The Vacuum Stress-Energy Tensor in General Relativity

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In General Relativity, the Einstein field equations with a massive term source are plagued with the non conservation of this energy-momentum tensor. To remedy this problem, a pseudo-tensor of the gravitational field is classically added so that the global right hand side is conserved. Using Cartan's calculus,we derive here the differential form of the Einstein field equations which provides the Landau-Lifshitz symmetric (pseudo) 3-form of the gravitational field. Assuming a slightly variable cosmological term, we then infer a vacuum energy-momentum (real) 3-form so that the global r.h.s. of the field equations eventually exhibits a full real 3-form that satisfies the conserved property. In the early phase of the universe's expansion it is suggested that the cosmological term was huge and constant, before becoming variable to yield the estimated value predicted to-day.

Notations

Space-time Greek indices $\alpha, \beta \dots$ are 0, 1, 2, 3 for local coordinates.
Space-time Latin indices $a, b \dots$ are 0, 1, 2, 3 for a general basis.
The space-time signature is -2.
Einstein's constant is denoted by $\varkappa$.
The velocity of light is $c = 1$.

1 Differential calculus

1.1 The classical field equations in GR (short overview)

In General Relativity (GR), any line element on the 4-pseudo-Riemannian manifold $(\mathrm{M}, \mathbf{g})$ is given by $ds^2 = g_{ab}\, dx^a dx^b$. By varying the action $\mathcal{S} = \mathcal{L}_{\mathrm{E}}\, d^4x$ with respect to the g_{ab} where the lagrangian density is given by

$$\mathcal{L}_{\mathrm{E}} = g^{ab}\sqrt{-g}\left[\{^e{}_{ab}\}\{^d{}_{de}\} - \{^d{}_{ae}\}\{^e{}_{bd}\}\right] \tag{1}$$

one infers the symmetric Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R\,, \tag{2}$$

where

$$R_{bc} = \partial_a\{^a{}_{bc}\} - \partial_c\{^a{}_{ba}\} + \{^d{}_{bc}\}\{^a{}_{da}\} - \{^d{}_{ba}\}\{^a{}_{dc}\} \tag{3}$$

is the (symmetric) Ricci tensor whose contraction gives the curvature scalar R. (Here $\{^e{}_{ab}\}$ denote the Christoffel symbols of the second kind.)

The source free field equations are

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R + \Lambda g_{ab} = 0\,, \tag{4}$$

where Λ is the Einstein cosmological constant.

The second rank tensor G_{ab} is symmetric and is only function of the metric tensor components g_{ab} and their first and second order derivatives. Due to Bianchi's identities the Einstein tensor is conceptually conserved

$$\nabla_a\, G^a_b = 0\,, \tag{5}$$

where ∇_a is the Riemann covariant derivative.

When a massive source is present, the field equations become

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\,(R - 2\Lambda) = \varkappa\, T_{ab}\,. \tag{6}$$

If ρ is the matter density, then T_{ab} is here the tensor describing a pressure free fluid

$$T_{ab} = \rho\, u_a u_b\,. \tag{7}$$

1.2 The Cartan structure equations

Let us now consider a 4-manifold M referred to a vector basis e_α. The dual basis θ^β of one-forms (pfaffian forms) are related to the local coordinates x^α by

$$\theta^\beta = a^\beta_a\, dx^a\,. \tag{8}$$

These are called *vierbein* or *tetrad fields* [1].

In a dual basis θ^a, to any parallel transported vector along a closed path, can be then associated the following 2-forms:

— A *rotation curvature form*

$$\Omega^\alpha_\beta = \frac{1}{2}\, R^\alpha_{\ \beta\gamma\delta}\, \theta^\gamma \wedge \theta^\delta\,; \tag{9}$$

— A *torsion form*

$$\Omega^\alpha = \frac{1}{2}\, T^a_{\gamma\delta}\, \theta^\gamma \wedge \theta^\delta. \tag{10}$$

Introducing the Cartan procedure one first defines the connection forms

$$\Gamma^\alpha_\beta = \{^\alpha{}_{\gamma\beta}\}\, \theta^\gamma. \tag{11}$$

The first Cartan structure equation is related to the torsion by [2, p.40]

$$\Omega^\alpha = \frac{1}{2}\, T^\alpha_{\gamma\delta}\, \theta^\gamma \wedge \theta^\delta = d\theta^\alpha + \Gamma^\alpha_\gamma \wedge \theta^\gamma\,. \tag{12}$$

The second Cartan structure equation is defined as follows [2, p.42]

$$\Omega^{\alpha}_{\beta} = \frac{1}{2} R^{\alpha}_{\ \beta\gamma\delta}\, \theta^{\gamma} \wedge \theta^{\delta} = d\Gamma^{\alpha}_{\beta} + \Gamma^{\alpha}_{\gamma} \wedge \Gamma^{\gamma}_{\beta}\,, \tag{13}$$

where $R^{\alpha}_{\ \beta\gamma\delta}$ are the components of the curvature tensor in the most general sense.

Within the Riemannian framework alone (torsion free), $R^{\alpha}_{\ \beta\gamma\delta}$ reduce to the Riemann curvature tensor components and the first structure equation (12) becomes:

$$d\theta^{\alpha} = -\Gamma^{\alpha}_{\gamma} \wedge \theta^{\gamma}. \tag{14}$$

2 Differential equations of General Relativity

2.1 The Einstein field equations

We first recall the Hodge star operator defnition for an oriented 4-dimensional pseudo-Riemannian manifold $(\mathrm{M}, \mathbf{g})$ with volume element determined by $\mathbf{g}$

$$\boldsymbol{\eta} = \sqrt{-g}\, \theta^{0} \wedge \theta^{1} \wedge \theta^{2} \wedge \theta^{3}. \tag{15}$$

Let $\Lambda_k(\mathbf{E})$ be the subspace of completely antisymmetric multilinear forms on the real vector space $\mathbf{E}$.

The Hodge star operator $*$ is a linear isomorphism, i.e., $L_k(\mathrm{M}) \to L_{n-k}(\mathrm{M})$ (where $k \leqslant n$).

If $\theta^0, \theta^1, \theta^2, \theta^3$ is an oriented basis of 1-forms, then this operator is defined by

$$^{*}\left(\theta^{i1} \wedge \theta^{i2} \wedge \theta^{ik}\right) = \\ = \frac{\sqrt{-g}}{(n-k)!}\, \varepsilon_{j1\ldots jn}\, g^{j1i1} \ldots g^{jkik}\, \theta^{jk+1} \wedge \cdots \wedge \theta^{jn}. \tag{16}$$

With this preparation, the Einstein action simply reads

$$^{*}R = R\,\boldsymbol{\eta}\,. \tag{17}$$

Varying this action with respect to $\delta\theta^{\beta}$ of the orthonormal tetrad fields, eventually leads to the field equations under the differential form

$$-\frac{1}{2}\, \eta_{\beta\gamma\delta} \wedge \Omega^{\gamma\delta} = \varkappa\, {}^{*}T_{\beta}\,, \tag{18}$$

where T_{α} is related to the energy-momentum tensor $T_{\alpha\beta}$ by $T_{\alpha} = T_{\alpha\beta}\, \theta^{\beta}$ and include all other contributions.

In the same manner, one has for the Einstein tensor $G_{\alpha} = G_{\alpha\beta}\, \theta^{\beta}$. For all detailed derivations refer to [3].

2.2 The energy-momentum tensor

In the field equations (18), we insert $\eta^{\alpha\beta\gamma} = \eta^{\alpha\beta\gamma\delta}\, \theta_{\delta}$. Then, we use the second structure equation under the following form

$$\Omega_{\beta\gamma} = d\Gamma_{\beta\gamma} - \Gamma_{\mu\beta} \wedge \Gamma^{\mu}_{\gamma}\,, \tag{19}$$

so that

$$-\frac{1}{2}\, \eta^{\alpha\beta\gamma\delta}\theta_{\delta} \wedge \left(d\Gamma_{\beta\gamma} - \Gamma_{\mu\beta} \wedge \Gamma^{\mu}_{\gamma}\right) = \varkappa\, {}^{*}T_{\alpha}\,, \tag{20}$$

leading to

$$-\frac{1}{2}\, \eta^{\alpha\beta\gamma\delta}\, d\left(\Gamma_{\beta\gamma} \wedge \theta_{\delta}\right) = \varkappa\, ({}^{*}T_{\alpha} + {}^{*}t_{\alpha})\,, \tag{21}$$

where [4]

$$^{*}t^{\alpha} = -\frac{1}{2\varkappa}\, \eta^{\alpha\beta\gamma\delta}\left(\Gamma_{\mu\beta} \wedge \Gamma^{\mu}_{\gamma} \wedge \theta_{\delta} - \Gamma_{\beta\gamma} \wedge \Gamma_{\mu\delta} \wedge \eta^{\mu}\right). \tag{22}$$

We see that ${}^{*}t^{\alpha}$ is unaffected by the exterior product terms in the bracket, therefore $t_{\alpha\beta}$ is symmetric.

In that case, we idendify ${}^{*}t^{\alpha}$ with the Landau-Lifshitz 3-form ${}^{*}t^{\alpha}_{\mathrm{L-L}}$ which yields the corresponding pseudo-tensor $t^{\alpha\beta}_{\mathrm{L-L}}$ [5, eq.101.7]

$$\begin{aligned}(-g)\, t^{\alpha\nu}_{\mathrm{L-L}} = \frac{1}{2\varkappa}\Big\{ & {}^{\#}g^{\alpha\nu}_{\ ,\lambda}\, {}^{\#}g^{\lambda\mu}_{\ ,\mu} - {}^{\#}g^{\alpha\lambda}_{\ ,\lambda}\, {}^{\#}g^{\nu\mu}_{\ ,\mu} + \\ & + \frac{1}{2}\, g^{\alpha\nu} g_{\lambda\mu}\, {}^{\#}g^{\lambda\theta}_{\ ,\rho}\, {}^{\#}g^{\rho\mu}_{\ ,\theta} - (g^{\alpha\lambda} g_{\mu\theta}\, {}^{\#}g^{\nu\theta}_{\ ,\rho}\, {}^{\#}g^{\mu\rho}_{\ ,\lambda} + \\ & + g^{\nu\lambda} g_{\mu\theta}\, {}^{\#}g^{\alpha\theta}_{\ ,\rho}\, {}^{\#}g^{\mu\rho}_{\ ,\lambda}) + g_{\mu\lambda}\, g^{\theta\rho}\, {}^{\#}g^{\alpha\lambda}_{\ ,\theta}\, {}^{\#}g^{\nu\mu}_{\ ,\rho} + \\ & + \frac{1}{8}\, (2g^{\alpha\lambda} g^{\nu\mu} - g^{\alpha\nu} g^{\lambda\mu})(2\, g_{\theta\rho}\, g_{\delta\tau} - g_{\rho\delta}\, g_{\theta\tau})\, {}^{\#}g^{\theta\tau}_{\ ,\lambda}\, {}^{\#}g^{\rho\delta}_{\ ,\mu}\Big\}, \end{aligned} \tag{23}$$

where

$$^{\#}g^{\alpha\nu} = \sqrt{-g}\, g^{\alpha\nu}. \tag{24}$$

3 The vacuum energy

3.1 The gravitational field tensor

In General Relativity, it is well known, that the Einstein tensor $G_{\alpha\beta}$ is intrinsically conserved, while the massive tensor $T_{\alpha\beta}$ is not. This is because the gravitational field is not included in $T_{\alpha\beta}$. If so, then one obtains the conservation law

$$\partial_{\beta}\, \sqrt{-g}\left(T^{\alpha\beta} + t^{\alpha\beta}\right) = 0\,. \tag{25}$$

The tensor $t_{\alpha\beta}$ describes the gravitational field, derived from the Einstein-Dirac pseudo-tensor density [6, p.61]

$$\sqrt{-g}\, t^{\beta}_{a} = \frac{1}{2\varkappa}\left\{\frac{\left(\partial_{\alpha}\, {}^{\#}g^{\sigma\tau}\right)\partial\mathcal{L}_{\mathrm{E}}}{\partial\left(\partial_{\beta}\, {}^{\#}g^{\sigma\tau}\right)} - \delta^{\beta}_{\alpha}\, \mathcal{L}_{\mathrm{E}}\right\}. \tag{26}$$

However, the Einstein field equations are yet unbalanced since they do not exhibit a full real tensor as a source.

To remedy this problem, we showed that a sligthly variable cosmolgical term Λ-term induces a stress energy tensor of vacuum which restores a true gravitational tensor on the r.h.s. of the equation (6) as it should be [7, 8].

This real vacuum tensor was given by

$$(t_{\alpha\beta})_{\text{vac}} = -\frac{1}{2\varkappa}\,\Lambda g_{\alpha\beta}\,, \tag{27}$$

where the term Λ was found to be [9]

$$\Lambda = \nabla_\alpha K^\alpha = \theta^2, \tag{28}$$

where K^α is a 4-vector, and

$$\theta = X^\alpha_{;\alpha} \tag{29}$$

is the space-time volume scalar expansion characterizing the vacuum stress-energy tensor $(t_{\alpha\beta})_{\text{vac}}$. X^α is a congruence of non-intersecting unit time lines: $X^\alpha X_\alpha = 1$

$$X^\alpha_{;\alpha} = h^{\alpha\beta}\theta_{\alpha\beta}\,, \tag{30}$$

where $\theta_{\alpha\beta}$ stands for the expansion tensor, and $h_{\alpha\beta} = g_{\alpha\beta} - X_\alpha X_\beta$ is the projection tensor. Due to the form of (28), the lagrangian (1) differs only from a divergence and varying its action generates the same Einstein equations.

The real tensor $t^{\alpha\beta}_{\text{vac}}$ corresponding to the vacuum stress energy tensor can be added to $t^{\alpha\beta}$ without affecting the Einstein tensor inferred from the variational principle. So the final (real) gravitational tensor density is given by

$$\sqrt{-g}\,(t^\beta_\alpha)_{\text{G}} = \frac{1}{2\varkappa}\left\{\frac{\left(\partial_\alpha{}^{\#}g^{\sigma\tau}\right)\partial\mathcal{L}_{\text{E}}}{\partial\left(\partial_\beta{}^{\#}g^{\sigma\tau}\right)} - \delta^\beta_\alpha\left(\mathcal{L}_{\text{E}} - \sqrt{-g}\,\Lambda\right)\right\}. \tag{31}$$

The real tensor $(t^\beta_\alpha)_{\text{G}}$ is afterwards conveniently symmetrized through the Belinfante procedure [10].

With this definition the field equations can be finally written

$$R^{\alpha\beta} - \frac{1}{2}\,g^{\alpha\beta}R = \varkappa\left(T^{\alpha\beta} + t^{\alpha\beta}_{\text{G}}\right). \tag{32}$$

Sufficiently far from this matter we always have

$$R^{\alpha\beta} - \frac{1}{2}\,g^{\alpha\beta}R = \varkappa\,t^{\alpha\beta}_{\text{vac}}\,. \tag{33}$$

Inspection shows that each energy-momentum tensor is conserved.

3.2 The vacuum stress-energy 3-form

Here we adopt the Landau-Lifshitz symmetric pseudo-3-form ${}^*t^\alpha_{\text{L-L}}$ instead of the Einstein-Dirac pseudo-density. We then determine a vacuum energy 3-form designed to render the r.h.s. of (21) fully real according to the previous derivation.

To this end, we first regard the variable cosmological term Λ as inducing a given space-time curvature. This is legitimized by the fact that the real tensor $(t_{\alpha\beta})_{\text{vac}}$ is *à priori* persistent throughout the vacuum.

Since Λ is a scalar, let us then set the resulting curvature 2-form as

$$\Omega = \frac{1}{2}\,R^{\delta}{}_{\sigma\delta\tau}\,\theta^\sigma \wedge \theta^\tau. \tag{34}$$

By analogy with the classical formalism (4) we then apply the quantity $g^{\gamma\delta}\Omega$ to the field equations (18) as follows

$$\left.\begin{aligned} -\frac{1}{2}\,\eta_{\alpha\mu\nu} \wedge (\Omega^{\mu\nu} + g^{\mu\nu}\Omega) = \varkappa\,{}^*T_\alpha \\ -\frac{1}{2}\,\eta^\mu_{\alpha\nu} \wedge \left(\Omega^\nu_\mu + \delta^\nu_\mu\Omega\right) = \varkappa\,{}^*T_\alpha \end{aligned}\right\}. \tag{35}$$

Using $\Omega^\nu_\mu = \frac{1}{2}\,R^\nu{}_{\beta\gamma\delta}\,\theta^\gamma \wedge \theta^\delta$, these equations can be written in the form

$$-\frac{1}{4}\,\eta^\mu_{\alpha\nu} \wedge \theta^\sigma \wedge \theta^\tau \left(R_\mu{}^\nu{}_{\sigma\tau} + \delta^\nu_\mu R^\delta{}_{\sigma\delta\tau}\right) = \varkappa\,T_{\alpha\beta}\eta^\beta. \tag{36}$$

To $R_\mu{}^\nu{}_{\sigma\tau}$ is now added a new 4th rank curvature tensor which is noted

$$2\Lambda_\mu{}^\nu{}_{\sigma\tau} = -\,\delta^\nu_\mu R^\delta{}_{\sigma\delta\tau}\,. \tag{37}$$

To make it apparent, we first use the following relations

$$\eta^\alpha \equiv {}^*\theta^\alpha \tag{38}$$

$$\eta_\alpha = \frac{1}{3!}\left(\eta_{\alpha\beta\gamma\delta}\,\theta^\beta \wedge \theta^\gamma \wedge \theta^\delta\right) = \frac{1}{3!}\,\theta^\beta \wedge \eta_{\alpha\beta}\,. \tag{39}$$

Then, we apply the following identities

$$\theta^\beta \wedge \eta_\alpha = \delta^\beta_\alpha\,\eta\,,$$

$$\theta^\gamma \wedge \eta_{\alpha\beta} = \delta^\gamma_\beta\,\eta_\alpha - \delta^\gamma_\alpha\,\eta_\beta\,,$$

$$\theta^\delta \wedge \eta_{\alpha\beta\gamma} = \delta^\delta_\gamma\,\eta_{\alpha\beta} + \delta^\delta_\beta\,\eta_{\gamma\alpha} + \delta^\delta_\alpha\,\eta_{\beta\gamma}\,,$$

$$\theta^\varepsilon \wedge \eta_{\alpha\beta\gamma\delta} = \delta^\varepsilon_\delta\,\eta_{\alpha\beta\gamma} - \delta^\varepsilon_\gamma\,\eta_{\delta\alpha\beta} + \delta^\varepsilon_\beta\,\eta_{\gamma\delta\alpha} - \delta^\varepsilon_\alpha\,\eta_{\beta\gamma\delta}\,.$$

With this preparation, (36) reads

$$\begin{aligned} &-\frac{1}{4}\left(R^{\mu\nu}_{\sigma\tau} - 2\Lambda^{\mu\nu}_{\sigma\tau}\right)\left[\delta^\tau_\nu\left(\delta^\sigma_\mu\eta_\alpha - \delta^\sigma_\nu\eta_\mu\right) + \right.\\ &\left.+\,\delta^\tau_\mu\left(\delta^\sigma_\alpha\eta_\nu - \delta^\sigma_\nu\eta_\alpha\right) + \delta^\tau_\alpha\left(\delta^\sigma_\nu\eta_\mu - \delta^\sigma_\mu\eta_\nu\right)\right] = \\ &= -\frac{1}{2}\left(R^{\mu\nu}_{\mu\nu} - 2\Lambda^{\mu\nu}_{\mu\nu}\right)\eta_\alpha + \left(R^{\mu\nu}_{\alpha\nu} - 2\Lambda^{\mu\nu}_{\alpha\nu}\right)\eta_\nu = \\ &= \left(R^{\beta\nu}_{\alpha\nu} - 2\Lambda^{\beta\nu}_{\alpha\nu}\right)\eta_\beta - \frac{1}{2}\,\delta^\beta_\alpha\left(R^{\mu\nu}_{\mu\nu} - 2\Lambda^{\mu\nu}_{\mu\nu}\right)\eta_\beta\,. \end{aligned} \tag{40}$$

As a contributing curvature tensor, $2\Lambda^{\beta\nu}_{\alpha\nu}$ must be included in $R^{\beta\nu}_{\alpha\nu}$ so that we eventually retrieve the classical field equations with a cosmological term

$$\left(R^\beta_\alpha - \frac{1}{2}\,\delta^\beta_\alpha\,R + \delta^\beta_\alpha\,\Lambda\right)\eta_\beta = \varkappa\,T^\beta_\alpha\,\eta_\beta\,. \tag{41}$$

Taking account of $\Omega_{\beta\gamma} = d\Gamma_{\beta\gamma} - \Gamma_{\mu\beta} \wedge \Gamma^\mu_\gamma$, we revert to the field equations (22) which are also expressed as

$$-\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,\theta_\delta \wedge \left(d\Gamma_{\beta\gamma} - \Gamma_{\mu\beta} \wedge \Gamma^\mu_\gamma\right) = \varkappa\,{}^*T^\alpha. \tag{42}$$

Adding the extra-curvature yields

$$-\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,\theta_\delta \wedge \left[\left(d\Gamma_{\beta\gamma} - \Gamma_{\mu\beta} \wedge \Gamma^\mu_\gamma\right) + g_{\beta\gamma}\Omega\right] = \varkappa\,{}^*T^\alpha. \tag{43}$$

that is according to (21)

$$-\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\left[d\left(\Gamma_{\beta\gamma}\wedge\theta_\delta\right)+\theta_\delta\wedge g_{\beta\gamma}\Omega\right]=\varkappa\left({}^*T^\alpha+{}^*t^\alpha_{\text{L–L}}\right). \quad (44)$$

Therefore

$$-\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,d\left(\Gamma_{\beta\gamma}\wedge\theta_\delta\right)=$$
$$=\varkappa\left[{}^*T^\alpha+{}^*t^\alpha_{\text{L–L}}+\frac{1}{2\varkappa}\,\eta^{\alpha\beta\gamma\delta}\left(\theta_\delta\wedge g_{\beta\gamma}\Omega\right)\right],$$
$$-\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,\delta\left(\Gamma_{\beta\gamma}\wedge\theta_\delta\right)=$$
$$=\varkappa\left[{}^*T^\alpha+{}^*t^\alpha_{\text{L–L}}-\frac{1}{4\varkappa}\,\eta^{\alpha\beta\gamma\delta}\left(\theta_\delta\wedge g_{\beta\gamma}R^\lambda_{\ \lambda\sigma\tau}\theta^\sigma\wedge\theta^\tau\right)\right].$$

In the expression $-\frac{1}{4\varkappa}\,\eta^{\alpha\beta\gamma\delta}\theta_\delta\wedge g_{\beta\gamma}R^\lambda_{\ \lambda\sigma\tau}\theta^\sigma\wedge\theta^\tau$, we make the substitution

$$-g_{\beta\gamma}R^\lambda_{\ \lambda\sigma\tau}=2\Lambda_{\beta\gamma\sigma\tau}\,. \quad (45)$$

We eventually find the *vacuum stress-energy momentum 3-form*

$${}^*t^\alpha_{\text{vac}}=\frac{1}{2\varkappa}\,\eta^{\alpha\beta\gamma\delta}\,\theta_\delta\wedge\Lambda_{\beta\gamma\sigma\tau}\theta^\sigma\wedge\theta^\tau. \quad (46)$$

Therefore the global gravitational field is described by the (real) 3-form

$${}^*t^\alpha_{\text{G}}=-\frac{1}{2\varkappa}\,\eta^{\alpha\beta\gamma\delta}\left[\left(\Gamma_{\mu\beta}\wedge\Gamma^\mu_\gamma\wedge\theta_\delta-\Gamma_{\beta\gamma}\wedge\Gamma_{\mu\delta}\wedge\theta^\mu\right)-\right.$$
$$\left.-\,\theta_\delta\wedge\Lambda_{\beta\gamma\sigma\tau}\theta^\sigma\wedge\theta^\tau\right]. \quad (47)$$

3.3 The complete Einstein equations

The field equations are

$$-\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,d\left(\Gamma_{\beta\gamma}\wedge\theta_\delta\right)=\varkappa\left({}^*T^{\,\alpha}+{}^*t^\alpha_{\text{G}}\right). \quad (48)$$

As per (33) far from matter, we always have

$$-\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,d\left(\Gamma_{\beta\gamma}\wedge\theta_\delta\right)=\varkappa\,{}^*t^\alpha_{\text{vac}}\,. \quad (49)$$

Now, let us multiply equation (48) with $\sqrt{-g}$, then taking into account $\eta_{\alpha\beta\gamma\delta}=-\frac{1}{2\sqrt{-g}}\,\varepsilon_{\alpha\beta\gamma\delta}$, we find a new form for the field equations

$$-d\left(\sqrt{-g}\,\eta^{\alpha\beta\gamma\delta}\,\Gamma_{\beta\gamma}\wedge\theta_\delta\right)=2\varkappa\sqrt{-g}\left({}^*T^{\,\alpha}+{}^*t^\alpha_{\text{G}}\right) \quad (50)$$

or

$$-d\left(\sqrt{-g}\,\Gamma^{\beta\gamma}\wedge\eta^\alpha_{\beta\gamma}\right)=2\varkappa\sqrt{-g}\left({}^*T^{\,\alpha}+{}^*t^\alpha_{\text{G}}\right). \quad (51)$$

From these equations follows immediately the differential conservation law

$$d\left[\sqrt{-g}\left({}^*T^{\,\alpha}+{}^*t^\alpha_{\text{G}}\right)\right]=0. \quad (52)$$

If we integrate equation (51) over a 3-dimensional space-like region D_3, then we obtain

$$P^\alpha=-\frac{1}{2\varkappa}\int\sqrt{-g}\,\Gamma^{\beta\gamma}\wedge\eta^\alpha_{\beta\gamma}\,, \quad (53)$$

which is the total 4-momentum of the isolated system. Inspection shows that P^α is gauge invariant in the following sense

$$\theta(x)\to A(x)\,\theta(x)\,, \quad (54)$$

$$\Gamma(x)\to A(x)\,\Gamma(x)A^{-1}(x)-dA(x)A^{-1}(x), \quad (55)$$

where $A(x)$ is a local transformation matrix (A^α_β).

General Relativity is invariant with respect to such transformations and is thus a non-abelian gauge theory.

3.4 The early cosmological expansion evolution

The singularity of our universe is generally set at 10^{-43} seconds corresponding to the Planck era.

At this epoch, the size of our universe is predicted to be 10^{-35} meters with an energy of 10^{19} GeV and a temperature amounting to 10^{32} K. We postulate that the cosmological term was present and constant in the early stage of the singularity possessing a huge value. As time was the very first parameter to appear, the cosmological constant Λ would be associated to a large "pre" 3-form time component ${}^*t^0_{\text{vac}}$ with no further explicit structure. At 10^{-35} seconds, strong force and electro-weak force decoupled and at 10^{-12} seconds, the electro-weak force splits into weak and electromagnetic forces. Over this period of time, the cosmological term drastically decreases and becomes slightly variable. These processes cause the Universe's expansion to accelerate and ${}^*t^0_{\text{vac}}$ would deploy according to equation (46)

$${}^*t^\alpha_{\text{vac}}=\frac{1}{2\varkappa}\,\eta^{\alpha\beta\gamma\delta}\,\theta_\delta\wedge\Lambda_{\beta\gamma\sigma\tau}\theta^\sigma\wedge\theta^\tau. \quad (56)$$

Such a hypothesis would lend support to the inflation scenario recently suggested by the astronomer Claude Poher. His theory is based on the detection of massless particles moving at the speed of light which are assumed to propagate throughout the entire vacuum [11, 12]. According to Poher, these particles act as a gravitational isotropic flux and each one bears an individual energy measured at Eu = 8.5×10^{-21} Joules [13–15]. Without invoking a quantum aspect, the corpuscular nature of this flux might well appear as a piecewise structure of the vacuum field we have inferred in the above.

Conclusion

If one relaxes our demand on the cosmological term constancy, it is possible to define a real homogeneous vacuum stress-energy tensor which is by essence a pervasive field. In our picture, the gravitational field of a matter appears as an excited state of this field. Far from its matter source, the gravitational field pseudo-tensor asymptotically decreases down to the level of the vacuum energy-momentum tensor leaving the field equations with a non-zero right hand side. In here, we have shown that starting with the Landau-Lifshitz 3-form, it is also possible to infer a real 3-form representing the vacuum energy-momentum to restore a real r.h.s. of the field equations. The vacuum energy field hypothesis is rewarding in terms of several physical advantages:

— The ill-defined gravitational pseudo-tensor remains here a true tensor restoring the consistency in the field equations with a massive source;

— The inferred global energy-momentum tensor always satisfies the conservation law as well as the vacuum tensor alone;

— Because of the nature of this vacuum tensor there is no need to introducing any other arbitrary ingredients or modification of the general theory of relativity. Despite its smallness, a cosmological term seems to be badly needed to ascertain some major astrophysical observations which are all related to the FLRW expanding model of universe.

The Lambda-CDM model, which uses the FLRW metric, currently measures the cosmological constant to be on the order of 10^{-52} m^{-2}. However, there is no reason "à priori" to consider this term as a constant everywhere which would constitute a strong physical evidence for the vacuum field to exist in General Relativity.

Submitted on May 31, 2024

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