A Note on the Barut Second-Order Equation

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The second-order equation in the $(1/2,0)\oplus(0,1/2)$ representation of the Lorentz group has been proposed by A. Barut in the 70s [1]. It permits to explain the mass splitting of leptons (e,μ,τ). The interest is growing in this model (see, for instance, the papers by S. Kruglov [2] and J. P. Vigier *et al.* [3, 4]). We note some additional points of this model.

The Barut main equation is

$$\left[i\gamma^\mu\partial_\mu + \alpha_2\partial^\mu\partial_\mu - \kappa\right]\Psi = 0\,, \tag{1}$$

where α_2 and κ are the constants later related to the anomalous magnetic moment and mass, respectively. The matrices γ^μ are defined by the anticommutation relation:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,, \tag{2}$$

$g^{\mu\nu}$ is the metrics of the Minkowski space, $\mu,\nu = 0,1,2,3$. The equation represents a theory with the conserved current that is linear in 15 generators of the 4-dimensional representation of the O(4, 2) group, $N_{ab} = \frac{i}{2}\gamma_a\gamma_b, \gamma_a = \{\gamma_\mu, \gamma_5, i\}$. Instead of 4 solutions, (1) has 8 solutions with the correct relativistic relation $E = \pm\sqrt{\mathbf{p}^2 + m_i^2}$. In fact, it describes states of different masses (the second one is $m_2 = 1/\alpha_2 - m_1 = m_e(1 + 3/2\alpha)$, α is the fine structure constant), provided that the certain physical condition is imposed on $\alpha_2 = (1/m_1)(2\alpha/3)/(1 + 4\alpha/3)$, the parameter (the anomalous magentic moment should be equal to $4\alpha/3$). One can also generalize the formalism to include the third state, the τ-lepton [1b]. Barut has indicated the possibility of including γ_5 terms (e.g. $\sim\gamma_5\kappa'$).

The most general form of spinor relations in the $(1/2,0)\oplus(0,1/2)$ representation has been given by Dvoeglazov [5]. It was possible to derive the Barut equation from first principles [6]. Let us reveal the connections with other models. For instance, in [3, 7] the following equation has been studied:

$$\begin{aligned}&\left[\left(i\hat\partial - e\hat A\right)\left(i\hat\partial - e\hat A\right) - m^2\right]\Psi = \\ &\left[\left(i\partial_\mu - eA_\mu\right)(i\partial^\mu - eA^\mu) - \tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu} - m^2\right]\Psi = 0\end{aligned} \tag{3}$$

for the 4-component spinor Ψ. $\hat A = \gamma^\mu A_\mu$; A_μ is the 4-vector potential; e is electric charge; $F_{\mu\nu}$ is the electromagnetic tensor. $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]_-$. This is the Feynman-Gell-Mann equation. In the free case we have the Lagrangian (see Eq. (9) of [3c]):

$$\mathcal{L}_0 = (\overline{i\hat\partial\Psi})(i\hat\partial\Psi) - m^2\overline\Psi\Psi\,. \tag{4}$$

Let us re-write (1) into the form:*

$$\left[i\gamma^\mu\partial_\mu + a\partial^\mu\partial_\mu + b\right]\Psi = 0\,. \tag{5}$$

So, one should calculate ($p^2 = p_0^2 - \mathbf{p}^2$)

$$\mathrm{Det}\begin{pmatrix} b - ap^2 & p_0 + \boldsymbol\sigma\cdot\mathbf{p} \\ p_0 - \boldsymbol\sigma\cdot\mathbf{p} & b - ap^2\end{pmatrix} = 0 \tag{6}$$

in order to find energy-momentum-mass relations. Thus, $[(b - ap^2)^2 - p^2]^2 = 0$ and if $a = 0$, $b = \pm m$ we come to the well-known relation $p^2 = p_0^2 - \mathbf{p}^2 = m^2$ with four Dirac solutions. However, in the general case $a \neq 0$ we have

$$p^2 = \frac{(2ab + 1) \pm \sqrt{4ab + 1}}{2a^2} > 0\,, \tag{7}$$

that signifies that we do not have tachyons. However, the above result implies that we cannot just put $a = 0$ in the solutions, while it was formally possible in (5). When $a \to 0$ then† $p^2 \to \infty$; when $a \to \pm\infty$ then $p^2 \to 0$. It should be stressed that *the limit in the equation does not always coincide with the limit in the solutions.* So, the questions arise when we consider limits, such as Dirac → Weyl, and Proca → Maxwell. The similar method has also been presented by S. Kruglov for bosons [8]. Other fact should be mentioned: when $4ab = -1$ we have only the solutions with $p^2 = 4b^2$. For instance, $b = m/2$, $a = -1/2m$, $p^2 = m^2$. Next, I just want to mention one Barut omission. While we can write

$$\frac{\sqrt{4ab + 1}}{a^2} = m_2^2 - m_1^2\,, \text{ and } \quad \frac{2ab + 1}{a^2} = m_2^2 + m_1^2\,, \tag{8}$$

but m_2 and m_1 should not necessarily be associated with $m_{\mu,e}$ (or $m_{\tau,\mu}$). They may be associated with their superpositions, and applied to neutrino mixing, or quark mixing.

The lepton mass splitting has also been studied by Markov [9] on using the concept of both positive and negative masses in the Dirac equation. Next, obviously we can calculate anomalous magnetic moments in this scheme (on using, for instance, methods of [10, 11]).

We previously noted:

- The Barut equation is a sum of the Dirac equation and the Feynman-Gell-Mann equation.
- Recently, it was suggested to associate an analogue of (4) with dark matter, provided that Ψ is composed of

*Of course, one could admit p^4, p^6 etc. in the Dirac equation too. The dispersion relations will be more complicated [6].

†a has dimensionality [1/m], b has dimensionality [m].

the self/anti-self charge conjugate spinors, and it has the dimension [energy]^1 in the unit system $c = \hbar = 1$. The interaction Lagrangian is $\mathcal{L}^H \sim g\bar{\Psi}\Psi\phi^2$, ϕ is a scalar field.

- The term $\sim \overline{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}$ will affect the photon propagation, and non-local terms will appear in higher orders.
- However, it was shown in [3b,c] that a) the Mott cross-section formula (which represents the Coulomb scattering up to the order $\sim e^2$) is still valid; b) the hydrogen spectrum is not much disturbed; if the electromagnetic field is weak the corrections are small.
- The solutions are the eigenstates of the γ^5 operator.
- In general, the current J_0 is not the positive-defined quantity, since the general solution $\Psi = c_1\Psi_+ + c_2\Psi_-$, where $[i\gamma^\mu\partial_\mu \pm m]\Psi_\pm = 0$, see also [9].
- We obtained the Barut-like equations of the 2nd order and 3rd order in derivatives.
- We obtained dynamical invariants for the free Barut field on the classical and quantum level.
- We found relations with other models (such as the Feynman-Gell-Mann equation).
- As a result of analysis of dynamical invariants, we can state that at the free level, the term $\sim \partial_\mu\overline{\Psi}\sigma_{\mu\nu}\partial_\nu\Psi$ in the Lagrangian does not contribute.
- However, the interaction terms $\sim \bar{\Psi}\sigma_{\mu\nu}\partial_\nu\Psi A_\mu$ will contribute when we construct the Feynman diagrams and the S-matrix. In the curved space (the 4-momentum Lobachevsky space), the influence of such terms has been investigated in the Skachkov work [10, 11]. Briefly, the contribution will be such as if the 4-potential were to interact with some "renormalized" spin. Perhaps, this explains why Barut used the classical anomalous magnetic moment $g \sim 4\alpha/3$ instead of $\alpha/2\pi$.

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