2)$, $\lambda \approx (1 - |\delta|)$, $(V/c) \approx (1 - |\delta|)$, $(\mathcal{E}''/E^2) \approx |\delta|$, $(T''^{\alpha\beta}/E^2) \approx \text{diag}(|\delta|, \delta, -\delta, |\delta|)$, and $\left(a''/E^2\right) \approx -\delta$.

†See McDonald [8] for calculation of the total field momentum of a similar example.

‡See, for example, Babson et al [1].

§For example, see Section 3 of Sebens [13].

¶Note that (24) can be written as $\mathbf{G} = \mathcal{M}_{\text{eff}}\mathbf{V}$ where, using (12), (14), and (23), $\mathcal{M}_{\text{eff}} = \mathcal{M}_{\text{rel}}/\lambda = 2\gamma^2(\mathcal{E}'/c^2)$ where $\gamma = (1 - V^2/c^2)^{-1/2}$.

‖See Part I, Chapter 2, Section 1 of Weinberg [16].

convention, it is written as $x'^{\alpha} = \Lambda^{\alpha}_{\ \beta}\, x^{\beta}$ where $\Lambda^{0}_{\ 0} = \gamma$, $\Lambda^{0}_{\ i} = \Lambda^{i}_{\ 0} = -\gamma V_i/c$, and $\Lambda^{i}_{\ j} = \delta_{ij} + (\gamma - 1)V_i V_j/V^2$. The δ_{ij} is the Kronecker delta function. Also $\gamma = \left(1 - V^2/c^2\right)^{-1/2}$.

The inverse boost $\underline{\Lambda}^{\alpha}_{\ \beta}$ is the same except for the substitution $V_i \to -V_i$. Thus the inverse boost vector is $(-\mathbf{V}')$ where $\mathbf{V}' \triangleq \mathbf{V}$.

Meaning of the boost velocity V

The velocity $\mathbf{V}$ that parameterizes the Lorentz boost is also the coordinate velocity, as measured from the unprimed laboratory system, of any point that is at rest in the primed system. In this sense, the entire primed system is moving with velocity $\mathbf{V}$ as observed from the laboratory system.

To see this, apply the inverse Lorentz boost to the differentials of a point at rest in the primed system, $dx'^{i} = 0$ for $i = 1, 2, 3$, but $dx'^{0} > 0$. The result is $dx^0 = \gamma dx'^{0}$ and $dx^i = \gamma\,(V_i/c)\,dx'^{0}$. Thus $dx^i/dt = V_i$, as was asserted.

Consequence of existence of a four-vector G

The discussion surrounding (2) shows that $\mathsf{G} = (\mathcal{E}/c)\,\mathsf{e}_0 + \mathbf{G}$ is *not* a four-vector, despite being written in four-vector form here. Its components instead transform as components of the energy-momentum tensor. But suppose for a moment that it is a four-vector. If so, then a Lorentz boost with a boost velocity $(\mathbf{u}_e/c) = \mathbf{G}/(\mathcal{E}/c)$ would make the transformed space part of G equal to zero.

As applied to a four-vector, the Lorentz boost transformation rule is $G'^{\alpha} = \Lambda^{\alpha}_{\ \beta}\, G^{\beta}$. Hence

$$\begin{aligned} G'^{i} &= \Lambda^{i}_{\ 0} G^0 + \Lambda^{i}_{\ j} G^j \\ &= -\gamma \frac{V_i}{c} G^0 + G^i + (\gamma - 1)\frac{V_i\left(V_j G^j\right)}{V^2}\,. \end{aligned} \tag{25}$$

Replacing V_i/c by $(u_e)_i/c = G^i/G^0$ in (25) makes $G'^{i} = 0$, as asserted.

Appendix B: Detailed derivation of Eq. (6) for S′

We have (1), (4) and (5) and $(\mathbf{V}\cdot\mathbf{E}) = (\mathbf{V}\cdot\mathbf{B}) = 0$. Inserting (5) into $\mathbf{S}' = c\,(\mathbf{E}'\times\mathbf{B}')$ gives

$$\mathbf{S}' = c\mathbf{E}'\times\mathbf{B}' \triangleq c\gamma^2\left\{(\mathbf{E}\times\mathbf{B}) + \mathbf{f} + \mathbf{g}\right\} \tag{26}$$

where, omitting zero terms,

$$\begin{aligned} \mathbf{f} &= -\mathbf{E}\times\left(\frac{\mathbf{V}}{c}\times\mathbf{E}\right) + \left(\frac{\mathbf{V}}{c}\times\mathbf{B}\right)\times\mathbf{B} \\ &= -\left(E^2 + B^2\right)\frac{\mathbf{V}}{c} = -\lambda\left(E^2 + B^2\right)\frac{\mathbf{u}_e}{c} \\ &= -\lambda\left(E^2 + B^2\right)\frac{2\,(\mathbf{E}\times\mathbf{B})}{(E^2 + B^2)} = -2\lambda\,(\mathbf{E}\times\mathbf{B}) \end{aligned} \tag{27}$$

and, again omitting zero terms,

$$\begin{aligned} \mathbf{g} &= -\left(\frac{\mathbf{V}}{c}\times\mathbf{B}\right)\times\left(\frac{\mathbf{V}}{c}\times\mathbf{E}\right) \\ &= -\frac{\mathbf{V}}{c}\left\{\left(\frac{\mathbf{V}}{c}\times\mathbf{B}\right)\cdot\mathbf{E}\right\} = \frac{\mathbf{V}}{c}\left\{\frac{\mathbf{V}}{c}\cdot(\mathbf{E}\times\mathbf{B})\right\} \\ &= \lambda^2\left\{\frac{2\,(\mathbf{E}\times\mathbf{B})}{(E^2 + B^2)}\right\}\left\{\frac{\mathbf{u}_e}{c}\cdot\left(\frac{E^2 + B^2}{2}\right)\frac{\mathbf{u}_e}{c}\right\} \\ &= \lambda^2\left(\frac{\mathbf{u}_e}{c}\cdot\frac{\mathbf{u}_e}{c}\right)(\mathbf{E}\times\mathbf{B}) = \lambda^2\left(\frac{u_e}{c}\right)^2(\mathbf{E}\times\mathbf{B})\,. \end{aligned} \tag{28}$$

Collect terms and factor out $\mathbf{E}\times\mathbf{B}$ to get

$$\mathbf{S}' = c\mathbf{E}'\times\mathbf{B}' \triangleq \gamma^2 c\,(\mathbf{E}\times\mathbf{B})\left\{\left(\frac{u_e}{c}\right)^2\lambda^2 - 2\lambda + 1\right\} \tag{29}$$

which is (6).

Appendix C: Detail of the comoving system

The comoving system is defined by $\mathbf{S}' = c\,(\mathbf{E}'\times\mathbf{B}') = 0$. Thus $|\mathbf{E}'\times\mathbf{B}'| = E'B'\sin\theta' = 0$ where θ' is the angle between $\mathbf{E}'$ and $\mathbf{B}'$ in the comoving system.

From Eqs (7.62) and (7.63) of Rindler [9], we have $\left(E'^2 - B'^2\right) = \left(E^2 - B^2\right)$ and $(\mathbf{E}'\cdot\mathbf{B}') = (\mathbf{E}\cdot\mathbf{B})$. It follows that:

(a) An event with $\mathbf{E}\cdot\mathbf{B} \neq 0$ has $E'B' \neq 0$ and therefore $\mathbf{E}'$ and $\mathbf{B}'$ must be either parallel or anti-parallel, $\theta' = 0$ or $\theta' = \pi$, at this event;

(b) An event with $0 = (\mathbf{E}\cdot\mathbf{B}) = (\mathbf{E}'\cdot\mathbf{B}') = E'B'\cos\theta'$ cannot have $E'B' \neq 0$ in the comoving system because that would require both $\cos\theta' = 0$ and $\sin\theta' = 0$. Thus $E'B' = 0$ and one of E' and B' must be zero. If $E > B$ then $E' > B'$ and hence $B' = 0$. If $E < B$ then $E' < B'$ and hence $E' = 0$;

(c) If both $0 = \mathbf{E}\cdot\mathbf{B}$ and $E = B \neq 0$ at an event, then both $E'B' = 0$ and $E' = B'$, and therefore $E' = B' = 0$. But (1) and (10) show that such an event also has $u_e/c = 1$ and hence $V/c = 1$. The case $E = B \neq 0$ and $0 = \mathbf{E}\cdot\mathbf{B}$ therefore must be approached as a limit.

Received on Sept. 23, 2020

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