Zitterbewegung and the Non-Holonomity of Pseudo-Riemannian Spacetime

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In this paper, we explore the connection between *zitterbewegung* for free particles, and the work of Rabounski and Borissova on Zelmanov's chronometric invariant formulation of General Relativity to calculate space and time physical observables [2, 6]. In the chr.inv.-analysis, the spin of a particle interacts with the space non-holonomity field of pseudo-Riemannian spacetime. From this, the particle gains an additional momentum which imparts a non-geodesic component to the particle's motion. The solution of the particle with spin chr.inv.-equation of motion is a spiral that can be visualized as being wound on a pulsating cylinder. Free electron oscillations occur at a frequency equal to the double angular velocity of the space rotation Ω, with fluctuations of the particle position on the order of its reduced Compton wavelength. We thus show that zitterbewegung is a direct manifestation of general relativistic space and time physical observables at the elementary particle level.

1 Introduction

In this paper, we explore the connection between *zitterbewegung*, German for "jittery" or "trembling motion", as calculated for Dirac free particles [1], and the work of Rabounski and Borissova on Zelmanov's chronometric invariant formulation of General Relativity to calculate space and time physical observables [2, 6]. We will show that zitterbewegung is a direct manifestation of general relativistic space and time physical observables at the elementary particle level.

2 Zitterbewegung

Zitterbewegung was first recognized by Breit [7] and further analyzed and the name coined by Schrödinger [8, 9]. This solution is obtained in the Heisenberg representation equation of motion for the velocity operator $\boldsymbol{\alpha}$ of the Dirac equation for a free particle

$$H_0 = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m\,, \tag{1}$$

where m and $\mathbf{p}$ are the mass and momentum of the free particle respectively, and the $\boldsymbol{\alpha}$ and β matrices are used instead of the γ^μ ($\beta = \gamma_0$ and $\alpha_i = \gamma_0\gamma_i$) [1, 10].

The space operator solution in the Heisenberg representation $\mathbf{x}(t)$ (i.e. $\boldsymbol{\alpha} = \dot{\mathbf{x}}$) is then given by

$$\mathbf{x}(t) = \mathbf{x}(0) + \frac{\mathbf{p}\,c^2}{H_0}\,t + \left(\boldsymbol{\alpha}(0) - \frac{\mathbf{p}\,c}{H_0}\right) \frac{i\hbar c}{2H_0}\, \exp\left(-2iH_0 t/\hbar\right)\,, \tag{2}$$

where the first two terms on the right hand side of (2) correspond to the classical equation of motion trajectory of the particle, with the third term corresponding to a rapid oscillatory motion (zitterbewegung) about the classical trajectory.

The angular frequency of these oscillations is of order $2mc^2/\hbar \sim 2\times 10^{21}\ \mathrm{s}^{-1}$ and their amplitude of order $\hbar/mc \equiv \lambdabar_C$, corresponding to fluctuations of the particle position on the order of its reduced Compton wavelength. Schrödinger found that the zitterbewegung results from the interference between positive and negative-energy state amplitudes. Consequently, there has been a tendency to dismiss zitterbewegung, as its expectation value vanishes for wave-packets consisting entirely of positive-energy or negative-energy waves. In addition, it has not been observed experimentally due to its high-frequency, low amplitude motion, although indirect evidence of its presence has been suggested in numerous areas by some investigators [11–14]. One is reminded of the situation with Brownian motion, where it has not been observed directly, but evidence of its presence is now unquestionably accepted.

However, zitterbewegung has been investigated by many researchers, and identified in many areas. Indeed, there is other evidence that points to the reality of zitterbewegung. For example, the Darwin term which provides a small correction in the fine-structure of the energy level of s-orbitals of the hydrogen atom can be shown to result from zitterbewegung [15]. In the 1990s, David Hestenes revived zitterbewegung as a physical process when he recast it in terms of his Geometric Algebra [16–19]. Since then. much work has been done on modelling and detecting zitterbewegung — see for example [20–25] among many others.

3 Physical observables in General Relativity

Many practitioners of General Relativity do not realize that the theory is based on a 4-dimensional pseudo-Riemannian representation of spacetime and that the calculations they perform give results in that particular spacetime description. The pseudo-Riemannian characterization refers to the three space and one time dimensions, described by a metric with signature (+−−−) or (−+++), which uniquely results in space-like and time-like intervals. To properly understand the results obtained, the 4-dimensional calculations in general covariant form must be projected onto the observer's 3+1 space and time dimensions separately as space and time physical observables.

This requires developing a mathematical theory to enable the calculation of observable components for any tensor. This work started in the 1930s — Landau and Lifshitz introduced the observable time interval and the observable three-dimensional interval in their classic *The Classical Theory of Fields* [26, §84]. Zelmanov, starting in 1941, developed such a comprehensive theory over many decades — it is known as the *theory of chronometric invariants* [2, 3]. The most complete description of the mathematical apparatus of physically observable quantities in General Relativity is given in the recent review article by Rabounski and Borissova [4]. It provides an up-to-date compendium of the results obtained by Zelmanov and the authors over the past decades, and allows for the calculation of the physical observable components of any tensor.

The basic approach consists in projecting a general covariant 4-dimensional tensor onto an observer's physical object frame of reference (e.g. the Earth's surface), consisting of a three-dimensional coordinate grid with "real" physical clocks (a spatial section $x_0 = ct = constant$, orthogonal to the observer's physical time line at time of observation t), known as the observer's *accompanying reference frame*.

The projection operator onto an observer's time line is the unit vector of the observer's four-dimensional velocity b^α with respect to his physical object frame of reference, which is tangential at each point of the observer's four-dimensional trajectory

$$b^\alpha = \frac{dx^\alpha}{ds} \,. \tag{3}$$

The projection of a tensor onto an observer's time line is given by its contraction with the vector b^α of his reference frame. In an observer's accompanying reference frame, his three-dimensional velocity with respect to his reference object is zero, $b^i = 0$, and its components are given by

$$b^0 = \frac{1}{\sqrt{g_{00}}}\,, \quad b_0 = g_{0\alpha} b^\alpha = \sqrt{g_{00}}\,, \quad b_i = g_{i\alpha} b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}}\,.$$

The projection operator onto an observer's three-dimensional spatial section is the four-dimensional symmetric tensor

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta\,, \qquad h^{\alpha\beta} = -g^{\alpha\beta} + b^\alpha b^\beta\,. \tag{4}$$

The projection of a tensor onto an observer's three-dimensional spatial section is given by its contraction with the tensor $h_{\alpha\beta}$ of his reference frame.

The observer's physical object reference frame has a gravitational field that can be rotated and deformed, and hence, the observer's local reference space can be inhomogeneous and anisotropic. If there is a spatial section everywhere orthogonal to the time lines, then the space is an *holonomic space*. If only spatial sections locally orthogonal to the time lines exist, then the space is a *non-holonomic space*.

Any coordinate grid that is at rest with respect to its reference physical object can be transformed to another coordinate grid through standard coordinate transformations, within the same spatial section. However, time transformations imply a change of spatial section (i.e. new clocks), and hence a change in the measurements of observable quantities. This requires that physical observable quantities in an observer's reference frame must be invariant with respect to time transformations throughout his three-dimensional spatial section $x^i = constant$, so these must be *chronometric invariant quantities*, and are named *chr.inv.-quantities* for short. Thus Zelmanov developed a general mathematical method to calculate physically observable chr.inv.-projections of any four-dimensional general covariant tensor (see [4] for details).

Accordingly, Zelmanov introduced chr.inv.-derivative operators with respect to time and the spatial coordinates given by

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0}\,, \tag{5}$$

where g_{00} and g_{0i} are components of the metric tensor $g_{\mu\nu}$, and the superscripted symbol ${}^*\partial$ indicates a chr.inv.-partial derivative. These are non-commutative: the order in which their second derivatives are taken gives different results, and their difference is not zero.

From these, three tensors can be defined:

1. A_{ik}: three-dimensional antisymmetric chr.inv.-tensor of the angular velocity with which the reference space of the observer rotates.
2. F_i: three-dimensional chr.inv.-vector of the gravitational inertial force.
3. D_{ik}: three-dimensional symmetric chr.inv.-tensor characterizing the rate of deformation of the observer's space.

Specifically, these tensors are explicitly given by:

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\,\left(F_i v_k - F_k v_i\right)\,, \tag{6}$$

where v_i is the tangential (linear) velocity of the rotation and c is the speed of light *in vacuo*,

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right) = \frac{1}{1-\frac{\mathrm{w}}{c^2}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \tag{7}$$

where $\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right)$ is the gravitational potential, originating from the gravitational field of the observer's reference object,

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \quad D = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \quad h = \det\|h_{ik}\|\,, \tag{8}$$

where h_{ik} is the physically observable chr.inv.-metric of the observer's space, $D = h^{ik} D_{ik} = D^m_m$, the trace of the tensor of the space deformation rate, is the relative dilatation rate of an elementary volume of the observer's space.

In addition, the tensor A_{ik} is further identified as the *space non-holonomity tensor*, which Zelmanov defined in the following theorem:

> *Zelmanov's theorem on the holonomity of space-time*: The identical equality to zero of the tensor A_{ik} in a four-dimensional region of space-time is the necessary and sufficient condition for the orthogonality of the spatial sections to the time lines everywhere in this region.
>
> In other words, $A_{ik} \neq 0$ in a non-holonomic space-time region, and $A_{ik} = 0$ in a holonomic one. [4, p. 7]

Rotating spaces ($A_{ik} \neq 0$) are non-holonomic, as three-dimensional spatial sections are non-orthogonal to time lines in rotating spaces.

This section has covered the basics of Zelmanov's chronometric invariants theory to generate physically observable quantities in General Relativity by projecting general covariant 4-dimensional tensors onto an observer's physical object frame of reference to obtain physically observable chr.inv.-projections. The reader is encouraged to consult the recent compendium article of Rabounski and Borissova [4] for a deeper complete coverage of the chr.inv.-theory.

4 Geodesic motion of particles in pseudo-Riemannian spacetime

We first apply this formalism to the equations of motion of a particle. The motion of a particle under the influence of gravitation is characterized as freely falling along a geodesic (shortest-distance) line, known as free or geodesic motion. Under the action of additional non-gravitational forces, the particle deviates from its geodesic trajectory, and its motion is known as non-geodesic.

In a four-dimensional pseudo-Riemannian spacetime, the motion of a particle is geometrically determined by the parallel transport of the four-dimensional vector Q^α tangential to the points along the particle's four-dimensional trajectory, given by [6, see p. 9]

$$\frac{\mathrm{D}Q^\alpha}{ds} = \frac{dQ^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\, Q^\mu\, \frac{dx^\nu}{ds}\,, \quad Q_\alpha Q^\alpha = constant\,, \tag{9}$$

where $\mathrm{D}Q^\alpha$ is the absolute differential of the transported vector Q^α along the trajectory, dQ^α is the differential of the vector and $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol of the second kind.

For a particle of rest mass m_0 and four-dimensional momentum vector P^α given by [6, see p. 12]

$$P^\alpha = m_0\, \frac{dx^\alpha}{ds}\,, \quad P_\alpha P^\alpha = m_0^2 = constant\,, \tag{10}$$

the equation of motion of the free particle is given by

$$\frac{dP^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\, P^\mu\, \frac{dx^\nu}{ds} = 0\,. \tag{11}$$

For a massless particle of four-dimensional wave vector K^α given by

$$K^\alpha = \frac{\omega}{c}\, \frac{dx^\alpha}{d\sigma}\,, \quad K_\alpha K^\alpha = 0\,, \tag{12}$$

where ω is the characteristic frequency of the massless particle and $d\sigma = h_{ik} dx^i dx^k$ is the three-dimensional chr.inv.-interval, the equation of motion of the free massless particle is given by

$$\frac{dK^\alpha}{d\sigma} + \Gamma^\alpha_{\mu\nu}\, K^\mu\, \frac{dx^\nu}{d\sigma} = 0\,. \tag{13}$$

The projection of the four-dimensional equation of motion (11) onto the time line and the spatial section of an observer for a free particle is then given respectively by [4, see p. 23]

$$\begin{aligned} &\frac{dm}{d\tau} - \frac{m}{c^2}\, F_i \mathrm{v}^i + \frac{m}{c^2}\, D_{ik} \mathrm{v}^i \mathrm{v}^k = 0\,, \\ &\frac{d(m\mathrm{v}^i)}{d\tau} + 2m(D^i_k + A^{\cdot i}_{k\cdot})\mathrm{v}^k - mF^i + m\Delta^i_{nk} \mathrm{v}^n \mathrm{v}^k = 0\,, \end{aligned} \tag{14}$$

where m is the relativistic mass of the particle, $d\tau$ is the physically observable time interval, v^i is the chr.inv.-vector of the physically observable velocity of the particle and Δ^i_{nk} is the chr.inv.-Christoffel symbol of the second kind, while the equivalent chr.inv.-equations of motion for a free massless particle are given by

$$\begin{aligned} &\frac{d\omega}{d\tau} - \frac{\omega}{c^2}\, F_i c^i + \frac{\omega}{c^2}\, D_{ik} c^i c^k = 0\,, \\ &\frac{d(\omega c^i)}{d\tau} + 2\omega(D^i_k + A^{\cdot i}_{k\cdot})c^k - \omega F^i + \omega\Delta^i_{nk} c^n c^k = 0\,, \end{aligned} \tag{15}$$

where c^i is the chr.inv.-vector of the physically observable velocity of light, with $c^i c_i = c^2$.

In the case where $Q_\alpha Q^\alpha \neq constant$, the trajectory of the particle is non-geodesic and the absolute derivative of the transported vector $\frac{\mathrm{D}Q^\alpha}{ds} = \Phi^\alpha$, which is a force that deviates the particle from a geodesic trajectory. The right hand side of (14) and (15) are set equal to the chr.inv.-projections of the deviating force Φ^α instead of 0. These are called the equations of non-geodesic motion.

5 Fields and charged spin particles in pseudo-Riemannian spaces

The previous section §4 has covered the necessary background on the calculation of equations of motion in the theory of chronometric invariants to permit their generalization to charged particles with spin. In their book *Fields, Vacuum and the Mirror Universe: Fields and particles in the space-time of General Relativity*, Rabounski and Borissova apply the chronometric invariants formalism to the analysis of fields and charged particles with spin [6, see Chapters 3 & 4].

Chapter 3 provides the chronometrically invariant theory of electrodynamics in a pseudo-Riemannian space. It takes into account the impact on the electromagnetic field of the physically observable chr.inv.-properties of the reference space, specifically the gravitational inertial force (i.e. acceleration) F_i, the space non-holonomity tensor of space rotation A_{ik}, and the rate of deformation of space tensor D_{ik}. This theory will not be covered here as it is beyond the scope of this paper.

Chapter 4 covers the chronometrically invariant theory of particles with spin in a pseudo-Riemannian space. It is based on the premise that spin is a fundamental property of matter, such as mass and charge. The analysis will show that the field of the space non-holonomity from the spatial rotation of the space A_{ik} interacts with the particle's spin and imparts it an additional momentum. From this will be derived the equations of motion of a particle with an internal rotation momentum (i.e. spin).

5.1 Spin particle equation of motion

Based on these considerations, the four-dimensional dynamic vector Q^α for the parallel transport equations is assumed to be given by [6, see pp. 155]

$$Q^\alpha = P^\alpha + S^\alpha \,, \tag{16}$$

where P^α is given by (10) and S^α is the spin momentum which the particle gains from its internal momentum resulting from the spin, thus making the motion of the particle non-geodesic.

To deduce the spin momentum vector S^α, we start from the known properties of the spin of elementary particles. Their numerical value is given by $\pm n\hbar$, where $\hbar$ is the reduced Planck constant which has units of angular momentum, and $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, with the $\pm$ sign indicating right-wise or left-wise internal rotation of the spin particle respectively. This suggests that the spin vector would be an antisymmetric tensor of the 2nd rank, similar to a tensor of angular momentum.

From Bohr's second postulate on the length of an electron orbit in an atom and the experimental finding that an electron has an internal magnetic moment proportional to its internal rotation spin momentum, Rabounski and Borissova make an argument to define a four-dimensional antisymmetric 2nd rank angular momentum-like tensor, which they call the Planck tensor and write as $\hbar^{\alpha\beta}$, given by [6, see pp. 155–156]

$$[r^i; p^k] = \frac{1}{2}\left(r^i p^k - r^k p^i\right) = k\hbar^{ik} \tag{17}$$

for some constant k, to characterize the spin of a particle in four-dimensional pseudo-Riemannian space.

The diagonal and space-time components of the Planck tensor are zero, while the non-diagonal spatial components are $\pm\hbar$, based on the spatial direction of the spin and the right- or left-handedness of the reference frame. Note that the antisymmetric Planck tensor $\hbar^{ik}$ is not to be confused with the symmetric physically observable chr.inv.-metric of the observer's space tensor h^{ik}.

This represents a general mathematical approach that requires no assumption on the internal structure of a particle's spin. Instead, it is based on a fundamental quantum space rotation. We have already encountered an antisymmetric rotation of space chr.inv.-tensor A_{ik} in §3, given by (6). In the absence of gravitational fields, the tensor of angular velocity A_{ik} is given by

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right), \tag{18}$$

which can be more specifically denoted as $A_{\alpha\beta} = \Omega_{\alpha\beta}$, with components

$$\Omega_{00} = 0 \quad \Omega_{0i} = -\Omega_{i0} = 0 \quad \Omega_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right). \tag{19}$$

The quantum principle of wave-particle duality results in a particle's energy being given by $E = mc^2 = \hbar\omega$ where ω is the characteristic frequency of the particle with relativistic mass m. Rabounski and Borissova suggest a generalization of that equation into the geometric tensor relation $mc^2 = \hbar^{\alpha\beta}\omega_{\alpha\beta}$.

The additional momentum S^α in (16) gained by a particle from its spin can be determined from the *action* $\mathcal{S}$ of a particle with spin. The action to displace a spin particle generated by the interaction of its spin with the space non-holonomity field $A_{\alpha\beta}$ is given by [6, see pp. 162]

$$\mathcal{S} = \alpha(S)\int_a^b \hbar^{\alpha\beta}A_{\alpha\beta}\,ds = \frac{n}{c}\int_a^b \hbar^{\alpha\beta}A_{\alpha\beta}\,ds\,, \tag{20}$$

where $\alpha(S)$ is a scalar constant characteristic of the particle in the spin interaction. One then obtains [6, see pp. 164]

$$S^\alpha = \frac{1}{c^2}\,n\,\hbar^{\mu\nu}A_{\mu\nu}\,\frac{dx^\alpha}{ds}\,, \tag{21}$$

such that the dynamic vector Q^α that characterizes the motion of the spin particle is given by

$$Q^\alpha = P^\alpha + S^\alpha = m_0\,\frac{dx^\alpha}{ds} + \frac{1}{c^2}\,n\,\hbar^{\mu\nu}A_{\mu\nu}\,\frac{dx^\alpha}{ds}\,, \tag{22}$$

where P^α is given by (10).

The equations of motion of a spin particle are obtained from the parallel transport equations of Q^α given by (22) along the trajectory of the particle

$$\frac{d}{ds}(P^\alpha + S^\alpha) + \Gamma^\alpha_{\mu\nu}\,(P^\mu + S^\mu)\,\frac{dx^\nu}{ds} = 0\,, \tag{23}$$

where $Q_\alpha Q^\alpha = constant$. The chr.inv.-equations of a particle with mass and spin is given by [6, see pp. 170]

$$\begin{aligned}&\frac{dm}{d\tau}-\frac{m}{c^2}F_i\mathrm{v}^i+\frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k=\\&=-\frac{1}{c^2}\frac{d\eta}{d\tau}+\frac{\eta}{c^4}F_i\mathrm{v}^i-\frac{\eta}{c^4}D_{ik}\mathrm{v}^i\mathrm{v}^k,\\&\frac{d(m\mathrm{v}^i)}{d\tau}+2m\,(D^i_k+A^{\cdot i}_{k\cdot})\mathrm{v}^k-m\,F^i+m\,\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\&=-\frac{1}{c^2}\frac{d(\eta\mathrm{v}^i)}{d\tau}-\frac{2\eta}{c^2}(D^i_k+A^{\cdot i}_{k\cdot})\mathrm{v}^k+\frac{\eta}{c^2}F^i-\frac{\eta}{c^2}\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k,\end{aligned}\qquad(24)$$

where η is given by

$$\eta=\frac{n\,\hbar^{\mu\nu}A_{\mu\nu}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,.\qquad(25)$$

The left hand side of equations (24) is the same as that of equations (14), and represents the geodesic part of a spinless particle's motion. However, while the right hand side of equations (14) are equal to zero, in the case of a particle with spin, the right hand side of equations (24) are non-zero, and thus represent the non-geodesic component of the motion of a particle with spin. That, is the component that gives rise to zitterbewegung, while the left hand side represents the classical geodesic trajectory of the particle.

Allowing for the weak gravitational interaction, compared to others, by setting $\mathrm{w}\to 0$ in (7) and $D=0$ in (8) [6, p. 176], results in the elimination of the F_i and D_{ik} terms, and a simplification of (24). The kinematic equations of motion (24) become

$$\frac{d\mathrm{v}^i}{d\tau}+2A^{\cdot i}_{k\cdot}\mathrm{v}^k+\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k=0\,.\qquad(26)$$

Assuming that the space rotates with a constant angular velocity Ω around the x^3-axis (z-axis), from (18) and (19) and the linear velocity of rotation of the space given by $v_i=\Omega_{ik}x^k$, then the space non-holonomity tensor A_{ik} has only two non-zero components,

$$A_{12}=-A_{21}=-\Omega\,,\qquad(27)$$

and the chr.inv.-vector equations of motion become

$$\frac{d\mathrm{v}^1}{d\tau}+2\Omega\mathrm{v}^2=0\,,\quad\frac{d\mathrm{v}^2}{d\tau}-2\Omega\mathrm{v}^1=0\,,\quad\frac{d\mathrm{v}^3}{d\tau}=0\,,\qquad(28)$$

where the superscripts are numerical vector indices.

Solving the equations of motion, we obtain the solutions [6, p. 179–183]

$$\mathrm{v}^1=\mathrm{v}^1_{(0)}\cos(2\Omega\tau)\,,\quad\mathrm{v}^2=\mathrm{v}^2_{(0)}\sin(2\Omega\tau)\,,\quad\mathrm{v}^3=\mathrm{v}^3_{(0)}\,,\qquad(29)$$

where the $\mathrm{v}^i_{(0)}$ represent the initial values of v^i. Integrating (29) with respect to $d\tau$, we obtain the particle's trajectory displacements

$$\begin{aligned}x^1&=x^1_{(0)}+\frac{\mathrm{v}^1_{(0)}}{2\Omega}\sin(2\Omega\tau)\\x^2&=x^2_{(0)}+\frac{\mathrm{v}^1_{(0)}}{2\Omega}-\frac{\mathrm{v}^1_{(0)}}{2\Omega}\cos(2\Omega\tau)\\x^3&=x^3_{(0)}+\mathrm{v}^3_{(0)}\tau\,,\end{aligned}\qquad(30)$$

where the $x^i_{(0)}$ represent the initial values of x^i.

Setting the initial displacement of the particle to be zero, $x^1_{(0)}=x^2_{(0)}=x^3_{(0)}=0$, (30) can be simplified as

$$\begin{aligned}x^1&=x=a\sin(2\Omega\tau)\\x^2&=y=a\,[1-\cos(2\Omega\tau)]\\x^3&=z=b\tau\,,\end{aligned}\qquad(31)$$

where $a=\frac{\mathrm{v}^1_{(0)}}{2\Omega}$ and $b=\mathrm{v}^3_{(0)}$. From this, we can move from the τ parametric representation to the coordinate representation of the solution to determine the shape of the three-dimensional trajectory covered by the particle. We obtain [6, p. 184]

$$x^2+y^2=2a^2\,[1-\cos(2\Omega\tau)]=4a^2\sin^2(\Omega\tau)\,,\qquad(32)$$

where $\tau=z/b$, which is similar to a spiral line equation $x^2+y^2=a^2$, $z=b\tau$. The particle has a constant velocity $b=\mathrm{v}^3_{(0)}$ along the axis of the spiral, with the radius of the particle's trajectory oscillating with a frequency Ω in the range 0 to $2a=\mathrm{v}^1_{(0)}/\Omega$ at distances $z=\frac{\pi kb}{2\Omega}$, for $k=0,1,2,3,\cdots$. The spiral can be visualized as being wound on a pulsating cylinder.

5.2 Charged spin particle in an electromagnetic field

For a charged spin particle in an electromagnetic field, the four-dimensional dynamic vector Q^α for the parallel transport equations takes the form [6, p. 186]

$$Q^\alpha=P^\alpha+\frac{e}{c^2}A^\alpha+S^\alpha\,,\qquad(33)$$

where e is the electric charge and A^α is the electromagnetic field potential. There is thus an additional momentum gained by the particle from the interaction of its charge with the electromagnetic field. The chr.inv.-scalar equation of motion of a charged spin particle in an electromagnetic field is then given by [6, p. 204]

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=-\frac{e}{c^2}E_i\mathrm{v}^i\,,\qquad(34)$$

where E_i is the i^{th} component of the electric field. Then for particles with mass,

$$m_0c^2=-n\hbar^{mn}A_{mn}\qquad(35)$$

where again $\hbar^{mn}$ is the Planck tensor and A_{mn} is the rotation of space chr.inv.-tensor. The right hand side of this equation (without the negative sign) characterizes the interaction energy of the particle's spin with the space non-holonomity field, i.e. the "spin energy". Rabounski and Borissova refer to (35) as the *law of quantization of the masses of elementary particles*:

> The rest-energy of any mass-bearing spin particle is equal to the energy of its spin interaction with the space non-holonomity field, taken with the opposite sign. [6, p. 205]

From (35), it can be shown that for any elementary particle with mass, the following relationship exists between its rest-mass m_0 and the angular velocity of the space rotation Ω [6, p. 207]:

$$\Omega = \frac{m_0 c^2}{2n\hbar}\,. \tag{36}$$

5.3 The Compton wavelength and zitterbewegung

The wavelength corresponding to the frequency of the space rotation Ω given by (36) can be calculated by assuming that the wave of the space non-holonomity propagates at the speed of light c [6, p. 209]:

$$\lambda\!\!\!^{-}_{\Omega} = \frac{c}{\Omega} = 2n\,\frac{\hbar}{m_0 c}\,. \tag{37}$$

For an electron, with $n = \frac{1}{2}$, (37) becomes

$$\lambda\!\!\!^{-}_{C} = \frac{\hbar}{m_0 c}\,, \tag{38}$$

i.e. the wavelength of the space non-holonomity rotation Ω is equal to the reduced Compton wavelength of the electron.

This confirms that (31) and (32) are the candidate equations to describe zitterbewegung: free electron oscillations occur at a frequency equal to the double angular velocity of the space rotation Ω given by (31), with fluctuations of the particle position on the order of its reduced Compton wavelength given by (38) while following a trajectory described by a pulsating spiral equation of motion.

6 Discussion and conclusion

In this paper, we have explored the connection between *zitterbewegung* for free particles, and the work of Rabounski and Borissova on Zelmanov's chronometric invariant formulation of General Relativity to calculate space and time physical observables [2,6]. They introduced a four-dimensional antisymmetric tensor of the 2nd rank they called the Planck tensor to characterize the spin of an elementary particle. In the chr.inv.-analysis, the spin of a particle interacts with the space non-holonomity field of pseudo-Riemannian spacetime.

From this, the particle gains an additional momentum which imparts a non-geodesic component to the particle's motion. The solution of the particle with spin chr.inv.-equation of motion is a spiral that can be visualized as being wound on a pulsating cylinder. It has a constant velocity $b = \mathrm{v}^3_{(0)}$ along the x^3-axis of the spiral, with the radius of the particle's trajectory oscillating with a frequency Ω in the range 0 to $2a = \mathrm{v}^1_{(0)}/\Omega$ at distances $z = \frac{\pi k b}{2\Omega}$, for $k = 0, 1, 2, 3, \cdots$. The wavelength of the space non-holonomity rotation Ω is equal to the reduced Compton wavelength of the electron.

Free electron oscillations occur at a frequency equal to the double angular velocity of the space rotation Ω, with fluctuations of the particle position on the order of its reduced Compton wavelength. Thus, we have shown that within the chr.inv.-equation of motion of particles with spin derived in Rabounski and Borissova's work [6], zitterbewegung is a direct manifestation of general relativistic space and time physical observables at the elementary particle level.

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