

FIELDS, VACUUM AND THE MIRROR UNIVERSE

The 3rd revised edition

L. Borissova and D. Rabounski



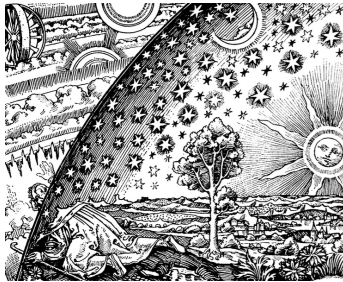
Fields, Vacuum and the Mirror Universe

Fields and particles
in the space-time of General Relativity

by Larissa Borissova and Dmitri Rabounski

The 3rd revised edition

First published in English in 2001
Translated into French in 2010



New Scientific Frontiers
London, 2023

Summary: — In this book, the authors create a theory of non-geodesic motion of particles in the space-time of General Relativity. A charged particle is considered in an electromagnetic field in the curved space-time (in contrast to the ordinary considerations held in the Minkowski space of Special Relativity). Spin particles are explained using the variational principle: this approach distinctly shows that elementary particles have masses in accordance with a special quantum relation. The physical vacuum and the forces of non-Newtonian gravitation acting in the physical vacuum are determined through the lambda-term of Einstein's field equations. A cosmological concept of the inversion explosion of the Universe from a compact object with the radius of an electron is proposed. The physical conditions inside the membrane that separates the space-time regions where the observable time flows to the future and to the past (our world and the mirror world), are examined.

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This book was typeset using the \LaTeX typesetting system.

New Scientific Frontiers is a publisher registered with Nielsen Book Services Ltd., Woking, Surrey, UK.

ISBN: 978-1-7392930-3-1

Published in the United Kingdom.

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Preface

This is a book written in 1999 by theoretical physicists Larissa Borissova and Dmitri Rabounski.

The book offers a new theoretical research that develops the theory of physical observables in General Relativity. In their famous book *The Classical Theory of Fields* Lev Landau and Evgeny Lifshitz described in detail the motion of particles in the electromagnetic and gravitational fields. However, in the 1930s, the methods of general covariant analysis did not yet take into account the concepts of physically observable quantities (chronometric invariants). Therefore, the authors extended the mathematical apparatus of chronometric invariants to the existing physical theory, applying it to the motion of particles in the electromagnetic and gravitational fields. In addition, Landau and Lifshitz did not consider the motion of a particle with an internal torque (spin). Therefore, a chapter in this book is devoted to the motion of particles with spin. In two other chapters, the authors introduce the theory of the physical vacuum and the theory of the mirror Universe. In another chapter, the authors outline the elements of tensor algebra and analysis in terms of chronometric invariants. All this makes this book a modern addition to *The Classical Theory of Fields*.

Paris, June 17, 2010

In the 3rd edition, the authors have added a list of chronometrically invariant derivatives, as well as references to their recent publications. We have also fixed typographical errors found in the previous editions.

Calais, January 10, 2023

Patrick Marquet

Acknowledgements

First, we would like to express our sincere gratitude to our mentors of General Relativity in Moscow. These are Abraham L. Zelmanov (1913–1987) and Kyril P. Stanyukovich (1916–1989). Many years of friendly acquaintance, individual training and countless hours of scientific conversations with them had sown the seeds of fundamental ideas, which by now have grown in our minds and are reflected in these pages. We are also very grateful to Kyril I. Dombrowski (1913–1997), a mathematician whose friendly conversations and discussions with us had a great influence on our scientific outlooks.

All preparation of the text of the first English edition of this book in 2001 was undertaken by Gregory V. Semionov (Moscow, Russia), to whom we are very grateful.

We also thank Ebenezer Chifu (Dutse, Nigeria) for his many hours of assistance in preparing the 2nd edition of the book in 2009.

We would also like to express our sincere gratitude to Patrick Marquet (Calais, France). His initiative to translate our books into French has opened the door for our books to the Francophonie world.

We are also very grateful to Pierre A. Millette (Ottawa, Canada), who volunteered his time to carefully edit the 2023 English and French editions of the book.

Special thanks go to Anatole V. Belyakov (Tverin Kariela, Russia), who has translated all our books from English into Russian.

This book was written in 1999. In the 3rd English edition, we have completely revised the entire text of the book and made many necessary corrections.

Puschino, April 21, 2023

L. Borissova and D. Rabounski

1.1 Geodesic motion of particles

Numerous experiments aimed at confirming the theoretical conclusions of the General Theory of Relativity have also proved that its basic space-time (four-dimensional pseudo-Riemannian space) is the basis of the geometry of our real world. This means that, despite the progress in experimental physics and astronomy, with the discovery of new effects, the four-dimensional pseudo-Riemannian space will remain the cornerstone for a further extension of the basic geometry of the world and will become one of its particular cases. Therefore, when creating the basic mathematical theory of the motion of particles, we must consider their motion in the four-dimensional pseudo-Riemannian space.

The following terminology must be taken into account here. Generally, the basic space-time of General Relativity is a *Riemannian space*^{*} of four dimensions having Minkowski's sign-alternating label (+---) or (-+++). The latter means a (3+1)-split of the coordinate axes of the Riemannian space into three spatial coordinate axes and the time axis. For convenience of calculations, we consider a Riemannian space of the signature (+---), where time is real while the spatial coordinates are imaginary. Some other researchers use the signature label (-+++), according to which time is imaginary and the spatial coordinates are real. In general, Riemannian spaces can have any number of dimensions and a non-alternating signature, e.g. (++++)). Therefore, a Riemannian space with an alternating signature label is commonly referred to as a *pseudo-Riemannian space*, to emphasize the split of the coordinate axes into two different types, referred to as time and spatial coordinates. Nonetheless, in this case, all its geometric properties are still properties of Riemannian

^{*}This is a metric space, the geometry of which is determined by the square metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ known also as the Riemann metric. Bernhard Riemann (1826–1866) was a German mathematician, the founder of Riemannian geometry (1854).

nian geometry and the prefix “pseudo” is not absolutely proper from the mathematical point of view. Nevertheless, we are going to use this notation as a long-established and traditionally understood one.

We will consider here particles travelling in the four-dimensional pseudo-Riemannian space. A particle affected by gravitation only falls freely thus travelling along a shortest (*geodesic*) line. Such motion is called *free motion* or *geodesic motion*. If the particle is also affected by additional non-gravitational forces, then the forces deviate the particle from its geodesic trajectory and its motion becomes *non-geodesic*.

From a geometric point of view, the motion of a particle in the four-dimensional pseudo-Riemannian space is the parallel transport of its own four-dimensional vector Q^α tangential to the particle’s trajectory in any of its points. Consequently, the equations of motion of such a particle actually determine the parallel transport of the particle’s vector Q^α along its four-dimensional trajectory and these are the equations of the absolute derivative of this vector with respect to a parameter ρ , which is non-zero along the trajectory

$$\frac{DQ^\alpha}{d\rho} = \frac{dQ^\alpha}{d\rho} + \Gamma_{\mu\nu}^\alpha Q^\mu \frac{dx^\nu}{d\rho}, \quad \alpha, \mu, \nu = 0, 1, 2, 3, \quad (1.1)$$

where $DQ^\alpha = dQ^\alpha + \Gamma_{\mu\nu}^\alpha Q^\mu dx^\nu$ is the absolute differential (absolute increment in the pseudo-Riemannian space) of the vector Q^α .

The absolute differential differs from the ordinary differential dQ^α by the presence of the Christoffel symbols of the 2nd kind $\Gamma_{\mu\nu}^\alpha$ (coherence coefficients of the Riemannian space), which are formulated through the Christoffel symbols (coherence coefficients) of the 1st kind $\Gamma_{\mu\nu,\rho}$ and they are functions of the first derivatives of the fundamental metric tensor $g_{\alpha\beta}$ of the space*

$$\Gamma_{\mu\nu}^\alpha = g^{\alpha\rho} \Gamma_{\mu\nu,\rho}, \quad \Gamma_{\mu\nu,\rho} = \frac{1}{2} \left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right). \quad (1.2)$$

When travelling along a geodesic trajectory (free motion), the parallel transport occurs in the sense of Levi-Civita. Here the absolute

*Coherence coefficients of a Riemannian space are named after Elwin Bruno Christoffel (1829–1900), the German mathematician who introduced them in 1869. In the space-time of Special Relativity (the Minkowski space), one can always set an inertial reference frame, where the matrix of the fundamental metric tensor becomes a unit diagonal, so all of the Christoffel symbols become zeroes.

derivative of any transported vector equals zero, in particular it is true for the four-dimensional vector of a particle*

$$\frac{dQ^\alpha}{d\rho} + \Gamma_{\mu\nu}^\alpha Q^\mu \frac{dx^\nu}{d\rho} = 0, \quad (1.3)$$

thus the square of the transported vector remains unchanged along the trajectory, i.e., $Q_\alpha Q^\alpha = \text{const}$. Such equations are called the equations of free motion.

Kinematic motion of a free particle is characterized by the four-dimensional vector of the velocity of the particle, called the *kinematic vector*

$$Q^\alpha = \frac{dx^\alpha}{d\rho}, \quad (1.4)$$

so the Levi-Civita parallel transport of the vector gives the equations of the four-dimensional trajectory of the particle (called the *equations of geodesic lines*)

$$\frac{d^2 x^\alpha}{d\rho^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{d\rho} \frac{dx^\nu}{d\rho} = 0. \quad (1.5)$$

The necessary condition $\rho \neq 0$ along the trajectory means that the derivation parameter ρ is not the same along trajectories of different kinds. In the pseudo-Riemannian space, the three kinds of trajectories are principally possible, each kind of which is corresponding to a specific kind of particles, namely:

- 1) *Non-isotropic real trajectories* lay “inside” the light cone. Along such trajectories, the square of the space-time interval is $ds^2 > 0$, thus, the interval ds is real. These are the trajectories of ordinary subluminal particles. Such particles have non-zero rest-masses and real relativistic masses;
- 2) *Non-isotropic imaginary trajectories* lay “outside” the light cone. Along such trajectories the square of the space-time interval is $ds^2 < 0$, hence, ds is imaginary. These are the trajectories of superluminal particles. Such particles have imaginary relativistic masses and are known as *tachyons*[†];

*Tullio Levi-Civita (1873–1941), an Italian mathematician, who was the first to study such a parallel transport [1].

[†]Tachyons — faster-than-light particles. The possibility of tachyons and faster-than-light signals was first considered in the framework of Special Relativity in 1958 by

- 3) *Isotropic trajectories* lay on the surface of the light cone. These are the trajectories of particles having zero rest-mass (massless light-like particles), which travel with the velocity of light. Along isotropic trajectories, the space-time interval is zero, $ds^2 = 0$, but the three-dimensional interval is not zero.

The space-time interval ds is commonly used as a derivation parameter along non-isotropic trajectories. On the other hand, it cannot be used as a derivation parameter for the trajectories of massless particles, because $ds = 0$ along isotropic trajectories.

For this reason, Zelmanov [9] had proposed another variable to be used as the derivation parameter, which does not turn into zero along isotropic trajectories. This is a three-dimensional (spatial) physically observable interval

$$d\sigma^2 = \left(-g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}} \right) dx^i dx^k, \quad (1.6)$$

which differs from a three-dimensional coordinate interval. Landau and Lifshitz had arrived at the same conclusion in their *The Classical Theory of Fields* [10, §84].

Substituting the above differentiation parameters into the general form of the equations of geodesic lines (1.5), we arrive at the equations of non-isotropic geodesic lines (trajectories of mass-bearing particles)

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \quad (1.7)$$

and the equations of isotropic geodesic lines (light-like particles)

$$\frac{d^2 x^\alpha}{d\sigma^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = 0. \quad (1.8)$$

But, in order to give the complete description of the motion of a particle, we have to build dynamic equations of motion, which contain the physical properties of the particle, such as its mass, energy, etc.

Tangerlini, in his PhD thesis [2]. As was pointed out by Malykins [3], most studies on the history of tachyons missed this fact. Meanwhile, the most important surveys on this topic such as [4, 5] referred to Tangerlini. Tachyons were first illuminated in the journal publications on the theory of relativity in 1960, the principal paper [6] authored by Terletsii, and then in 1962, in the more detailed paper [7] published by Bilaniuk, Deshpande, and Sudarshan. The term “tachyons” was first introduced later, in 1967 by Feinberg [8]. See Malykins’ survey [3] for detail.

Motion of a free mass-bearing particle (non-isotropic geodesic line) is characterized by its own four-dimensional momentum vector

$$P^\alpha = m_0 \frac{dx^\alpha}{ds}, \quad (1.9)$$

where m_0 is the rest-mass of the particle. From a geometric point of view, the Levi-Civita parallel transport of the vector P^α gives the dynamic equations of motion of the mass-bearing particle

$$\frac{dP^\alpha}{ds} + \Gamma_{\mu\nu}^\alpha P^\mu \frac{dx^\nu}{ds} = 0, \quad P_\alpha P^\alpha = m_0^2 = \text{const}. \quad (1.10)$$

Motion of a massless light-like particle (an isotropic geodesic line) is characterized by its own four-dimensional wave vector

$$K^\alpha = \frac{\omega}{c} \frac{dx^\alpha}{d\sigma}, \quad (1.11)$$

where ω is a cyclic frequency characteristic of the massless particle. Respectively, the Levi-Civita parallel transport of the vector K^α gives the dynamic equations of motion of the massless particle

$$\frac{dK^\alpha}{d\sigma} + \Gamma_{\mu\nu}^\alpha K^\mu \frac{dx^\nu}{d\sigma} = 0, \quad K_\alpha K^\alpha = 0. \quad (1.12)$$

So, we have got the dynamic equations of motion for free particles. Here, the equations are presented in the four-dimensional general covariant form. This form has its own advantage as well as a substantial drawback. The advantage is their invariance in all transitions from one reference frame to another. The drawback is that, in the general covariant form, the terms of the equations do not contain actual three-dimensional quantities, which can be measured in experiments or observations (namely — *physically observable quantities*). This means that, in the general covariant form, the equations of motion are merely an intermediate theoretical result, not applicable in practice.

Therefore, in order to get the results of any physical mathematical theory applicable in practice, we need to formulate the equations of the theory through physically observable quantities. In particular, — we need to formulate the general covariant equations of motion of particles through the physically observable properties characteristic of the actual physical reference frame of an observer.

At the same time, to define physically observable quantities is not a trivial problem. For instance, for a four-dimensional vector Q^α (it has 4 components) we can *heuristically assume* that its three spatial components form a three-dimensional observable vector, and its time component is the observable potential of the vector field (which generally does not prove that these quantities can be actually observed). However, a contravariant tensor of the 2nd rank $Q^{\alpha\beta}$ (it has as many as 16 components) makes the problem much more indefinite. For tensors of higher ranks the problem of the heuristic definition of their observable components is much more complicated. Besides, there is an obstacle related to the definition of the observable components of covariant tensors (with lower indices) and mixed type tensors (with both lower and upper indices).

Therefore, the most reasonable way out of the labyrinth of heuristic guesses is creating a strict mathematical theory to enable calculating observable components for any tensor quantity. Such a theory was created in 1944 by Zelmanov [9]. It should be noted that other researchers were also working on the theory of observable quantities in the 1930s. For example, Landau and Lifshitz in their famous *The Classical Theory of Fields* [10, §84] introduced the observable time and observable three-dimensional interval similar to those introduced by Zelmanov. Meanwhile, they limited themselves only to this particular case and they did not arrive at general mathematical methods to determine physically observable quantities in a pseudo-Riemannian space.

Over the next decades, Zelmanov improved his mathematical apparatus of physically observable quantities, called the *theory of chronometric invariants* [11–13]. A similar result had also been obtained by Cattaneo [14–17], an Italian mathematician, independently from Zelmanov. However, Cattaneo published his first study on the theme only in 1958 [14] and his study was far from a complete theory.

In §1.2, we will give an overview of the Zelmanov theory of physically observable quantities, which is necessary for understanding this subject and using these mathematical methods in practice.

In §1.3, we present the results of our study of the geodesic motion of particles using the mathematical methods of chronometric invariants. In §1.4 we will focus on the formulation of the problem of creating the equations of motion along non-geodesic trajectories, i.e., under the action of non-gravitational external forces.

1.2 Physical observable quantities

This section introduces the basics of Zelmanov's mathematical apparatus of chronometric invariants*.

To determine which components of any four-dimensional quantity are physically observable, we consider a real reference frame of a real observer, which includes a *coordinate grid*, spanned over his *reference body* (which is a real physical body near him), at each point of which a *real clock* is installed. The reference body, being a real physical body has a gravitational field, can be rotating and deforming, thereby making the reference space inhomogeneous and anisotropic. Actually, the reference body and its associated reference space can be considered as a set of real physical references, to which the observer compares all results of his measurements. Therefore, physically observable quantities registered by an observer must be obtained as a result of projecting four-dimensional quantities onto the time lines and the three-dimensional space of the observer's reference body.

From a geometric point of view, the observer's three-dimensional space is the *spatial section* $x^0 = ct = const$. At any point of the space-time, a local spatial section (local space) can be placed orthogonally to the *time line*. If there exists a space-time enveloping curve to such local spaces, then it is a spatial section everywhere orthogonal to the time lines. Such a space is known as a *holonomic space*. If no enveloping curve exists to such local spaces, but only spatial sections locally orthogonal to the time lines exist, then such a space is known as a *non-holonomic space*.

We assume that the observer is at rest with respect to his physical references (his reference body). The reference frame of such an observer everywhere accompanies his reference body and, hence, his reference space in any displacements. Therefore, such a reference frame is called the *accompanying reference frame*.

Any coordinate grid that is at rest with respect to the same reference body is related to another one within the same spatial section (three-

*To date, the most complete description (compendium) of the mathematical apparatus of physically observable quantities in General Relativity is given in our recent article. In this article, we have collected everything (or almost everything) that we know on this topic from Zelmanov and what has been obtained over the past decades: Rabounski D. and Borissova L. Physical observables in General Relativity and the Zelmanov chronometric invariants. *Progress in Physics*, 2023, vol. 19, no. 1, 3–29.

dimensional reference space) through the transformation

$$\left. \begin{aligned} \tilde{x}^0 &= \tilde{x}^0(x^0, x^1, x^2, x^3) \\ \tilde{x}^i &= \tilde{x}^i(x^1, x^2, x^3), \quad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \end{aligned} \right\}, \quad (1.13)$$

where the latter equation means that the spatial coordinates in the tilde-marked grid are independent of time in the non-tilde grid, which is equivalent to setting a coordinate grid of fixed time lines $x^i = const$ at any of its points. The transformation of the spatial coordinates is nothing but only the transition from one coordinate grid to another within the same spatial section. The transformation of time means changing the whole set of clocks, so this is the transition to another spatial section (another three-dimensional reference space). In practice, this means the replacement of one reference body with all of its physical references with another reference body that has its own physical references. But, when using different references, the observer will obtain different results of measurements (other observable quantities). Therefore, physically observable quantities must be invariant with respect to the transformations of time, so they must be *chronometrically invariant quantities*.

Since the transformations (1.13) determine a set of fixed time lines, chronometric invariants (physical observables) are all those quantities, which are invariant with respect to the transformations.

In practice, to obtain physically observable quantities in the accompanying reference frame of a real observer, we have to calculate the chronometrically invariant projections of four-dimensional quantities onto the time line and the spatial section of his physical reference body, then formulate the chr.inv.-projections with the chronometrically invariant (physically observable) properties of his reference space.

We project four-dimensional quantities onto the time line and the spatial section of an observer using the projection operators, characterizing the properties of the observer's reference space. The operator b^α projecting onto the time line is a unit vector of the four-dimensional velocity of the observer with respect to his reference body

$$b^\alpha = \frac{dx^\alpha}{ds}, \quad (1.14)$$

which is tangential to the observer's world-trajectory at every point. Because any reference frame is described by its own tangential unit vec-

tor b^α , Zelmanov referred to the b^α as the *monad vector*. The operator projecting onto the spatial section associated with the observer is the four-dimensional symmetric tensor

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta, \quad h^{\alpha\beta} = -g^{\alpha\beta} + b^\alpha b^\beta, \quad (1.15)$$

the mixed components of which are

$$h_\alpha^\beta = -g_\alpha^\beta + b_\alpha b^\beta. \quad (1.16)$$

Zelmanov [9] had showed that the vector b^α and the tensor $h_{\alpha\beta}$ have all properties necessary to projection operators, namely — the properties $b_\alpha b^\alpha = 1$ and $h_\alpha^\beta b^\alpha = 0$. As a result, the projection of a tensor quantity onto the time line is a result of its contraction with the monad vector b^α . The projection onto the spatial section is its contraction with the tensor $h_{\alpha\beta}$.

The observer's three-dimensional velocity with respect to his reference body in the accompanying reference frame is zero: $b^i = 0$. The other components of the monad vector are

$$b^0 = \frac{1}{\sqrt{g_{00}}}, \quad b_0 = g_{0\alpha} b^\alpha = \sqrt{g_{00}}, \quad b_i = g_{i\alpha} b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}}. \quad (1.17)$$

Therefore, in the accompanying reference frame ($b^i = 0$), the components of the operator projecting onto the spatial section are

$$\left. \begin{aligned} h_{00} &= 0, & h^{00} &= -g^{00} + \frac{1}{g_{00}}, & h_0^0 &= 0 \\ h_{0i} &= 0, & h^{0i} &= -g^{0i}, & h_0^i &= \delta_0^i = 0 \\ h_{i0} &= 0, & h^{i0} &= -g^{i0}, & h_i^0 &= \frac{g_{i0}}{g_{00}} \\ h_{ik} &= -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}, & h^{ik} &= -g^{ik}, & h_k^i &= -g_k^i = \delta_k^i \end{aligned} \right\} \cdot (1.18)$$

The tensor $h_{\alpha\beta}$ in the three-dimensional space of the reference frame accompanying the observer has all properties characteristic of the fundamental metric tensor

$$h_\alpha^i h_k^\alpha = \delta_k^i - b_k b^i = \delta_k^i, \quad \delta_k^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1.19)$$

where δ_k^i is the unit three-dimensional tensor*. For this reason, in the accompanying reference frame, the three-dimensional chr.inv.-tensor h_{ik} can lift or lower indices in chr.inv.-quantities.

The projections of an arbitrary vector Q^α onto the time line and the spatial section in the accompanying reference frame ($b^i = 0$) of an observer are

$$T = b^\alpha Q_\alpha = b^0 Q_0 = \frac{Q_0}{\sqrt{g_{00}}}, \quad (1.20)$$

$$L^0 = h_\beta^0 Q^\beta = -\frac{g_{0k}}{g_{00}} Q^k, \quad L^i = h_\beta^i Q^\beta = \delta_k^i Q^k = Q^k. \quad (1.21)$$

The projections of an arbitrary tensor of the 2nd rank $Q^{\alpha\beta}$ are

$$T = b^\alpha b^\beta Q_{\alpha\beta} = b^0 b^0 Q_{00} = \frac{Q_{00}}{g_{00}}, \quad (1.22)$$

$$L^{00} = h_\alpha^0 h_\beta^0 Q^{\alpha\beta} = -\frac{g_{0i} g_{0k}}{g_{00}^2} Q^{ik}, \quad L^{ik} = h_\alpha^i h_\beta^k Q^{\alpha\beta} = Q^{ik}. \quad (1.23)$$

After testing the obtained quantities by the transformations (1.13), we see that chronometrically invariant (physically observable) quantities are the projection onto the time line and the spatial components of the projection onto the spatial section. We will refer to these observable quantities as the *chr.inv.-projections*.

Projecting the four-dimensional coordinates x^α in the accompanying reference frame, we obtain the chr.inv.-invariant of the *physically observable time*

$$\tau = \sqrt{g_{00}} t + \frac{g_{0i}}{c \sqrt{g_{00}}} x^i, \quad (1.24)$$

and the chr.inv.-vector of the *physically observable coordinates*, which coincide the spatial coordinates x^i . Thus, projecting an elementary four-dimensional coordinate interval dx^α gives an elementary interval of the physically observable time, which is the chr.inv.-invariant

$$d\tau = \sqrt{g_{00}} dt + \frac{g_{0i}}{c \sqrt{g_{00}}} dx^i, \quad (1.25)$$

*This tensor δ_k^i is the three-dimensional part of the four-dimensional unit tensor δ_β^α , which can be used to replace (lift and lower) indices in four-dimensional quantities.

and also the chr.inv.-vector of an elementary interval of the physically observable coordinates dx^i . As a result, the *physically observable velocity* of a particle is the three-dimensional chr.inv.-vector

$$v^i = \frac{dx^i}{d\tau}, \quad (1.26)$$

which differs from its coordinate velocity $u^i = \frac{dx^i}{dt}$.

Projecting the fundamental metric tensor $g_{\alpha\beta}$, we obtain that h_{ik} is the *chr.inv.-metric tensor*, or, in other words, the *observable metric tensor* in the accompanying reference frame

$$h_{\alpha}^i h_{\beta}^k g^{\alpha\beta} = g^{ik} = -h^{ik}, \quad h_i^{\alpha} h_k^{\beta} g_{\alpha\beta} = g_{ik} - b_i b_k = -h_{ik}, \quad (1.27a)$$

the components of which are

$$h_{ik} = -g_{ik} + b_i b_k, \quad h^{ik} = -g^{ik}, \quad h_k^i = -g_k^i = \delta_k^i. \quad (1.27b)$$

Therefore, the square of any observable spatial interval $d\sigma$ is

$$d\sigma^2 = h_{ik} dx^i dx^k. \quad (1.28)$$

The space-time interval ds expressed through physically observable quantities is obtained by substituting $g_{\alpha\beta}$ from (1.15), namely

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \quad (1.29)$$

Apart from their projections onto the time line and the spatial section of an observer, four-dimensional quantities of the 2nd rank and higher ranks also have mixed components that have both upper and lower indices at the same time. How do we find physically observable quantities among them, if any? The best approach is to develop a general mathematical method to calculate physically observable quantities, based solely on their property of chronometric invariance. Such a method had been developed by Zelmanov, who formulated it in the form of a theorem, which we call *Zelmanov's theorem*:

Zelmanov's theorem

We assume that $Q_{00\dots 0}^{ik\dots p}$ are components of a four-dimensional tensor $Q_{\alpha\beta\dots\sigma}^{\mu\nu\dots\rho}$ of r -th rank, in which all upper indices are not zero, while all m lower indices are zeroes. Then tensor quantities

$$T^{ik\dots p} = (g_{00})^{-\frac{m}{2}} Q_{00\dots 0}^{ik\dots p} \quad (1.30)$$

make up three-dimensional contravariant chr.inv.-tensor of $(r-m)$ -th rank. Hence, the tensor $T^{ik\dots p}$ is a result of m -fold projection on time lines by indices $\alpha, \beta \dots \sigma$ and also, projection on the spatial section by $r-m$ indices $\mu, \nu \dots \rho$ of the initial tensor $Q_{\alpha\beta\dots\sigma}^{\mu\nu\dots\rho}$.

An immediate result of this theorem is that, for any vector Q^α the following two quantities are physically observable

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}, \quad h_\alpha^i Q^\alpha = Q^i, \quad (1.31)$$

and for any symmetric tensor of the 2nd rank $Q^{\alpha\beta}$, the following three quantities are physically observable

$$b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}, \quad h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q_0^i}{\sqrt{g_{00}}}, \quad h_\alpha^i h_\beta^k Q^{\alpha\beta} = Q^{ik}, \quad (1.32)$$

while in an antisymmetric tensor of the 2nd rank, the first quantity is zero, because $Q_{00} = Q^{00} = 0$.

All physically observable chr.inv.-projections must be compared to the observer's references — the physically observable properties characteristic of his reference body and local space, and with which the final equations of theory must be formulated. These physically observable properties are obtained using the chr.inv.-derivation operators with respect to time and the spatial coordinates. The mentioned operators had been introduced by Zelmanov as follows [9]

$$\frac{*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}, \quad \frac{*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial x^0}, \quad (1.33)$$

and they are non-commutative, so the difference between the 2nd derivatives is not zero

$$\frac{*\partial^2}{\partial x^i \partial t} - \frac{*\partial^2}{\partial t \partial x^i} = \frac{1}{c^2} F_i \frac{*\partial}{\partial t}, \quad (1.34)$$

$$\frac{*\partial^2}{\partial x^i \partial x^k} - \frac{*\partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{*\partial}{\partial t}. \quad (1.35)$$

Here, A_{ik} is the *three-dimensional antisymmetric chr.inv.-invariant tensor of the angular velocity with which the space rotates*

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} (F_i v_k - F_k v_i), \quad (1.36)$$

where v_i is the linear velocity of this rotation

$$\left. \begin{aligned} v_i &= -c \frac{g_{0i}}{\sqrt{g_{00}}}, & v^i &= -c g^{0i} \sqrt{g_{00}} \\ v_i &= h_{ik} v^k, & v^2 &= v_k v^k = h_{ik} v^i v^k \end{aligned} \right\}. \quad (1.37)$$

The tensor A_{ik} , equated to zero, is the necessary and sufficient condition of the holonomy of space [9]. In this case, $g_{0i} = 0$ and $v_i = 0$. In a non-holonomic space, $A_{ik} \neq 0$. For this reason, the tensor A_{ik} is also the tensor of the space non-holonomy*.

The quantity F_i is the *three-dimensional chr.inv.-vector of the gravitational inertial force*

$$F_i = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \quad (1.38)$$

where w is a gravitational potential

$$w = c^2 (1 - \sqrt{g_{00}}), \quad (1.38a)$$

the origin of which is the gravitational field of the observer's reference body[†]. In quasi-Newtonian approximation, i.e., in a weak gravitational field at velocities much lower than the light velocity and in the absence of rotations of the space, the quantity F_i becomes a non-relativistic gravitational force

$$F_i = \frac{\partial w}{\partial x^i}. \quad (1.39)$$

The reference body of any observer is a real physical body, which can deform. As a result, the coordinate grid spanned over it can deform, and also the real reference space associated with the reference body can deform as well. Therefore, real physical references must take the space deformations into account.

In particular, as a result of the deformations, the observable metric h_{ik} of the reference space is non-stationary. This is taken into account by

*The space-time of Special Relativity (Minkowski space) in a Galilean reference frame, as well as numerous cases of the space-time of General Relativity are examples of holonomic spaces ($A_{ik} = 0$).

[†]The quantities w and v_i do not have the property of chronometric invariance, while the gravitational inertial force vector and the tensor of the angular velocity of the space rotation, created using them, are chr.inv.-quantities.

introducing the *three-dimensional symmetric chr.inv.-tensor of the rate of the space deformations*

$$\left. \begin{aligned} D_{ik} &= \frac{1}{2} \frac{\partial h_{ik}}{\partial t}, & D^{ik} &= -\frac{1}{2} \frac{\partial h^{ik}}{\partial t} \\ D &= h^{ik} D_{ik} = D_n^n = \frac{\partial \ln \sqrt{h}}{\partial t} \\ h &= \det \|h_{ik}\| \end{aligned} \right\}. \quad (1.40)$$

With the definitions above, we can generally express any property of a geometric object located in a space through the observable properties of the space.

For instance, the Christoffel symbols, which appear in the equations of motion, are non-tensorial geometric objects [18]. Nevertheless, they can be formulated with physically observable quantities. The formulae obtained by Zelmanov [9] are

$$\Gamma_{00}^0 = -\frac{1}{c^3} \left[\frac{1}{1 - \frac{w}{c^2}} \frac{\partial w}{\partial t} + \left(1 - \frac{w}{c^2}\right) v_k F^k \right], \quad (1.41)$$

$$\Gamma_{00}^k = -\frac{1}{c^2} \left(1 - \frac{w}{c^2}\right)^2 F^k, \quad (1.42)$$

$$\Gamma_{0i}^0 = \frac{1}{c^2} \left[-\frac{1}{1 - \frac{w}{c^2}} \frac{\partial w}{\partial x^i} + v_k \left(D_i^k + A_i^k + \frac{1}{c^2} v_i F^k \right) \right], \quad (1.43)$$

$$\Gamma_{0i}^k = \frac{1}{c} \left(1 - \frac{w}{c^2}\right) \left(D_i^k + A_i^k + \frac{1}{c^2} v_i F^k \right), \quad (1.44)$$

$$\begin{aligned} \Gamma_{ij}^0 &= -\frac{1}{c \left(1 - \frac{w}{c^2}\right)} \left\{ -D_{ij} + \frac{1}{c^2} v_n \times \right. \\ &\quad \times \left[v_j \left(D_i^n + A_i^n \right) + v_i \left(D_j^n + A_j^n \right) + \frac{1}{c^2} v_i v_j F^n \right] + \\ &\quad \left. + \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} \right) - \frac{1}{2c^2} \left(F_i v_j + F_j v_i \right) - \Delta_{ij}^n v_n \right\}, \end{aligned} \quad (1.45)$$

$$\Gamma_{ij}^k = \Delta_{ij}^k - \frac{1}{c^2} \left[v_i \left(D_j^k + A_j^k \right) + v_j \left(D_i^k + A_i^k \right) + \frac{1}{c^2} v_i v_j F^k \right], \quad (1.46)$$

where Δ_{ij}^k are the *chr.inv.-Christoffel symbols*, which are defined similarly to the ordinary Christoffel symbols (1.2) but using the chr.inv.-metric tensor h_{ik} and the chr.inv.-derivation operators

$$\Delta_{jk}^i = h^{im} \Delta_{jk,m} = \frac{1}{2} h^{im} \left(\frac{* \partial h_{jm}}{\partial x^k} + \frac{* \partial h_{km}}{\partial x^j} - \frac{* \partial h_{jk}}{\partial x^m} \right). \quad (1.47)$$

By analogy with the respective absolute derivatives, Zelmanov had also introduced the chr.inv.-derivatives

$$*\nabla_i Q_k = \frac{* \partial Q_k}{\partial x^i} - \Delta_{ik}^l Q_l, \quad (1.48)$$

$$*\nabla_i Q^k = \frac{* \partial Q^k}{\partial x^i} + \Delta_{il}^k Q^l, \quad (1.49)$$

$$*\nabla_i Q_{jk} = \frac{* \partial Q_{jk}}{\partial x^i} - \Delta_{ij}^l Q_{lk} - \Delta_{ik}^l Q_{jl}, \quad (1.50)$$

$$*\nabla_i Q_j^k = \frac{* \partial Q_j^k}{\partial x^i} - \Delta_{ij}^l Q_l^k + \Delta_{il}^k Q_j^l, \quad (1.51)$$

$$*\nabla_i Q^{jk} = \frac{* \partial Q^{jk}}{\partial x^i} + \Delta_{il}^j Q^{lk} + \Delta_{il}^k Q^{jl}, \quad (1.52)$$

$$*\nabla_i Q^i = \frac{* \partial Q^i}{\partial x^i} + \Delta_{ji}^j Q^i, \quad (1.53)$$

$$*\nabla_i Q^{ji} = \frac{* \partial Q^{ji}}{\partial x^i} + \Delta_{il}^j Q^{il} + \Delta_{li}^j Q^{ji}, \quad (1.54)$$

where, as Zelmanov had proved,

$$\Delta_{li}^l = \frac{* \partial \ln \sqrt{h}}{\partial x^i}. \quad (1.55)$$

So, we have explained the basics of the mathematical apparatus of chronometric invariants. Now, having any equations obtained using general covariant methods we can calculate their chr.inv.-projections onto the time line and spatial section associated with any particular reference observer and formulate the obtained chr.inv.-projections with the real physically observable properties of his reference space. Following this way, we arrive at the equations containing only the quantities measurable in practice.

Naturally, the first possible application of this mathematical apparatus that comes to our mind is deducing the chr.inv.-equations of motion of free particles and studying the results. A particular solution to this problem had been obtained by Zelmanov [9]. The next section, §1.3, will focus on the general solution to the problem.

1.3 The dynamic equations of motion of a free particle

The absolute derivative of the four-dimensional vector of a particle with respect to a non-zero scalar parameter along its trajectory is actually a four-dimensional vector

$$N^\alpha = \frac{dQ^\alpha}{d\rho} + \Gamma_{\mu\nu}^\alpha Q^\mu \frac{dx^\nu}{d\rho}, \quad (1.56)$$

the chr.inv.-projections of which are determined in the same way as the projections of any four-dimensional vector (1.31)

$$\frac{N_0}{\sqrt{g_{00}}} = \frac{g_{0\alpha} N^\alpha}{\sqrt{g_{00}}} = \frac{1}{\sqrt{g_{00}}} (g_{00} N^0 + g_{0i} N^i), \quad (1.57)$$

$$N^i = h_\beta^i N^\beta = h_0^i N^0 + h_k^i N^k. \quad (1.58)$$

From a geometric point of view, these are the projections of the vector N^α on the time line and the spatial components of its projection on the spatial section in the accompanying reference frame. Projecting the general covariant dynamic equations of motion of a free mass-bearing particle (1.10) and those of a free massless particle (1.12), we obtain the chr.inv.-equations of motion of the free mass-bearing particle

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = 0, \quad (1.59)$$

$$\frac{d(mv^i)}{d\tau} + 2m(D_k^i + A_k^i)v^k - mF^i + m\Delta_{nk}^i v^n v^k = 0, \quad (1.60)$$

while for the free massless particle we have

$$\frac{dk}{d\tau} - \frac{k}{c^2} F_i c^i + \frac{k}{c^2} D_{ik} c^i c^k = 0, \quad (1.61)$$

$$\frac{d(kc^i)}{d\tau} + 2k(D_k^i + A_k^i)c^k - kF^i + k\Delta_{nk}^i c^n c^k = 0, \quad (1.62)$$

where m is the relativistic mass of the mass-bearing particle, $k = \frac{\omega}{c}$ is the wave number of the massless particle, and c^i is the three-dimensional chr.inv.-vector of the light velocity.

It is easy to see that, in contrast to the general covariant dynamic equations of motion (1.10) and (1.12), the chr.inv.-equations have a single derivation parameter for both mass-bearing and massless particles. This universal parameter is the physically observable time τ .

These chr.inv.-equations were first obtained by Zelmanov [9]. As we have showed in our first book [19], the Zelmanov chr.inv.-equations of motion above include the strictly positive time function $\frac{dt}{d\tau} > 0$. Therefore, the above equations manifest a case, where the physically observable time has a strictly direct flow ($d\tau > 0$).

The flow of the coordinate time dt shows the change of the time coordinate of the particle $x^0 = ct$ with respect to the clock associated with the observer (his reference clock). Hence, the sign of the time function shows the direction along the time axis at which the particle travels with respect to the observer.

The time function $\frac{dt}{d\tau}$ is derived from the geometric condition, according to which the square of the four-dimensional velocity of the particle, transported parallel to itself, remains unchanged along its world-trajectory $u_\alpha u^\alpha = g_{\alpha\beta} u^\alpha u^\beta = const$. We showed [19] that the time function equation $\frac{dt}{d\tau}$ is the same for both subluminal mass-bearing particles, massless (light-like) particles and superluminal mass-bearing particles. The equations have two solutions which are given here by the common formula according to [19]

$$\left(\frac{dt}{d\tau}\right)_{1,2} = \frac{v_i v^i \pm c^2}{c^2 \left(1 - \frac{w}{c^2}\right)}. \quad (1.63)$$

We showed [19] that time has a direct flow, if $v_i v^i \pm c^2 > 0$, time has a reverse flow, if $v_i v^i \pm c^2 < 0$, and the flow of time stops, if $v_i v^i \pm c^2 = 0$. Therefore, there exists a whole range of solutions for various kinds of particles and the directions they travel in time with respect to the observer. For instance, the relativistic mass of a mass-bearing particle* $\frac{P_0}{\sqrt{g_{00}}} = \pm m$ is positive, if the particle travels to the future, and it is negative, if the particle travels to the past. Respectively, the wave number

*The relativistic mass is the projection of the particle's four-dimensional vector onto the observer's time line.

of a massless particle $\frac{K_0}{\sqrt{g_{00}}} = \pm k$ is positive, when the massless particle travels to the future, and is negative, when it travels to the past.

As a result, for a free mass-bearing particle, which travels to the past, we obtain the chr.inv.-equations of motion

$$-\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = 0, \quad (1.64)$$

$$\frac{d(mv^i)}{d\tau} + mF^i + m\Delta_{nk}^i v^n v^k = 0, \quad (1.65)$$

while for a free massless particle, travelling to the past, we have

$$-\frac{dk}{d\tau} - \frac{k}{c^2} F_i c^i + \frac{k}{c^2} D_{ik} c^i c^k = 0, \quad (1.66)$$

$$\frac{d(kc^i)}{d\tau} + kF^i + k\Delta_{nk}^i c^n c^k = 0. \quad (1.67)$$

For a superluminal mass-bearing particle, the chr.inv.-equations of motion are similar to those of a mass-bearing particle travelling with a subluminal velocity, except that the relativistic mass m is multiplied by imaginary unit i [19].

It is easy to see that the chr.inv.-equations of motion to the future and to the past are not symmetric due to the different physical conditions in the cases of the direct and reverse flow of time, therefore some terms in the equations vanish.

Besides, we considered mass-bearing and massless particles according to the wave-particle concept by assuming their motion as wave propagation in the framework of the approximation of geometric optics [19]. As is well-known from *The Classical Theory of Fields* [10], in the framework of the wave-particle concept and the geometric optics approximation, the dynamic vector of a massless particle has the form

$$K_\alpha = \frac{\partial\psi}{\partial x^\alpha}, \quad (1.68)$$

where ψ is the wave phase (eikonal). In the same way, we had introduced the dynamic wave vector of a mass-bearing particle

$$P_\alpha = \frac{\hbar}{c} \frac{\partial\psi}{\partial x^\alpha}, \quad (1.69)$$

where \hbar is Planck's constant. Since the wave phase equation (eikonal equation) is the condition $K_\alpha K^\alpha = 0$ [10], we had obtained the chr.inv.-eikonal equation for a massless particle

$$\frac{1}{c^2} \left(\frac{*\partial\psi}{\partial t} \right)^2 - h^{ik} \frac{*\partial\psi}{\partial x^i} \frac{*\partial\psi}{\partial x^k} = 0, \quad (1.70)$$

and for a mass-bearing particle

$$\frac{1}{c^2} \left(\frac{*\partial\psi}{\partial t} \right)^2 - h^{ik} \frac{*\partial\psi}{\partial x^i} \frac{*\partial\psi}{\partial x^k} = \frac{m_0^2 c^2}{\hbar^2}, \quad (1.71)$$

then, after substituting the dynamic wave vector into the general covariant equations of motion (1.10, 1.12), then projecting them onto the time line and the spatial section in the accompanying reference frame, we had obtained the “wave form” of the chr.inv.-equations of motion of a mass-bearing particle [19]

$$\pm \frac{d}{d\tau} \left(\frac{*\partial\psi}{\partial t} \right) + F^i \frac{*\partial\psi}{\partial x^i} - D_k^i v^k \frac{*\partial\psi}{\partial x^i} = 0, \quad (1.72)$$

$$\begin{aligned} \frac{d}{d\tau} \left(h^{ik} \frac{*\partial\psi}{\partial x^k} \right) - \left(D_k^i + A_k^i \right) \left(\pm \frac{1}{c^2} \frac{*\partial\psi}{\partial t} v^k - h^{km} \frac{*\partial\psi}{\partial x^m} \right) \pm \\ \pm \frac{1}{c^2} \frac{*\partial\psi}{\partial t} F^i + h^{mn} \Delta_{mk}^i v^k \frac{*\partial\psi}{\partial x^n} = 0, \end{aligned} \quad (1.73)$$

where “plus” in the alternating terms stands for the particle's motion from the past to the future (direct flow of time), while “minus” stands for its motion to the past (reverse flow of time). Noteworthy, in contrast to the “corpuscular form” of the chr.inv.-equations of motion (1.59, 1.60) and (1.64, 1.65), the “wave equations” (1.72, 1.73) are symmetric with respect to the direction of motion in time. For a massless particle, the “wave form” of the chr.inv.-equations of motion include the chr.inv.-vector of the light velocity c^i instead of the subluminal chr.inv.-velocity v^i of a mass-bearing particle.

The fact that the corpuscular equations of motion to the past and to the future are asymmetric had led us to the conclusion that in the four-dimensional space-time of General Relativity there exists a fundamental asymmetry of the directions in time. To understand the physical sense of this fundamental asymmetry, we had introduced the *mirror principle* or, in other words — the *observable effect of the mirror Universe* [19].

Let us imagine a *mirror* in the four-dimensional space-time, which coincides with the spatial section (three-dimensional space) associated with an observer, so that the mirror separates the past from the future. Then, particles and waves travelling from the past to the future (positive relativistic masses and frequencies) hit the mirror and bounce back in time to the past. As a result, their properties take negative numerical values. Conversely, particles and waves travelling to the past (negative relativistic masses and frequencies) bounce from the mirror to give positive numerical values to their properties and begin travelling to the future. When bouncing from the mirror, the quantity $\frac{\partial \psi}{dt}$ changes sign, and so the equations of wave propagation to the future become the equations of wave propagation to the past (and vice versa). Noteworthy, when reflecting from the mirror, the chr.inv.-equations of wave propagation transform into each other *completely* without contracting or adding new terms. In other words, the wave form of matter undergoes *complete reflection* from the mirror. On the contrary, the “corpuscular” chr.inv.-equations of motion *do not transform completely* upon reflection from the mirror: the spatial projections of the equations for mass-bearing and massless particles, travelling from the past to the future, have the additional term

$$2m(D_k^i + A_k^i)v^k, \quad 2k(D_k^i + A_k^i)c^k, \quad (1.74)$$

not found in the equations of motion from the future to the past. In other words, the equations of motion to the past gain the additional term upon the reflection, while the equations of motion to the future lose the term when the particle hits the mirror. This means that, either in the case of motion of a ball-particle (the corpuscular equations) as well as in the case of wave propagation, we come across a situation that is not a simple “bouncing” from the mirror, but rather *passing* through the mirror itself into another world — a *world beyond the mirror*.

In the *mirror world*, all particles have negative masses or frequencies, so they travel (from our point of view) from the future to the past. The wave form of matter in our world does not affect events in the mirror world, and the mirror world wave matter does not affect events in our world. On the contrary, the corpuscular form of matter (particles) in our world can produce a significant effect on events in the mirror world, while the mirror world particles can affect events in our world. Our world and the mirror world are completely isolated from each other (no

mutual effect between particles from the two worlds) under the obvious condition $D_k^i v^k = -A_k^i v^k$, at which the additional term in the corpuscular chr.inv.-equations of motion becomes zero. This becomes true, in particular, when $D_k^i = 0$ and $A_k^i = 0$, i.e., when the space does not rotate or deform [19].

So, we have considered the motion of particles along non-isotropic trajectories, where $ds^2 = c^2 d\tau^2 - d\sigma^2 > 0$, and the motion along isotropic (light-like) trajectories, where $ds^2 = 0$ and $c^2 d\tau^2 = d\sigma^2 \neq 0$. Besides, we have considered the third kind of trajectories [19], which, apart from $ds^2 = 0$, meet even more strict conditions $c^2 d\tau^2 = d\sigma^2 = 0$

$$d\tau = \left[1 - \frac{1}{c^2} (\mathbf{w} + v_i u^i) \right] dt = 0, \quad (1.75)$$

$$d\sigma^2 = h_{ik} dx^i dx^k = 0. \quad (1.76)$$

We called such completely degenerate trajectories *zero-trajectories*, because from the point of view of an ordinary subluminal observer, any physically observable time intervals and any physically observable spatial intervals are zeroes along them. We also showed that along zero-trajectories the determinant of the fundamental metric tensor $g_{\alpha\beta}$ is zero ($g = 0$), while as is known, in Riemannian spaces, by their definition, there is $g < 0$, so the Riemannian metric is strictly non-degenerate. Therefore, we called a space, the metric of which is completely degenerate, *zero-space*. For the same reason, we called particles hosted by such a completely degenerate space (zero-space) and travelling along trajectories in it *zero-particles* [19].

Actually, formulae (1.75, 1.76) show the physical conditions, under which the complete degeneration of the four-dimensional space-time occurs. Re-write the *physical conditions of degeneration* as follows

$$\mathbf{w} + v_i u^i = c^2, \quad (1.77)$$

$$g_{ik} u^i u^k = c^2 \left(1 - \frac{\mathbf{w}}{c^2} \right)^2. \quad (1.78)$$

Thus, we had obtained the formula for the mass of a zero-particle M , which includes the degeneration conditions

$$M = \frac{m}{1 - \frac{1}{c^2} (\mathbf{w} + v_i u^i)}, \quad (1.79)$$

which differs from the relativistic mass m of an ordinary particle, located in a non-degenerate space-time region. The M is the ratio between two scalar quantities, m and $1 - \frac{1}{c^2}(w + v_i u^i)$, each one equals zero in the case where the metric is degenerate, but the ratio is not zero*.

The dynamic vector of a zero-particle, represented in the corpuscular and wave forms, is

$$P^\alpha = \frac{M}{c} \frac{dx^\alpha}{dt}, \quad P_\alpha = \frac{\hbar}{c} \frac{\partial \psi}{\partial x^\alpha}. \quad (1.80)$$

Then, the dynamic chr.inv.-equations of motion in the zero-space, taken in their corpuscular form, are

$$MD_{ik} u^i u^k = 0, \quad (1.81)$$

$$\frac{d}{dt}(Mu^i) + M\Delta_{nk}^i u^n u^k = 0, \quad (1.82)$$

while the wave form of the equations is

$$D_k^m u^k \frac{* \partial \psi}{\partial x^m} = 0, \quad (1.83)$$

$$\frac{d}{dt} \left(h^{ik} \frac{* \partial \psi}{\partial x^k} \right) + h^{mn} \Delta_{mk}^i u^k \frac{* \partial \psi}{\partial x^n} = 0. \quad (1.84)$$

The chr.inv.-eikonal equation for a zero-particle takes the form

$$h^{ik} \frac{* \partial \psi}{\partial x^i} \frac{* \partial \psi}{\partial x^k} = 0, \quad (1.85)$$

which is a standing wave equation, i.e., the zero-particle has the form of a standing light-like wave (information ring). This result means that, from the viewpoint of an ordinary observer like us, the entire zero-space is filled with a system of standing light-like waves (zero-particles) — a *standing-light hologram*. Besides, in the zero-space, the physically observable time has the same numerical value for any two events (1.75). This means that, from the viewpoint of an ordinary observer, the velocity of any zero-particle is infinite, and zero-particles can instantly transfer information from one point of our ordinary world to another, thereby performing the *long-range action* [19].

*This is similar to the case of massless particles, because given $v^2 = c^2$ we have that $m_0 = 0$ and $\sqrt{1 - v^2/c^2} = 0$, but their ratio is non-zero, i.e., $m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \neq 0$.

1.4 Non-geodesic motion of particles. Problem statement

It is known that, when a particle travels freely in a Riemannian space, the absolute derivative of its dynamic world-vector (its four-dimensional momentum P^α) remains equal to zero, and the square of the vector remains unchanged along the motion trajectory. In other words, the vector is transported parallel to itself in the sense of Levi-Civita.

If the motion of a particle is non-free (non-geodesic), then the absolute derivative of its four-dimensional momentum is not zero, but the absolute derivative of the sum of its four-dimensional momentum P^α and an additional momentum vector L^α , gained by the particle from an external field that deviates its motion from geodesic line, is zero. Superposition of any number of vectors can be subjected to parallel transport [18]. Hence, when creating the equations of non-geodesic motion, we first of all require the definition of perturbing non-gravitational fields.

Naturally, an external field will only interact with a particle and deviate it from its geodesic line, if the particle has a physical property of the same kind as the external field does. As of today, we know of three fundamental physical properties of particles, not related to each other. These are *mass*, *electric charge* and *spin*. If the fundamental character of the former two was under no doubt, the spin of an electron over a few years after experiments by Stern and Gerlach (1921) and their interpretation by Goudsmit and Uhlenbeck (1925), was considered as a specific momentum of the electron caused by its rotation around its own axis. But experiments done over the next decades, in particular, the discovery of the spin in other elementary particles, proved that the views of spin particles as rotating gyroscopes were wrong. Spin proved to be a fundamental property of particles just like mass and electric charge, although it has the dimension of angular momentum and in interactions manifests its as a specific rotation momentum inside the particle.

Gravitational fields by now have received a geometric interpretation due to Einstein's equations. In the theory of chronometric invariants, the gravitational force and potential (1.38) are obtained as functions of only the geometric properties of the space itself. Therefore, considering the motion of a particle in a pseudo-Riemannian space, we actually consider its motion in a gravitational field.

But we still do not know whether the electromagnetic Lorentz force and the electromagnetic field potential can be expressed through the ge-

ometric properties of the space. Therefore, electromagnetic fields at the moment have no geometric interpretation. An electromagnetic field is introduced into a pseudo-Riemannian space as an external tensor field (the field of Maxwell's tensor). By now the main equations of the electromagnetic field theory have been obtained in the general covariant form*. In this theory, a charged particle gains a four-dimensional momentum $\frac{e}{c^2} A^\alpha$ from an acting electromagnetic field, where A^α is the four-dimensional potential of the field, and e is the electric charge of the particle [10, 20]. By adding this additional momentum to the particle's own momentum vector and applying the Levi-Civita parallel transport to the summary vector, we can obtain the general covariant equations of motion of the particle in a space filled with the gravitational and electromagnetic fields.

The case of spin particles is far more complicated. To deduce a momentum that a particle gains due to its spin, we need to define the external field that interacts with the spin. Initially, this problem was approached using the methods of Quantum Mechanics (Dirac's equations, 1928). The geometric methods of the General Theory of Relativity were first used by Papapetrou [21] and then together with Corinaldesi [22] in the attempt to solve the problem of spin particles. Their approach relied on the general view of particles as mechanical monopoles and dipoles. From this point of view, an ordinary mass-bearing particle is a *mechanical monopole*. If a particle is represented as two masses co-rotating around a common centre of gravity, then the particle is a *mechanical dipole*. Proceeding from the representation of a spin particle as a rotating gyroscope, they considered it as a mechanical dipole, where the centre of gravity is under the particle's surface. They considered the motion of such a mechanic dipole in a pseudo-Riemannian space with the Schwarzschild metric — a particular case, where the space does not rotate or deform (the latter means that the space metric is stationary, i.e., the tensor of the space deformation rate is zero).

There is no doubt that Papapetrou's method is noteworthy, but it has a significant drawback. Being developed in the 1940s, it fully relied on

*Despite this positive fact, due to the complicated calculation of the electromagnetic field energy-momentum tensor in the space-time of General Relativity, specific problems are usually solved either for particular cases of General Relativity, or, most often, in a Galilean reference frame in the Minkowski space, i.e., in the space-time of Special Relativity.

the view of spin particles as swiftly rotating gyroscopes, which does not match experimental data of the recent decades*.

There is another way to solve the problem of motion of spin particles. In Riemannian spaces, the fundamental metric tensor is symmetric, $g_{\alpha\beta} = g_{\beta\alpha}$. Nevertheless, we can create a space in which the metric tensor has an arbitrary form $g_{\alpha\beta} \neq g_{\beta\alpha}$ (the geometry of such a space is non-Riemannian). Then, a non-zero antisymmetric part can be found in the metric tensor[†]. Then corresponding additions will appear in the Christoffel symbols $\Gamma_{\mu\nu}^{\alpha}$ and in the Riemann-Christoffel curvature tensor $R_{\alpha\beta\mu\nu}$. These additions will be the result of the fact that, a vector transported along a closed contour does not return to its initial point, so the trajectory becomes twisted like a spiral. Such a space is known as a twisted space. In such a space, the spin rotation of a particle can be considered as the transport of the rotation vector along the contour over the particle's surface, which generates a local field of the space twist.

Nonetheless, this method has got significant drawbacks as well. Firstly, if we have $g_{\alpha\beta} \neq g_{\beta\alpha}$, then functions of the $g_{\alpha\beta}$ components with different order of indices can vary. The functions are somehow fixed in order to set a specific field of this rotation, which very narrows the range of possible solutions, allowing you to create equations only for a number of specific cases. Secondly, this method is completely based on the assumption that the spin rotation of a particle is a local twist field created by the transport of the particle's rotation vector along the closed contour. This again means the view of spin particles as rotating tiny mechanical gyroscopes (similarly to Papapetrou's method), which is inconsistent with experimental data.

Nevertheless, there is no doubt that an additional momentum gained by a spin particle can be represented using the methods of the General Theory of Relativity. Adding the gained momentum to the dynamic vector of the particle (which is the effect of gravitation) and applying

*Indeed, considering an electron as a tiny ball with a radius of $r_e = 2.8 \times 10^{-13}$ cm means that the linear velocity of its rotation on the surface is $u = \frac{\hbar}{2m_0 r_e} = 2 \times 10^{11}$ cm/sec, which is ~ 70 times greater than the light velocity. Experiments show that electrons do not have such rotation speeds.

[†]Generally, in any tensor of the 2nd rank and of high ranks symmetric and antisymmetric parts can be distinguished. For instance, in the fundamental metric tensor $g_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta} + g_{\beta\alpha}) + \frac{1}{2}(g_{\alpha\beta} - g_{\beta\alpha}) = S_{\alpha\beta} + N_{\alpha\beta}$ we have the symmetric part $S_{\alpha\beta}$ and the antisymmetric part $N_{\alpha\beta}$. Because the metric tensor of any Riemannian space is symmetric $g_{\alpha\beta} = g_{\beta\alpha}$, its antisymmetric part is zero.

the Levi-Civita parallel transport of the summary vector, we can obtain the general covariant equations of motion of the spin particle*.

Having obtained the general covariant equations of motion of a spin particle and an electrically charged particle, we shall project them onto the time line and the spatial section of an observer, and then express the obtained chr.inv.-projections in terms of the physically observable properties of his reference space. As a result, we will arrive at the chr.inv.-equations of non-geodesic motion.

Therefore, the problem that we are going to solve in this book falls into several stages. In Chapter 3, we will create a chr.inv.-theory of the electromagnetic field in the four-dimensional pseudo-Riemannian space. We will also obtain the chr.inv.-equations of motion of a charged particle in the electromagnetic field.

In Chapter 4, we will create a theory of the motion of spin particles. We will approach this problem in its most general form, assuming that spin is a fundamental property of matter (like mass or electric charge). A detailed study will show that the field of the space non-holonomy (spatial rotation of the space) interacts with the particle's spin, giving it an additional momentum.

In Chapter 5 we are going to discuss the chr.inv.-projections of Einstein's equations. Based on them, we will derive the properties of the physical vacuum and how they are applied to cosmology.

In Chapter 6, we will consider the theory of the mirror world, as well as the physical conditions for entering it through the membrane that separates it from us.

Before starting this research, in Chapter 2 we will give tensor algebra and analysis in terms of physically observable quantities (chronometric invariants). We recommend Chapter 2 to those readers, who want to use the chronometrically invariant formalism in their research.

*We wrote this in the mid-1990s, in the 1st edition of this book. In 2007, Suhendro [23, 24] developed a new and highly original approach to spin particles, which is based on the view of the spin as an elementary curl of the space itself. We agree that, since his approach is purely geometric in nature, it is closer to Einstein's approach (the geometrization of matter and interactions) than our approach, implemented in Chapter 4 of this book based on the Lagrange method.

2.1 Tensors and tensor algebra

We assume a space (not necessarily a metric one) with an arbitrary reference frame x^α located in it. In an area of this space, there exists an object G defined by n functions f_n of the coordinates x^α . We know the transformation rule to calculate these n functions in any other reference frame \tilde{x}^α in this space. If the n functions f_n and also the transformation rule have been given, then G is a *geometric object*, which in the system x^α has axial components $f_n(x^\alpha)$, while in any other system \tilde{x}^α it has components $\tilde{f}_n(\tilde{x}^\alpha)$.

We assume that a tensor object (*tensor*) of zero rank is any geometric object φ , transformable according to the rule

$$\tilde{\varphi} = \varphi \frac{\partial x^\alpha}{\partial \tilde{x}^\alpha}, \quad (2.1)$$

where the index sequentially takes the numbers of all coordinate axes (this notation is also known as *component notation* or *tensor notation*). Any tensor of zero rank has a single component and is also known as a *scalar*. From a geometric point of view, any scalar is a point to which a certain number is attributed. A scalar field* is a set of points, which have a common property. For instance, a point mass is a scalar, and a distributed mass (a gas, for instance) makes up a scalar field.

Contravariant tensors of the 1st rank A^α are geometric objects with components, transformable according to the rule

$$\tilde{A}^\alpha = A^\mu \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}. \quad (2.2)$$

*Algebraic notations for a tensor and a tensor field are the same. The field of a tensor is represented as the tensor in a given point of the space, but its presence in other points of the space is assumed.

From a geometric point of view, such an object is an n -dimensional vector. For instance, the vector of an elementary displacement dx^α is a contravariant tensor of the 1st rank.

Contravariant tensors of the 2nd rank $A^{\alpha\beta}$ are geometric objects with components, transformable according to the rule

$$\tilde{A}^{\alpha\beta} = A^{\mu\nu} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu}. \quad (2.3)$$

From a geometric point of view, such an object is an area (parallelogram) constructed by two vectors. For this reason, contravariant tensors of the 2nd rank are also known as *bivectors*.

Thus, contravariant tensors of higher ranks are geometric objects, transformable according to the rule

$$\tilde{A}^{\alpha\dots\sigma} = A^{\mu\dots\tau} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \dots \frac{\partial \tilde{x}^\sigma}{\partial x^\tau}. \quad (2.4)$$

A vector field or a higher rank tensor field are space distributions of the tensor quantities. For instance, because a mechanical strength characterizes both its own magnitude and the direction, its distribution in a physical body can be presented by a vector field.

Covariant (i.e., lower-index) tensors of the 1st rank A_α are geometric objects, transformable according to the rule

$$\tilde{A}_\alpha = A_\mu \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}. \quad (2.5)$$

The gradient of a scalar field φ , i.e., the quantity $A_\alpha = \frac{\partial \varphi}{\partial x^\alpha}$, is a covariant tensor of the 1st rank. That is, since for an ordinary invariant we have $\tilde{\varphi} = \varphi$, then

$$\frac{\partial \tilde{\varphi}}{\partial \tilde{x}^\alpha} = \frac{\partial \tilde{\varphi}}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} = \frac{\partial \varphi}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}. \quad (2.6)$$

Covariant tensors of the 2nd rank $A_{\alpha\beta}$ are geometric objects with the transformation rule

$$\tilde{A}_{\alpha\beta} = A_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta}. \quad (2.7)$$

Hence, covariant tensors of higher ranks are

$$\tilde{A}_{\alpha\dots\sigma} = A_{\mu\dots\tau} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \dots \frac{\partial x^\tau}{\partial \tilde{x}^\sigma}. \quad (2.8)$$

Mixed tensors are tensors of the 2nd rank or of higher ranks with both upper and lower indices. For instance, a mixed symmetric tensor A_{β}^{α} is a geometric object, transformable according to the rule

$$\tilde{A}_{\beta}^{\alpha} = A_{\nu}^{\mu} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}}. \quad (2.9)$$

Tensor objects exist both in metric and non-metric spaces*. Any tensor has a^n components, where a is its dimension and n is the rank. For instance, a four-dimensional tensor of zero rank has 1 component, a tensor of the 1st rank has 4 components, a tensor of the 2nd rank has 16 components and so on.

Indices in a geometric object, marking its axial components, are found not in tensors only, but in other geometric objects as well. For this reason, if we come across a quantity in component notation, this is not necessarily a tensor quantity.

In practice, to know whether a given object is a tensor or not, we need to know a formula for this object in a reference frame and then transform it to any other reference frame. For instance, consider the following classic question: are the Christoffel symbols (i.e., the coherence coefficients of space) tensors? To answer this question, we need to calculate the quantities in a tilde-marked reference frame

$$\tilde{\Gamma}_{\mu\nu}^{\alpha} = \tilde{g}^{\alpha\sigma} \tilde{\Gamma}_{\mu\nu,\sigma}, \quad \tilde{\Gamma}_{\mu\nu,\sigma} = \frac{1}{2} \left(\frac{\partial \tilde{g}_{\mu\sigma}}{\partial \tilde{x}^{\nu}} + \frac{\partial \tilde{g}_{\nu\sigma}}{\partial \tilde{x}^{\mu}} - \frac{\partial \tilde{g}_{\mu\nu}}{\partial \tilde{x}^{\sigma}} \right) \quad (2.10)$$

proceeding from the quantities in a non-marked reference frame.

First, calculate the terms in brackets (2.10). The fundamental metric tensor like any other covariant tensor of the 2nd rank, is transformable to the tilde-marked reference frame according to the rule

$$\tilde{g}_{\mu\sigma} = g_{\varepsilon\tau} \frac{\partial x^{\varepsilon}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\tau}}{\partial \tilde{x}^{\sigma}}. \quad (2.11)$$

Because the $g_{\varepsilon\tau}$ depends on non-tilde-marked coordinates, its derivative with respect to tilde-marked coordinates (which are functions of

*In non-metric spaces, as is known, the distance between any two points cannot be measured. This is in contrast to metric spaces. In the theories of space-time-matter, such as the General Theory of Relativity and its extensions, metric spaces are taken under consideration. This is due to the fact that the core of these theories is the measurement of time durations and spatial lengths, which is nonsense in a non-metric space.

non-tilde ones) is calculated according to the rule

$$\frac{\partial g_{\varepsilon\tau}}{\partial \tilde{x}^\nu} = \frac{\partial g_{\varepsilon\tau}}{\partial x^\rho} \frac{\partial x^\rho}{\partial \tilde{x}^\nu}. \quad (2.12)$$

Then the first term in brackets (2.10), taking the rule of transformation of the fundamental metric tensor into account, is

$$\frac{\partial \tilde{g}_{\mu\sigma}}{\partial \tilde{x}^\nu} = \frac{\partial g_{\varepsilon\tau}}{\partial x^\rho} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial x^\tau}{\partial \tilde{x}^\sigma} + g_{\varepsilon\tau} \left(\frac{\partial x^\tau}{\partial \tilde{x}^\sigma} \frac{\partial^2 x^\varepsilon}{\partial \tilde{x}^\nu \partial \tilde{x}^\mu} + \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial^2 x^\tau}{\partial \tilde{x}^\nu \partial \tilde{x}^\sigma} \right). \quad (2.13)$$

Hence, calculating the remaining terms of the tilde-marked Christoffel symbols (2.10), after transitioning the free indices we obtain

$$\tilde{\Gamma}_{\mu\nu,\sigma} = \Gamma_{\varepsilon\rho,\tau} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} \frac{\partial x^\tau}{\partial \tilde{x}^\sigma} + g_{\varepsilon\tau} \frac{\partial x^\tau}{\partial \tilde{x}^\sigma} \frac{\partial^2 x^\varepsilon}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}, \quad (2.14)$$

$$\tilde{\Gamma}_{\mu\nu}^\alpha = \Gamma_{\varepsilon\rho}^\gamma \frac{\partial \tilde{x}^\alpha}{\partial x^\gamma} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} + \frac{\partial \tilde{x}^\alpha}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}, \quad (2.15)$$

so, we see that the Christoffel symbols are not transformed in the same way as tensors, hence they are not tensors.

Tensors can be represented as matrices. But in practice, this form can be possible for only tensors of the 1st or 2nd rank (single-row and flat matrices, respectively). For instance, the tensor of an elementary four-dimensional displacement is

$$dx^\alpha = (dx^0, dx^1, dx^2, dx^3), \quad (2.16)$$

while the four-dimensional fundamental metric tensor is

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \quad (2.17)$$

Tensors of the 3rd rank are three-dimensional matrices. Representing tensors of higher ranks as matrices is more problematic.

Now we turn to tensor algebra — a section of tensor calculus, which focuses on algebraic operations over tensors.

Only same-type tensors of the same rank with indices in the same position can be added or subtracted. The *adding up* two same-type ten-

sors gives a new tensor of the same type and rank with the components being the sums of the corresponding components of these tensors

$$A^\alpha + B^\alpha = D^\alpha, \quad A_\beta^\alpha + B_\beta^\alpha = D_\beta^\alpha. \quad (2.18)$$

Multiplication is permitted not only for same-type, but for any tensors of any ranks. The *external multiplication* of tensors of the n -th rank and the m -th rank gives a tensor of the $(n + m)$ -th rank

$$A_{\alpha\beta} B_\gamma = D_{\alpha\beta\gamma}, \quad A_\alpha B^{\beta\gamma} = D_\alpha^{\beta\gamma}. \quad (2.19)$$

Contraction is the multiplication of the same-rank tensors, when their indices are the same. The contraction by all indices gives a scalar

$$A_\alpha B^\alpha = C, \quad A_{\alpha\beta}^\gamma B_\gamma^{\alpha\beta} = D. \quad (2.20)$$

Often multiplication of tensors means contraction of some indices. Such multiplication is known as *internal multiplication*, which means contraction of some indices inside the multiplication. This is an example of internal multiplication

$$A_{\alpha\sigma} B^\sigma = D_\alpha, \quad A_{\alpha\sigma}^\gamma B_\gamma^{\sigma\alpha} = D_\alpha^\beta. \quad (2.21)$$

Using internal multiplication of geometric objects, we can determine whether they are tensors or not. This is the so-called theorem of fractions, which is given here according to [9]:

Theorem of fractions

If $B^{\sigma\beta}$ is a tensor and its internal multiplication with a geometric object $A(\alpha, \sigma)$ is a tensor D_α^β

$$A(\alpha, \sigma) B^{\sigma\beta} = D_\alpha^\beta, \quad (2.22)$$

then the object $A(\alpha, \sigma)$ is also a tensor.

According to the theorem, if the internal multiplication of an object $A_{\alpha\sigma}$ with a tensor $B^{\sigma\beta}$ gives a tensor $D^{\alpha\beta}$

$$A_{\alpha\sigma} B^{\sigma\beta} = D_\alpha^\beta, \quad (2.23)$$

then the object $A_{\alpha\sigma}^\alpha$ is a tensor. Or, if the internal multiplication of an object A_σ^α with a tensor $B^{\sigma\beta}$ gives a tensor D_α^β

$$A_\sigma^\alpha B^{\sigma\beta} = D_\alpha^\beta, \quad (2.24)$$

then the object A_{σ}^{α} is a tensor.

The geometric properties of any metric space are determined by its fundamental metric tensor $g_{\alpha\beta}$, which can lift and lower indices in geometric objects of this metric space*. For instance,

$$g_{\alpha\beta}A^{\beta} = A_{\alpha}, \quad g^{\mu\nu}g^{\sigma\rho}A_{\mu\nu\sigma} = A^{\rho}. \quad (2.25)$$

In Riemannian spaces, the mixed fundamental metric tensor g_{α}^{β} is equal to the unit tensor $g_{\alpha}^{\beta} = g_{\alpha\sigma}g^{\sigma\beta} = \delta_{\alpha}^{\beta}$. The diagonal components of the unit tensor are units, and its other components are zeroes. Using the unit tensor, we can replace indices in four-dimensional quantities, so that

$$\delta_{\alpha}^{\beta}A_{\beta} = A_{\alpha}, \quad \delta_{\mu}^{\nu}\delta_{\rho}^{\sigma}A^{\mu\rho} = A^{\nu\sigma}. \quad (2.26)$$

Contraction of any tensor of the 2nd rank with the fundamental metric tensor gives a scalar, known as the *tensor spur* or the *tensor trace*

$$g^{\alpha\beta}A_{\alpha\beta} = A_{\sigma}^{\sigma}. \quad (2.27)$$

For instance, the trace of the fundamental metric tensor in a four-dimensional pseudo-Riemannian space is

$$g_{\alpha\beta}g^{\alpha\beta} = g_{\sigma}^{\sigma} = g_0^0 + g_1^1 + g_2^2 + g_3^3 = \delta_0^0 + \delta_1^1 + \delta_2^2 + \delta_3^3 = 4. \quad (2.28)$$

The chr.inv.-metric tensor h_{ik} (1.27) has all properties of the fundamental metric tensor $g_{\alpha\beta}$ in the observer's three-dimensional space. Therefore, h_{ik} can lower, lift or replace indices in chr.inv.-quantities. Respectively, the trace of a three-dimensional chr.inv.-tensor is obtained by means of its contraction with the chr.inv.-metric tensor h_{ik} .

For instance, the trace of the tensor of the space deformation rate D_{ik} (1.40) is

$$h^{ik}D_{ik} = D_m^m, \quad (2.29)$$

the physical sense of which is the relative expansion rate of an elementary volume of the space.

Of course, the above very brief account cannot fully cover such a vast field like tensor algebra. Moreover, there is even no need in doing that here. Detailed accounts of tensor algebra can be found in many

*In Riemannian spaces, the metric has the square form $ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$, known also as the *Riemannian metric form*. Therefore, the fundamental metric tensor of a Riemannian space is the tensor of the 2nd rank, $g_{\alpha\beta}$.

mathematical books not related to the General Theory of Relativity. Besides, many specific techniques of this science, which occupy a substantial part of mathematical textbooks, are not used in theoretical physics. Therefore, our contribution has been to provide only the basic introduction to tensors and tensor algebra needed to understand this book. For the same reasons, we have not covered issues such as the weight of tensors and many others not used in the calculations in this book.

2.2 Scalar product of two vectors

The *scalar product* of two vectors A^α and B^α in a four-dimensional pseudo-Riemannian space is

$$g_{\alpha\beta} A^\alpha B^\beta = A_\alpha B^\alpha = A_0 B^0 + A_i B^i. \quad (2.30)$$

Scalar product is a contraction, because multiplication of vectors contracts all of their indices. Therefore, the scalar product of two vectors (tensors of the 1st rank) is always a scalar (tensor of zero rank). If both of the vectors are the same, their scalar product

$$g_{\alpha\beta} A^\alpha A^\beta = A_\alpha A^\alpha = A_0 A^0 + A_i A^i \quad (2.31)$$

is the square of the given vector A^α . Consequently, the length of this vector A^α is

$$A = |A^\alpha| = \sqrt{g_{\alpha\beta} A^\alpha A^\beta}. \quad (2.32)$$

Since the four-dimensional pseudo-Riemannian space of General Relativity by definition has the sign-alternating metric of the signature (+---) or (-+++), the lengths of a four-dimensional vector in the space can be real, imaginary or zero. Vectors having non-zero (real or imaginary) lengths are known as *non-isotropic vectors*. Vectors having zero length are known as *isotropic vectors*. Isotropic vectors are tangential to the trajectories of light-like particles (isotropic trajectories).

In three-dimensional Euclidean space, the scalar product of two vectors is a scalar quantity with a magnitude equal to the product of their lengths, multiplied by the cosine of the angle between them

$$A_i B^i = |A^i| |B^i| \cos(A^i; B^i). \quad (2.33)$$

Theoretically, at every point of any Riemannian space a tangential flat space can be set, the basis vectors of which are tangential to the

basis vectors of the Riemannian space at that point. In this case, the metric of the tangential flat space is the same as the metric of the Riemannian space at that point. This statement is also true in the Riemannian space, if we take the angle between the coordinate lines into account and replace Roman (three-dimensional) indices with Greek (four-dimensional) ones.

From here, we can see that the scalar product of two vectors is zero, if the vectors are orthogonal to each other. In other words, the scalar product from a geometric point of view is the projection of one vector onto the other. If the vectors are the same, then the vector is projected onto itself, so the result of this projection is the square of its length.

Denote the chr.inv.-projections of arbitrary vectors A^α and B^α as follows

$$a = \frac{A_0}{\sqrt{g_{00}}}, \quad a^i = A^i, \quad (2.34)$$

$$b = \frac{B_0}{\sqrt{g_{00}}}, \quad b^i = B^i, \quad (2.35)$$

then their remaining components are

$$A^0 = \frac{a + \frac{1}{c} v_i a^i}{1 - \frac{w}{c^2}}, \quad A_i = -a_i - \frac{a}{c} v_i, \quad (2.36)$$

$$B^0 = \frac{b + \frac{1}{c} v_i b^i}{1 - \frac{w}{c^2}}, \quad B_i = -b_i - \frac{b}{c} v_i. \quad (2.37)$$

Substituting the chr.inv.-projections into the formulae for $A_\alpha B^\alpha$ and $A_\alpha A^\alpha$, we obtain

$$A_\alpha B^\alpha = ab - a_i b^i = ab - h_{ik} a^i b^k, \quad (2.38)$$

$$A_\alpha A^\alpha = a^2 - a_i a^i = a^2 - h_{ik} a^i a^k. \quad (2.39)$$

From here, we see that the square of the length of any vector is the difference between the squares of the lengths of its time and spatial chr.inv.-projections. If both of the projections are equal, then the vector's length is zero, so the vector is isotropic. Hence, any isotropic vector equally belongs to the time line and the spatial section. The equality of the time and spatial chr.inv.-projections also means that the vector

is orthogonal to itself. If its time projection is “longer” than the spatial one, then the vector is real. If the spatial projection is “longer”, then the vector is imaginary.

The scalar product of any four-dimensional vector with itself can be illustrated by the square of the length of the space-time interval

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dx_\alpha dx^\alpha = dx_0 dx^0 + dx_i dx^i. \quad (2.40)$$

In terms of physically observable quantities, it can be represented as follows

$$ds^2 = c^2 d\tau^2 - dx_i dx^i = c^2 d\tau^2 - h_{ik} dx^i dx^k = c^2 d\tau^2 - d\sigma^2. \quad (2.41)$$

Its length $ds = \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}$ can be real, imaginary or zero, depending on whether ds is time-like $c^2 d\tau^2 > d\sigma^2$ (subluminal real trajectories), space-like $c^2 d\tau^2 < d\sigma^2$ (imaginary superluminal trajectories), or isotropic $c^2 d\tau^2 = d\sigma^2$ (light-like trajectories).

2.3 Vector product of two vectors. Antisymmetric tensors and pseudotensors

The *vector product* of two vectors A^α and B^α is a tensor of the 2nd rank $V^{\alpha\beta}$, obtained from their external multiplication according to the rule

$$V^{\alpha\beta} = [A^\alpha; B^\beta] = \frac{1}{2} (A^\alpha B^\beta - A^\beta B^\alpha) = \frac{1}{2} \begin{vmatrix} A^\alpha & A^\beta \\ B^\alpha & B^\beta \end{vmatrix}. \quad (2.42)$$

As is easy to see, the order in which vectors are multiplied matters, i.e., the order in which we write down tensor indices is important. Therefore, tensors obtained as vector products are *antisymmetric*. In an antisymmetric tensor $V^{\alpha\beta} = -V^{\beta\alpha}$; its indices being moved “reserve” their places as dots, $g_{\alpha\sigma} V^{\sigma\beta} = V_{\alpha}^{\cdot\beta}$, thereby showing from where the index was moved. In symmetric tensors, there is no need to “reserve” places for moved indices, because the order in which they appear does not matter. In particular, the fundamental metric tensor is symmetric $g_{\alpha\beta} = g_{\beta\alpha}$, while the tensor of the space curvature $R_{\cdot\beta\gamma\delta}^{\alpha\cdots}$ is symmetric in respect to the transposition by pair of its indices, but is antisymmetric inside each pair of the indices.

It is obvious that only tensors of the 2nd rank or of higher ranks can be symmetric or antisymmetric.

All diagonal components of any antisymmetric tensor by definition are zeroes. So, in an antisymmetric tensor of the 2nd rank, we have

$$V^{\alpha\alpha} = [A^\alpha; B^\alpha] = \frac{1}{2} (A^\alpha B^\alpha - A^\alpha B^\alpha) = 0. \quad (2.43)$$

In the three-dimensional Euclidean space, the vector product of two vectors is the area of the parallelogram they make and is equal to the product of their moduli, multiplied by the sine of the angle between them

$$V^{ik} = |A^i| |B^k| \sin(A^i; B^k). \quad (2.44)$$

This means that the vector product of two vectors (i.e., an antisymmetric tensor of the 2nd rank) is an area, oriented in the space according to the directions of its forming vectors.

Contraction of an antisymmetric tensor $V_{\alpha\beta}$ with any symmetric tensor $A^{\alpha\beta} = A^\alpha A^\beta$ is zero, because $V_{\alpha\alpha} = 0$ and $V_{\alpha\beta} = -V_{\beta\alpha}$. For example,

$$V_{\alpha\beta} A^\alpha A^\beta = V_{00} A^0 A^0 + V_{0i} A^0 A^i + V_{i0} A^i A^0 + V_{ik} A^i A^k = 0. \quad (2.45)$$

According to the theory of chronometric invariants, the chr.inv.-projections of an antisymmetric tensor of the 2nd rank $V^{\alpha\beta}$ are

$$\frac{V_{0\cdot}^i}{\sqrt{g_{00}}} = -\frac{V_{\cdot 0}^i}{\sqrt{g_{00}}} = \frac{1}{2} (a^i b^i - b^i a^i), \quad (2.46)$$

$$V^{ik} = \frac{1}{2} (a^i b^k - a^k b^i), \quad (2.47)$$

where the third chr.inv.-projection $\frac{V_{00}}{g_{00}}$ (1.32) is zero, because in any antisymmetric tensor all diagonal components are zeroes.

The physically observable chr.inv.-projection V^{ik} of the tensor $V^{\alpha\beta}$ onto the observer's spatial section is analogous to a vector product in a three-dimensional space, but the quantity $\frac{V_{0\cdot}^i}{\sqrt{g_{00}}}$, which is the space-time (mixed) chr.inv.-projection of the tensor $V^{\alpha\beta}$, has no equivalent among components of an ordinary three-dimensional vector product.

The square of an antisymmetric tensor of the 2nd rank, formulated with the chr.inv.-projections of its forming vectors, is

$$\begin{aligned} V_{\alpha\beta} V^{\alpha\beta} = & \frac{1}{2} (a_i a^i b_k b^k - a_i b^i a_k b^k) + \\ & + a b a_i b^i - \frac{1}{2} (a^2 b_i b^i - b^2 a_i a^i). \end{aligned} \quad (2.48)$$

The last two terms in this formula contain the quantities a (2.34) and b (2.35), which are the chr.inv.-projections of the multiplied vectors A^α and B^α onto the observer's time line, so these terms have no equivalent in the three-dimensional Euclidean space.

Asymmetry of tensor fields is defined by reference to antisymmetric tensors. In a Galilean reference frame* such antisymmetric references are the Levi-Civita tensors: for four-dimensional quantities, this is the *four-dimensional completely antisymmetric unit tensor* $e^{\alpha\beta\mu\nu}$, while for three-dimensional quantities, this is the *three-dimensional completely antisymmetric unit tensor* e^{ikm} . The components of the Levi-Civita tensors, which have all indices different, are either $+1$ or -1 depending on the number of transpositions of their indices. All of the remaining components, i.e., those having at least two coinciding indices, are zeroes. Moreover, for the signature $(+---)$ we are using, all the non-zero components have a sign opposite to their corresponding covariant components[†]. For instance, in the Minkowski space we have

$$\left. \begin{aligned} g_{\alpha\sigma} g_{\beta\rho} g_{\mu\tau} g_{\nu\gamma} e^{\sigma\rho\tau\gamma} &= g_{00} g_{11} g_{22} g_{33} e^{0123} = -e^{0123} \\ g_{i\alpha} g_{k\beta} g_{m\gamma} e^{\alpha\beta\gamma} &= g_{11} g_{22} g_{33} e^{123} = -e^{123} \end{aligned} \right\} \quad (2.49)$$

due to the signature conditions $g_{00} = 1$ and $g_{11} = g_{22} = g_{33} = -1$ we have accepted. Therefore, the components of the tensor $e^{\alpha\beta\mu\nu}$ are

$$\left. \begin{aligned} e^{0123} &= +1, & e^{1023} &= -1, & e^{1203} &= +1, & e^{1230} &= -1 \\ e_{0123} &= -1, & e_{1023} &= +1, & e_{1203} &= -1, & e_{1230} &= +1 \end{aligned} \right\} \quad (2.50)$$

and the components of the tensor e^{ikm} are

$$\left. \begin{aligned} e^{123} &= +1, & e^{213} &= -1, & e^{231} &= +1 \\ e_{123} &= -1, & e_{213} &= +1, & e_{231} &= -1 \end{aligned} \right\}. \quad (2.51)$$

Because we have an arbitrary choice for the sign of the first component, we can also assume $e^{0123} = -1$ and $e^{123} = -1$. Consequently, the

* A Galilean reference frame is the one that does not rotate or deform and falls freely in the flat space-time of Special Relativity (Minkowski space). In a Galilean frame, the time lines are linear and so are three-dimensional coordinate axes.

[†] If the space-time signature is $(-+++)$, the said is true for only the four-dimensional tensor $e^{\alpha\beta\mu\nu}$. The components of the three-dimensional tensor e^{ikm} will have the same sign as the corresponding components of e_{ikm} .

remaining components will change. In general, the tensor $e^{\alpha\beta\mu\nu}$ is related to the tensor e^{ikm} as follows $e^{0ikm} = e^{ikm}$.

Multiplying the four-dimensional antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$ by itself we obtain an ordinary tensor of the 8th rank with non-zero components, which are given in the matrix

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\rho\gamma} = - \begin{pmatrix} \delta_{\sigma}^{\alpha} & \delta_{\tau}^{\alpha} & \delta_{\rho}^{\alpha} & \delta_{\gamma}^{\alpha} \\ \delta_{\sigma}^{\beta} & \delta_{\tau}^{\beta} & \delta_{\rho}^{\beta} & \delta_{\gamma}^{\beta} \\ \delta_{\sigma}^{\mu} & \delta_{\tau}^{\mu} & \delta_{\rho}^{\mu} & \delta_{\gamma}^{\mu} \\ \delta_{\sigma}^{\nu} & \delta_{\tau}^{\nu} & \delta_{\rho}^{\nu} & \delta_{\gamma}^{\nu} \end{pmatrix}. \quad (2.52)$$

The remaining properties of the tensor $e^{\alpha\beta\mu\nu}$ are derived from the previous by means of the contraction of its indices

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\rho\nu} = - \begin{pmatrix} \delta_{\sigma}^{\alpha} & \delta_{\tau}^{\alpha} & \delta_{\rho}^{\alpha} \\ \delta_{\sigma}^{\beta} & \delta_{\tau}^{\beta} & \delta_{\rho}^{\beta} \\ \delta_{\sigma}^{\mu} & \delta_{\tau}^{\mu} & \delta_{\rho}^{\mu} \end{pmatrix}, \quad (2.53)$$

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\mu\nu} = -2 \begin{pmatrix} \delta_{\sigma}^{\alpha} & \delta_{\tau}^{\alpha} \\ \delta_{\sigma}^{\beta} & \delta_{\tau}^{\beta} \end{pmatrix} = -2 (\delta_{\sigma}^{\alpha} \delta_{\tau}^{\beta} - \delta_{\sigma}^{\beta} \delta_{\tau}^{\alpha}), \quad (2.54)$$

$$e^{\alpha\beta\mu\nu} e_{\sigma\beta\mu\nu} = -6 \delta_{\sigma}^{\alpha}, \quad e^{\alpha\beta\mu\nu} e_{\alpha\beta\mu\nu} = -6 \delta_{\alpha}^{\alpha} = -24. \quad (2.55)$$

Multiplying the three-dimensional antisymmetric unit tensor e^{ikm} by itself we obtain an ordinary tensor of the 6th rank

$$e^{ikm} e_{rst} = \begin{pmatrix} \delta_r^i & \delta_s^i & \delta_t^i \\ \delta_r^k & \delta_s^k & \delta_t^k \\ \delta_r^m & \delta_s^m & \delta_t^m \end{pmatrix}. \quad (2.56)$$

The remaining properties of the tensor e^{ikm} are

$$e^{ikm} e_{rsm} = - \begin{pmatrix} \delta_r^i & \delta_s^i \\ \delta_r^k & \delta_s^k \end{pmatrix} = \delta_s^i \delta_r^k - \delta_r^i \delta_s^k, \quad (2.57)$$

$$e^{ikm} e_{rkm} = 2 \delta_r^i, \quad e^{ikm} e_{ikm} = 2 \delta_i^i = 6. \quad (2.58)$$

The completely antisymmetric unit tensor defines for a tensor object its corresponding *pseudotensor*, marked with asterisk. For instance, any four-dimensional scalar, vector and tensors of the 2nd, 3rd, and 4th

ranks have corresponding four-dimensional pseudotensors of the following ranks

$$\left. \begin{aligned} V^{*\alpha\beta\mu\nu} &= e^{\alpha\beta\mu\nu} V, & V^{*\alpha\beta\mu} &= e^{\alpha\beta\mu\nu} V_\nu \\ V^{*\alpha\beta} &= \frac{1}{2} e^{\alpha\beta\mu\nu} V_{\mu\nu}, & V^{*\alpha} &= \frac{1}{6} e^{\alpha\beta\mu\nu} V_{\beta\mu\nu} \\ V^* &= \frac{1}{24} e^{\alpha\beta\mu\nu} V_{\alpha\beta\mu\nu} \end{aligned} \right\}. \quad (2.59)$$

Pseudotensors of the 1st rank, such as $V^{*\alpha}$, are called *pseudovectors*, and pseudotensors of zero rank, such as V^* , are called *pseudoscalars*. Any tensor and its corresponding pseudotensor are known as *dual* to each other to emphasize their common genesis.

Three-dimensional tensors have corresponding three-dimensional pseudotensors as follows

$$\left. \begin{aligned} V^{*ikm} &= e^{ikm} V, & V^{*ik} &= e^{ikm} V_m \\ V^{*i} &= \frac{1}{2} e^{ikm} V_{km}, & V^* &= \frac{1}{6} e^{ikm} V_{ikm} \end{aligned} \right\}. \quad (2.60)$$

Pseudotensors are called such because, in contrast to ordinary tensors, they do not change their sign when reflected with respect to one of the coordinate axes. For instance, when the coordinates are reflected with respect to the abscissa axis, we have $x^1 = -\tilde{x}^1$, $x^2 = \tilde{x}^2$, $x^3 = \tilde{x}^3$. In this case, the reflected component of an antisymmetric tensor V_{ik} , which is orthogonal to x^1 , is $\tilde{V}_{23} = -V_{23}$, but its corresponding component of the dual pseudovector V^{*i} is

$$\left. \begin{aligned} V^{*1} &= \frac{1}{2} e^{1km} V_{km} = \frac{1}{2} (e^{123} V_{23} + e^{132} V_{32}) = V_{23} \\ \tilde{V}^{*1} &= \frac{1}{2} \tilde{e}^{1km} \tilde{V}_{km} = \frac{1}{2} e^{k1m} \tilde{V}_{km} = \\ &= \frac{1}{2} (e^{213} \tilde{V}_{23} + e^{312} \tilde{V}_{32}) = V_{23} \end{aligned} \right\}. \quad (2.61)$$

Because a four-dimensional antisymmetric tensor of the 2nd rank and its dual pseudotensor are of the same rank, their contraction gives a pseudoscalar, so that we have

$$V_{\alpha\beta} V^{*\alpha\beta} = V_{\alpha\beta} e^{\alpha\beta\mu\nu} V_{\mu\nu} = e^{\alpha\beta\mu\nu} B_{\alpha\beta\mu\nu} = B^*. \quad (2.62)$$

The square of a pseudotensor $V^{*\alpha\beta}$ and the square of a pseudovector V^{*i} , expressed through their dual tensors, are

$$V_{*\alpha\beta}V^{*\alpha\beta} = e_{\alpha\beta\mu\nu}V^{\mu\nu}e^{\alpha\beta\rho\sigma}V_{\rho\sigma} = -24V_{\mu\nu}V^{\mu\nu}, \quad (2.63)$$

$$V_{*i}V^{*i} = e_{ikm}V^{km}e^{ipq}V_{pq} = 6V_{km}V^{km}. \quad (2.64)$$

In inhomogeneous anisotropic pseudo-Riemannian spaces, we cannot set a Galilean reference frame, so the asymmetry references of tensor fields will depend on the inhomogeneity and anisotropy of the space, which are defined by the fundamental metric tensor. In this general case, a reference antisymmetric tensor is the *four-dimensional completely antisymmetric discriminant tensor*

$$E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}, \quad E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}. \quad (2.65)$$

Here is the proof. The transformation of the completely antisymmetric unit tensor from a Galilean (non-tilde-marked) reference frame into an arbitrary (tilde-marked) reference frame is

$$\tilde{e}_{\alpha\beta\mu\nu} = \frac{\partial x^\sigma}{\partial \tilde{x}^\alpha} \frac{\partial x^\gamma}{\partial \tilde{x}^\beta} \frac{\partial x^\epsilon}{\partial \tilde{x}^\mu} \frac{\partial x^\tau}{\partial \tilde{x}^\nu} e_{\sigma\gamma\epsilon\tau} = J e_{\alpha\beta\mu\nu}, \quad (2.66)$$

where $J = \det \left\| \frac{\partial x^\alpha}{\partial \tilde{x}^\sigma} \right\|$ is called the *Jacobian of the transformation* (determinant of the Jacobi matrix)

$$J = \det \left\| \frac{\partial x^\alpha}{\partial \tilde{x}^\sigma} \right\| = \det \left\| \begin{array}{cccc} \frac{\partial x^0}{\partial \tilde{x}^0} & \frac{\partial x^0}{\partial \tilde{x}^1} & \frac{\partial x^0}{\partial \tilde{x}^2} & \frac{\partial x^0}{\partial \tilde{x}^3} \\ \frac{\partial x^1}{\partial \tilde{x}^0} & \frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} & \frac{\partial x^1}{\partial \tilde{x}^3} \\ \frac{\partial x^2}{\partial \tilde{x}^0} & \frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} & \frac{\partial x^2}{\partial \tilde{x}^3} \\ \frac{\partial x^3}{\partial \tilde{x}^0} & \frac{\partial x^3}{\partial \tilde{x}^1} & \frac{\partial x^3}{\partial \tilde{x}^2} & \frac{\partial x^3}{\partial \tilde{x}^3} \end{array} \right\|. \quad (2.67)$$

Because the fundamental metric tensor $g_{\alpha\beta}$ is transformable according to the rule

$$\tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu}, \quad (2.68)$$

its determinant in the tilde-marked reference frame is

$$\tilde{g} = \det \left\| \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu} \right\| = J^2 g. \quad (2.69)$$

Because in the Galilean (non-tilde-marked) reference frame

$$g = \det \|g_{\alpha\beta}\| = \det \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1, \quad (2.70)$$

then $J^2 = -\tilde{g}^2$. Expressing $\tilde{e}_{\alpha\beta\mu\nu}$ in an arbitrary reference frame as $E_{\alpha\beta\mu\nu}$ and writing down the metric tensor in an ordinary non-tilde-marked form, we obtain $E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}$ (2.65). In the same way, we obtain the transformation rules for the $E^{\alpha\beta\mu\nu}$ components, because for them $g = \tilde{g}\tilde{J}^2$, where $\tilde{J} = \det \left\| \frac{\partial \tilde{x}^\alpha}{\partial x^\sigma} \right\|$.

The discriminant tensor $E^{\alpha\beta\mu\nu}$ is not a physically observable quantity. A physically observable reference of the asymmetry of tensor fields is the *three-dimensional discriminant chr.inv.-tensor*

$$\varepsilon^{\alpha\beta\gamma} = h_\mu^\alpha h_\nu^\beta h_\rho^\gamma b_\sigma E^{\sigma\mu\nu\rho} = b_\sigma E^{\sigma\alpha\beta\gamma}, \quad (2.71)$$

$$\varepsilon_{\alpha\beta\gamma} = h_\alpha^\mu h_\beta^\nu h_\gamma^\rho b^\sigma E_{\sigma\mu\nu\rho} = b^\sigma E_{\sigma\alpha\beta\gamma}, \quad (2.72)$$

which in the accompanying reference frame ($b^i = 0$), taking into account that $\sqrt{-g} = \sqrt{h}\sqrt{g_{00}}$, takes the form

$$\varepsilon^{ikm} = b_0 E^{0ikm} = \sqrt{g_{00}} E^{0ikm} = \frac{e^{ikm}}{\sqrt{h}}, \quad (2.73)$$

$$\varepsilon_{ikm} = b^0 E_{0ikm} = \frac{E_{0ikm}}{\sqrt{g_{00}}} = e_{ikm} \sqrt{h}. \quad (2.74)$$

Using this tensor, we can transform chr.inv.-tensors into chr.inv.-pseudotensors. For instance, based on the antisymmetric chr.inv.-tensor of the angular velocity with which the space rotates, A_{ik} (1.36), we obtain the chr.inv.-pseudovector of this rotation $\Omega^{*i} = \frac{1}{2} \varepsilon^{ikm} A_{km}$.

2.4 Differential and directional derivative

In geometry, the *differential* of a function is its variation between two infinitely close points, the coordinates of which are x^α and $x^\alpha + dx^\alpha$. Respectively, the *absolute differential* in an n -dimensional space is the variation of an n -dimensional quantity between two infinitely close points in this space. For continuous functions, which we commonly deal with in practice, their variations between infinitely close points are infinite-

simil. But in order to define an infinitesimal variation of a tensor quantity, we cannot use simply the “difference” between its numerical values in the points x^α and $x^\alpha + dx^\alpha$, since tensor algebra does not define the ratio between the numerical values of a tensor in different points of a space. This ratio can be defined only using the rules transforming tensors from one reference frame to another. Therefore, differential operators and the results of their application to tensors must be tensors.

Thus, the absolute differential of a tensor quantity is a tensor of the same rank as the original tensor itself. For a scalar φ it is the scalar

$$D\varphi = \frac{\partial\varphi}{\partial x^\alpha} dx^\alpha, \quad (2.75)$$

which in the accompanying reference frame ($b^i = 0$) is

$$D\varphi = \frac{*\partial\varphi}{\partial t} d\tau + \frac{*\partial\varphi}{\partial x^i} dx^i. \quad (2.76)$$

It is easy to see that, apart from the three-dimensional observable differential, there is an additional term that takes into account the dependence of the absolute displacement $D\varphi$ on the flow of the physically observable time $d\tau$.

The absolute differential of a contravariant vector A^α , formulated with the absolute derivation operator ∇ (nabla), is

$$\begin{aligned} DA^\alpha &= \nabla_\sigma A^\alpha dx^\sigma = \frac{\partial A^\alpha}{\partial x^\sigma} dx^\sigma + \Gamma_{\mu\sigma}^\alpha A^\mu dx^\sigma = \\ &= dA^\alpha + \Gamma_{\mu\sigma}^\alpha A^\mu dx^\sigma, \end{aligned} \quad (2.77)$$

where $\nabla_\sigma A^\alpha$ is the absolute derivative of A^α with respect to x^σ , and d stands for the ordinary differential

$$\nabla_\sigma A^\alpha = \frac{\partial A^\alpha}{\partial x^\sigma} + \Gamma_{\mu\sigma}^\alpha A^\mu, \quad (2.78)$$

$$d = \frac{\partial}{\partial x^\alpha} dx^\alpha. \quad (2.79)$$

Formulating the absolute differential with physical observables is equivalent to projecting its general covariant form onto the time line and the spatial section in the accompanying reference frame

$$T = b_\alpha DA^\alpha = \frac{g_{0\alpha} DA^\alpha}{\sqrt{g_{00}}}, \quad B^i = h^i_\alpha DA^\alpha. \quad (2.80)$$

Denoting the chr.inv.-projections of the vector A^α as

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}, \quad q^i = A^i, \quad (2.81)$$

we obtain its remaining components

$$A_0 = \varphi \left(1 - \frac{w}{c^2}\right), \quad A^0 = \frac{\varphi + \frac{1}{c} v_i q^i}{1 - \frac{w}{c^2}}, \quad A_i = -q_i - \frac{\varphi}{c} v_i. \quad (2.82)$$

Because the ordinary differential in the chr.inv.-form is

$$d = \frac{* \partial}{\partial t} d\tau + \frac{* \partial}{\partial x^i} dx^i, \quad (2.83)$$

after substituting it and the Christoffel symbols, taken in the accompanying reference frame (1.41–1.46), into the formulae for the chr.inv.-projections T and B^i (2.80) of an arbitrary vector A^α , we obtain

$$T = b_\alpha DA^\alpha = d\varphi + \frac{1}{c} \left(-F_i q^i d\tau + D_{ik} q^i dx^k\right), \quad (2.84)$$

$$B^i = h^i_\sigma DA^\sigma = dq^i + \left(\frac{\varphi}{c} dx^k + q^k d\tau\right) \left(D_k^i + A_{k.}^i\right) - \frac{\varphi}{c} F^i d\tau + \Delta_{mk}^i q^m dx^k. \quad (2.85)$$

To create the chr.inv.-equations of motion, we need the chr.inv.-projections of the absolute derivative of a vector to the direction, tangential to the trajectory. From a geometric point of view, the *directional derivative* of a function is its change with respect to an elementary displacement along the given direction. The *absolute directional derivative* in an n -dimensional space is the change of an n -dimensional quantity with respect to an elementary n -dimensional interval along the given direction. For instance, the absolute derivative of a scalar function φ along a curve $x^\alpha = x^\alpha(\rho)$, where ρ is a non-zero monotone parameter along the curve, shows the “rate” of the change of this function

$$\frac{D\varphi}{d\rho} = \frac{d\varphi}{d\rho}. \quad (2.86)$$

In the accompanying reference frame it is

$$\frac{D\varphi}{d\rho} = \frac{* \partial \varphi}{\partial t} \frac{d\tau}{d\rho} + \frac{* \partial \varphi}{\partial x^i} \frac{dx^i}{d\rho}. \quad (2.87)$$

The absolute directional derivative of an arbitrary vector A^α along a curve $x^\alpha = x^\alpha(\rho)$ is

$$\frac{DA^\alpha}{d\rho} = \nabla_\sigma A^\alpha \frac{dx^\sigma}{d\rho} = \frac{dA^\alpha}{d\rho} + \Gamma_{\mu\sigma}^\alpha A^\mu \frac{dx^\sigma}{d\rho}, \quad (2.88)$$

and its chr.inv.-projections are

$$b_\alpha \frac{DA^\alpha}{d\rho} = \frac{d\varphi}{d\rho} + \frac{1}{c} \left(-F_i q^i \frac{d\tau}{d\rho} + D_{ik} q^i \frac{dx^k}{d\rho} \right), \quad (2.89)$$

$$h_\sigma^i \frac{DA^\sigma}{d\rho} = \frac{dq^i}{d\rho} + \left(\frac{\varphi}{c} \frac{dx^k}{d\rho} + q^k \frac{d\tau}{d\rho} \right) (D_k^i + A_{k\cdot}^i) - \frac{\varphi}{c} F^i \frac{d\tau}{d\rho} + \Delta_{mk}^i q^m \frac{dx^k}{d\rho}. \quad (2.90)$$

Actually, the above chr.inv.-projections are the “generic” chr.inv.-equations of motion of a particle in the space. Once we define a particular vector characterizing the motion of a particle, we calculate its chr.inv.-projections and substitute them into the above equations (2.90, 2.91), we immediately obtain the chr.inv.-equations of the motion of the particle.

2.5 Divergence and curl

The *divergence* of a tensor field is its “change” along a coordinate axis. Respectively, the *absolute divergence* of an n -dimensional tensor field is its divergence in an n -dimensional space. The divergence of a tensor field is the result of the contraction of the field tensor with the absolute derivation operator ∇ . The divergence of a vector field is a scalar

$$\nabla_\sigma A^\sigma = \frac{\partial A^\sigma}{\partial x^\sigma} + \Gamma_{\sigma\mu}^\sigma A^\mu, \quad (2.91)$$

and the divergence of a 2nd rank tensor field is a vector

$$\nabla_\sigma F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma_{\sigma\mu}^\sigma F^{\alpha\mu} + \Gamma_{\sigma\mu}^\alpha F^{\sigma\mu}, \quad (2.92)$$

where, as it can be proved, $\Gamma_{\sigma\mu}^\sigma$ is

$$\Gamma_{\sigma\mu}^\sigma = \frac{\partial \ln \sqrt{-g}}{\partial x^\mu}. \quad (2.93)$$

To prove (2.93), we will use the definition of the Christoffel symbols. Write the definition of $\Gamma_{\sigma\mu}^{\sigma}$ in detail

$$\Gamma_{\sigma\mu}^{\sigma} = g^{\sigma\rho} \Gamma_{\mu\sigma,\rho} = \frac{1}{2} g^{\sigma\rho} \left(\frac{\partial g_{\mu\rho}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\rho}} \right). \quad (2.94)$$

Because σ and ρ are free indices here, they can change their sites. As a result, after the contraction with the tensor $g^{\rho\sigma}$, the first and last terms cancel each other, so $\Gamma_{\sigma\mu}^{\sigma}$ takes the form

$$\Gamma_{\sigma\mu}^{\sigma} = \frac{1}{2} g^{\rho\sigma} \frac{\partial g_{\rho\sigma}}{\partial x^{\mu}}. \quad (2.95)$$

The quantities $g^{\rho\sigma}$ are the components of a tensor reciprocal to the tensor $g_{\rho\sigma}$. Therefore, each component of the matrix $g^{\rho\sigma}$ is

$$g^{\rho\sigma} = \frac{a^{\rho\sigma}}{g}, \quad g = \det \|g_{\rho\sigma}\|, \quad (2.96)$$

where $a^{\rho\sigma}$ is the algebraic co-factor of the matrix element with indices $\rho\sigma$, equal to $(-1)^{\rho+\sigma}$, multiplied by the determinant of the matrix obtained by crossing the row and the column with the numbers σ and ρ out of the matrix $g_{\rho\sigma}$. As a result, we have $a^{\rho\sigma} = g g^{\rho\sigma}$. Since the determinant of the fundamental metric tensor $g = \det \|g_{\rho\sigma}\|$ by definition is

$$g = \sum_{\alpha_0 \dots \alpha_3} (-1)^{N(\alpha_0 \dots \alpha_3)} g_{0(\alpha_0)} g_{1(\alpha_1)} g_{2(\alpha_2)} g_{3(\alpha_3)}, \quad (2.97)$$

then the quantity dg will be $dg = a^{\rho\sigma} dg_{\rho\sigma} = g g^{\rho\sigma} dg_{\rho\sigma}$, or

$$\frac{dg}{g} = g^{\rho\sigma} dg_{\rho\sigma}. \quad (2.98)$$

Integrating the left hand side, we get $\ln(-g)$, because the g is negative while logarithms are defined for only positive functions. Then, we have $d \ln(-g) = \frac{dg}{g}$. Since $(-g)^{1/2} = \frac{1}{2} \ln(-g)$, we obtain

$$d \ln \sqrt{-g} = \frac{1}{2} g^{\rho\sigma} dg_{\rho\sigma}, \quad (2.99)$$

so $\Gamma_{\sigma\mu}^{\sigma}$ (2.95) takes the form

$$\Gamma_{\sigma\mu}^{\sigma} = \frac{1}{2} g^{\rho\sigma} \frac{\partial g_{\rho\sigma}}{\partial x^{\mu}} = \frac{\partial \ln \sqrt{-g}}{\partial x^{\mu}}, \quad (2.100)$$

which has been proved (2.93).

Now, we are going to deduce the chr.inv.-projections of the divergence of a vector field (2.91) and of a tensor field of the 2nd rank (2.92). The divergence of a vector field A^α is a scalar, consequently the divergence $\nabla_\sigma A^\sigma$ cannot be projected onto the time line and the spatial section, but, this is enough to express it through the chr.inv.-projections of the A^α and through the observable properties of the reference space. Besides, the ordinary derivation operators must be replaced with the chr.inv.-derivation operators.

Assuming the notations φ and q^i for chr.inv.-projections of the vector A^α (2.81), we express the remaining components of the vector A^α through them (2.82). Then, substituting the ordinary derivation operators in the form, expressed through the chr.inv.-derivation operators

$$\frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t} = \frac{* \partial}{\partial t}, \quad \sqrt{g_{00}} = 1 - \frac{w}{c^2}, \quad (2.101)$$

$$\frac{* \partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2} v_i \frac{* \partial}{\partial t} \quad (2.102)$$

into (2.91), and taking into account that $\sqrt{-g} = \sqrt{h} \sqrt{g_{00}}$, we obtain

$$\nabla_\sigma A^\sigma = \frac{1}{c} \left(\frac{* \partial \varphi}{\partial t} + \varphi D \right) + \frac{* \partial q^i}{\partial x^i} + q^i \frac{* \partial \ln \sqrt{h}}{\partial x^i} - \frac{1}{c^2} F_i q^i. \quad (2.103)$$

In the third term, the quantity

$$\frac{* \partial \ln \sqrt{h}}{\partial x^i} = \Delta_{ji}^j \quad (2.104)$$

stands for the chr.inv.-Christoffel symbols Δ_{ji}^k (1.47), contracted by two indices. Hence, similar to the definition of the absolute divergence of a vector field (2.91), the quantity

$$\frac{* \partial q^i}{\partial x^i} + q^i \frac{* \partial \ln \sqrt{h}}{\partial x^i} = \frac{* \partial q^i}{\partial x^i} + q^i \Delta_{ji}^j = {}^* \nabla_i q^i \quad (2.105)$$

is the *chr.inv.-divergence* of a three-dimensional vector field q^i .

Consequently, we call the *physical chr.inv.-divergence* of the vector field q^i the following chr.inv.-quantity

$${}^* \widetilde{\nabla}_i q^i = {}^* \nabla_i q^i - \frac{1}{c^2} F_i q^i, \quad (2.106)$$

in which the 2nd term takes into account the fact that the flow of time is different at the opposite walls of an elementary volume of the space [9]. As a matter of fact that, when calculating the divergence we consider an elementary volume of the space, so we calculate the difference between the amounts of a “substance” that flows in and out of the volume over an elementary time interval. But the presence of the gravitational inertial force F^i (1.38) results in a different flow of time at different points in the space. Therefore, when we measure time intervals on the clocks installed at the opposite walls of the volume, the beginnings of the time intervals will not coincide, thereby making the measured time intervals invalid for comparison. The clock synchronization at the opposite walls of the volume will give the true picture — the measured time durations will be different.

The final equation for the divergence $\nabla_\sigma A^\sigma$ is

$$\nabla_\sigma A^\sigma = \frac{1}{c} \left(\frac{{}^* \partial \varphi}{\partial t} + \varphi D \right) + {}^* \widetilde{\nabla}_i q^i. \quad (2.107)$$

The second term in this formula is a physically observable analogy to the ordinary divergence in the observer’s three-dimensional space.

The first term (in brackets) has no equivalent. It is the sum of the two functions: $\frac{{}^* \partial \varphi}{\partial t}$ is the variation in time of the time projection φ of the vector A^α , while φD is the variation in time of the volume of the three-dimensional vector field q^i . The latter is because the trace of the chr.inv.-tensor of the space deformation rate $D = h^{ik} D_{ik} = D_n^n$ is the rate of the relative expansion of an elementary volume of the space.

Applying $\nabla_\sigma A^\sigma = 0$ to the four-dimensional vector potential A^α of an electromagnetic field gives the Lorenz condition for the field. As a result, the Lorenz condition in the chr.inv.-form is

$${}^* \widetilde{\nabla}_i q^i = - \frac{1}{c} \left(\frac{{}^* \partial \varphi}{\partial t} + \varphi D \right). \quad (2.108)$$

Now we are going to deduce chr.inv.-projections of the divergence of an arbitrary antisymmetric tensor $F^{\alpha\beta} = -F^{\beta\alpha}$ (later we will need them to obtain Maxwell’s equations in the chr.inv.-form)

$$\begin{aligned} \nabla_\sigma F^{\sigma\alpha} &= \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma_{\sigma\mu}^\sigma F^{\alpha\mu} + \Gamma_{\sigma\mu}^\alpha F^{\sigma\mu} = \\ &= \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \frac{\partial \ln \sqrt{-g}}{\partial x^\mu} F^{\alpha\mu}, \end{aligned} \quad (2.109)$$

where the third term $\Gamma_{\sigma\mu}^{\alpha} F^{\sigma\mu}$ is zero, because the contraction of the Christoffel symbols $\Gamma_{\sigma\mu}^{\alpha}$ (which are symmetric by their lower indices) with the antisymmetric tensor $F^{\sigma\mu}$ is zero as in the case of any symmetric and antisymmetric tensors.

The divergence $\nabla_{\sigma} F^{\sigma\alpha}$ is a four-dimensional vector, so its chr.inv.-projections are calculated as for a vector, i.e.

$$T = b_{\alpha} \nabla_{\sigma} F^{\sigma\alpha}, \quad B^i = h_{\alpha}^i \nabla_{\sigma} F^{\sigma\alpha} = \nabla_{\sigma} F^{\sigma i}. \quad (2.110)$$

We denote chr.inv.-projections of the tensor $F^{\alpha\beta}$ as follows

$$E^i = \frac{F_{0\cdot}^i}{\sqrt{g_{00}}}, \quad H^{ik} = F^{ik}, \quad (2.111)$$

then the remaining non-zero components of the tensor are

$$F_{0\cdot}^0 = \frac{1}{c} v_k E^k, \quad (2.112)$$

$$F_{k\cdot}^0 = \frac{1}{\sqrt{g_{00}}} \left(E_k - \frac{1}{c} v_n H_{k\cdot}^n - \frac{1}{c^2} v_k v_n E^n \right), \quad (2.113)$$

$$F^{0i} = \frac{E^i - \frac{1}{c} v_k H^{ik}}{\sqrt{g_{00}}}, \quad F_{0i} = -\sqrt{g_{00}} E_i, \quad (2.114)$$

$$F_{i\cdot}^k = -H_{i\cdot}^k - \frac{1}{c} v_i E^k, \quad F_{ik} = H_{ik} + \frac{1}{c} (v_i E_k - v_k E_i), \quad (2.115)$$

and the square of this tensor $F^{\alpha\beta}$ is

$$F_{\alpha\beta} F^{\alpha\beta} = H_{ik} H^{ik} - 2E_i E^i. \quad (2.116)$$

Substituting the components into (2.110) and replacing the ordinary derivation operators with the chr.inv.-derivation operators, after some algebra we obtain

$$T = \frac{\nabla_{\sigma} F_{0\cdot}^{\sigma}}{\sqrt{g_{00}}} = \frac{* \partial E^i}{\partial x^i} + E^i \frac{* \partial \ln \sqrt{h}}{\partial x^i} - \frac{1}{c} H^{ik} A_{ik}, \quad (2.117)$$

$$B^i = \nabla_{\sigma} F^{\sigma i} = \frac{* \partial H^{ik}}{\partial x^k} + H^{ik} \frac{* \partial \ln \sqrt{h}}{\partial x^k} - \frac{1}{c^2} F_k H^{ik} - \frac{1}{c} \left(\frac{* \partial E^i}{\partial t} + DE^i \right), \quad (2.118)$$

where A_{ik} is the antisymmetric tensor of non-holonomy of the space. Taking into account that

$$\frac{* \partial E^i}{\partial x^i} + E^i \frac{* \partial \ln \sqrt{h}}{\partial x^i} = * \nabla_i E^i \quad (2.119)$$

is the chr.inv.-divergence of the vector E^i , and also that

$$* \nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} = * \widetilde{\nabla}_k H^{ik} \quad (2.120)$$

is the physical chr.inv.-divergence of the tensor H^{ik} , we arrive at the final equations for the chr.inv.-projections of the divergence of an arbitrary antisymmetric tensor $F^{\alpha\beta}$

$$T = * \nabla_i E^i - \frac{1}{c} H^{ik} A_{ik}, \quad (2.121)$$

$$B^i = * \widetilde{\nabla}_k H^{ik} - \frac{1}{c} \left(\frac{* \partial E^i}{\partial t} + D E^i \right). \quad (2.122)$$

So forth, we calculate the chr.inv.-projections of the divergence of the pseudotensor $F^{*\alpha\beta}$

$$F^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} F_{\mu\nu}, \quad F_{*\alpha\beta} = \frac{1}{2} E_{\alpha\beta\mu\nu} F^{\mu\nu}, \quad (2.123)$$

which is dual to the given antisymmetric tensor $F^{\alpha\beta}$.

We denote its chr.inv.-projections as follows

$$H^{*i} = \frac{F_{0 \cdot}^{*i}}{\sqrt{g_{00}}}, \quad E^{*ik} = F^{*ik}, \quad (2.124)$$

so there are the obvious relations $H^{*i} \sim H^{ik}$ and $E^{*ik} \sim E^i$ between the above chr.inv.-quantities and chr.inv.-projections of the antisymmetric tensor $F^{\alpha\beta}$ (2.111), because of the duality of $F^{\alpha\beta}$ and $F^{*\alpha\beta}$.

Therefore, given that

$$\frac{F_{0 \cdot}^{*i}}{\sqrt{g_{00}}} = \frac{1}{2} \varepsilon^{ipq} H_{pq}, \quad F^{*ik} = -\varepsilon^{ikp} E_p, \quad (2.125)$$

the remaining components of the pseudotensor $F^{*\alpha\beta}$, formulated with the chr.inv.-projections of its dual tensor $F^{\alpha\beta}$ (2.111) are

$$F_{0 \cdot}^{*0} = \frac{1}{2c} v_k \varepsilon^{kpq} \left[H_{pq} + \frac{1}{c} (v_p E_q - v_q E_p) \right], \quad (2.126)$$

$$F_i^{*0} = \frac{1}{2\sqrt{g_{00}}} \left[\varepsilon_i^{pq} H_{pq} + \frac{1}{c} \varepsilon_i^{pq} (v_p E_q - v_q E_p) - \frac{1}{c^2} \varepsilon^{kpq} v_i v_k H_{pq} - \frac{1}{c^3} \varepsilon^{kpq} v_i v_k (v_p E_q - v_q E_p) \right], \quad (2.127)$$

$$F^{*0i} = \frac{1}{2\sqrt{g_{00}}} \varepsilon^{ipq} \left[H_{pq} + \frac{1}{c} (v_p E_q - v_q E_p) \right], \quad (2.128)$$

$$F_{*0i} = \frac{1}{2} \sqrt{g_{00}} \varepsilon_{ipq} H^{pq}, \quad (2.129)$$

$$F_i^{*k} = \varepsilon_i^{kp} E_p - \frac{1}{2c} v_i \varepsilon^{kpq} H_{pq} - \frac{1}{c^2} v_i v_m \varepsilon^{mkp} E_p, \quad (2.130)$$

$$F_{*ik} = \varepsilon_{ikp} \left(E^p - \frac{1}{c} v_q H^{pq} \right), \quad (2.131)$$

while its square is

$$F_{*\alpha\beta} F^{*\alpha\beta} = \varepsilon^{ipq} (E_p H_{iq} - E_i H_{pq}), \quad (2.132)$$

where ε^{ipq} is the three-dimensional discriminant chr.inv.-tensor (2.73, 2.74). Then the chr.inv.-projections of the divergence of the pseudotensor $F^{*\alpha\beta}$ are written as

$$\frac{\nabla_\sigma F_{0\cdot}^{*\sigma}}{\sqrt{g_{00}}} = \frac{* \partial H^{*i}}{\partial x^i} + H^{*i} \frac{* \partial \ln \sqrt{h}}{\partial x^i} - \frac{1}{c} E^{*ik} A_{ik}, \quad (2.133)$$

$$\begin{aligned} \nabla_\sigma F^{*\sigma i} &= \frac{* \partial E^{*ik}}{\partial x^i} + E^{*ik} \frac{* \partial \ln \sqrt{h}}{\partial x^k} - \\ &- \frac{1}{c^2} F_k E^{*ik} - \frac{1}{c} \left(\frac{* \partial H^{*i}}{\partial t} + D H^{*i} \right), \end{aligned} \quad (2.134)$$

or, using the respective formulae for the chr.inv.-divergence $* \nabla_i H^{*i}$ and also the physical chr.inv.-divergence $* \widetilde{\nabla}_k E^{*ik}$, as well as (2.119, 2.120), we obtain

$$\frac{\nabla_\sigma F_{0\cdot}^{*\sigma}}{\sqrt{g_{00}}} = * \nabla_i H^{*i} - \frac{1}{c} E^{*ik} A_{ik}, \quad (2.135)$$

$$\nabla_\sigma F^{*\sigma i} = * \widetilde{\nabla}_k E^{*ik} - \frac{1}{c} \left(\frac{* \partial H^{*i}}{\partial t} + D H^{*i} \right). \quad (2.136)$$

Apart from the divergence of a vector, antisymmetric tensor and pseudotensor of the 2nd rank, we need to deduce the chr.inv.-projections of the divergence of a symmetric tensor of the 2nd rank (we will need them to obtain the conservation law in the chr.inv.-form). We will import them from Zelmanov [9]. Like Zelmanov did in his theory, we denote chr.inv.-projections of a symmetric tensor $T^{\alpha\beta}$ as follows

$$\frac{T_{00}}{g_{00}} = \rho, \quad \frac{T_0^i}{\sqrt{g_{00}}} = K^i, \quad T^{ik} = N^{ik}, \quad (2.137)$$

then, according to [9], we have

$$\frac{\nabla_\sigma T_0^\sigma}{\sqrt{g_{00}}} = \frac{* \partial \rho}{\partial t} + \rho D + D_{ik} N^{ik} + c * \nabla_i K^i - \frac{2}{c} F_i K^i, \quad (2.138)$$

$$\begin{aligned} \nabla_\sigma T^{\sigma i} = c \frac{* \partial K^i}{\partial t} + c D K^i + 2c (D_k^i + A_k^i) K^k + \\ + c^2 * \nabla_k N^{ik} - F_k N^{ik} - \rho F^i. \end{aligned} \quad (2.139)$$

So forth, consider the *curl* of a tensor field — the difference between the covariant derivatives of the tensor. From a geometric point of view, it is the vortex (rotation) of the field. The *absolute curl* is the curl of an n -dimensional tensor field in an n -dimensional space. The curl of an arbitrary four-dimensional vector field A^α is a covariant antisymmetric tensor of the 2nd rank, which is defined as follows*

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}, \quad (2.140)$$

where $\nabla_\mu A_\nu$ is the absolute derivative of A_α with respect to the coordinate x^μ

$$\nabla_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma_{\nu\mu}^\sigma A_\sigma. \quad (2.141)$$

The curl, contracted with the four-dimensional absolutely antisymmetric discriminant tensor $E^{\alpha\beta\mu\nu}$ (2.65), is the pseudotensor

$$F^{*\alpha\beta} = E^{\alpha\beta\mu\nu} (\nabla_\mu A_\nu - \nabla_\nu A_\mu) = E^{\alpha\beta\mu\nu} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right). \quad (2.142)$$

*See §98 in the well-known book authored by Peter Raschewski [18]. Actually, the curl of a tensor field is not the tensor (2.140), but its dual pseudotensor (2.142), because the invariance with respect to reflection is necessary for any rotations.

In electrodynamics, $F_{\mu\nu}$ (2.140) is the electromagnetic field tensor (known also as the Maxwell tensor). It is the curl of the four-dimensional electromagnetic field potential A^α . Therefore, when considering electrodynamics in terms of chronometric invariants, we will need formulae for the chr.inv.-projections of the four-dimensional curl $F_{\mu\nu}$ and its dual pseudotensor $F^{*\alpha\beta}$, expressed through the chr.inv.-projections of the four-dimensional vector potential A^α (2.81) that formed them.

Let us calculate the components of the curl $F_{\mu\nu}$, taking into account that $F_{00} = F^{00} = 0$ just like for any other antisymmetric tensor. As a result, after some algebra, we obtain

$$F_{0i} = \left(1 - \frac{w}{c^2}\right) \left(\frac{\varphi}{c^2} F_i - \frac{*\partial\varphi}{\partial x^i} - \frac{1}{c} \frac{*\partial q_i}{\partial t} \right), \quad (2.143)$$

$$F_{ik} = \frac{*\partial q_i}{\partial x^k} - \frac{*\partial q_k}{\partial x^i} + \frac{\varphi}{c} \left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i} \right) + \frac{1}{c} \left(v_i \frac{*\partial\varphi}{\partial x^k} - v_k \frac{*\partial\varphi}{\partial x^i} \right) + \frac{1}{c^2} \left(v_i \frac{*\partial q_k}{\partial t} - v_k \frac{*\partial q_i}{\partial t} \right), \quad (2.144)$$

$$F_{0\cdot}^{\cdot 0} = -\frac{\varphi}{c^3} v_k F^k + \frac{1}{c} v^k \left(\frac{*\partial\varphi}{\partial x^k} + \frac{1}{c} \frac{*\partial q_k}{\partial t} \right), \quad (2.145)$$

$$F_{k\cdot}^{\cdot 0} = -\frac{1}{\sqrt{g_{00}}} \left[\frac{\varphi}{c^2} F_k - \frac{*\partial\varphi}{\partial x^k} - \frac{1}{c} \frac{*\partial q_k}{\partial t} + \frac{2\varphi}{c^2} v^m A_{mk} + \frac{1}{c^2} v_k v^m \left(\frac{*\partial\varphi}{\partial x^m} + \frac{1}{c} \frac{*\partial q_m}{\partial t} \right) - \frac{1}{c} v^m \left(\frac{*\partial q_m}{\partial x^k} - \frac{*\partial q_k}{\partial x^m} \right) - \frac{\varphi}{c^4} v_k v_m F^m \right], \quad (2.146)$$

$$F_{k\cdot}^{\cdot i} = h^{im} \left(\frac{*\partial q_m}{\partial x^k} - \frac{*\partial q_k}{\partial x^m} \right) - \frac{1}{c} h^{im} v_k \frac{*\partial\varphi}{\partial x^m} - \frac{1}{c^2} h^{im} v_k \frac{*\partial q_m}{\partial t} + \frac{\varphi}{c^3} v_k F^i + \frac{2\varphi}{c} A_{k\cdot}^{\cdot i}, \quad (2.147)$$

$$F^{0k} = \frac{1}{\sqrt{g_{00}}} \left[h^{km} \left(\frac{*\partial\varphi}{\partial x^m} + \frac{1}{c} \frac{*\partial q_m}{\partial t} \right) - \frac{\varphi}{c^2} F^k + \frac{1}{c} v^n h^{mk} \left(\frac{*\partial q_n}{\partial x^m} - \frac{*\partial q_m}{\partial x^n} \right) - \frac{2\varphi}{c^2} v_m A^{mk} \right], \quad (2.148)$$

$$\frac{F_{0.}^i}{\sqrt{g_{00}}} = \frac{g^{i\alpha} F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik} \left(\frac{* \partial \varphi}{\partial x^k} + \frac{1}{c} \frac{* \partial q_k}{\partial t} \right) - \frac{\varphi}{c^2} F^i, \quad (2.149)$$

$$F^{ik} = g^{i\alpha} g^{k\beta} F_{\alpha\beta} = h^{im} h^{kn} \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) - \frac{2\varphi}{c} A^{ik}, \quad (2.150)$$

where (2.149, 2.150) are the chr.inv.-projections of the curl $F_{\mu\nu}$. Respectively, the chr.inv.-projections of its dual pseudotensor $F^{*\alpha\beta}$ are

$$\frac{F_{0.}^{*i}}{\sqrt{g_{00}}} = \frac{g_{0\alpha} F^{*\alpha i}}{\sqrt{g_{00}}} = \varepsilon^{ikm} \left[\frac{1}{2} \left(\frac{* \partial q_k}{\partial x^m} - \frac{* \partial q_m}{\partial x^k} \right) - \frac{\varphi}{c} A_{km} \right], \quad (2.151)$$

$$F^{*ik} = \varepsilon^{ikm} \left(\frac{\varphi}{c^2} F_m - \frac{* \partial \varphi}{\partial x^m} - \frac{1}{c} \frac{* \partial q_m}{\partial t} \right), \quad (2.152)$$

where $F_{0.}^{*i} = g_{0\alpha} F^{*\alpha i} = g_{0\alpha} E^{\alpha i \mu \nu} F_{\mu \nu}$ can be calculated using the mentioned components of the curl $F_{\mu \nu}$ (2.143–2.148).

2.6 Laplace's operator and d'Alembert's operator

Laplace's operator, known as the Laplacian, is the three-dimensional derivation operator having the following form

$$\Delta = \nabla \nabla = \nabla^2 = -g^{ik} \nabla_i \nabla_k. \quad (2.153)$$

Its four-dimensional generalization in a pseudo-Riemannian space is *d'Alembert's general covariant operator*

$$\square = g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \quad (2.154)$$

In the Minkowski space, the operators take the form

$$\Delta = \frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} + \frac{\partial^2}{\partial x^3 \partial x^3}, \quad (2.155)$$

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^1 \partial x^1} - \frac{\partial^2}{\partial x^2 \partial x^2} - \frac{\partial^2}{\partial x^3 \partial x^3} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta. \quad (2.156)$$

Our task is to apply d'Alembert's operator to scalar and vector fields in a pseudo-Riemannian space, and also to present the results in the chr. inv.-form. At first, we apply d'Alembert's operator to a four-dimensional scalar field φ , because in this case the calculations are much simpler (the

absolute derivative of a scalar field $\nabla_\alpha \varphi$ does not contain the Christoffel symbols, so it becomes the ordinary derivative)

$$\square \varphi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi = g^{\alpha\beta} \frac{\partial \varphi}{\partial x^\alpha} \left(\frac{\partial \varphi}{\partial x^\beta} \right) = g^{\alpha\beta} \frac{\partial^2 \varphi}{\partial x^\alpha \partial x^\beta}. \quad (2.157)$$

At first, we formulate the components of the fundamental metric tensor in terms of chronometric invariants. For the g^{ik} component, according to (1.18), we have $g^{ik} = -h^{ik}$. The g^{0i} components are obtained from the linear velocity of the space rotation $v^i = -c g^{0i} \sqrt{g_{00}}$

$$g^{0i} = -\frac{1}{c \sqrt{g_{00}}} v^i. \quad (2.158)$$

The g^{00} component can be obtained, based on the main property of the fundamental metric tensor, which is $g_{\alpha\sigma} g^{\beta\sigma} = g_\alpha^\beta$. Setting $\alpha = \beta = 0$ in the mentioned property, we obtain

$$g_{0\sigma} g^{0\sigma} = g_{00} g^{00} + g_{0i} g^{0i} = \delta_0^0 = 1, \quad (2.159)$$

then, taking into account that

$$g_{00} = \left(1 - \frac{w}{c^2}\right)^2, \quad g_{0i} = -\frac{1}{c} v_i \left(1 - \frac{w}{c^2}\right), \quad (2.160)$$

we obtain the formula

$$g^{00} = \frac{1}{\left(1 - \frac{w}{c^2}\right)^2} \left(1 - \frac{1}{c^2} v_i v^i\right). \quad (2.161)$$

Substituting the obtained formulae into $\square \varphi$ (2.157) and replacing the ordinary derivation operators with the chr.inv.-derivation operators, we obtain the d'Alembertian of the scalar field in the form, expressed through only chronometrically invariant quantities

$$\square \varphi = \frac{1}{c^2} \frac{* \partial^2 \varphi}{\partial t^2} - h^{ik} \frac{* \partial^2 \varphi}{\partial x^i \partial x^k} = * \square \varphi, \quad (2.162)$$

where, in contrast to the ordinary operators, $* \square$ is the chr.inv.-d'Alembert operator, and $* \Delta$ is the chr.inv.-Laplace operator

$$* \square = \frac{1}{c^2} \frac{* \partial^2}{\partial t^2} - h^{ik} \frac{* \partial^2}{\partial x^i \partial x^k}, \quad (2.163)$$

$$*\Delta = -g^{ik} * \nabla_i * \nabla_k = h^{ik} \frac{* \partial^2}{\partial x^i \partial x^k}. \quad (2.164)$$

So forth, we are going to apply d'Alembert's operator to an arbitrary four-dimensional vector field A^α

$$\square A^\alpha = g^{\mu\nu} \nabla_\mu \nabla_\nu A^\alpha. \quad (2.165)$$

Since $\square A^\alpha$ is a four-dimensional vector, the chr.inv.-projections of this quantity are calculated as for any vector

$$T = b_\sigma \square A^\sigma = b_\sigma g^{\mu\nu} \nabla_\mu \nabla_\nu A^\sigma, \quad (2.166)$$

$$B^i = h^i_\sigma \square A^\sigma = h^i_\sigma g^{\mu\nu} \nabla_\mu \nabla_\nu A^\sigma. \quad (2.167)$$

In general, to obtain the d'Alembertian in the chr.inv.-form for a vector field in a pseudo-Riemannian space is not a trivial task, because the Christoffel symbols are not zeroes in a pseudo-Riemannian space, so the auxiliary formulae for the chr.inv.-projections of the second derivatives take dozens of pages*.

After some difficult algebra, we obtain the required formulae for the chr.inv.-projections of the d'Alembertian of the vector field A^α in the four-dimensional pseudo-Riemannian space (space-time of General Relativity). The formulae that we have obtained have the form

$$\begin{aligned} T = & * \square \varphi - \frac{1}{c^3} \frac{* \partial}{\partial t} (F_k q^k) - \frac{1}{c^3} F_i \frac{* \partial q^i}{\partial t} + \frac{1}{c^2} F_i \frac{* \partial \varphi}{\partial x^i} + \\ & + h^{ik} \Delta_{ik}^m \frac{* \partial \varphi}{\partial x^m} - h^{ik} \frac{1}{c} \frac{* \partial}{\partial x^i} [(D_{kn} + A_{kn}) q^n] + \frac{D}{c^2} \frac{* \partial \varphi}{\partial t} - \\ & - \frac{1}{c} D_m^k \frac{* \partial q^m}{\partial x^k} + \frac{2}{c^3} A_{ik} F^i q^k + \frac{\varphi}{c^4} F_i F^i - \frac{\varphi}{c^2} D_{mk} D^{mk} - \\ & - \frac{D}{c^3} F_m q^m - \frac{1}{c} \Delta_{kn}^m D_m^k q^n + \frac{1}{c} h^{ik} \Delta_{ik}^m (D_{mn} + A_{mn}) q^n, \end{aligned} \quad (2.168)$$

*This is one of the reasons why practical applications and theoretical problems of the electromagnetic field theory are in most cases calculated in a Galilean reference frame in the Minkowski space (space-time of Special Relativity), where the Christoffel symbols are zeroes. As a matter of fact, the general covariant notation hardly permits unambiguous interpretation of calculation results, unless they are formulated with physically observable quantities (chronometric invariants) or demoted to a simple specific case, like that in the Minkowski space, for instance.

$$\begin{aligned}
B^i = & * \square A^i + \frac{1}{c^2} \frac{* \partial}{\partial t} \left[(D_k^i + A_{k.}^i) q^k \right] + \frac{D}{c^2} \frac{* \partial q^i}{\partial t} + \\
& + \frac{1}{c^2} (D_k^i + A_{k.}^i) \frac{* \partial q^k}{\partial t} - \frac{1}{c^3} \frac{* \partial}{\partial t} (\varphi F^i) - \frac{1}{c^3} F^i \frac{* \partial \varphi}{\partial t} + \\
& + \frac{1}{c^2} F^k \frac{* \partial q^i}{\partial x^k} - \frac{1}{c} (D^{mi} + A^{mi}) \frac{* \partial \varphi}{\partial x^m} + \frac{1}{c^4} q^k F_k F^i + \\
& + \frac{1}{c^2} \Delta_{km}^i q^m F^k - \frac{\varphi}{c^3} D F^i + \frac{D}{c^2} (D_n^i + A_{n.}^i) q^n - \quad (2.169) \\
& - h^{km} \left\{ \frac{* \partial}{\partial x^k} (\Delta_{mn}^i q^n) + \frac{1}{c} \frac{* \partial}{\partial x^k} [\varphi (D_m^i + A_{m.}^i)] + \right. \\
& + (\Delta_{kn}^i \Delta_{mp}^n - \Delta_{km}^n \Delta_{np}^i) q^p + \frac{\varphi}{c} [\Delta_{kn}^i (D_m^n + A_{m.}^n) - \\
& \left. - \Delta_{km}^n (D_n^i + A_{n.}^i)] + \Delta_{kn}^i \frac{* \partial q^n}{\partial x^m} - \Delta_{km}^n \frac{* \partial q^i}{\partial x^n} \right\},
\end{aligned}$$

where $* \square \varphi$ and $* \square q^i$ are the results of applying the chr.inv.-d'Alembert operator (2.163) to the quantities $\varphi = \frac{A_0}{\sqrt{g_{00}}}$ and $q^i = A^i$, which, in turn, are the chr.inv.-projections of the vector A^α ,

$$* \square \varphi = \frac{1}{c^2} \frac{* \partial^2 \varphi}{\partial t^2} - h^{ik} \frac{* \partial^2 \varphi}{\partial x^i \partial x^k}, \quad (2.170)$$

$$* \square q^i = \frac{1}{c^2} \frac{* \partial^2 q^i}{\partial t^2} - h^{km} \frac{* \partial^2 q^i}{\partial x^k \partial x^m}. \quad (2.171)$$

The main criterion for correct calculations in such a complicated case as here (the chr.inv.-projections of the d'Alembertian of a vector field, which resulted in the formulae 2.168 and 2.169) is Zelmanov's rule of chronometric invariance: "Correct calculations make all the terms in the final equations chronometrically invariant quantities. That is, they consist only of chr.inv.-quantities, their chr.inv.-derivatives, and also of the chr.inv.-properties of the reference space. If at least one error was made in the calculations, the terms of the final equations will not be chronometric invariants."

The d'Alembertian of a tensor field, equated to zero or not zero, gives the *d'Alembert equations* for this field. From a physical point of view, these are the equations of propagation of the field waves. If the

d'Alembertian of a field is not zero, these are the equations of wave propagation enforced by the field-inducing sources (the so-called *d'Alembert equations with sources*). For instance, the sources of an electromagnetic field are electric charges and currents. If the d'Alembert operator of a field is zero, then these are the equations of free wave propagation not related to any sources (the *d'Alembert equations without sources*). If the space-time region under consideration, besides the tensor field in the question, is also filled with another medium, then the d'Alembert equations will have an additional term or terms characterizing the medium, which can be obtained from the equations that determine it.

2.7 Conclusions

We are now ready to outline the results of this Chapter. Apart from general knowledge of tensors and tensor algebra, we have obtained some tools to facilitate our calculations in the next Chapters. The equality to zero of the absolute directional derivative of the dynamic vector of a particle along its trajectory sets the equations of motion of the particle. The equality to zero of the divergence of a vector field sets the Lorenz condition and the continuity equation for the field. The equality to zero of the divergence of a 2nd rank symmetric tensor sets the conservation law, and the equality to zero of a 2nd rank antisymmetric tensor (and also of its dual pseudotensor) set the Maxwell equations. The curl of a vector field, applied to an electromagnetic field, is the electromagnetic field tensor (Maxwell tensor). The d'Alembert equations for a field are the equations of propagation of the field waves.

This is a short list of possible applications of the mathematical apparatus at our disposal. Therefore, if we now come across an antisymmetric tensor or a differential operator, we can simply use the templates we have already obtained in this Chapter.

3.1 Problem statement

In this Chapter, we will create a theory of the electromagnetic field and charged particles in the four-dimensional pseudo-Riemannian space, which is the basic space-time of General Relativity. The peculiarity that makes our theory different from the ordinary relativistic electrodynamics, is that all equations of the theory will be given in the chr.inv.-form, i.e., expressed through physically observable quantities.

An electromagnetic field is usually considered as a vector field of the electromagnetic four-dimensional potential A^α in the four-dimensional pseudo-Riemannian space. Its time component is known as the *scalar electromagnetic potential* φ , and its spatial components make up the so-called *vector electromagnetic potential* A^i . The four-dimensional electromagnetic potential A^α in CGSE and Gaussian systems of units has the dimension

$$A^\alpha [\text{gram}^{1/2} \text{cm}^{1/2} \text{sec}^{-1}]. \quad (3.1)$$

It is obvious that the components φ and A^i have the same dimension. Therefore, when studying an electromagnetic field, we have a substantial difference from studying a gravitational field: according to the theory of chronometric invariants, the gravitational inertial force F^i and the gravitational potential w (1.38) are only functions of the geometric properties of the space, while electromagnetic fields (fields of the electromagnetic potential A^α) have not yet received a “geometric interpretation”, so we have to study an electromagnetic field as an external vector field introduced into the space.

The equations of Classical Electrodynamics — Maxwell’s equations that determine the relationship between the electric and magnetic components of the electromagnetic field — were obtained long before theo-

retical physics adopted the terms of Riemannian geometry and even the Minkowski space of Special Relativity. Later, when electrodynamics was set forth in Minkowski space under the name *relativistic electrodynamics*, Maxwell's equations were obtained in a four-dimensional form. Then Maxwell's equations were obtained in the general covariant form, acceptable for any pseudo-Riemannian space. But, having the general covariant form, Maxwell's equations became less visual, which was the advantage of Classical Electrodynamics. On the other hand, four-dimensional equations in the Minkowski space can simply be represented in terms of their scalar (time) and vector (spatial) components, since in a Galilean reference frame they are observable quantities by definition. But when we consider an inhomogeneous, anisotropic, curved, rotating and deforming pseudo-Riemannian space, the problem of comparing the vector and scalar components of the general covariant equations with the equations of Classical Electrodynamics becomes non-trivial. Then the following question arises: what quantities are physically observable in relativistic electrodynamics?

Therefore, the equations of relativistic electrodynamics must be formulated in the pseudo-Riemannian space, in terms of the physically observable components of the electromagnetic field potential as well as the physically observable properties of the space. We will solve this problem using the mathematical apparatus of chronometric invariants, i.e., projecting general covariant quantities onto the time line and the spatial section associated with a real observer. The result that we are going to get with this method will be an *observable generalization* of the fundamental quantities and laws of relativistic electrodynamics. In addition, Classical Electrodynamics will be obtained as a special case taking into account the effects of the physical and geometric properties of the reference space of the observer.

3.2 The observable components of the electromagnetic field tensor. The field invariants

In accordance with the basics of electrodynamics, the tensor of an electromagnetic field is the curl of the four-dimensional potential A^α of the field. The electromagnetic field tensor is also referred to as *Maxwell's tensor*

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. \quad (3.2)$$

It is easy to see that this formula is a general covariant generalization of the three-dimensional quantities of Classical Electrodynamics

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{H} = \text{rot}\vec{A}, \quad (3.3)$$

where \vec{E} and \vec{H} are, respectively, the strength vectors of the electric and magnetic field components, the scalar φ is the scalar potential of the electromagnetic field, the vector \vec{A} is the spatial vector-potential of the electromagnetic field, and

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \quad (3.4)$$

is the gradient operator in the three-dimensional Euclidean space.

At first, we are going to determine those components of the electromagnetic field tensor $F_{\alpha\beta}$, which are physically observable quantities in the four-dimensional pseudo-Riemannian space. Then, we will find a relationship between the observable quantities and the electric strength \vec{E} and the magnetic strength \vec{H} of the electromagnetic field in the framework of Classical Electrodynamics. Then the strength vectors will be obtained in the pseudo-Riemannian space, which in general is inhomogeneous, anisotropic, curved, rotating and deforming.

It is important to pay attention to the following. Since in the Minkowski space, i.e., in the space-time of Special Relativity, in an inertial reference frame (the one that moves linearly with a constant velocity) the metric is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \quad (3.5)$$

and, hence, the components of the fundamental metric tensor are

$$g_{00} = 1, \quad g_{0i} = 0, \quad g_{11} = g_{22} = g_{33} = -1. \quad (3.6)$$

there is no difference between the covariant and contravariant components of A^α (in particular, this is why all calculations in the Minkowski space are much simpler)

$$\varphi = A_0 = A^0, \quad A_i = -A^i. \quad (3.7)$$

In the pseudo-Riemannian space (and in Riemannian spaces in general) there is a difference, because the metric has a general form. Therefore, the scalar potential and vector-potential of an electromagnetic field

must be defined as the chr.inv.-projections (physically observable components) of the four-dimensional electromagnetic field potential A^α

$$\varphi = b^\alpha A_\alpha = \frac{A_0}{\sqrt{g_{00}}}, \quad q^i = h_\sigma^i A^\sigma = A^i. \quad (3.8)$$

The other components of A^α , are not chr.inv.-quantities. They are formulated with the φ and q^i as follows

$$A^0 = \frac{1}{1 - \frac{w}{c^2}} \left(\varphi + \frac{1}{c} v_i q^i \right), \quad A_i = -q_i - \frac{\varphi}{c} v_i. \quad (3.9)$$

Note that, according to the theory of chronometric invariants, the covariant chr.inv.-vector q_i is obtained from the contravariant chr.inv.-vector q^i by lowering the index using the chr.inv.-metric tensor h_{ik} , i.e., $q_i = h_{ik} q^k$. On the contrary, the ordinary covariant vector A_i , which is not a chr.inv.-quantity, is obtained as a result of lowering the index using the fundamental metric tensor: $A_i = g_{i\alpha} A^\alpha$.

According to the formula for the square of an arbitrary vector (2.39), the square of the potential A^α in the accompanying reference frame is

$$A_\alpha A^\alpha = g_{\alpha\beta} A^\alpha A^\beta = \varphi^2 - h_{ik} q^i q^k = \varphi^2 - q^2, \quad (3.10)$$

and is real if $\varphi^2 > q^2$, imaginary if $\varphi^2 < q^2$, and zero if $\varphi^2 = q^2$.

Now, using the components of the potential A^α (3.8, 3.9) in the definition of the electromagnetic field tensor $F_{\alpha\beta}$ (3.2), then formulating the ordinary derivatives with the chr.inv.-derivatives (1.33) and using the components of the curl of an arbitrary vector field (2.143–2.150), we obtain the chr.inv.-projections of the field tensor $F_{\alpha\beta}$

$$\frac{F_{0\cdot}^i}{\sqrt{g_{00}}} = \frac{g^{i\alpha} F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik} \left(\frac{* \partial \varphi}{\partial x^k} + \frac{1}{c} \frac{* \partial q_k}{\partial t} \right) - \frac{\varphi}{c^2} F^i, \quad (3.11)$$

$$F^{ik} = g^{i\alpha} g^{k\beta} F_{\alpha\beta} = h^{im} h^{kn} \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) - \frac{2\varphi}{c} A^{ik}. \quad (3.12)$$

Let us denote the chr.inv.-projections of the electromagnetic field tensor, as in Classical Electrodynamics

$$E^i = \frac{F_{0\cdot}^i}{\sqrt{g_{00}}}, \quad H^{ik} = F^{ik}, \quad (3.13)$$

so their covariant (lower-index) chr.inv.-counterparts are

$$E_i = h_{ik} E^k = \frac{* \partial \varphi}{\partial x^i} + \frac{1}{c} \frac{* \partial q_i}{\partial t} - \frac{\varphi}{c^2} F_i, \quad (3.14)$$

$$H_{ik} = h_{im} h_{kn} H^{mn} = \frac{* \partial q_i}{\partial x^k} - \frac{* \partial q_k}{\partial x^i} - \frac{2\varphi}{c} A_{ik}, \quad (3.15)$$

while the mixed components $H_{\cdot k}^{\cdot m} = -H_{\cdot k}^{m \cdot}$ are obtained from H^{ik} using the chr.inv.-metric tensor h_{ik} , so that $H_{\cdot k}^{\cdot m} = h_{ki} H^{im}$. The space deformation tensor $D_{ik} = \frac{1}{2} \frac{* \partial h_{ik}}{\partial t}$ (1.40) is also present in the formulae, but in a hidden form: it appears in the formulae, when we substitute the components $q_k = h_{km} q^m$ into the time derivatives.

We can also formulate other components of the electromagnetic field tensor $F_{\alpha\beta}$ with its chr.inv.-projections E^i and H^{ik} (3.11), using the formulae for the components of an arbitrary antisymmetric tensor (2.112–2.115). We can do it, since the general formulae (2.112–2.115) contain E^i and H^{ik} in “implicit form”, regardless of whether they are components of a curl or any other kind of antisymmetric tensor.

In the Minkowski space, since there is no acceleration F^i , rotation A_{ik} and deformations D_{ik} , the formula for E_i becomes

$$E_i = \frac{\partial \varphi}{\partial x^i} + \frac{1}{c} \frac{\partial A_i}{\partial t}, \quad (3.16)$$

or, in the three-dimensional vector form,

$$\vec{E} = \vec{\nabla} \varphi + \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad (3.17)$$

which, apart from the sign, matches the formula for \vec{E} in Classical Electrodynamics.

Now, we formulate the electric and magnetic strengths through the components of the field pseudotensor $F^{*\alpha\beta}$, which is dual to the Maxwell tensor $F^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} F_{\mu\nu}$ (2.123). So forth, in accordance with (2.124), the chr.inv.-projections of the pseudotensor $F^{*\alpha\beta}$ are

$$H^{*i} = \frac{F_{0 \cdot}^{*i}}{\sqrt{g_{00}}}, \quad E^{*ik} = F^{*ik}. \quad (3.18)$$

Using the formulae for the components of an arbitrary pseudotensor $F^{*\alpha\beta}$, which we have obtained in Chapter 2 (2.125–2.131), and also the

above formulae for E_i and H_{ik} (3.14, 3.15), we obtain expanded formulae for H^{*i} and E^{*ik} , which have the form

$$H^{*i} = \frac{1}{2} \varepsilon^{imn} \left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} - \frac{2\varphi}{c} A_{mn} \right) = \frac{1}{2} \varepsilon^{imn} H_{mn}, \quad (3.19)$$

$$E^{*ik} = \varepsilon^{ikn} \left(\frac{\varphi}{c^2} F_n - \frac{\partial \varphi}{\partial x^n} - \frac{1}{c} \frac{\partial q_n}{\partial t} \right) = -\varepsilon^{ikn} E_n. \quad (3.20)$$

It is easy to see that the following pairs of tensors are dual conjugates: H^{*i} and H_{mn} , E^{*ik} and E_m . The chr.inv.-pseudovector H^{*i} (3.19) includes the term

$$\frac{1}{2} \varepsilon^{imn} \left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} \right) = \frac{1}{2} \varepsilon^{imn} (*\nabla_n q_m - *\nabla_m q_n), \quad (3.21)$$

which is the chr.inv.-curl of the three-dimensional vector field q_m . There is also the term

$$\frac{1}{2} \varepsilon^{imn} \frac{2\varphi}{c} A_{mn} = \frac{2\varphi}{c} \Omega^{*i}, \quad (3.22)$$

where $\Omega^{*i} = \frac{1}{2} \varepsilon^{imn} A_{mn}$ is the chr.inv.-pseudovector of the angular velocity with which the space rotates. In a Galilean reference frame in the Minkowski space (since there is no acceleration, rotation and deformations), the obtained formula for the magnetic strength chr.inv.-pseudovector H^{*i} (3.19) takes the form

$$H^{*i} = \frac{1}{2} \varepsilon^{imn} \left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} \right), \quad (3.23)$$

which in the three-dimensional vector form is

$$\vec{H} = \text{rot} \vec{A}. \quad (3.24)$$

We see that the structure of a pseudo-Riemannian space affects an electromagnetic field, located in it. As a result, the physically observable chr.inv.-vectors of the electric strength E_i (3.14) and the magnetic strength H^{*i} (3.19) of the electromagnetic field depend on the gravitational potential and rotation of the space.

The same effect will as well appear in the Minkowski space, if a non-inertial reference frame (which rotates and moves with acceleration) is assumed to be the reference frame of the observer. But in the Minkow-

ski space, we can always find a Galilean reference frame (which is not true in a pseudo-Riemannian space), because the Minkowski space itself does not accelerate reference frames and neither rotates nor deforms it. Therefore, such effects in the Minkowski space are strictly relative and, therefore, can be removed by coordinate transformations.

In relativistic electrodynamics, there are two invariants characterizing the electromagnetic field. They are called the *electromagnetic field invariants* and formulated as follows

$$J_1 = F_{\mu\nu} F^{\mu\nu} = 2F_{0i} F^{0i} + F_{ik} F^{ik}, \quad (3.25)$$

$$J_2 = F_{\mu\nu} F^{*\mu\nu} = 2F_{0i} F^{*0i} + F_{ik} F^{*ik}. \quad (3.26)$$

The first invariant is a scalar, while the second is a pseudoscalar. Formulating them with the components of the electromagnetic field tensor, we obtain

$$J_1 = H_{ik} H^{ik} - 2E_i E^i, \quad J_2 = \varepsilon^{imn} (E_m H_{in} - E_i H_{nm}), \quad (3.27)$$

and, using the formulae for the components of the field pseudotensor $F^{*\mu\nu}$, which we have obtained in Chapter 2, we can re-write the field invariants in the following form

$$J_1 = -2(E_i E^i - H_{*i} H^{*i}), \quad J_2 = -4E_i H^{*i}. \quad (3.28)$$

Since the above quantities J_1 and J_2 are invariants, we arrive at the following conclusions:

- a) If the squares of the electric and magnetic strengths are equal $E^2 = H^{*2}$ in one reference frame, then this equality remains valid in any other reference frame;
- b) If the electric and magnetic strengths are orthogonal $E_i H^{*i} = 0$ in one reference frame, then this orthogonality remains valid in any other reference frame.

An electromagnetic field, where the condition $E^2 = H^{*2}$ and/or the condition $E_i H^{*i} = 0$ are true, i.e., one or both of the field invariants (3.28) are zeroes, is known as an *isotropic electromagnetic field*. In this case, the term “isotropic” does not mean the location of this field in the light-like region of the pseudo-Riemannian space (as is assumed in geometry), but rather the property of the field to radiate equally in any direction in the three-dimensional space (spatial section).

The electromagnetic field invariants can be also formulated with the chr.inv.-derivatives of the scalar chr.inv.-potential φ and the vector chr. inv.-potential q^i (3.8) as well as the chr.inv.-properties of the reference space of the observer. After some algebra based on the formulae (3.27), we obtain the desired formulae

$$J_1 = 2 \left[h^{im} h^{kn} \left(\frac{* \partial q_i}{\partial x^k} - \frac{* \partial q_k}{\partial x^i} \right) \frac{* \partial q_m}{\partial x^n} - h^{ik} \frac{* \partial \varphi}{\partial x^i} \frac{* \partial \varphi}{\partial x^k} - \right. \\ \left. - \frac{2}{c} h^{ik} \frac{* \partial \varphi}{\partial x^i} \frac{* \partial q_k}{\partial t} - \frac{1}{c^2} h^{ik} \frac{* \partial q_i}{\partial t} \frac{* \partial q_k}{\partial t} + \frac{8\varphi}{c^2} \Omega_{*i} \Omega^{*i} - \right. \\ \left. - \frac{2\varphi}{c} \varepsilon^{imn} \Omega_{*m} \frac{* \partial q_i}{\partial x^n} + \frac{2\varphi}{c^2} \frac{* \partial \varphi}{\partial x^i} F^i + \frac{2\varphi}{c^3} \frac{* \partial q_i}{\partial t} F^i - \frac{\varphi}{c^4} F_i F^i \right], \quad (3.29)$$

$$J_2 = \frac{1}{2} \left[\varepsilon^{imn} \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) - \frac{4\varphi}{c} \Omega^{*i} \right] \times \\ \times \left(\frac{* \partial \varphi}{\partial x^i} + \frac{1}{c} \frac{* \partial q_i}{\partial t} - \frac{\varphi}{c^2} F_i \right). \quad (3.30)$$

We can find physical conditions specific of isotropic electromagnetic fields, by setting the formulae (3.29, 3.30) equal to zero. Doing this, we see that the conditions for the equality of the electric and magnetic strengths $E^2 = H^{*2}$ and their orthogonality $E_i H^{*i} = 0$ in a pseudo-Riemannian space depend not only on the properties of the electromagnetic field itself (the scalar potential φ and the vector potential q^i), but also on the acceleration F^i , rotation A_{ik} and deformation D_{ik} of the space itself. In particular, the vectors E_i and H^{*i} are orthogonal, if the space is holonomic $\Omega^{*i} = 0$, and the field of the electromagnetic vector potential q^i does not rotate $\varepsilon^{imn} \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) = 0$.

3.3 Maxwell's equations and their observable components. Conservation of electric charge. Lorenz' condition

In Classical Electrodynamics, the correlations of the electric strength \vec{E} [gram^{1/2} cm^{-1/2} sec⁻¹] of an electromagnetic field to its magnetic strength \vec{H} [gram^{1/2} cm^{-1/2} sec⁻¹] are determined by *Maxwell's equations*, which had originally been derived from a generalization of experimental data. In the middle of the 19th century, Maxwell showed that if an electromagnetic field is induced in emptiness by given charges

and currents, then the resulting field is determined by the two groups of equations [20]

$$\left. \begin{aligned} \operatorname{rot} \vec{H} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} &= \frac{4\pi}{c} \vec{j} \\ \operatorname{div} \vec{E} &= 4\pi\rho \end{aligned} \right\} \text{I,} \quad (3.31a)$$

$$\left. \begin{aligned} \operatorname{rot} \vec{E} + \frac{1}{c} \frac{\partial \vec{H}}{\partial t} &= 0 \\ \operatorname{div} \vec{H} &= 0 \end{aligned} \right\} \text{II,} \quad (3.31b)$$

where ρ [$\text{gram}^{1/2} \text{cm}^{-3/2} \text{sec}^{-1}$] is the electric charge density (namely — the amount of the charge e [$\text{gram}^{1/2} \text{cm}^{3/2} \text{sec}^{-1}$] within 1cm^3) and \vec{j} [$\text{gram}^{1/2} \text{cm}^{-1/2} \text{sec}^{-2}$] is the current density vector. The equations containing the field-inducing sources ρ and \vec{j} are known as the *1st group of the Maxwell equations*, and the equations that do not contain the field sources are known as the *2nd group of the Maxwell equations*.

The first equation in the 1st group is Biot-Savart's law, the second is Gauss' theorem, both in differential notation. The first equation in the 2nd group is the differential notations of Faraday's law of electromagnetic induction, and the second is the condition according to which no magnetic charges exist. In total, there are 8 equations (four vector and four scalar ones) in 10 unknowns: three components of \vec{E} , three components of \vec{H} , three components of \vec{j} , and one component of ρ .

A correlation between the field sources ρ and \vec{j} is set by the *law of conservation of electric charge*

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0, \quad (3.32)$$

which is a mathematical notation of the experimental fact that an electric charge cannot be destroyed, but is merely re-distributed between charged bodies in contact.

Now we have a system of 9 equations in 10 unknowns, so the system defining the field and its sources is still indefinite. The 10th equation that makes the system definite (the number of equations and unknowns must be the same) is *Lorenz' condition* that connects the scalar and vector potentials of the field as follows

$$\frac{1}{c} \frac{\partial \varphi}{\partial t} + \operatorname{div} \vec{A} = 0. \quad (3.33)$$

The Lorenz condition is derived from the fact that the scalar potential φ and the vector potential \vec{A} of any electromagnetic field, related to the strength vectors \vec{E} and \vec{H} with (3.3), are defined ambiguously: \vec{E} and \vec{H} in (3.3) remain unchanged, if we replace

$$\vec{A} = \vec{A}' + \vec{\nabla}\Psi, \quad \varphi = \varphi' - \frac{1}{c} \frac{\partial\Psi}{\partial t}, \quad (3.34)$$

where Ψ is an arbitrary scalar. Obviously, the ambiguous definition of the φ and \vec{A} permits other correlations between the quantities except for the Lorenz condition. Nevertheless, it is the Lorenz condition, which enables the transformation of the Maxwell equations into wave equations. This is how the Lorenz condition does the transformation.

The equation $\text{div}\vec{H} = 0$ (3.31) is satisfied, if we assume $\vec{H} = \text{curl}\vec{A}$. In this case, the first equation in the 1st group (3.31) takes the form

$$\text{rot} \left(\vec{E} + \frac{1}{c} \frac{\partial\vec{A}}{\partial t} \right) = 0, \quad (3.35)$$

which has the solution

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c} \frac{\partial\vec{A}}{\partial t}. \quad (3.36)$$

Substituting $\vec{H} = \text{curl}\vec{A}$ and \vec{E} (3.36) into the 1st group of the Maxwell equations, we obtain

$$\Delta\vec{A} - \frac{1}{c^2} \frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla} \left(\text{div}\vec{A} + \frac{1}{c} \frac{\partial\varphi}{\partial t} \right) = -\frac{4\pi}{c} \vec{j}, \quad (3.37)$$

$$\Delta\varphi + \frac{1}{c} \frac{\partial}{\partial t} (\text{div}\vec{A}) = -4\pi\rho, \quad (3.38)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the ordinary Laplace operator.

Imposing the Lorenz condition (3.33) on the potentials φ and \vec{A} , we transform the equations of the 1st group to the form

$$\square\varphi = -4\pi\rho, \quad (3.39)$$

$$\square\vec{A} = -\frac{4\pi}{c} \vec{j}, \quad (3.40)$$

where $\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta$ is the ordinary d'Alembert operator.

Applying the d'Alembert operator to a field gives the equations of propagation of the field waves (see §2.6). For this reason, the obtained result means that if the Lorenz condition is true, then the 1st group of the Maxwell equations (3.31) is a system of the equations of propagation of waves of the scalar and vector electromagnetic field potentials (in the presence of the field-inducing sources — electric charges and currents). The equations will be obtained in the next section, §3.4.

Next, we are going to consider the Maxwell equations in the four-dimensional pseudo-Riemannian space to obtain them in the chr.inv.-form, i.e., formulated with physically observable quantities.

In the four-dimensional pseudo-Riemannian space, the Lorenz condition has the general covariant form

$$\nabla_{\sigma} A^{\sigma} = \frac{\partial A^{\sigma}}{\partial x^{\sigma}} + \Gamma_{\sigma\mu}^{\sigma} A^{\mu} = 0, \quad (3.41)$$

which is the condition of conservation of the four-dimensional electromagnetic field potential. The law of conservation of electric charge (*continuity equation*) is

$$\nabla_{\sigma} j^{\sigma} = 0, \quad (3.42)$$

where j^{α} is the four-dimensional *current vector* known as the *shift current*. The chr.inv.-projections of the current vector j^{α} are the electric charge density

$$\rho = \frac{1}{c} \frac{j_0}{\sqrt{g_{00}}}, \quad (3.43)$$

and the spatial current density j^i . Using the chr.inv.-formula for the divergence of a vector field (2.107), we obtain the Lorenz condition (3.41) and the continuity equation (3.42) in the chr.inv.-form

$$\frac{1}{c} \frac{* \partial \varphi}{\partial t} + \frac{\varphi}{c} D + * \nabla_i q^i - \frac{1}{c^2} F_i q^i = 0, \quad (3.44)$$

$$\frac{* \partial \rho}{\partial t} + \rho D + * \nabla_i j^i - \frac{1}{c^2} F_i j^i = 0. \quad (3.45)$$

Here, $D = h^{ik} D_{ik} = D_n^n = \frac{* \partial \ln \sqrt{h}}{\partial t}$ is the trace of the space deformations rate tensor (1.40), the physical sense of which is the relative expansion rate of an elementary volume. The sign $* \nabla$ stands for a chr.inv.-derivative, determined by analogy with the sign ∇ of a general covariant (absolute) derivative, see formulae (1.48–1.54).

Because F_i (1.38) contains the first derivative of gravitational potential $w = c^2(1 - \sqrt{g_{00}})$, the term $\frac{1}{c^2} F_i q^i$ takes into account the fact that the flow of time is different at the opposite walls of an elementary volume. The formula for the gravitational inertial force F_i (1.38) also takes into account the non-stationarity of the space rotation (if any).

Besides, since the chr.inv.-derivation operators (1.33) have the form

$$\frac{* \partial}{\partial t} = \frac{1}{1 - \frac{w}{c^2}} \frac{\partial}{\partial t}, \quad \frac{* \partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{1}{c^2} v_i \frac{* \partial}{\partial t}, \quad (3.46)$$

the condition of conservation of the vector field A^α , namely — the equations (3.44, 3.45), directly depend on the gravitational potential and the velocity with which the space rotates.

The chr.inv.-derivatives $\frac{* \partial \varphi}{\partial t}$ and $\frac{* \partial \rho}{\partial t}$ are the observed time variations of the chr.inv.-quantities φ and ρ . The chr.inv.-quantities φD and ρD are the observed time variations of the spatial volume of the φ and ρ .

If there are no gravitational inertial forces, and the space does not rotate or deform, then the obtained chr.inv.-formulae for the Lorenz condition (3.44) and the charge conservation law (3.45) take the form

$$\frac{1}{c} \frac{\partial \varphi}{\partial t} + \frac{\partial q^i}{\partial x^i} - \frac{\partial \ln \sqrt{h}}{\partial x^i} q^i = 0, \quad (3.47)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial j^i}{\partial x^i} - \frac{\partial \ln \sqrt{h}}{\partial x^i} j^i = 0, \quad (3.48)$$

which in a Galilean reference frame in the Minkowski space become

$$\frac{1}{c} \frac{\partial \varphi}{\partial t} + \frac{\partial q^i}{\partial x^i} = 0, \quad \frac{\partial \rho}{\partial t} + \frac{\partial j^i}{\partial x^i} = 0, \quad (3.49)$$

or, in the ordinary vector notation

$$\frac{1}{c} \frac{\partial \varphi}{\partial t} + \operatorname{div} \vec{A} = 0, \quad \frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0, \quad (3.50)$$

which completely matches the Lorenz condition (3.33) and the charge conservation law (3.32) in Classical Electrodynamics.

Let us turn to the Maxwell equations. In a pseudo-Riemannian space each pair of the equations merge into a single general covariant equation

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c} j^\mu, \quad \nabla_\sigma F^{*\mu\sigma} = 0, \quad (3.51)$$

where $F^{\mu\sigma}$ is the contravariant (upper-index) form of the electromagnetic field tensor, and $F^{*\mu\sigma}$ is its dual pseudotensor. Using the chr.inv.-formulae for the divergence of an antisymmetric tensor of the 2nd rank (2.121, 2.122) and for its dual pseudotensor (2.135, 2.136), we arrive at the Maxwell equations in the chr.inv.-form

$$\left. \begin{aligned} * \nabla_i E^i - \frac{1}{c} H^{ik} A_{ik} &= 4\pi\rho \\ * \nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} - \frac{1}{c} \left(\frac{\partial E^i}{\partial t} + D E^i \right) &= \frac{4\pi}{c} j^i \end{aligned} \right\} \text{I,} \quad (3.52)$$

$$\left. \begin{aligned} * \nabla_i H^{*i} - \frac{1}{c} E^{*ik} A_{ik} &= 0 \\ * \nabla_k E^{*ik} - \frac{1}{c^2} F_k E^{*ik} - \frac{1}{c} \left(\frac{\partial H^{*i}}{\partial t} + D H^{*i} \right) &= 0 \end{aligned} \right\} \text{II.} \quad (3.53)$$

The above chr.inv.-Maxwell equations were first obtained, independently, by José del Prado and Nikolai Pavlov [25] (Zelmanov asked these students to do it, and explained how to do it).

Now, we transform the chr.inv.-Maxwell equations to express them through E^i and H^{*i} as unknowns. Getting the E^i and H^{*i} from their definitions (2.111, 2.124)

$$H_{*i} = \frac{1}{2} \varepsilon_{imn} H^{mn}, \quad (3.54)$$

$$E^{*ik} = \varepsilon^{ikm} \left(\frac{\varphi}{c^2} F_m - \frac{\partial \varphi}{\partial x^m} - \frac{1}{c} \frac{\partial q_m}{\partial t} \right) = -\varepsilon^{ikm} E_m, \quad (3.55)$$

and multiplying the first equation by ε^{ipq} , we obtain

$$\varepsilon^{ipq} H_{*i} = \frac{1}{2} \varepsilon^{ipq} \varepsilon_{imn} H^{mn} = \frac{1}{2} (\delta_m^p \delta_n^q - \delta_m^q \delta_n^p) H^{mn} = H^{pq}. \quad (3.56)$$

Substituting the result as $H^{ik} = \varepsilon^{mik} H_{*m}$ into the first equation of the 1st group (3.52), we bring it to the form

$$* \nabla_i E^i - \frac{2}{c} \Omega_{*m} H^{*m} = 4\pi\rho, \quad (3.57)$$

where $\Omega^{*i} = \frac{1}{2} \varepsilon^{imn} A_{mn}$ is the chr.inv.-pseudovector of the angular velocity with which the space rotates. Substituting $E^{*ik} = -\varepsilon^{ikm} E_m$ (3.55)

into the first equation of the 2nd group (3.53), we obtain

$${}^*\nabla_i H^{*i} + \frac{2}{c} \Omega_{*m} E^m = 0. \quad (3.58)$$

Then, substituting $H^{ik} = \varepsilon^{mik} H_{*m}$ into the second equation of the 2nd group (3.52) we obtain

$$\begin{aligned} {}^*\nabla_k (\varepsilon^{mik} H_{*m}) - \frac{1}{c^2} F_k \varepsilon^{mik} H_{*m} - \\ - \frac{1}{c} \left(\frac{{}^*\partial E^i}{\partial t} + \frac{{}^*\partial \ln \sqrt{h}}{\partial t} E^i \right) = \frac{4\pi}{c} j^i \end{aligned} \quad (3.59)$$

and, multiplying both sides of the equation by \sqrt{h} and taking ${}^*\nabla_k \varepsilon^{mik} = 0$ into account, we bring this formula (3.59) to the form

$$\begin{aligned} \varepsilon^{ikm} {}^*\nabla_k (H_{*m} \sqrt{h}) - \frac{1}{c^2} \varepsilon^{ikm} F_k H_{*m} \sqrt{h} - \\ - \frac{1}{c} \frac{{}^*\partial}{\partial t} (E^i \sqrt{h}) = \frac{4\pi}{c} j^i \sqrt{h} \end{aligned} \quad (3.60)$$

or, in the other notation

$$\varepsilon^{ikm} {}^*\widetilde{\nabla}_k (H_{*m} \sqrt{h}) - \frac{1}{c} \frac{{}^*\partial}{\partial t} (E^i \sqrt{h}) = \frac{4\pi}{c} j^i \sqrt{h}, \quad (3.61)$$

where $j^i \sqrt{h}$ is the volume density of the current j^i , and ${}^*\widetilde{\nabla}_k = {}^*\nabla_k - \frac{1}{c^2} F_k$ is the physical chr.inv.-divergence (2.106), which takes into account the fact that the flow of time is different at the opposite walls of an elementary volume.

The obtained equation (3.60) is the chr.inv.-notation for the Biot-Savart law in the pseudo-Riemannian space.

Substituting $E^{*ik} = -\varepsilon^{ikm} E_m$ (3.55) into the second equation of the 2nd group (3.53), after similar transformations we obtain

$$\varepsilon^{ikm} {}^*\widetilde{\nabla}_k (E_m \sqrt{h}) + \frac{1}{c} \frac{{}^*\partial}{\partial t} (H^{*i} \sqrt{h}) = 0, \quad (3.62)$$

which is the chr.inv.-notation for the Faraday law of electromagnetic induction in the pseudo-Riemannian space.

So, the final system of 10 chr.inv.-equations in 10 unknowns (two groups of the Maxwell equations, the Lorenz condition, and the continuity equation), which completely determine an electromagnetic field

and its sources in the pseudo-Riemannian space, is

$$\left. \begin{aligned} {}^* \nabla_i E^i - \frac{2}{c} \Omega_{*m} H^{*m} &= 4\pi\rho \\ \varepsilon^{ikm} {}^* \widetilde{\nabla}_k (H_{*m} \sqrt{h}) - \frac{1}{c} \frac{{}^* \partial}{\partial t} (E^i \sqrt{h}) &= \frac{4\pi}{c} j^i \sqrt{h} \end{aligned} \right\} \text{I,} \quad (3.63)$$

$$\left. \begin{aligned} {}^* \nabla_i H^{*i} + \frac{2}{c} \Omega_{*m} E^m &= 0 \\ \varepsilon^{ikm} {}^* \widetilde{\nabla}_k (E_m \sqrt{h}) + \frac{1}{c} \frac{{}^* \partial}{\partial t} (H^{*i} \sqrt{h}) &= 0 \end{aligned} \right\} \text{II,} \quad (3.64)$$

$$\frac{1}{c} \frac{{}^* \partial \varphi}{\partial t} + \frac{\varphi}{c} D + {}^* \widetilde{\nabla}_i q^i = 0 \quad \text{the Lorenz condition,} \quad (3.65)$$

$$\frac{{}^* \partial \rho}{\partial t} + \rho D + {}^* \widetilde{\nabla}_i j^i = 0 \quad \text{the continuity equation.} \quad (3.66)$$

In a Galilean reference frame in the Minkowski space, the determinant of the chr.inv.-metric tensor is $\sqrt{h} = 1$, so the space does not rotate ($\Omega_{*m} = 0$) or deform ($D_{ik} = 0$), and it does not contain gravitational fields ($F_i = 0$). In this case, the chr.inv.-Maxwell equations (3.63, 3.64) that we have obtained in the pseudo-Riemannian space of General Relativity transform into the Maxwell equations in Classical Electrodynamics written in the tensor form

$$\left. \begin{aligned} \frac{\partial E^i}{\partial x^i} &= 4\pi\rho \\ e^{ikm} \left(\frac{\partial H_{*m}}{\partial x^k} - \frac{\partial H_{*k}}{\partial x^m} \right) - \frac{1}{c} \frac{\partial E^i}{\partial t} &= \frac{4\pi}{c} j^i \end{aligned} \right\} \text{I,} \quad (3.67)$$

$$\left. \begin{aligned} \frac{\partial H^{*i}}{\partial x^i} &= 0 \\ e^{ikm} \left(\frac{\partial E_m}{\partial x^k} - \frac{\partial E_k}{\partial x^m} \right) - \frac{1}{c} \frac{\partial H^{*i}}{\partial t} &= 0 \end{aligned} \right\} \text{II.} \quad (3.68)$$

The same equations, but written in the ordinary vector notation, are similar to the classic Maxwell equations in the three-dimensional Euclidean space (3.31). Besides, the chr.inv.-Maxwell equations obtained in the four-dimensional pseudo-Riemannian space (3.64) show that if the space does not rotate, then the chr.inv.-divergence of the magnetic field strength is zero ${}^* \nabla_i H^{*i} = 0$. In other words, the magnetic compo-

ment of an electromagnetic field remains unchanged, if the space is holonomic. At the same time, the divergence of the electric field strength in this case is not zero ${}^* \nabla_i E^i = 4\pi\rho$ (3.63), so the electric component is linked directly to the charge density ρ . Hence, a conclusion on “magnetic charge”, if it actually exists, should be linked directly to the field of rotation of the space itself.

3.4 D’Alembert’s equations for the electromagnetic potential, and their observable components

As we have already mentioned in Chapter 2, d’Alembert’s operator applied to a field gives the equations of propagation of the field waves. For this reason, the d’Alembert equations for the scalar electromagnetic potential φ are the wave propagation equations for the scalar field φ , while for the spatial vector-potential \vec{A} these are the wave propagation equations for the vector field \vec{A} .

The general covariant d’Alembert equations for the electromagnetic field potential A^α in the four-dimensional pseudo-Riemannian space were obtained in the end-1950s by Stanyukovich [26] using the 1st group of the general covariant Maxwell equations $\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c} j^\mu$ (3.51) and the Lorenz condition $\nabla_\sigma A^\sigma = 0$ (3.41). Stanyukovich’s equations are

$$\square A^\alpha - R^\alpha_\beta A^\beta = -\frac{4\pi}{c} j^\alpha, \quad (3.69)$$

where $R^\alpha_\beta = g^{\alpha\mu} R^\sigma_{\cdot\mu\beta\sigma}$ is Ricci’s tensor (the contraction of the Riemann-Christoffel curvature tensor $R^\alpha_{\cdot\mu\beta\sigma}$). The term $R^\alpha_\beta A^\beta$ vanishes from the left hand side of the equations, if the Ricci tensor is zero, so the space metric satisfies Einstein’s field equations away from gravitating masses. This term can be neglected in the case, where the space curvature is not significant. But, even in the Minkowski space, the problems of physics can be considered in the presence of acceleration and rotation. Therefore, even in the framework of this approximation, it is possible to reveal, for example, the influence of the rotation of the observer’s reference body and the acting gravitational inertial force on the observed propagation velocity of electromagnetic waves.

The reason for simplifications is that the chr.inv.-projections of the d’Alembert equations in their complete form are a very difficult task to deduce. The resulting equations will be so bulky to make any unambigu-

ous conclusions. Therefore, we will limit the scope of our work to transforming the d'Alembert equations into the chr.inv.-tensor form for an electromagnetic field in a non-inertial reference frame in the Minkowski space. But this does not affect the other sections in this Chapter, where we go back to the pseudo-Riemannian space of General Relativity.

Calculating the chr.inv.-projections of the d'Alembert equations

$$\square A^\alpha = -\frac{4\pi}{c} j^\alpha \quad (3.70)$$

based on their general formulae (2.168, 2.169), we obtain

$$\begin{aligned} * \square \varphi - \frac{1}{c^3} \frac{* \partial}{\partial t} (F_k q^k) - \frac{1}{c^3} F_i \frac{* \partial q^i}{\partial t} + \frac{1}{c^2} F^i \frac{* \partial \varphi}{\partial x^i} + h^{ik} \Delta_{ik}^m \frac{* \partial \varphi}{\partial x^m} - \\ - h^{ik} \frac{1}{c} \frac{* \partial}{\partial x^i} (A_{kn} q^n) + \frac{1}{c} h^{ik} \Delta_{ik}^m A_{mn} q^n = 4\pi \rho, \end{aligned} \quad (3.71)$$

$$\begin{aligned} * \square A^i + \frac{1}{c^2} \frac{* \partial}{\partial t} (A_k^i q^k) + \frac{1}{c^2} A_k^i \frac{* \partial q^k}{\partial t} - \frac{1}{c^3} \frac{* \partial (\varphi F^i)}{\partial t} - \\ - \frac{1}{c^3} F^i \frac{* \partial \varphi}{\partial t} + \frac{1}{c^2} F^k \frac{* \partial q^i}{\partial x^k} - \frac{1}{c} A^{mi} \frac{* \partial \varphi}{\partial x^m} + \frac{1}{c^2} \Delta_{km}^i q^m F^k - \\ - h^{km} \left\{ \frac{* \partial}{\partial x^k} (\Delta_{mn}^i q^n) + \frac{1}{c} \frac{* \partial}{\partial x^k} (\varphi A_m^i) \right\} + \\ + (\Delta_{kn}^i \Delta_{mp}^n - \Delta_{km}^n \Delta_{np}^i) q^p + \frac{\varphi}{c} (\Delta_{kn}^i A_m^n - \Delta_{km}^n A_n^i) + \\ + \Delta_{kn}^i \frac{* \partial q^n}{\partial x^m} - \Delta_{km}^n \frac{* \partial q^i}{\partial x^n} \left\} = \frac{4\pi}{c} j^i, \end{aligned} \quad (3.72)$$

where we take into account the observable charge density $\rho = \frac{g_{0\alpha} j^\alpha}{c \sqrt{g_{00}}}$ in the space that does not deform, and in the linear approximation (with higher-order terms withheld, because we assume that the field of gravitation and the field of the space rotation are weak).

We see that the physically observable chr.inv.-properties of the reference space (i.e., the quantities F^i , A_{ik} , D_{ik} , Δ_{km}^i) constitute some additional "sources" that together with the electromagnetic field sources φ and j^i induce waves travelling along the electromagnetic field.

Let us now analyse the results. At first, consider the obtained equations (3.71, 3.72) in a Galilean reference frame in the Minkowski space.

Here the metric takes the form as in the formula (3.5) and, therefore, the chr.inv.-d'Alembert operator ${}^*\square$ (2.163) transforms into the ordinary d'Alembert operator ${}^*\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta = \square$. Then the obtained equations (3.71, 3.72) take the simplest form

$$\square \varphi = 4\pi\rho, \quad \square q^i = -\frac{4\pi}{c} j^i, \quad (3.73)$$

which completely matches the corresponding equations in Classical Electrodynamics (3.39, 3.40).

Now we return to the obtained chr.inv.-d'Alembert equations (3.71, 3.72). To make their analysis easier we denote all terms on the left hand side of the scalar equation (3.71) as T and those of the vector equation (3.72) as B^i . Transpositioning the variables into their rightful positions and expanding the formulae for ${}^*\square$ (2.163), we obtain

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - h^{ik} {}^*\nabla_i {}^*\nabla_k \varphi = T + 4\pi\rho, \quad (3.74)$$

$$\frac{1}{c^2} \frac{\partial^2 q^i}{\partial t^2} - h^{mk} {}^*\nabla_m {}^*\nabla_k q^i = B^i + \frac{4\pi}{c} j^i, \quad (3.75)$$

where $h^{ik} {}^*\nabla_i {}^*\nabla_k = {}^*\Delta$ is the chr.inv.-Laplace operator. If the field potentials φ and q^i are stationary, then the d'Alembert equations become the Laplace equations

$${}^*\Delta \varphi = T + 4\pi\rho, \quad (3.76)$$

$${}^*\Delta q^i = B^i + \frac{4\pi}{c} j^i, \quad (3.77)$$

i.e., they characterize static states of the field.

A field is homogeneous along a direction, if its ordinary derivative with respect to this direction is zero. A field in a Riemannian space is homogeneous, if its general covariant derivative is zero. If a field is considered in the accompanying reference frame, then the observable inhomogeneity of the field is characterized by a non-zero chr.inv.-derivative ${}^*\nabla_i$ of the field potential [9, 11–13]. On the contrary, if the chr.inv.-derivative ${}^*\nabla_i$ is non-zero, then the field is *observed* as homogeneous.

So, the chr.inv.-d'Alembert operator ${}^*\square$ is the difference between the term characterizing the observable field non-stationarity and the term characterizing the observable field inhomogeneity. If the electro-

magnetic field is stationary and homogeneous, then the left hand side of the d'Alembert equations (3.74, 3.75) is zero: the field does not generate electromagnetic waves (it is not a wave field).

In an inhomogeneous stationary field (where ${}^*\nabla_i \neq 0$ and $\frac{1}{c} \frac{\partial}{\partial t} = 0$), the d'Alembert equations (3.74, 3.75) characterize a standing wave

$$-h^{ik} {}^*\nabla_i {}^*\nabla_k \varphi = T + 4\pi\rho, \quad (3.78)$$

$$-h^{mk} {}^*\nabla_m {}^*\nabla_k q^i = B^i + \frac{4\pi}{c} j^i. \quad (3.79)$$

In a homogeneous non-stationary field (where ${}^*\nabla_i = 0$ and $\frac{1}{c} \frac{\partial}{\partial t} \neq 0$), the d'Alembert equations describe the field change with time depending on the field-inducing sources (charges and currents)

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = T + 4\pi\rho, \quad (3.80)$$

$$\frac{1}{c^2} \frac{\partial^2 q^i}{\partial t^2} = B^i + \frac{4\pi}{c} j^i. \quad (3.81)$$

In an inertial reference frame (where the Christoffel symbols are zero), the general covariant derivative is equal to the ordinary derivative ${}^*\nabla_i \varphi = \frac{\partial \varphi}{\partial x^i}$, so the d'Alembert chr.inv.-scalar equation (3.74) is

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - h^{ik} \frac{\partial^2 \varphi}{\partial x^i \partial x^k} = T + 4\pi\rho. \quad (3.82)$$

As is known from the oscillation theory in mathematical physics, the term a in the ordinary d'Alembert equations

$$\square \varphi = \frac{1}{a^2} \frac{\partial^2 \varphi}{\partial t^2} + g^{ik} \frac{\partial^2 \varphi}{\partial x^i \partial x^k} \quad (3.83)$$

is the absolute value of the three-dimensional velocity of elastic oscillations propagating along the field φ .

Expanding the chr.inv.-derivatives (3.46), we bring the d'Alembert scalar equation (3.82) to the form

$$\begin{aligned} & \frac{1}{c^2} \left(1 - \frac{v^2}{c^2}\right) \frac{\partial^2 \varphi}{\partial t^2} - h^{ik} \frac{\partial^2 \varphi}{\partial x^i \partial x^k} + \frac{2v^k}{c^2 - w} \frac{\partial^2 \varphi}{\partial x^k \partial t} + \\ & + \frac{1}{c^2 - w} h^{ik} \frac{\partial v_k}{\partial x^i} \frac{\partial \varphi}{\partial t} + \frac{1}{c^2} v^k F_k \frac{\partial \varphi}{\partial t} = T + 4\pi\rho, \end{aligned} \quad (3.84)$$

where $v^2 = h_{ik} v^i v^k$, and the second chr.inv.-derivative with respect to time formulates with the ordinary derivatives as follows

$$\frac{{}^* \partial^2 \varphi}{\partial t^2} = \frac{1}{\left(1 - \frac{w}{c^2}\right)^2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{1}{c^2 \left(1 - \frac{w}{c^2}\right)^3} \frac{\partial w}{\partial t} \frac{\partial \varphi}{\partial t}. \quad (3.85)$$

We can now see that the square of the linear velocity v^2 with which the space rotates has a greater effect on the propagation of the field waves, than the observable non-stationarity of the field, i.e., the term $\frac{{}^* \partial \varphi}{\partial t}$. In the limiting case, where $v \rightarrow c$, the d'Alembert operator becomes the Laplace operator, therefore, the d'Alembert wave equations become the Laplace stationary equations. At low velocities of the space rotation ($v \ll c$), observable electromagnetic waves propagate with the velocity of light.

In general, the modulus of the observable wave velocity of the scalar electromagnetic potential $v_{(\varphi)}$ takes the form

$$v_{(\varphi)} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (3.86)$$

It is obvious that the chr.inv.-quantity (3.85), which is the observable acceleration of the scalar potential φ , is quite different from the analogous "coordinate" quantity; the stronger the gravitational potential, the greater the charge rate of the gravitational potential with time

$$\frac{\partial^2 \varphi}{\partial t^2} = \left(1 - \frac{w}{c^2}\right)^2 \frac{{}^* \partial^2 \varphi}{\partial t^2} + \frac{1}{c^2 - w} \frac{\partial w}{\partial t} \frac{\partial \varphi}{\partial t}. \quad (3.87)$$

In the limiting case, where $w \rightarrow c^2$ (approaching the state of gravitational collapse as the state on the surface of a gravitational collapse), the observable acceleration of the scalar electromagnetic potential (3.85) becomes infinitesimal, while the coordinate rate of the scalar potential growth (3.87), to the contrary, becomes infinitely large. But under ordinary conditions, the gravitational potential w needs only smaller corrections to the acceleration and the rate of the electromagnetic scalar potential growth.

All that has been concluded above about the chr.inv.-scalar quantity $\frac{{}^* \partial^2 \varphi}{\partial t^2}$ is also true for the chr.inv.-vector $\frac{{}^* \partial^2 q^i}{\partial t^2}$, because the chr.inv.-d'Alembert operator ${}^* \square = \frac{1}{c^2} \frac{{}^* \partial^2}{\partial t^2} - h^{ik} \frac{{}^* \partial^2}{\partial x^i \partial x^k}$ is different from the men-

tioned scalar and vector functions in only the second term — the Laplace operator, in which the chr.inv.-derivatives of the scalar and vector quantities are different from each other, i.e.

$${}^*\nabla_i \varphi = \frac{{}^*\partial \varphi}{\partial x^i}, \quad {}^*\nabla_i q^k = \frac{{}^*\partial q^k}{\partial x^i} + \Delta_{im}^k q^m. \quad (3.88)$$

If the gravitational potential and the velocity with which the space rotates are infinitesimal, then the chr.inv.-d'Alembert operator for the scalar electromagnetic potential becomes the ordinary d'Alembert operator

$${}^*\square \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - h^{ik} \frac{\partial^2 \varphi}{\partial x^i \partial x^k}, \quad (3.89)$$

so in this case electromagnetic waves, produced by the scalar potential φ , propagate with the velocity of light.

3.5 The Lorentz force. The energy-momentum tensor of an electromagnetic field

Now we are going to deduce the chr.inv.-projections (physically observable components) of the four-dimensional force, which is the result of the action of an electromagnetic field on an electric charge in a pseudo-Riemannian space.

This problem will be solved for the two cases: a) for a point charge; b) for a charge distributed in the space. In addition, we will deduce the chr.inv.-projections of the energy-momentum tensor for an electromagnetic field.

In the three-dimensional Euclidean space of Classical Electrodynamics, the motion of a charged particle in an electromagnetic field is described by the vector equation

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c} [\vec{u}; \vec{H}], \quad (3.90)$$

where $\vec{p} = m\vec{u}$ is the three-dimensional momentum vector of the particle, and m is the particle's relativistic mass. The right hand side of this equation is referred to as the *Lorentz force*.

The equation, characterizing the change of the kinetic (relativistic) energy of the particle

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \quad (3.91)$$

due to the work accomplished by the electric field strength to displace it, takes the three-dimensional vector form (in the framework of Classical Electrodynamics)

$$\frac{dE}{dt} = e \vec{E} \vec{u}, \quad (3.92)$$

and is also known as the *live forces theorem*.

In the four-dimensional form, thanks to the unification of energy and momentum, in a Galilean reference frame in the Minkowski space, the equations (3.90) and (3.92) take the joint form

$$m_0 c \frac{dU^\alpha}{ds} = \frac{e}{c} F_{\cdot\sigma}^{\alpha} U^\sigma, \quad U^\alpha = \frac{dx^\alpha}{ds}, \quad (3.93)$$

and are known as the *Minkowski equations* ($F_{\cdot\sigma}^{\alpha}$ is the electromagnetic field tensor). Because the metric here is diagonal (3.5),

$$ds = c dt \sqrt{1 - \frac{u^2}{c^2}}, \quad u^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2, \quad (3.94)$$

and the components of the particle's four-dimensional velocity U^α are

$$U^0 = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad U^i = \frac{u^i}{c \sqrt{1 - \frac{u^2}{c^2}}}, \quad (3.95)$$

where $u^i = \frac{dx^i}{dt}$ is its three-dimensional coordinate velocity. Because the components of $\frac{e}{c} F_{\cdot\sigma}^{\alpha} U^\sigma$ in the Galilean reference frame are

$$\frac{e}{c} F_{\cdot\sigma}^{0} U^\sigma = -\frac{e}{c^2} \frac{E_i u^i}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad (3.96)$$

$$\frac{e}{c} F_{\cdot\sigma}^{i} U^\sigma = -\frac{1}{c \sqrt{1 - \frac{u^2}{c^2}}} \left(e E^i + \frac{e}{c} e^{ikm} u_k H_{*m} \right), \quad (3.97)$$

then, in the Galilean reference frame, the time and spatial components of the Minkowski equations (3.93) take the form

$$\frac{dE}{dt} = -e E_i u^i, \quad (3.98)$$

$$\frac{dp^i}{dt} = -\left(e E^i + \frac{e}{c} e^{ikm} u_k H_{*m} \right), \quad p^i = m u^i. \quad (3.99)$$

The above relativistic equations, except for the sign on the right hand side, match the live forces theorem and the equations of motion of a charged particle in Classical Electrodynamics (3.90, 3.91). Note that the difference in the sign of the right hand side of the equations is determined only by the choice of the space signature. We use the signature (+---). But, if we assume the signature (-+++), then the sign of the right hand side of the equations will be the opposite.

Let us now consider this problem not in the Minkowski space, but in the pseudo-Riemannian space of General Relativity.

The chr.inv.-projections of the four-dimensional momentum vector $\Phi^\alpha = \frac{e}{c} F_{\cdot\sigma}^\alpha U^\sigma$ gained by a charged particle in the pseudo-Riemannian space from the interaction of the charge e of the particle with the electromagnetic field that fills the space, are

$$T = \frac{e}{c} \frac{F_{0\sigma} U^\sigma}{\sqrt{g_{00}}}, \quad (3.100)$$

$$B^i = \frac{e}{c} F_{\cdot\sigma}^i U^\sigma = \frac{e}{c} (F_{\cdot 0}^i U^0 + F_{\cdot k}^i U^k). \quad (3.101)$$

Given that the components of the U^α are

$$U^0 = \frac{\frac{1}{c^2} v_i v^i \pm 1}{\sqrt{1 - \frac{v^2}{c^2} \left(1 - \frac{w}{c^2}\right)}}, \quad U^i = \frac{v^i}{c \sqrt{1 - \frac{v^2}{c^2}}}, \quad (3.102)$$

and taking into account the formulae for the chr.inv.-components of an arbitrary curl (2.143–2.150), we obtain

$$T = - \frac{e}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} \left(\frac{\partial \varphi}{\partial x^i} + \frac{1}{c} \frac{\partial q_i}{\partial t} - \frac{\varphi}{c^2} F_i \right) v^i, \quad (3.103)$$

$$B^i = - \frac{e}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} \left\{ \pm \left(\frac{\partial \varphi}{\partial x^k} + \frac{1}{c} \frac{\partial q_k}{\partial t} - \frac{\varphi}{c^2} F_k \right) h^{ik} + \left[h^{im} h^{kn} \left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} \right) - \frac{2\varphi}{c} A^{ik} \right] v_k \right\}. \quad (3.104)$$

The chr.inv.-scalar T , to within the multiplier $-\frac{1}{c^2}$, is the work done by the electromagnetic field to displace the charge e . The chr.inv.-vector B^i , to within the multiplier $\frac{1}{c}$, represents the chr.inv.-force below acting

on the charged particle due to the electromagnetic field and called the physically observable *chr.inv.-Lorentz force*

$$\Phi^i = c B^i = -e \left(E^i + \frac{1}{c} \varepsilon^{ikm} H_{*m} v_k \right). \quad (3.105)$$

The alternating sign appears here because the square equation with respect to $\frac{dt}{d\tau}$ has two roots (1.63) in the pseudo-Riemannian space. “Plus” in the Lorentz force stands for the particle’s motion to the future (with respect to the observer), and “minus” denotes the particle’s motion to the past. In a Galilean reference frame in the Minkowski space, there is no difference between the physically observable time τ and the coordinate time t . Therefore, the Lorentz force (3.99) obtained from the Minkowski equations has no alternating signs.

If the electric charge is not a point, but a distributed matter, then the Lorentz force $\Phi^\alpha = \frac{e}{c} F_{\cdot\sigma}^\alpha U^\sigma$ in the Minkowski equations (3.93) is replaced by the four-dimensional vector of the *Lorentz force density*

$$f^\alpha = \frac{1}{c} F_{\cdot\sigma}^\alpha j^\sigma, \quad (3.106)$$

where the four-dimensional current density $j^\sigma = \{c\rho; j^i\}$ is determined by the 1st group of the Maxwell equations (3.51)

$$j^\sigma = \frac{c}{4\pi} \nabla_\mu F^{\sigma\mu}. \quad (3.107)$$

The *chr.inv.-projections* of the Lorentz force density f^α are

$$\frac{f_0}{\sqrt{g_{00}}} = -\frac{1}{c} E_i j^i, \quad (3.108)$$

$$f^i = -\left(\rho E^i + \frac{1}{c} H_{\cdot k}^i j^k \right) = -\left(\rho E^i + \frac{1}{c} \varepsilon^{ikm} H_{*m} j_k \right), \quad (3.109)$$

and in the three-dimensional Euclidean space they take the form

$$\frac{f_0}{\sqrt{g_{00}}} = \frac{q}{c} = \frac{1}{c} \vec{E} \vec{j}, \quad (3.110)$$

$$\vec{f} = \rho \vec{E} + \frac{1}{c} [\vec{j}; \vec{H}], \quad (3.111)$$

where q is the heat power density released in the current conductor.

Transform the Lorentz force density (3.106), using the Maxwell equations. Substituting j^σ (3.107) we arrive at

$$\begin{aligned} f_\nu &= \frac{1}{c} F_{\nu\sigma} j^\sigma = \frac{1}{4\pi} F_{\nu\sigma} \nabla_\mu F^{\sigma\mu} = \\ &= \frac{1}{4\pi} \left[\nabla_\mu (F_{\nu\sigma} F^{\sigma\mu}) - F^{\sigma\mu} \nabla_\mu F_{\nu\sigma} \right]. \end{aligned} \quad (3.112)$$

Transpositioning the mute indices and using the antisymmetry of the Maxwell tensor $F_{\alpha\beta}$, we transform the second term to the form

$$\begin{aligned} F^{\sigma\mu} \nabla_\mu F_{\nu\sigma} &= \frac{1}{2} F^{\sigma\mu} (\nabla_\mu F_{\nu\sigma} + \nabla_\sigma F_{\mu\nu}) = \\ &= -\frac{1}{2} F^{\sigma\mu} \nabla_\nu F_{\mu\sigma} = \frac{1}{2} F^{\sigma\mu} \nabla_\nu F_{\sigma\mu}. \end{aligned} \quad (3.113)$$

As a result, for f_ν (3.112) and its contravariant form we obtain

$$f_\nu = \frac{1}{4\pi} \nabla_\mu \left(-F^{\mu\sigma} F_{\nu\sigma} + \frac{1}{4} \delta_\nu^\mu F^{\alpha\beta} F_{\alpha\beta} \right), \quad (3.114)$$

$$f^\nu = \frac{1}{4\pi} \nabla_\mu \left(-F^{\mu\sigma} F_{\cdot\sigma}^\nu + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right). \quad (3.115)$$

Introducing the notation

$$\frac{1}{4\pi} \left(-F^{\mu\sigma} F_{\cdot\sigma}^\nu + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) = T^{\mu\nu}, \quad (3.116)$$

we obtain the formula

$$f^\nu = \nabla_\mu T^{\mu\nu}, \quad (3.117)$$

according to which the four-dimensional vector of the Lorentz force density f^ν is equal to the absolute divergence of a quantity $T^{\mu\nu}$ called the *energy-momentum tensor* of the electromagnetic field. The tensor $T^{\mu\nu}$ is symmetric $T^{\mu\nu} = T^{\nu\mu}$, and its trace (given that the trace of the fundamental metric tensor is $g_{\mu\nu} g^{\mu\nu} = \delta_\nu^\nu = 4$) is zero

$$\begin{aligned} T_\nu^\nu &= g_{\mu\nu} T^{\mu\nu} = \frac{1}{4\pi} \left(-F^{\mu\sigma} F_{\mu\sigma} + \frac{1}{4} g_{\mu\nu} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) = \\ &= \frac{1}{4\pi} \left(-F^{\mu\sigma} F_{\mu\sigma} + F^{\alpha\beta} F_{\alpha\beta} \right) = 0. \end{aligned} \quad (3.118)$$

The chr.inv.-projections of the energy-momentum tensor are

$$q = \frac{T_{00}}{g_{00}}, \quad J^i = \frac{c T_0^i}{\sqrt{g_{00}}}, \quad U^{ik} = c^2 T^{ik}, \quad (3.119)$$

where the chr.inv.-scalar q is the observable field density, the chr.inv.-vector J^i is the observable density of the field momentum, and the chr. inv.-tensor U^{ik} is the observable density of the field momentum flux.

For the electromagnetic field energy-momentum tensor (3.116) we obtain

$$q = \frac{E^2 + H^{*2}}{8\pi}, \quad (3.120)$$

$$J^i = \frac{c}{4\pi} \varepsilon^{ikm} E_k H_{*m}, \quad (3.121)$$

$$U^{ik} = q c^2 h^{ik} - \frac{c^2}{4\pi} (E^i E^k + H^{*i} H^{*k}), \quad (3.122)$$

where $E^2 = h_{ik} E^i E^k$ and $H^{*2} = h_{ik} H^{*i} H^{*k}$.

Comparing the obtained formula for q (3.120) with that for the energy density in Classical Electrodynamics, we obtain

$$W = \frac{E^2 + H^2}{8\pi}, \quad (3.123)$$

where $E^2 = (\vec{E}; \vec{E})$ and $H^2 = (\vec{H}; \vec{H})$. We see that the chr.inv.-quantity q is the *observable energy density* of the electromagnetic field in the pseudo-Riemannian space.

Comparing the obtained formula for the chr.inv.-vector J^i (3.121) with that for Poynting's vector in Classical Electrodynamics we have

$$\vec{S} = \frac{c}{4\pi} (\vec{E}; \vec{H}), \quad (3.124)$$

from which we can see that the J^i is the *Poynting observable vector* in the pseudo-Riemannian space.

The correspondence of the third observable component U^{ik} (3.122) to the quantities of Classical Electrodynamics can be established using analogies with continuum mechanics, where a similar tensor is the three-dimensional stress tensor of an elementary volume of a medium. Therefore, the above U^{ik} is the *observable stress tensor* of the electromagnetic field in the pseudo-Riemannian space.

Expressing the left hand side of the identities for the Lorentz force density (3.108, 3.109) through the chr.inv.-components of the electromagnetic field energy-momentum tensor (3.120–3.122), we take into account the equation $f^\nu = \nabla_\mu T^{\mu\nu}$ (3.117) and the formulae for chr.inv.-components of the absolute divergence of an arbitrary symmetric tensor of the 2nd rank (2.138, 2.139). Thus, we obtain

$$\frac{* \partial q}{\partial t} + qD + \frac{1}{c^2} D_{ij} U^{ij} + {}^* \widetilde{\nabla}_i J^i - \frac{1}{c^2} F_i J^i = -\frac{1}{c} E_i J^i, \quad (3.125)$$

$$\begin{aligned} \frac{* \partial J^k}{\partial t} + DJ^k + 2(D_i^k + A_{;i}^k) J^i + {}^* \widetilde{\nabla}_i U^{ik} - qF^k = \\ = -\left(\rho E^k + \frac{1}{c} \varepsilon^{kim} H_{*i} j_m \right). \end{aligned} \quad (3.126)$$

The first chr.inv.-identity (3.125) shows that the observable change in time of the electromagnetic field density q with time depends on:

- The rate of change of the observable volume of the space, filled with the electromagnetic field (the term qD);
- The force caused by the space deformation (the term $D_{ij} U^{ij}$);
- The effect of the gravitational inertial force on the electromagnetic field momentum density (the term $F_i J^i$);
- The observable spatial variation (physical divergence) of the electromagnetic field momentum density (the term ${}^* \widetilde{\nabla}_i J^i$);
- The magnitudes and mutual orientation of the current density vector j^i and the electric strength vector E^i (on the right hand side of the identity).

The second chr.inv.-identity (3.126) shows the observable change in time of the electromagnetic field momentum density J^k depending on:

- The rate of change of the observable volume of the space, filled with the electromagnetic field (the term DJ^k);
- The force caused by the space deformation and the Coriolis force, which are expressed by the term $2(D_i^k + A_{;i}^k) J^i$;
- The effect of the gravitational inertial force on the observable density of the electromagnetic field (the term qF^k);
- The observable spatial variation of the field stress ${}^* \widetilde{\nabla}_i U^{ik}$;
- The effect of the observable Lorentz force density — the quantity $f^k = -\left(\rho E^k + \frac{1}{c} \varepsilon^{kim} H_{*i} j_m \right)$ on the right hand side.

In conclusion, we consider a particular case, where the electromagnetic field is isotropic. A formal definition of isotropic fields made using the Maxwell tensor [20] is a set of the two conditions

$$F_{\mu\nu}F^{\mu\nu} = 0, \quad F_{\mu\nu}F^{*\mu\nu} = 0, \quad (3.127)$$

which mean that the field invariants $J_1 = F_{\mu\nu}F^{\mu\nu}$ and $J_2 = F_{\mu\nu}F^{*\mu\nu}$ (3.25, 3.26) are zeroes. In the chr.inv.-notation, taking (3.28) into account, the conditions take the form

$$E^2 = H^{*2}, \quad E_i H^{*i} = 0. \quad (3.128)$$

So, an electromagnetic field in a pseudo-Riemannian space is observed as isotropic, if the observable lengths of its electric and magnetic strength vectors are equal, and the Poynting vector J^i (3.121) is zero

$$J^i = \frac{c}{4\pi} \varepsilon^{ikm} E_k H_{*m} = 0. \quad (3.129)$$

In terms of the chr.inv.-components of the energy-momentum tensor (3.120, 3.121), the obtained conditions (3.128) also mean that

$$J = cq, \quad (3.130)$$

where $J = \sqrt{J^2}$ and $J^2 = h_{ik} J^i J^k$. In other words, the observable momentum density J of any isotropic electromagnetic field depends only on the field density q .

3.6 The equations of motion of a charged particle, obtained by the parallel transport method

In this section, we will obtain the chr.inv.-equations of motion of a charged mass-bearing test-particle in an electromagnetic field, located in a four-dimensional pseudo-Riemannian space*.

*Generally speaking, using the method described herein we can also obtain equations of motion for a particle, which is not a test one. A test particle is one with charge and mass so small that they do not affect an electromagnetic or gravitational field, in which it moves.

There is also another approach to particle motion in the pseudo-Riemannian space. It is based on the elastodynamics of the space-time continuum — an extension of General Relativity, which was introduced a decade ago by Pierre A. Millette based on the analysis of the deformation of the space-time in terms of continuum mechanics. In

The desired equations are the chr.inv.-projections of the Levi-Civita parallel transport equations of the four-dimensional summary vector of a charged mass-bearing particle

$$Q^\alpha = P^\alpha + \frac{e}{c^2} A^\alpha, \quad (3.131)$$

where $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ is the four-dimensional momentum vector of the particle, and $\frac{e}{c^2} A^\alpha$ is an additional four-dimensional momentum that the particle gains from the interaction of its charge e with the electromagnetic field potential A^α deviating its trajectory from a geodesic line. Given this problem statement, the parallel transport of the superposition on the particle's non-geodesic momentum vector and the deviating vector is also geodesic, so that we have

$$\frac{d}{ds} \left(P^\alpha + \frac{e}{c^2} A^\alpha \right) + \Gamma_{\mu\nu}^\alpha \left(P^\mu + \frac{e}{c^2} A^\mu \right) \frac{dx^\nu}{ds} = 0. \quad (3.132)$$

By definition, a geodesic line is a *constant direction line*. This means that any vector tangential to such a line at a given point will remain tangential to this line along its entire length, when transported parallel to itself [9].

The equations of motion can also be obtained in another way — by considering the motion along a line of the least (shortest) length using the least action principle. Least length lines are also constant direction lines. But, for instance, in spaces with non-metric geometry, length is not defined as category. In this case, least length lines are neither defined and, therefore, we cannot use the least action method to obtain the equations of motion. Nevertheless, even in non-metric spaces we can define constant direction lines and a non-zero derivation parameter along them. Hence, we can assume that in metric spaces, to which Riemannian spaces belong, least length lines are merely a particular case of constant direction lines.

In accordance with the general formulae that we have obtained in Chapter 2, the chr.inv.-projections of the parallel transport equations

particular, he showed that the massive body itself is part of the spacetime fabric that is rotating. See his extensive paper and subsequent monograph on this subject: Millette P. A. Elastodynamics of the spacetime continuum. *The Abraham Zelmanov Journal*, 2012, vol. 5, 221–277. Millette P. A. Elastodynamics of the Spacetime Continuum. The 2nd expanded edition, American Research Press, Rehoboth (New Mexico), 2019, 415 pages.

(3.132) are defined as follows

$$\frac{d\tilde{\varphi}}{ds} + \frac{1}{c} \left(-F_i \tilde{q}^i \frac{d\tau}{ds} + D_{ik} \tilde{q}^i \frac{dx^k}{ds} \right) = 0, \quad (3.133)$$

$$\begin{aligned} \frac{d\tilde{q}^i}{ds} + \left(\frac{\tilde{\varphi}}{c} \frac{dx^k}{ds} + \tilde{q}^k \frac{d\tau}{ds} \right) (D_k^i + A_k^i) - \\ - \frac{\tilde{\varphi}}{c} F^i \frac{d\tau}{ds} + \Delta_{mk}^i \tilde{q}^m \frac{dx^k}{ds} = 0, \end{aligned} \quad (3.134)$$

where the space-time interval ds is assumed to be the derivation parameter along the trajectory, while $\tilde{\varphi}$ and \tilde{q}^i are the chr.inv.-projections of the dynamic vector Q^α (3.131) of the particle

$$\tilde{\varphi} = b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}} = \frac{1}{\sqrt{g_{00}}} \left(P_0 + \frac{e}{c^2} A_0 \right), \quad (3.135)$$

$$\tilde{q}^i = h_\alpha^i Q^\alpha = Q^i = P^i + \frac{e}{c^2} A^i. \quad (3.136)$$

The chr.inv.-projections of the momentum vector are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m, \quad P^i = \frac{1}{c} m v^i = \frac{1}{c} p^i, \quad (3.137)$$

where “plus” stands for the motion to the future (with respect to the observer), “minus” appears if the particle travels to the past, and $p^i = m \frac{dx^i}{d\tau}$ is the three-dimensional chr.inv.-momentum vector of the particle. The chr.inv.-projections of the additional momentum vector $\frac{e}{c^2} A^\alpha$ have the form

$$\frac{e}{c^2} \frac{A_0}{\sqrt{g_{00}}} = \frac{e}{c^2} \varphi, \quad \frac{e}{c^2} A^i = \frac{e}{c^2} q^i, \quad (3.138)$$

where φ is the scalar potential and q^i is the vector-potential of the acting electromagnetic field, which are the chr.inv.-components of the four-dimensional field potential A^α (3.8). Then $\tilde{\varphi}$ (3.135) and \tilde{q}^i (3.136), which are the chr.inv.-projections of the summary vector Q^α , are

$$\tilde{\varphi} = \pm m + \frac{e}{c^2} \varphi, \quad (3.139)$$

$$\tilde{q}^i = \frac{1}{c} \left(p^i + \frac{e}{c^2} q^i \right). \quad (3.140)$$

Substitute the quantities $\tilde{\varphi}$ and \tilde{q}^i into the general formulae for the chr.inv.-equations of motion (3.133, 3.134). Moving the terms characteristic of electromagnetic interaction to the right hand side, we arrive at the chr.inv.-equations of motion for a charged particle in our world (it travels to the future with respect to an ordinary observer)

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = -\frac{e}{c^2} \frac{d\varphi}{d\tau} + \frac{e}{c^3} (F_i q^i - D_{ik} q^i v^k), \quad (3.141)$$

$$\begin{aligned} \frac{d(mv^i)}{d\tau} - mF^i + 2m(D_k^i + A_k^i)v^k + m\Delta_{nk}^i v^n v^k = \\ = -\frac{e}{c} \frac{dq^i}{d\tau} - \frac{e}{c} \left(\frac{\varphi}{c} v^k + q^k \right) (D_k^i + A_k^i) + \frac{e\varphi}{c^2} F^i - \frac{e}{c} \Delta_{nk}^i q^n v^k. \end{aligned} \quad (3.142)$$

while for an analogous particle located in the mirror world (it travels to the past with respect to the observer) the equations have the form

$$-\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = -\frac{e}{c^2} \frac{d\varphi}{d\tau} + \frac{e}{c^3} (F_i q^i - D_{ik} q^i v^k), \quad (3.143)$$

$$\begin{aligned} \frac{d(mv^i)}{d\tau} + mF^i + m\Delta_{nk}^i v^n v^k = \\ = -\frac{e}{c} \frac{dq^i}{d\tau} - \frac{e}{c} \left(\frac{\varphi}{c} v^k + q^k \right) (D_k^i + A_k^i) + \frac{e\varphi}{c^2} F^i - \frac{e}{c} \Delta_{nk}^i q^n v^k. \end{aligned} \quad (3.144)$$

It is easy to see that the left hand side of the equations completely matches that of the chr.inv.-equations of motion of a free particle. The only difference is that the above equations include the right hand terms that characterize non-geodesic motion. Therefore, the right hand side is non-zero here; they take into account the influence of the electromagnetic field on the particle, as well as the influence of the physical and geometric properties of the space itself (F^i , A_{ik} , D_{ik} , Δ_{nk}^i). It is obvious that, if the particle is charge-free ($e = 0$), the right hand side terms turn to zero and the resulting equations completely match the chr.inv.-equations of motion of a free mass-bearing particle (see formulae 1.59, 1.60 and also 1.64, 1.65).

Let us consider the right hand side terms in detail. The obtained equations are absolutely symmetric for the motion either to the future or to the past and they change their sign once the charge sign changes. We denote the right hand side of the chr.inv.-scalar equations of motion

(3.141, 3.143) as T . Given that

$$\frac{d\varphi}{d\tau} = \frac{* \partial \varphi}{\partial t} + v^i \frac{* \partial \varphi}{\partial x^i}, \quad (3.145)$$

then using the formula for the covariant form of the electric strength E_i (3.14), we can represent T as follows

$$\begin{aligned} T = & -\frac{e}{c^2} E_i v^i - \frac{e}{c^2} \frac{* \partial \varphi}{\partial t} + \\ & + \frac{e}{c^3} \left(\frac{* \partial q_i}{\partial t} - D_{ik} q^k \right) v^i + \frac{e}{c^3} \left(q^i - \frac{\varphi}{c} v^i \right) F_i. \end{aligned} \quad (3.146)$$

Substituting this formula into (3.141, 3.143) and multiplying the results by c^2 , we obtain the equation for the relativistic energy $E = \pm mc^2$ of a charged particle travelling to the future and to the past

$$\begin{aligned} \frac{dE}{d\tau} - m F_i v^i + m D_{ik} v^i v^k = & -e E_i v^i - e \frac{* \partial \varphi}{\partial t} + \\ & + \frac{e}{c} \left(\frac{* \partial q_i}{\partial t} - D_{ik} q^k \right) v^i + \frac{e}{c} \left(q^i - \frac{\varphi}{c} v^i \right) F_i, \end{aligned} \quad (3.147)$$

$$\begin{aligned} -\frac{dE}{d\tau} - m F_i v^i + m D_{ik} v^i v^k = & -e E_i v^i - e \frac{* \partial \varphi}{\partial t} + \\ & + \frac{e}{c} \left(\frac{* \partial q_i}{\partial t} - D_{ik} q^k \right) v^i + \frac{e}{c} \left(q^i - \frac{\varphi}{c} v^i \right) F_i, \end{aligned} \quad (3.148)$$

where $e E_i v^i$ is the work done by the electric component of the electromagnetic field to displace the particle per unit time.

The obtained chr.inv.-scalar equations of motion of a charged particle (3.147, 3.148) is the *live forces theorem* in the pseudo-Riemannian space, represented in the chr.inv.-form. It is easy to see that, in a Galilean reference frame in the Minkowski space, the scalar equation of motion for the particle travelling to the future (3.147) matches the time component of the Minkowski equations (3.98). In the three-dimensional Euclidean space, the equation (3.147) transforms into the live forces theorem in Classical Electrodynamics which is $\frac{dE}{dt} = e \vec{E} \vec{u}$ (3.92).

Let us turn to the right hand side of the chr.inv.-vector equations of motion (3.142, 3.144). Denote them by M^i . Since

$$\frac{dq^i}{d\tau} = \frac{* \partial q^i}{\partial t} + v^k \frac{* \partial q^i}{\partial x^k}, \quad (3.149)$$

and taking into account that $\frac{* \partial h^{ik}}{\partial t} = -2D^{ik}$ (1.40),

$$\frac{* \partial q^i}{\partial t} = \frac{* \partial}{\partial t} (h^{ik} q_k) = -2D_k^i q^k + h^{ik} \frac{* \partial q_k}{\partial t}, \quad (3.150)$$

we obtain the M^i in the form

$$\begin{aligned} M^i = & -\frac{e}{c} h^{ik} \frac{* \partial q_k}{\partial t} + \frac{e \varphi}{c^2} (F^i + A^{ik} v_k) + \frac{e}{c} A^{ik} q_k + \\ & + \frac{e}{c} \left(q^k - \frac{\varphi}{c} v^k \right) D_k^i - \frac{e}{c} v^k \frac{* \partial q^i}{\partial x^k} - \frac{e}{c} \Delta_{nk}^i q^n v^k. \end{aligned} \quad (3.151)$$

Using the formulae for the chr.inv.-components E^i (3.11) and H^{ik} (3.12) of the Maxwell tensor $F_{\alpha\beta}$, we write down the first two terms and the third term from M^i (3.151) as follows

$$-\frac{e}{c} h^{ik} \frac{* \partial q_k}{\partial t} + \frac{e \varphi}{c^2} F^i = -e E^i + e h^{ik} \frac{* \partial \varphi}{\partial x^k}, \quad (3.152)$$

$$\frac{e \varphi}{c^2} A^{ik} v_k = \frac{e}{2c} h^{im} v^n \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) - \frac{e}{2c} H^{ik} v_k. \quad (3.153)$$

We write down the quantity H^{ik} as $H^{ik} = \varepsilon^{mik} H_{*m}$ (3.56). Then we have the following

$$\frac{e \varphi}{c^2} A^{ik} v_k = \frac{e}{2c} h^{im} v^n \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) - \frac{e}{2c} \varepsilon^{ikm} H_{*m} v_k, \quad (3.154)$$

$$\begin{aligned} M^i = & -e \left(E^i + \frac{1}{2c} \varepsilon^{ikm} v_k H_{*m} \right) + \frac{e}{c} \left(q^k - \frac{\varphi}{c} v^k \right) D_k^i + \\ & + e h^{ik} \frac{* \partial \varphi}{\partial x^k} + \frac{e}{c} A^{ik} q_k + \frac{e}{2c} h^{im} v^k \left(\frac{* \partial q_m}{\partial x^k} - \frac{* \partial q_k}{\partial x^m} \right) - \\ & - \frac{e}{c} v^k \frac{* \partial q^i}{\partial x^k} - \frac{e}{c} \Delta_{nk}^i q^n v^k, \end{aligned} \quad (3.155)$$

and the sum of the latter three terms in M^i is equal to

$$\begin{aligned} & \frac{e}{2c} h^{im} v^k \left(\frac{* \partial q_m}{\partial x^k} - \frac{* \partial q_k}{\partial x^m} \right) - \frac{e}{c} v^k \frac{* \partial q^i}{\partial x^k} - \frac{e}{c} \Delta_{nk}^i q^n v^k = \\ & = -\frac{e}{2c} h^{im} v_k \frac{* \partial q^k}{\partial x^m} - \frac{e}{2c} v^k \frac{* \partial q^i}{\partial x^k} - \frac{e}{2c} h^{im} q^n v^k \frac{* \partial h_{km}}{\partial x^n}. \end{aligned} \quad (3.156)$$

Finally, the chr.inv.-vector equations of motion of a charged particle (3.142, 3.144) that travels to the future and to the past take the following form, respectively

$$\begin{aligned}
& \frac{d(mv^i)}{d\tau} - mF^i + 2m(D_k^i + A_{k\cdot}^i)v^k + m\Delta_{nk}^i v^n v^k = \\
& = -e\left(E^i + \frac{1}{2c}\varepsilon^{ikm}v_k H_{*m}\right) + \\
& + \frac{e}{c}\left(q^k - \frac{\varphi}{c}v^k\right)D_k^i + eh^{ik}\frac{*\partial\varphi}{\partial x^k} + \frac{e}{c}A^{ik}q_k - \\
& - \frac{e}{2c}h^{im}v_k\frac{*\partial q^k}{\partial x^m} - \frac{e}{2c}v^k\frac{*\partial q^i}{\partial x^k} - \frac{e}{2c}h^{im}q^n v^k\frac{*\partial h_{km}}{\partial x^n},
\end{aligned} \tag{3.157a}$$

$$\begin{aligned}
& \frac{d(mv^i)}{d\tau} + mF^i + m\Delta_{nk}^i v^n v^k = \\
& = -e\left(E^i + \frac{1}{2c}\varepsilon^{ikm}v_k H_{*m}\right) + \\
& + \frac{e}{c}\left(q^k - \frac{\varphi}{c}v^k\right)D_k^i + eh^{ik}\frac{*\partial\varphi}{\partial x^k} + \frac{e}{c}A^{ik}q_k - \\
& - \frac{e}{2c}h^{im}v_k\frac{*\partial q^k}{\partial x^m} - \frac{e}{2c}v^k\frac{*\partial q^i}{\partial x^k} - \frac{e}{2c}h^{im}q^n v^k\frac{*\partial h_{km}}{\partial x^n}.
\end{aligned} \tag{3.157b}$$

Here the first term $-e\left(E^i + \frac{1}{2c}\varepsilon^{ikm}v_k H_{*m}\right)$ on the right hand side is different from the chr.inv.-Lorentz force $\Phi^i = -e\left(E^i + \frac{1}{c}\varepsilon^{ikm}v_k H_{*m}\right)$ by the coefficient $\frac{1}{2}$ in the term that stands for the magnetic component of the force. This fact is very surprising, because the ordinary equations of motion of a charged particle, which are the three-dimensional components of the general covariant equations of motion, contain the Lorentz force without any change. In §3.9 we will look for such a structure of the electromagnetic field potential A^α , with which the other terms in the M^i completely compensate the coefficient $\frac{1}{2}$ in the Lorentz force.

3.7 The equations of motion, obtained using the least action principle as a particular case of the previous equations

In this section, we are going to deduce chr.inv.-equations of motion of a mass-bearing charged particle, using the least action principle. The

principle says that an action S to displace a particle along a shortest trajectory is the least, so the variation of the action is zero

$$\delta \int_a^b dS = 0. \quad (3.158)$$

Therefore, the equations of motion, obtained from the least action principle are the equations of shortest length lines.

The action of a gravitational field and an electromagnetic field to displace a charged particle at an elementary interval ds is [10]

$$dS = -m_0 c ds - \frac{e}{c} A_\alpha dx^\alpha. \quad (3.159)$$

We see that this quantity is only applicable to particles that travel along non-isotropic trajectories ($ds \neq 0$). On the other hand, obtaining the equations of motion using the parallel transport method (constant direction lines) is applicable to both non-isotropic ($ds \neq 0$) and isotropic trajectories ($ds = 0$). Moreover, the parallel transport method is applicable to non-metric geometries, in particular, to particles that travel in a completely degenerate space-time (zero-space). Therefore, the equations of shortest length lines, since they are obtained using the least action method, are a narrow particular case of the equations of constant direction lines, which result from the parallel transport method.

Return to the least action principle (3.158). For a charged mass-bearing particle, this condition takes the form

$$\delta \int_a^b dS = -\delta \int_a^b m_0 c ds - \delta \int_a^b \frac{e}{c} A_\alpha dx^\alpha = 0, \quad (3.160)$$

where the first term can be expressed as follows

$$\begin{aligned} -\delta \int_a^b m_0 c ds &= -\int_a^b m_0 c DU_\alpha \delta x^\alpha = \\ &= \int_a^b m_0 c (dU_\alpha ds - \Gamma_{\alpha,\mu\nu} U^\mu dx^\nu) \delta x^\alpha. \end{aligned} \quad (3.161)$$

We represent the variation of the second integral from the initial formula (3.160) as the sum

$$-\frac{e}{c} \delta \int_a^b A_\alpha dx^\alpha = -\frac{e}{c} \left(\int_a^b \delta A_\alpha dx^\alpha + \int_a^b A_\alpha d\delta x^\alpha \right). \quad (3.162)$$

Integrating the second term, we obtain

$$\int_a^b A_\alpha d\delta x^\alpha = A_\alpha \delta x^\alpha \Big|_a^b - \int_a^b dA_\alpha \delta x^\alpha. \quad (3.163)$$

Here the first term is equal to zero, since the integral varies for given numerical values of the coordinates (integration limits). Taking into account the fact that the variation of any covariant vector is

$$\delta A_\alpha = \frac{\partial A_\alpha}{\partial x^\beta} \delta x^\beta, \quad dA_\alpha = \frac{\partial A_\alpha}{\partial x^\beta} dx^\beta, \quad (3.164)$$

we obtain the variation of the electromagnetic part of the action

$$-\frac{e}{c} \delta \int_a^b A_\alpha dx^\alpha = -\frac{e}{c} \int_a^b \left(\frac{\partial A_\alpha}{\partial x^\beta} dx^\alpha \delta x^\beta - \frac{\partial A_\alpha}{\partial x^\beta} \delta x^\alpha dx^\beta \right). \quad (3.165)$$

Transpositioning the free indices α and β in the first term of (3.165) and taking the variation of the gravitational part of the action (3.161) into account, we obtain the variation of the total action (3.160)

$$\delta \int_a^b dS = \int_a^b \left[m_0 c (dU_\alpha - \Gamma_{\alpha,\mu\nu} U^\mu dx^\nu) - \frac{e}{c} F_{\alpha\beta} dx^\beta \right] \delta x^\alpha, \quad (3.166)$$

where $F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}$ is the Maxwell tensor, and $U^\mu = \frac{dx^\mu}{ds}$ is the four-dimensional velocity of the particle. Since the quantity δx^α is arbitrary, the formula under the integral is always zero. Finally, we arrive at the general covariant equations of motion of the charged particle in their covariant (lower-index) form

$$m_0 c \left(\frac{dU_\alpha}{ds} - \Gamma_{\alpha,\mu\nu} U^\mu U^\nu \right) = \frac{e}{c} F_{\alpha\beta} U^\beta, \quad (3.167)$$

or, lifting the index α , at the contravariant form of the equations

$$m_0 c \left(\frac{dU^\alpha}{ds} + \Gamma_{\mu\nu}^\alpha U^\mu U^\nu \right) = \frac{e}{c} F^{\alpha\cdot} U^\beta. \quad (3.168)$$

These are actually the Minkowski equations (3.93) in the pseudo-Riemannian space. Therefore, their chr.inv.-projections can be called the chr.inv.-Minkowski equations in the pseudo-Riemannian space. For

an our-world charged particle (it travels to the future with respect to an ordinary observer), the chr.inv.-Minkowski equations have the form

$$\frac{dE}{d\tau} - mF_i v^i + mD_{ik} v^i v^k = -eE_i v^i, \quad (3.169)$$

$$\begin{aligned} \frac{d(mv^i)}{d\tau} - mF^i + 2m(D_k^i + A_k^i)v^k + m\Delta_{nk}^i v^n v^k = \\ = -e\left(E^i + \frac{1}{c}\varepsilon^{ikm}v_k H_{*m}\right), \end{aligned} \quad (3.170)$$

and for an analogous particle in the mirror world (it travels to the past) the equations have the form

$$-\frac{dE}{d\tau} - mF_i v^i + mD_{ik} v^i v^k = -eE_i v^i, \quad (3.171)$$

$$\frac{d(mv^i)}{d\tau} + mF^i + m\Delta_{nk}^i v^n v^k = -e\left(E^i + \frac{1}{c}\varepsilon^{ikm}v_k H_{*m}\right). \quad (3.172)$$

The chr.inv.-scalar equations of motion, both in our world and in the mirror world, represent the live forces theorem. The right hand side of the chr.inv.-vector equations represents the chr.inv.-Lorentz force in the pseudo-Riemannian space.

It is easy to see that, in a Galilean reference frame in the Minkowski space, the obtained chr.inv.-equations of motion become the ordinary live forces theorem (3.92) and the ordinary three-dimensional equations of motion (3.90) accepted in Classical Electrodynamics.

As is seen from the obtained chr.inv.-equations of motion, the right hand side of the equations (3.169–3.172), obtained using the least action method, is different from the right hand side of the equations (3.146, 3.157), obtained using the parallel transport method. The difference here is the absence in (3.169–3.172) of numerous terms, which characterize the structure of the acting electromagnetic field and the space itself. But as we have already mentioned above, shortest length lines are only a particular case of constant direction lines, determined by parallel transport.

Therefore, there is no surprise in that the parallel transport equations, as more general ones, have additional terms, which take into account the structure of the acting electromagnetic field and the structure of the space.

3.8 The geometric structure of the four-dimensional electromagnetic potential

In this section, we are going to find such a structure of the acting electromagnetic field potential A^α , under which the length of the summary vector $Q^\alpha = P^\alpha + \frac{e}{c^2} A^\alpha$ characteristic of a charged mass-bearing particle remains unchanged in the Levi-Civita parallel transport along the particle's trajectory. So, the four-dimensional pseudo-Riemannian space of General Relativity is assumed.

As is known, the Levi-Civita parallel transport preserves the length of any transported vector Q^α , therefore, the condition $Q_\alpha Q^\alpha = const$ is true along the entire trajectory of parallel transport. Since the square of the length of any n -dimensional vector is invariant in the n -dimensional pseudo-Riemannian space, in which the vector is located, the above condition is true in any reference frame, including any observer who accompanies his reference body. Hence, we can analyse the condition $Q_\alpha Q^\alpha = const$, formulating it with physically observable quantities in the accompanying reference frame, i.e., in the chr.inv.-form.

The components of the summary vector Q^α of a charged particle in the accompanying reference frame have the form

$$Q_0 = \left(1 - \frac{w}{c^2}\right) \left(\pm m + \frac{e\varphi}{c^2}\right), \quad (3.173)$$

$$Q^0 = \frac{1}{1 - \frac{w}{c^2}} \left[\left(\pm m + \frac{e\varphi}{c^2}\right) + \frac{1}{c^2} v_i \left(m v^i + \frac{e}{c} q^i\right) \right], \quad (3.174)$$

$$Q_i = -\frac{1}{c} \left(m v_i + \frac{e}{c} q_i\right) - \frac{1}{c} \left(\pm m + \frac{e\varphi}{c^2}\right) v_i, \quad (3.175)$$

$$Q^i = \frac{1}{c} \left(m v^i + \frac{e}{c} q^i\right), \quad (3.176)$$

and its square is

$$Q_\alpha Q^\alpha = m_0^2 + \frac{e^2}{c^4} (\varphi^2 - q_i q^i) + \frac{2me}{c^2} \left(\pm\varphi - \frac{1}{c} v_i q^i\right). \quad (3.177)$$

From here, we can see that the square of the summary momentum of a charged particle consists of the three quantities:

- a) The square of the four-dimensional momentum of the particle, which is the term $P_\alpha P^\alpha = m_0^2$;

- b) The square of the four-dimensional additional momentum $\frac{e}{c^2} A^\alpha$ that the particle gains from the acting electromagnetic field (it is the second term);
- c) The term $\frac{2me}{c^2} \left(\pm \varphi - \frac{1}{c} v_i q^i \right)$ that describes the interaction between the mass of this particle m and its electric charge e .

In the above formula for $Q_\alpha Q^\alpha$ (3.177), the first term m_0^2 remains unchanged. In other words, this term is an invariant and, therefore, it does not depend on the reference frame. Our task is to deduce such conditions, under which the entire formula (3.177) remains unchanged.

Propose that the field vector-potential q^i has the following structure

$$q^i = \frac{\varphi}{c} v^i. \quad (3.178)$$

In this case* the second term of (3.177) is

$$\frac{e^2}{c^4} A_\alpha A^\alpha = \frac{e^2 \varphi^2}{c^4} \left(1 - \frac{v^2}{c^2} \right). \quad (3.179)$$

Transforming the third term in the same way, we obtain the square of the vector Q^α (3.177) in the form

$$Q_\alpha Q^\alpha = m_0^2 + \frac{e^2 \varphi^2}{c^4} \left(1 - \frac{v^2}{c^2} \right) + \frac{2m_0 e}{c^2} \varphi \sqrt{1 - \frac{v^2}{c^2}}. \quad (3.180)$$

Then, introducing the following notation for the scalar potential

$$\varphi = \frac{\varphi_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (3.181)$$

we can represent the obtained formula (3.180) as follows

$$Q_\alpha Q^\alpha = m_0^2 + \frac{e^2 \varphi_0^2}{c^4} + \frac{2m_0 e \varphi_0}{c^2} = \text{const.} \quad (3.182)$$

So, the length of the summary vector Q^α remains unchanged in its parallel transport, if the observable potentials φ and q^i of the field are

*A similar problem could be solved, assuming that $q^i = \pm \frac{e}{c} v^i$. But in comparative analysis of two groups of the equations only positive numerical values of $q^i = \frac{e}{c} v^i$ will be important, because the observer's physical time τ , by definition, flows from the past to the future only, so the physically observable time intervals $d\tau$ are always positive.

related to its four-dimensional potential A^α as follows

$$\frac{A_0}{\sqrt{g_{00}}} = \varphi = \frac{\varphi_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad A^i = q^i = \frac{\varphi}{c} v^i. \quad (3.183)$$

In this case, the vector $\frac{e}{c^2} A^\alpha$ that characterizes the interaction of the particle's charge with the electromagnetic field has the form

$$\frac{e}{c^2} \frac{A_0}{\sqrt{g_{00}}} = \frac{e\varphi_0}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}, \quad \frac{e}{c^2} A^i = \frac{e\varphi_0}{c^3} \frac{v^i}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (3.184)$$

The dimensions of the vectors $\frac{e}{c^2} A^\alpha$ and $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ in CGSE and Gaussian systems of units are the same as that of mass m [gram].

Comparing the chr.inv.-projections of the above vectors, we can see the same quantity in the interaction of the particle's charge with the acting electromagnetic field

$$\frac{e\varphi}{c^2} = \frac{e\varphi_0}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}, \quad (3.185)$$

where $e\varphi$ is the potential energy of the particle travelling with the observable velocity $v^i = \frac{dx^i}{dt}$ in the acting electromagnetic field (this particle is at rest with respect to the observer and his reference body). In general, the scalar potential φ is the potential energy of the field, divided by unit charge. Then, $e\varphi$ is the potential *relativistic-energy* of the particle having a charge e and travelling in the electromagnetic field, and $e\varphi_0$ is its rest-energy in the field. When the particle rests in the field, its potential rest-energy is equal to the potential relativistic-energy.

Comparing $E = mc^2$ and $W = e\varphi$, we arrive at the same conclusion. Respectively, $\frac{W_0}{c^2} = \frac{e\varphi_0}{c^2}$ is an electromagnetic quantity analogous to the rest-mass m_0 . Then, the chr.inv.-quantity $\frac{e}{c^2} A^i = \frac{e\varphi}{c^2} v^i$ is similar to the observable chr.inv.-momentum vector $p^i = m v^i$. Therefore, when the particle rests in the electromagnetic field, its "electromagnetic projection" onto the observer's spatial section (it is a chr.inv.-vector) is zero, while only the time projection (potential rest-energy $e\varphi_0 = \text{const}$) is observable. But if the particle travels in the field, having a non-zero velocity v^i , its observable "electromagnetic projections" become the potential relativistic-energy $e\varphi$ and the three-dimensional momentum $\frac{e\varphi}{c^2} v^i$.

Having the chr.inv.-projections of the vector $\frac{e}{c^2} A^\alpha$ calculated for the given field structure (3.183), we can restore the vector A^α in the general covariant form. Taking into account that

$$A^i = q^i = \frac{\varphi}{c} v^i = \frac{\varphi}{c \sqrt{1 - \frac{v^2}{c^2}}} \frac{dx^i}{d\tau} = \varphi_0 \frac{dx^i}{ds}, \quad (3.186)$$

we obtain the desired general covariant notation for A^α

$$A^\alpha = \varphi_0 \frac{dx^\alpha}{ds}, \quad \frac{e}{c^2} A^\alpha = \frac{e\varphi_0}{c^2} \frac{dx^\alpha}{ds}, \quad (3.187)$$

the chr.inv.-projections of which are

$$\frac{A_0}{\sqrt{g_{00}}} = \pm \varphi = \pm \frac{\varphi_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad A^i = q^i = \frac{\varphi}{c} v^i, \quad (3.188)$$

where the alternating sign appears in the time chr.inv.-projection, which is not the case in the initial formula (3.183).

Naturally, a question arises: how did the scalar observable component of the vector A^α , initially defined as φ , acquire the alternating sign under the given structure of the A^α (3.187)? The answer is that, in the first case, the quantities φ and q^i were *defined* based on the general rule of building chr.inv.-quantities. But without knowing the structure of the projected vector A^α , we cannot calculate them. Therefore, in the formulae for the time and spatial projections (3.183), the symbols φ and q^i merely *denote* the quantities without revealing their structure. On the contrary, in the formulae (3.188) the quantities φ and q^i were *calculated* using the formulae $\varphi = \sqrt{g_{00}} A^0 + \frac{g_{0i}}{\sqrt{g_{00}}} A^i$ and $q^i = A^i$, where the components A^0 and A^i were given. Hence, in the second case, the quantity $\pm \varphi$ results from the calculation that sets forth the specific formula

$$\varphi = \pm \frac{\varphi_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (3.189)$$

As a result, the *calculated* chr.inv.-projections of the vector $\frac{e}{c^2} A^\alpha$ have the following formulation

$$\frac{e}{c^2} \frac{A_0}{\sqrt{g_{00}}} = \pm \frac{e\varphi}{c^2} = \pm \frac{e\varphi_0}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}, \quad \frac{e}{c^2} A^i = \frac{e\varphi}{c^3} v^i, \quad (3.190)$$

where “plus” stands for a particle located in our world, so travelling from the past to the future, while “minus” stands for a particle located in the mirror world, which travels to the past with respect to us. The square of the vector’s length is

$$\frac{e^2}{c^4} A_\alpha A^\alpha = \frac{e^2 \varphi^2}{c^4} \left(1 - \frac{v^2}{c^2} \right) = \frac{e^2 \varphi_0^2}{c^4} = \text{const.} \quad (3.191)$$

The vector $\frac{e}{c^2} A^\alpha$ has a real length at $v^2 < c^2$, zero length at $v^2 = c^2$ and an imaginary length at $v^2 > c^2$. However, we limit our study to the real form of the vector (subluminal velocities), because light-like and superluminal charged particles are unknown.

Comparing the formulae for $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ and $\frac{e}{c^2} A^\alpha = \frac{e\varphi_0}{c^2} \frac{dx^\alpha}{ds}$, we can see that these vectors are collinear, so they are tangential to the same non-isotropic trajectory, to which the derivation parameter ds is assumed. Hence, in this case, the momentum vector of the particle P^α is co-directed with the acting electromagnetic field, so the particle is travelling “along” the field.

Consider a general case, where the vectors are not collinear. The third term in the square of the summary vector $Q_\alpha Q^\alpha$ (3.177) is the doubled scalar product of the vectors P^α and $\frac{e}{c^2} A^\alpha$. The Levi-Civita parallel transport leaves their scalar product unchanged

$$D(P_\alpha A^\alpha) = A^\alpha DP_\alpha + P_\alpha DA^\alpha = 0, \quad (3.192)$$

therefore, we obtain

$$\frac{2e}{c^2} P_\alpha A^\alpha = \frac{2me}{c^2} \left(\pm \varphi - \frac{1}{c} v_i q^i \right) = \text{const.}, \quad (3.193)$$

i.e., the scalar product of P^α and $\frac{e}{c^2} A^\alpha$ remains unchanged. Consequently, the lengths of the vectors remain unchanged. In particular,

$$A_\alpha A^\alpha = \varphi^2 - q_i q^i = \text{const.} \quad (3.194)$$

The scalar product of two vectors is the product of their lengths multiplied by the cosine of the angle between them. Therefore, the Levi-Civita parallel transport leaves the angle between the transported vectors unchanged

$$\cos(P^\alpha; A^\alpha) = \frac{P_\alpha A^\alpha}{m_0 \sqrt{\varphi^2 - q_i q^i}} = \text{const.} \quad (3.195)$$

Taking into account the formula for the relativistic mass m , we can re-write the condition (3.193) as follows

$$\frac{2e}{c^2} P_\alpha A^\alpha = \pm \frac{2m_0 e}{c^2} \frac{\varphi}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{2m_0 e}{c^2} \frac{v_i q^i}{c \sqrt{1 - \frac{v^2}{c^2}}} = \text{const}, \quad (3.196)$$

or as the relationship between the scalar and vector potentials

$$\pm \frac{\varphi}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{v_i q^i}{c \sqrt{1 - \frac{v^2}{c^2}}} = \text{const}. \quad (3.197)$$

For instance, we can find the relationship between the potentials φ and q^i for the case, where the momentum vector of the particle P^α is orthogonal to the additional momentum $\frac{e}{c^2} A^\alpha$ gained from the electromagnetic field. Since the parallel transport leaves the angle between transported vectors unchanged (3.195), the cosine of the angle between the above two transported orthogonal vectors is zero. So, we have

$$P_\alpha A^\alpha = \pm \varphi - \frac{1}{c} v_i q^i = 0, \quad (3.198)$$

i.e., if the particle travels in the electromagnetic field so that the vectors P^α and A^α are orthogonal, then the field scalar potential is

$$\varphi = \pm \frac{1}{c} v_i q^i, \quad (3.199)$$

so it is the scalar product of the particle's observable velocity v^i and the spatial observable vector-potential of the field q^i .

Now, we are going to obtain a formula for the square of the summary vector Q^α , assuming that the structure of the electromagnetic field potential is $A^\alpha = \varphi_0 \frac{dx^\alpha}{ds}$ (3.187). So, the field potential A^α is co-directed with the particle's momentum vector P^α . Then

$$Q_\alpha Q^\alpha = m^2 - \frac{m^2}{c^2} v_i v^i + \frac{e^2}{c^4} (\varphi^2 - q_i q^i) = m_0^2 + \frac{e^2}{c^4} \varphi_0^2. \quad (3.200)$$

Multiplying both sides of the equation by c^4 and denoting the relativistic energy of the particle as $E = mc^2$, we obtain

$$E^2 - c^2 p^2 + e^2 \varphi^2 - e^2 q_i q^i = E_0^2 + e^2 \varphi_0^2. \quad (3.201)$$

3.9 Minkowski's equations as a particular case

In §3.6 we considered a charged mass-bearing particle in a pseudo-Riemannian space. There, the general covariant equations of its motion were obtained using the *parallel transport method*. We have also obtained the chr.inv.-projections of the general covariant equations.

We have showed that their time chr.inv.-projection (3.147) in a Galilean reference frame takes the form of the time component of the Minkowski equations (3.98), becoming the live forces theorem in the three-dimensional Euclidean space of Classical Electrodynamics (3.92). On the other hand, the right hand side of the spatial chr.inv.-projection has the term $-e\left(E^i + \frac{1}{2c} \varepsilon^{ikm} v_k H_{*m}\right)$ instead of the chr.inv.-Lorentz force, which is $\Phi^i = -e\left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m}\right)$, and also several other additional terms that depend on the observable characteristics of the acting electromagnetic field and of the space itself.

Therefore, for the spatial chr.inv.-projections of the equations of motion of a charged particle in a pseudo-Riemannian space, the correspondence principle with the three-dimensional components of the Minkowski equations is set non-trivially.

On the other hand, the equations of constant direction lines, obtained using the parallel transport method in a pseudo-Riemannian space, are a more general case of the equations of shortest length lines, obtained using the least action principle. The equations of motion, obtained using the least action principle in §3.7, have the structure matching that of the Minkowski equations. Consequently, we can suppose that the chr.inv.-projections of the equations of motion obtained in §3.6 are more general ones; in a particular case, i.e., under specific conditions, they can be transformed into the chr.inv.-projections of the equations of motion, obtained using the least action principle in §3.7.

To find exactly under what conditions this can be true, we are going to consider the spatial chr.inv.-projections of the equations of motion (3.157), which contain the mismatch with the Lorentz force.

For the convenience of analysis, we considered the right hand side of (3.157) as a separate formula denoted as M^i . Substituting the magnetic strength H^{ik} (3.12) into the term $\frac{e\varphi}{c^2} A^{ik} v_k$ of the formula for the M^i , we write down the term as follows

$$\frac{e\varphi}{c^2} A^{ik} v_k = \frac{e}{2c} h^{im} v^n \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) - \frac{e}{2c} \varepsilon^{ikm} H_{*m} v_k, \quad (3.202)$$

where $\varepsilon^{ikm}H_{*m} = H^{ik}$ according to the chronometrically invariant formalism (see Chapter 2).

So forth, we substitute the chr.inv.-components of the electromagnetic field potential A^α (3.188) into (3.157). With the field potential A^α (3.188), the additional momentum vector $\frac{e}{c^2}A^\alpha$ that the electrically charged particle gains from the electromagnetic field is tangential to the particle's trajectory.

Using the first formula, $q_m = \frac{\varphi}{c}v_m$, we arrive at the dependence of the right hand side under consideration on only the scalar potential of the electromagnetic field

$$M^i = -e \left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m} \right) + e h^{ik} \left(1 - \frac{v^2}{c^2} \right) \frac{* \partial \varphi}{\partial x^k} + \frac{e \varphi}{2} h^{ik} \frac{* \partial}{\partial x^k} \left(1 - \frac{v^2}{c^2} \right). \quad (3.203)$$

Substituting the obtained relativistic formula of the scalar electromagnetic potential φ (3.181) into this formula, we see that the sum of the last two terms becomes zero

$$- \frac{e \varphi}{2} h^{ik} \frac{* \partial}{\partial x^k} \left(1 - \frac{v^2}{c^2} \right) + \frac{e \varphi}{2} h^{ik} \frac{* \partial}{\partial x^k} \left(1 - \frac{v^2}{c^2} \right) = 0. \quad (3.204)$$

Then M^i takes the form of the chr.inv.-Lorentz force

$$M^i = -e \left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m} \right), \quad (3.205)$$

which is exactly what we had to prove.

Now, consider the right hand side $c^2 T$ of the chr.inv.-scalar equation of motion (3.147) under the condition, according to which the vector A^α has the structure that is mentioned above and is tangential to the particle's trajectory. Substituting the formulae for the chr.inv.-projections φ and q^i of the vector A^α having the given structure into (3.146), we transform the quantity $c^2 T$ to the form

$$\begin{aligned} c^2 T &= -e E_i v^i - e \frac{* \partial \varphi}{\partial t} + \frac{e}{c^2} \left[\frac{* \partial}{\partial t} (\varphi h_{ik} v^k) - \varphi D_{ik} q^k \right] v^i = \\ &= -e E_i v^i - e \frac{* \partial \varphi}{\partial t} \left(1 - \frac{v^2}{c^2} \right) + \frac{e \varphi}{c^2} D_{ik} v^i v^k + \frac{e \varphi}{c^2} v_k \frac{* \partial v^k}{\partial t}. \end{aligned} \quad (3.206)$$

Using the relativistic formula for φ (3.181) in the first derivative, then returning to φ again after derivation, we obtain

$$\begin{aligned} c^2 T &= -e E_i v^i - \frac{e\varphi}{2c^2} \frac{*}{\partial t} (h_{ik} v^i v^k) + \frac{e\varphi}{c^2} D_{ik} v^i v^k + \frac{e\varphi}{c^2} v_k \frac{*}{\partial t} v^k = \\ &= -e E_i v^i - \frac{e\varphi}{2c^2} \left(\frac{*}{\partial t} h_{ik} v^i v^k + 2 v_k \frac{*}{\partial t} v^k \right) + \\ &\quad + \frac{e\varphi}{c^2} D_{ik} v^i v^k + \frac{e\varphi}{c^2} v_k \frac{*}{\partial t} v^k = -e E_i v^i, \end{aligned} \quad (3.207)$$

because we took into account that $\frac{*}{\partial t} h_{ik} = 2 D_{ik}$ by definition of the space deformation tensor D_{ik} (1.40).

So, the chr.inv.-equations of motion of a charged mass-bearing particle, obtained in the four-dimensional pseudo-Riemannian space using the parallel transport method, match the equations, obtained using the least action principle in the particular case, where:

- a) The electromagnetic field potential A^α has the following structure $A^\alpha = \varphi_0 \frac{dx^\alpha}{ds}$ (3.187);
- b) The field potential A^α is tangential to the four-dimensional trajectory of the travelling particle.

In particular, for the above structure of the electromagnetic field potential in a Galilean reference frame in the Minkowski space, the obtained chr.inv.-equations of motion completely match the live forces theorem (chr.inv.-scalar equation of motion) and the Minkowski equations (chr.inv.-vector equations) in the three-dimensional Euclidean space, thus taking the form known in Classical Electrodynamics.

Noteworthy, this is another illustration of the geometric fact that the shortest length lines (determined by the least action principle) are merely a narrow particular case of constant direction lines (resulting from the Levi-Civita parallel transport).

3.10 Structure of a space filled with a stationary electromagnetic field

It is obvious that, when assuming a particular structure of the electromagnetic field, we impose a certain limit on the motion of electrically charged particles, which, in its turn, imposes a limitation on the structure of the pseudo-Riemannian space, in which the charged particles travel. We are going to find out what kind of structure the pseudo-

Riemannian space should have so that a charged particle can travel in a stationary electromagnetic field.

As we have obtained earlier in this Chapter, the chr.inv.-equations of motion of a charged particle in our world have the form

$$\frac{dE}{d\tau} - mF_i v^i + mD_{ik} v^i v^k = -e \frac{d\varphi}{d\tau} + \frac{e}{c} (F_i q^i - D_{ik} q^i v^k), \quad (3.208)$$

$$\begin{aligned} \frac{d(mv^i)}{d\tau} - mF^i + 2m(D_k^i + A_{k.}^i) v^k + m\Delta_{nk}^i v^n v^k = \\ = -\frac{e}{c} \frac{dq^i}{d\tau} - \frac{e}{c} \left(\frac{\varphi}{c} v^k + q^k \right) (D_k^i + A_{k.}^i) + \frac{e\varphi}{c^2} F^i - \frac{e}{c} \Delta_{nk}^i q^n v^k. \end{aligned} \quad (3.209)$$

Since we assume the electromagnetic field to be stationary, the field potentials φ and q^i do not depend on time. In this case, the chr.inv.-components of the electromagnetic field tensor are

$$E_i = \frac{* \partial \varphi}{\partial x^i} - \frac{\varphi}{c^2} F_i = \frac{\partial \varphi}{\partial x^i} - \varphi \frac{\partial}{\partial x^i} \ln \left(1 - \frac{w}{c^2} \right), \quad (3.210)$$

$$H^{*i} = \frac{1}{2} \varepsilon^{imn} H_{mn} = \frac{1}{2} \varepsilon^{imn} \left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} - \frac{2\varphi}{c} A_{mn} \right). \quad (3.211)$$

From the above, we can find the limitations imposed on the space metric due to the stationary state of the acting electromagnetic field.

The formulae for E_i and H^{*i} , together with the chr.inv.-derivatives of the scalar and vector electromagnetic potentials, include the observable properties of the space such as the chr.inv.-vector of the gravitational inertial force F_i and the chr.inv.-tensor of the space non-holonomy A_{ik} . It is obvious that, in a stationary electromagnetic field, the mentioned properties of the space must be stationary as well

$$\frac{* \partial F_i}{\partial t} = 0, \quad \frac{* \partial F^i}{\partial t} = 0, \quad \frac{* \partial A_{ik}}{\partial t} = 0, \quad \frac{* \partial A^{ik}}{\partial t} = 0. \quad (3.212)$$

From the above definitions, we see that the quantities F_i and A_{ik} are stationary (they do not depend on time), if the linear velocity with which the space rotates is stationary, $\frac{\partial v_i}{\partial t} = 0$. Therefore, the condition $\frac{\partial v_i}{\partial t} = 0$, i.e., the stationary rotation of the space, turns the chr.inv.-derivative with respect to spatial coordinates into the ordinary derivative

$$\frac{* \partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{1}{c^2} \frac{* \partial}{\partial t} = \frac{\partial}{\partial x^i}. \quad (3.213)$$

Since the chr.inv.-derivative with respect to time is different from the ordinary derivative only by the multiplier $\frac{\partial}{\partial t} = \left(1 - \frac{w}{c^2}\right) \frac{* \partial}{\partial t}$, the ordinary derivative of a stationary quantity is zero as well.

For the space deformations rate tensor D_{ik} , under a stationary rotation of the space we have

$$\frac{* \partial D_{ik}}{\partial t} = \frac{1}{2} \frac{* \partial h_{ik}}{\partial t} = \frac{1}{2} \frac{* \partial}{\partial t} \left(-g_{ik} + \frac{1}{c^2} v_i v_k \right) = -\frac{1}{2} \frac{* \partial g_{ik}}{\partial t}. \quad (3.214)$$

Because in the stationary case under consideration the right hand side of the equations of motion is stationary, the left hand side must be stationary too. This means that the space does not deform. Then, according to (3.124), the three-dimensional coordinate metric g_{ik} does not depend on time, so the chr.inv.-Christoffel symbols Δ_{jk}^i (1.47) are stationary too.

Using the chr.inv.-components of the Maxwell tensor (3.210, 3.211), we transform the Maxwell equations (3.63, 3.64) to the case of the stationary electromagnetic field. As a result, we have

$$\left. \begin{aligned} \frac{\partial E^i}{\partial x^i} + \frac{\partial \ln \sqrt{h}}{\partial x^i} E^i - \frac{2}{c} \Omega_{*m} H^{*m} &= 4\pi\rho \\ \varepsilon^{ikm} \widetilde{\nabla}_k (H_{*m} \sqrt{h}) &= \frac{4\pi}{c} j^i \sqrt{h} \end{aligned} \right\} \text{I}, \quad (3.215)$$

$$\left. \begin{aligned} \frac{\partial H^{*i}}{\partial x^i} + \frac{\partial \ln \sqrt{h}}{\partial x^i} H^{*i} + \frac{2}{c} \Omega_{*m} E^m &= 0 \\ \varepsilon^{ikm} \widetilde{\nabla}_k (E_m \sqrt{h}) &= 0 \end{aligned} \right\} \text{II}. \quad (3.216)$$

Then the Lorenz condition (3.65) and the continuity equation (3.66), respectively, take the form

$$*\widetilde{\nabla}_i q^i = 0, \quad *\widetilde{\nabla}_i j^i = 0. \quad (3.217)$$

So, we have found the way in which any stationary state of an electromagnetic field that fills a pseudo-Riemannian space affects the physically observable properties of the space itself and, hence, the main equations of electrodynamics.

In the next sections, §3.11–§3.13, we will use the above results for solving the equations of motion of a charged particle (3.208, 3.209) in stationary electromagnetic fields of the three kinds:

- 1) A stationary electric field (the magnetic strength is zero);
- 2) A stationary magnetic field (the electric strength is zero);
- 3) A stationary electromagnetic field (both the magnetic and electric components of the field are non-zero).

3.11 Motion in a stationary electric field

We are going to consider the motion of a charged mass-bearing particle in a pseudo-Riemannian space, filled with a stationary electromagnetic field of the strictly electric kind (the magnetic component of the field is zero in this case).

What conditions should the space satisfy to allow the existence of a stationary electromagnetic field of the strictly electric kind? From the formula for a stationary state of the magnetic strength

$$H_{ik} = \frac{\partial q_i}{\partial x^k} - \frac{\partial q_k}{\partial x^i} - \frac{2\varphi}{c} A_{ik} \quad (3.218)$$

we see that $H_{ik} = 0$ is satisfied under the two conditions:

- a) The vector-potential q^i is irrotational $\frac{\partial q_i}{\partial x^k} = \frac{\partial q_k}{\partial x^i}$;
- b) The space is holonomic $A_{ik} = 0$.

The stationary electric strength E_i (3.210) is the sum of the spatial derivative of the scalar electromagnetic potential φ and the term $\frac{\varphi}{c^2} F_i$. But in a real Earth-bound laboratory, the ratio between the gravitational potential and the square of the light velocity is nothing, but only

$$\frac{w}{c^2} = \frac{GM_{\oplus}}{c^2 R_{\oplus}} \approx 10^{-10}, \quad (3.219)$$

therefore the second term in (3.210) can be neglected, so the E_i depend only on the spatial distribution of the scalar potential

$$E_i = \frac{\partial \varphi}{\partial x^i}. \quad (3.220)$$

Because the right hand side of the equations of motion (it stands for the Lorentz force) is stationary, the left hand side must be stationary too. This is true under the conditions that we are considering, if the space deformation tensor is zero (the space does not deform). So, if a stationary electromagnetic field has the non-zero electric component and zero magnetic component, then the pseudo-Riemannian space, which is filled with the field, must satisfy the following conditions:

- a) The potential w of the acting gravitational field is negligible $w \approx 0$;
- b) The space does not rotate $A_{ik} = 0$;
- c) The space does not deform $D_{ik} = 0$.

To simplify further calculations, assume that the observer's three-dimensional space is similar to the Euclidean one, so we assume $\Delta_{nk}^i \approx 0$. Then the chr.inv.-equations of motion of a particle having an electric charge e (3.208, 3.209) take the form

$$\frac{dm}{d\tau} = -\frac{e}{c^2} \frac{d\varphi}{d\tau}, \quad (3.221)$$

$$\frac{d}{d\tau}(mv^i) = -\frac{e}{c} \frac{dq^i}{d\tau}. \quad (3.222)$$

From the chr.inv.-scalar equation of motion (live forces theorem), we see that the change of the particle's relativistic energy $E = mc^2$ is due to the work done by the electric component E_i of the electromagnetic field to displace the particle.

From the chr.inv.-vector equations of motion, we see that the particle's observable momentum changes due to the change of the electromagnetic field vector-potential q^i . Assuming that the four-dimensional electromagnetic field potential is tangential to the four-dimensional trajectory of the particle, we obtain the three-dimensional Lorentz force

$$\Phi^i = -eE^i \quad (3.223)$$

on the right hand side. That is, in this case, the particle's observable momentum changes under the action of the electric strength of the electromagnetic field.

Both of the groups of the chr.inv.-Maxwell equations for a stationary electromagnetic field (3.215, 3.216) in this case become simple

$$\left. \begin{array}{l} \frac{\partial E^i}{\partial x^i} = 4\pi\rho \\ j^i = 0 \end{array} \right\} \text{I}, \quad \left. \varepsilon^{ikm} \frac{\partial E_m}{\partial x^k} = 0 \right\} \text{II}. \quad (3.224)$$

Integrating the chr.inv.-scalar equation of motion (live forces theorem), we arrive at the so-called *live forces integral*

$$m + \frac{e\varphi}{c^2} = B = \text{const}, \quad (3.225)$$

where B is an integration constant.

Another consequence from the chr.inv.-Maxwell equations is that, in the present case, the scalar potential of the field satisfies:

- 1) Poisson's equation $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 4\pi\rho$, if $\rho \neq 0$;
- 2) Laplace's equation $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$, if $\rho = 0$.

So, we have found such properties of the pseudo-Riemannian space, which allow charged particles to travel in a stationary electric field. It would be natural now to obtain exact solutions to the chr.inv.-equations of motion for such a particle (3.221, 3.222). But, unless a particular structure of the electromagnetic field itself is determined by the Maxwell equations, this cannot be done. For this reason, to simplify further calculations, we assume that the given field is homogeneous.

Assume that the covariant chr.inv.-vector of the electric strength E_i is directed along the x axis. Following Landau and Lifshitz (see §20 of *The Classical Theory of Fields* [10]), we are going to consider a charged particle that is *repulsed* by the field — the case of a negative numerical value of the electric strength and the increasing coordinate x of the particle*. Then the components of the vector E_i are

$$E_1 = E_x = -E = \text{const}, \quad E_2 = E_3 = 0. \quad (3.226)$$

Because the field homogeneity means $E_i = \frac{\partial \varphi}{\partial x^i} = \text{const}$, the scalar potential φ is a function of x , which satisfies the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial E}{\partial x} = 0. \quad (3.227)$$

This means that the homogeneous stationary electric field satisfies the condition of the absence of the field-inducing charges $\rho = 0$.

Let a charged particle travel along the electric strength vector E_i , i.e., along x . Then the chr.inv.-equations of its motion have the form

$$\frac{dm}{d\tau} = -\frac{e}{c^2} \frac{d\varphi}{d\tau} = -\frac{e}{c^2} \frac{d\varphi}{dx^i} v^i = \frac{e}{c^2} E \frac{dx}{d\tau}, \quad (3.228)$$

$$\frac{d}{d\tau} \left(m \frac{dx}{d\tau} \right) = eE, \quad \frac{d}{d\tau} \left(m \frac{dy}{d\tau} \right) = 0, \quad \frac{d}{d\tau} \left(m \frac{dz}{d\tau} \right) = 0. \quad (3.229)$$

*Naturally, if a charged particle is *attracted* by the field, then the electric strength is positive, while the particle's coordinate decreases.

Integrating the chr.inv.-scalar equation of motion (live forces theorem), we arrive at the live forces integral

$$m = \frac{eE}{c^2} x + B, \quad B = \text{const.} \quad (3.230)$$

The integration constant B can be obtained from the initial conditions $m|_{\tau=0} = m_{(0)}$ and $x|_{\tau=0} = x_{(0)}$. So, we obtain

$$B = m_{(0)} - \frac{eE}{c^2} x_{(0)}, \quad (3.231)$$

therefore the solution (3.230) takes the form

$$m = \frac{eE}{c^2} (x - x_{(0)}) + m_{(0)}. \quad (3.232)$$

Substituting the obtained solution into the chr.inv.-vector equations of motion (3.229), we bring them to the form*

$$\left. \begin{aligned} \frac{eE}{c^2} \dot{x}^2 + \left(B + \frac{eE}{c^2} x \right) \ddot{x} &= eE \\ \frac{eE}{c^2} \dot{x}\dot{y} + \left(B + \frac{eE}{c^2} x \right) \ddot{y} &= 0 \\ \frac{eE}{c^2} \dot{x}\dot{z} + \left(B + \frac{eE}{c^2} x \right) \ddot{z} &= 0 \end{aligned} \right\}. \quad (3.233)$$

From here, we realize that the last two equations in (3.233) are ordinary equations with separable variables, which have the form

$$\frac{\ddot{y}}{\dot{y}} = \frac{-\frac{eE}{c^2} \dot{x}}{B + \frac{eE}{c^2} x}, \quad \frac{\ddot{z}}{\dot{z}} = \frac{-\frac{eE}{c^2} \dot{x}}{B + \frac{eE}{c^2} x}, \quad (3.234)$$

and are easy to integrate. Their solutions are

$$\dot{y} = \frac{C_1}{B + \frac{eE}{c^2} x}, \quad \dot{z} = \frac{C_2}{B + \frac{eE}{c^2} x}, \quad (3.235)$$

where C_1 and C_2 are integration constants. They can be found by setting the initial conditions $\dot{y}|_{\tau=0} = \dot{y}_{(0)}$ and $\dot{z}|_{\tau=0} = \dot{z}_{(0)}$ and using the formula for B (3.121). As a result, we obtain

$$C_1 = m_{(0)} \dot{y}_{(0)}, \quad C_2 = m_{(0)} \dot{z}_{(0)}. \quad (3.236)$$

*The dot stands for the derivation with respect to the physically observable time τ .

Let us solve the equation of motion along x , which is the first of the equations (3.233). Denote $\dot{x} = \frac{dx}{d\tau} = p$. Then

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dp}{dt} = \frac{dp}{dx} \frac{dx}{dt} = pp', \quad (3.237)$$

and, therefore, the above equation of motion along x can be transformed into an equation with separable variables, which has the form

$$\frac{p dp}{1 - \frac{p^2}{c^2}} = \frac{eE dx}{B + \frac{eE}{c^2} x}, \quad (3.238)$$

and is solved as a standard integral. After integrating the above equation, we arrive at the following solution

$$\sqrt{1 - \frac{p^2}{c^2}} = \frac{C_3}{B + \frac{eE}{c^2} x}, \quad C_3 = \text{const.} \quad (3.239)$$

Assuming $p = \dot{x}|_{\tau=0} = \dot{x}_{(0)}$ and substituting B from (3.231), we obtain the following formula for the integration constant C_3

$$C_3 = m_{(0)} \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}}. \quad (3.240)$$

In the case under consideration, we can replace the interval of the physically observable time $d\tau$ with the coordinate time interval dt . We will explain why in the next section.

In *The Classical Theory of Fields* [10], Landau and Lifshitz solved the equations of motion of a charged particle in a Galilean reference frame in the Minkowski space (space-time of Special Relativity). Naturally, in order to compare our solutions in the pseudo-Riemannian space with theirs, consider the same particular case of the motion in a homogeneous stationary electric field as they did (see §20 in *The Classical Theory of Fields*). To do this, we should place $F_i = 0$ and $A_{ik} = 0$ in our equations. As a result, we obtain that in this case

$$d\tau = \left(1 - \frac{w}{c^2}\right) dt - \frac{1}{c^2} v_i dx^i = dt. \quad (3.241)$$

In other words, in the four-dimensional region, where the particle travels, the three-dimensional metric is Galilean.

Substituting the variable $p = \frac{dx}{dt}$ into the formula (3.239) we arrive at the equation with separable variables

$$\frac{dx}{dt} = c \frac{\sqrt{\left(B + \frac{eE}{c^2}x\right)^2 - C_3^2}}{B + \frac{eE}{c^2}x}, \quad (3.242)$$

the solution of which is the function

$$ct = \frac{c^2}{eE} \sqrt{\left(B + \frac{eE}{c^2}x\right)^2 - C_3^2} + C_4, \quad C_4 = \text{const}, \quad (3.243)$$

where the integration constant C_4 , taking into account the initial conditions at the moment of time $t = 0$, is

$$C_4 = -\frac{m_{(0)}c}{eE} \dot{x}_{(0)}. \quad (3.244)$$

Now, formulating the coordinate x explicitly from (3.243) with t , we obtain the final solution to the spatial chr.inv.-equations of motion of the charged particle along x

$$x = \frac{c^2}{eE} \left[\sqrt{\frac{e^2 E^2}{c^4} (ct - C_4)^2 + C_3^2} - B \right], \quad (3.245)$$

or, after substituting the found integration constants,

$$x = \sqrt{\left(ct + \frac{m_{(0)}c \dot{x}_{(0)}}{eE}\right)^2 + \left(\frac{m_{(0)}c^2}{eE}\right)^2 \left(1 - \frac{\dot{x}_{(0)}^2}{c^2}\right)} - \frac{m_{(0)}c^2}{eE} + x_{(0)}. \quad (3.246)$$

If the field *attracts* the particle (i.e., the electric strength is positive $E_1 = E_x = E = \text{const}$), we will obtain the same solution for x but having the opposite sign

$$x = \frac{c^2}{eE} \left[B - \sqrt{\frac{e^2 E^2}{c^4} (ct - C_4)^2 + C_3^2} \right]. \quad (3.247)$$

In *The Classical Theory of Fields* [10], the same problem is considered. But, in contrast to our solution in the pseudo-Riemannian space,

Landau and Lifshitz solved this problem through integrating the three-dimensional components of the general covariant equations of motion (the Minkowski three-dimensional equations) without accounting for the live forces theorem. Their formula for x is

$$x = \frac{1}{eE} \sqrt{(m_0 c^2)^2 + (ceEt)^2}. \quad (3.248)$$

This formula matches our solution (3.245), if $x_{(0)} - \frac{m_{(0)}c^2}{eE} = 0$ and the initial velocity of the particle is zero $\dot{x}_{(0)} = 0$. The latter manifests the significant simplifications assumed in *The Classical Theory of Fields*, according to which some integration constants are zero.

It is easy to see that, even when solving the equations of motion in a Galilean reference frame in the Minkowski space, the mathematical methods of chronometric invariants give a certain advantage revealing the hidden factors that are left unnoticed when solving the three-dimensional components of the general covariant equations of motion. This means that, even when physically observable quantities coincide with coordinate quantities, it is geometrically correct to solve a *system* of the chr.inv.-equations of motion, because the live forces theorem, being their scalar part, inevitably affects the solution to the vector equations.

Of course, in the case of an inhomogeneous non-stationary electric field, some additional terms will appear in our solution to reveal the more complicated and time varying field structure.

Let us now calculate the three-dimensional trajectory of the particle in the homogeneous stationary electric field that we are considering. To do this, we integrate the equations of motion along the axes y and z (3.235), then express time from there and substitute it into the solution for x , which we have obtained.

First, substituting the obtained solution for x (3.245) into the equation for \dot{y} , we obtain the equation with separable variables

$$\frac{dy}{dt} = \frac{C_1}{\sqrt{\frac{e^2 E^2}{c^4} (ct - C_4)^2 + C_3^2}}, \quad (3.249)$$

integrating which we have

$$y = \frac{m_{(0)} \dot{y}_{(0)} c}{eE} \operatorname{arc} \sinh \frac{eEt + m_{(0)} \dot{x}_{(0)}}{m_{(0)} c \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}}} + C_5, \quad (3.250)$$

where C_5 is an integration constant. From $y = y_{(0)}$ at $t = 0$, we find

$$C_5 = y_{(0)} - \frac{m_{(0)} \dot{y}_{(0)} c}{eE} \operatorname{arc} \sinh \frac{\dot{x}_{(0)}}{c \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}}}. \quad (3.251)$$

Substituting the constant into y (3.250) we finally have

$$y = y_{(0)} + \frac{m_{(0)} \dot{y}_{(0)} c}{eE} \times \left\{ \operatorname{arc} \sinh \frac{eEt + m_{(0)} \dot{x}_{(0)}}{m_{(0)} c \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}}} - \operatorname{arc} \sinh \frac{\dot{x}_{(0)}}{c \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}}} \right\}. \quad (3.252)$$

Formulating from here t with y and $y_{(0)}$ and taking into account that $a = \operatorname{arc} \sinh b$ if $b = \sinh a$, after substituting $\operatorname{arc} \sinh b = \ln(b + \sqrt{b^2 + 1})$ into the second term we have

$$t = \frac{1}{eE} \left\{ m_{(0)} c \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}} \times \sinh \left[\frac{y - y_{(0)}}{m_{(0)} \dot{y}_{(0)} c} eE + \ln \frac{\dot{x}_{(0)} + c}{c \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}}} - m_{(0)} \dot{x}_{(0)} \right] \right\}. \quad (3.253)$$

Substitute it into our solution for x (3.246). As a result we obtain the desired equation for the three-dimensional trajectory of the particle

$$x = x_{(0)} + \frac{m_{(0)} c^2}{eE} \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}} \times \cosh \left\{ \frac{y - y_{(0)}}{m_{(0)} \dot{y}_{(0)} c} eE + \ln \frac{\dot{x}_{(0)} + c}{c \sqrt{1 - \frac{\dot{x}_{(0)}^2}{c^2}}} - \frac{m_{(0)} c^2}{eE} \right\}. \quad (3.254)$$

The obtained formula means that a charged particle in a homogeneous stationary electric field, located in our world, travels along a curve

based on a *chain line*, while the factors that deviate the particle from the “purely” chain line are functions of the initial conditions.

Our formula (3.254) completely matches the result from *The Classical Theory of Fields*, which is formula 20.5 in [10]

$$x = \frac{m_{(0)}c^2}{eE} \cosh \frac{eEy}{m_{(0)}\dot{y}_{(0)}c}, \quad (3.255)$$

once we assume that $x_{(0)} - \frac{m_{(0)}c^2}{eE} = 0$, and the initial velocity of the particle is $\dot{x}_{(0)} = 0$. The latter condition assumes that the integration constant in the chr.inv.-scalar equation of motion (live forces theorem) is zero, which is obviously not always true, but can be assumed only in a very narrow particular case.

At low velocities, after equating the relativistic terms to zero and expanding the hyperbolic cosine into series $\cosh b = 1 + \frac{b^2}{2!} + \frac{b^4}{4!} + \frac{b^6}{6!} + \dots$, our formula for the three-dimensional trajectory of the particle (3.254), with higher-order terms withheld, takes the form

$$x = x_{(0)} + \frac{eE(y - y_{(0)})^2}{2m_{(0)}\dot{y}_{(0)}^2}, \quad (3.256)$$

so the particle travels along a *parabola*. Thus, once the initial coordinates of the particle are assumed zeroes, our solution (3.256) completely matches the result from *The Classical Theory of Fields*, which is

$$x = \frac{eE y^2}{2m_{(0)}\dot{y}_{(0)}^2}. \quad (3.257)$$

Integrating the equation of motion along the z axis gives the same results. This is because the only difference between the equations with respect to \dot{y} and \dot{z} (3.235) is a fixed coefficient — the integration constant (3.236), which is equal to the initial momentum of the particle along y (in the equation for \dot{y}) and along z (in the equation for \dot{z}).

Let us find the properties of the particle (such as its energy and momentum) affected by the acting homogeneous stationary electric field. Calculating the relativistic square root (with the above assumptions)

$$\sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}} = \frac{m_{(0)} \sqrt{1 - \frac{\dot{x}_{(0)}^2 + \dot{y}_{(0)}^2 + \dot{z}_{(0)}^2}{c^2}}}{m_{(0)} + \frac{eE}{c^2}(x - x_{(0)})}, \quad (3.258)$$

we obtain the energy of the particle

$$E = \frac{m_{(0)}c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_{(0)}c^2 + eE(x - x_{(0)})}{\sqrt{1 - \frac{\dot{x}_{(0)}^2 + \dot{y}_{(0)}^2 + \dot{z}_{(0)}^2}{c^2}}}, \quad (3.259)$$

which at the velocity much lower than the light velocity is

$$E = m_{(0)}c^2 + eE(x - x_{(0)}). \quad (3.260)$$

The relativistic momentum of the particle is obtained in the same way, but since the formula is bulky we would not include it here.

So, we have studied the motion of a charged particle in a homogeneous stationary electric field, located in our world. So forth, we will consider the motion of an analogous particle of the mirror world under the same conditions.

The chr.inv.-equations of motion of a charged particle in a stationary electric field that fills the mirror world, taking into account the above constraints imposed on the geometric structure of the space, have the form

$$\frac{dm}{d\tau} = \frac{e}{c^2} \frac{d\varphi}{d\tau}, \quad (3.261)$$

$$\frac{d}{d\tau}(mv^i) = -\frac{e}{c} \frac{dq^i}{d\tau}. \quad (3.262)$$

The difference from the equations of motion in our world (3.221, 3.222) is only the sign in the live forces theorem.

Assume that the electric strength is negative (i.e., the field *repulses* the charged particle) and that the particle travels along the field strength, so its motion is co-directed with the x axis.

Then, integrating the live forces theorem for the mirror-world particle (3.261), we obtain the live forces integral

$$m = -\frac{eE}{c^2} x + B, \quad (3.263)$$

where the integration constant $B = const$, calculated from the initial conditions, is

$$B = m_{(0)} + \frac{eE}{c^2} x_{(0)}. \quad (3.264)$$

Substituting the results into the chr.inv.-vector equations of motion (3.262), we have (compare these equations with 3.233)

$$\left. \begin{aligned} -\frac{eE}{c^2} \dot{x}^2 + \left(B - \frac{eE}{c^2} x \right) \ddot{x} &= eE \\ -\frac{eE}{c^2} \dot{x}\dot{y} + \left(B - \frac{eE}{c^2} x \right) \ddot{y} &= 0 \\ -\frac{eE}{c^2} \dot{x}\dot{z} + \left(B - \frac{eE}{c^2} x \right) \ddot{z} &= 0 \end{aligned} \right\}. \quad (3.265)$$

After some algebra similar to that done to obtain the trajectory of an analogous charged particle in our world, we arrive at

$$x = \frac{c^2}{eE} \left[B - \sqrt{C_3^2 - \frac{e^2 E^2}{c^4} (ct - C_4)^2} \right], \quad (3.266)$$

where $C_3 = m_{(0)} \sqrt{1 + \frac{\dot{x}_{(0)}^2}{c^2}}$ and $C_4 = -\frac{cm_{(0)}\dot{x}_{(0)}}{eE}$. Or,

$$\begin{aligned} x = -\sqrt{\left(\frac{m_{(0)}c^2}{eE}\right)^2 \left(1 + \frac{\dot{x}_{(0)}^2}{c^2}\right)} - \left(ct + \frac{m_{(0)}c\dot{x}_{(0)}}{eE}\right)^2 + \\ + \frac{m_{(0)}c^2}{eE} + x_{(0)}. \end{aligned} \quad (3.267)$$

The obtained coordinate x of the mirror-world charged particle, *repulsed* by the stationary electric field, is similar to that for an analogous particle of our world, which is *attracted* by the field (3.247), i.e., when the electric strength is positive $E_1 = E_x = E = const$. Consequently, we arrive at the interesting conclusion: the transition of a charged particle from our world into the mirror world (where time flows in the opposite direction) is the same as changing the sign of its charge.

Noteworthy, we had arrived at the same conclusion about masses of particles [19]: the purported transition of a particle from our world into the mirror world is the same as changing the sign of its mass. Hence, our-world particles and mirror-world particles are mass and charge complementary.

Let us find the three-dimensional trajectory of the charged particle in the homogeneous stationary electric field that fills the mirror world.

Calculating y in the same way as for the our-world particle, we have

$$y = y_{(0)} + \frac{m_{(0)}\dot{y}_{(0)}c}{eE} \times \left\{ \arcsin \frac{eEt + m_{(0)}\dot{x}_{(0)}}{m_{(0)}c \sqrt{1 + \frac{\dot{x}_{(0)}^2}{c^2}}} - \arcsin \frac{\dot{x}_{(0)}}{c \sqrt{1 + \frac{\dot{x}_{(0)}^2}{c^2}}} \right\}. \quad (3.268)$$

In contrast to the formula for the our-world particle (3.252), this formula has an ordinary arcsine and “plus” under the square root.

Formulating time t from here with the coordinates y and $y_{(0)}$

$$t = \frac{1}{eE} \left\{ m_{(0)}c \sqrt{1 + \frac{\dot{x}_{(0)}^2}{c^2}} \times \sin \left[\frac{y - y_{(0)}}{m_{(0)}\dot{y}_{(0)}c} eE + \ln \frac{\dot{x}_{(0)} + c}{c \sqrt{1 + \frac{\dot{x}_{(0)}^2}{c^2}}} - m_{(0)}\dot{x}_{(0)} \right] \right\}, \quad (3.269)$$

and substituting it into our formula for x (3.267), we obtain the final formula for the trajectory

$$x = x_{(0)} - \frac{m_{(0)}c^2}{eE} \sqrt{1 + \frac{\dot{x}_{(0)}^2}{c^2}} \times \cos \left\{ \frac{y - y_{(0)}}{m_{(0)}\dot{y}_{(0)}c} eE + \arcsin \frac{\dot{x}_{(0)}}{c \sqrt{1 + \frac{\dot{x}_{(0)}^2}{c^2}}} \right\} - \frac{m_{(0)}c^2}{eE}. \quad (3.270)$$

In other words, the motion of the particle is *harmonic oscillation*. Once we assume the initial coordinates of the particle equal to zero, as well as its initial velocity $\dot{x}_{(0)} = 0$ and the integration constant $B = 0$, the obtained equation for the trajectory takes the much simpler form

$$x = -\frac{m_{(0)}c^2}{eE} \cos \frac{eEy}{m_{(0)}\dot{y}_{(0)}c}. \quad (3.271)$$

At low velocities, after equating the relativistic terms to zero and expanding the cosine into series $\cos b = 1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \dots \approx 1 - \frac{b^2}{2!}$ (this is always possible within a smaller part of the trajectory), our formula (3.270) becomes

$$x = x_{(0)} + \frac{eE(y - y_{(0)})^2}{2m_{(0)}\dot{y}_{(0)}^2}, \quad (3.272)$$

which is the equation of a *parabola*. So, the charged particle in the mirror world at low velocity travels along a parabola, as the our-world particle does under the same conditions in the field.

Therefore, a charged particle of our world travels in a homogeneous stationary electric field along a chain line, which at low velocities becomes a parabola. An analogous mirror-world particle travels along a harmonic trajectory, each small part of which at low velocities is a parabola (as in the case of the our-world particle).

3.12 Motion in a stationary magnetic field

Let us consider the motion of a charged particle in the case, where the electric component of the electromagnetic field is zero, while the magnetic component is non-zero and stationary. In this case, the chr.inv.-electric and magnetic strengths are

$$E_i = \frac{* \partial \varphi}{\partial x^i} - \frac{\varphi}{c^2} F_i = \frac{\partial \varphi}{\partial x^i} - \frac{\varphi}{c^2} \frac{1}{1 - \frac{w}{c^2}} \frac{\partial w}{\partial x^i} = 0, \quad (3.273)$$

$$H^{*i} = \frac{1}{2} \varepsilon^{imn} H_{mn} = \frac{1}{2} \varepsilon^{imn} \left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} - \frac{2\varphi}{c} A_{mn} \right) \neq 0, \quad (3.274)$$

because, if the field is strictly magnetic ($\varphi = const$ and $E_i = 0$), then the effect of gravitation can be neglected. From (3.274) we can see that the magnetic strength H^{*i} is not zero, if at least one of the following conditions is true:

- a) The vector potential q^i of the field is rotational;
- b) The space is non-holonomic $A_{ik} \neq 0$.

We will consider the motion of the particle in a general case, where both of the above conditions are true (we will also use a non-holonomic space later as the basic space for spin particles). As in the previous section §3.11, we assume deformations of the space to be zero and the

three-dimensional metric to be Euclidean $g_{ik} = \delta_{ik}$. However, the observable metric $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ in this case is not Galilean, because in a non-holonomic space we always have $h_{ik} \neq -g_{ik}$.

Assume that the space rotates around the z axis with a constant angular velocity $\Omega_{12} = -\Omega_{21} = \Omega$. Then the linear velocity of this rotation $v_i = \Omega_{ik} x^k$ has the two non-zero components $v_1 = \Omega y$ and $v_2 = -\Omega x$, and the non-holonomy tensor has the only non-zero component $A_{12} = -A_{21} = -\Omega$. In this case, the space metric takes the form

$$ds^2 = c^2 dt^2 - 2\Omega y dt dx + 2\Omega x dt dy - dx^2 - dy^2 - dz^2, \quad (3.275)$$

where $F_i = 0$ and $D_{ik} = 0$. In §3.11, which focused on a charged particle in a stationary electric field, we could assume that the Christoffel symbols are zeroes, i.e., consider the particle's motion in a Galilean reference frame in the Minkowski space. In contrast, in this section, the three-dimensional observable metric h_{ik} is not Euclidean due to the space rotation, and the Christoffel symbols Δ^i_{jk} (1.47) are not zeroes.

If the linear velocity with which the space rotates is not infinitesimal compared to the velocity of light, then the components of the chr.inv.-metric tensor h_{ik} are

$$h_{11} = 1 + \frac{\Omega^2 y^2}{c^2}, \quad h_{22} = 1 + \frac{\Omega^2 x^2}{c^2}, \quad h_{12} = -\frac{\Omega^2 xy}{c^2}, \quad h_{33} = 1, \quad (3.276)$$

and its determinant and the corresponding h^{ik} components are

$$h = \det \|h_{ik}\| = h_{11} h_{22} - h_{12}^2 = 1 + \frac{\Omega^2 (x^2 + y^2)}{c^2}, \quad (3.277)$$

$$\left. \begin{aligned} h^{11} &= \frac{1}{h} \left(1 + \frac{\Omega^2 x^2}{c^2} \right), & h^{22} &= \frac{1}{h} \left(1 + \frac{\Omega^2 y^2}{c^2} \right) \\ h^{12} &= \frac{\Omega^2 xy}{hc^2}, & h^{33} &= 1 \end{aligned} \right\}. \quad (3.278)$$

Based on the h_{ik} components, we obtain the non-zero components of the chr.inv.-Christoffel symbols Δ^i_{jk} (1.47)

$$\Delta^1_{11} = -\frac{2\Omega^4 xy^2}{c^4 \left[1 + \frac{\Omega^2 (x^2 + y^2)}{c^2} \right]}, \quad (3.279)$$

$$\Delta_{12}^1 = \frac{\Omega^2 y \left(1 + \frac{2\Omega^2 x^2}{c^2}\right)}{c^2 \left[1 + \frac{\Omega^2(x^2 + y^2)}{c^2}\right]}, \quad (3.280)$$

$$\Delta_{22}^1 = -\frac{2\Omega^2 x}{c^2} \frac{1 + \frac{\Omega^2 x^2}{c^2}}{1 + \frac{\Omega^2(x^2 + y^2)}{c^2}}, \quad (3.281)$$

$$\Delta_{11}^2 = -\frac{2\Omega^2 y}{c^2} \frac{1 + \frac{\Omega^2 y^2}{c^2}}{1 + \frac{\Omega^2(x^2 + y^2)}{c^2}}, \quad (3.282)$$

$$\Delta_{12}^2 = \frac{\Omega^2 x \left(1 + \frac{\Omega^2 y^2}{c^2}\right)}{c^2 \left[1 + \frac{\Omega^2(x^2 + y^2)}{c^2}\right]}, \quad (3.283)$$

$$\Delta_{22}^2 = -\frac{2\Omega^4 x^2 y}{c^4 \left[1 + \frac{\Omega^2(x^2 + y^2)}{c^2}\right]}. \quad (3.284)$$

We will solve the chr.inv.-equations of motion of a charged particle in the stationary magnetic field that fills the pseudo-Riemannian space. To make the calculations easier, assume that the four-dimensional field potential A^α is tangential to the four-dimensional trajectory of the particle. Since the electric field component is zero $E_i = 0$, it does not perform any work, so the right hand side of the chr.inv.-scalar equation of motion turns into zero.

Then, the chr.inv.-equations of motion of a charged particle (3.208, 3.209) belonging to our world take the following form

$$\frac{dm}{d\tau} = 0, \quad (3.285)$$

$$\frac{d}{d\tau}(m v^i) + 2m A_k^i v^k + m \Delta_{nk}^i v^n v^k = -\frac{e}{c} \varepsilon^{ikm} v_k H_{*m}, \quad (3.286)$$

while for an analogous charged particle travelling in the same stationary magnetic field, located in the mirror world, we have

$$-\frac{dm}{d\tau} = 0, \quad (3.287)$$

$$\frac{d}{d\tau}(m v^i) + m \Delta_{nk}^i v^n v^k = -\frac{e}{c} \varepsilon^{ikm} v_k H_{*m}. \quad (3.288)$$

Integrating the live forces theorem for the our-world particle and the mirror-world particle we obtain, respectively

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \text{const} = B, \quad -m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \text{const} = \widetilde{B}, \quad (3.289)$$

where B and \widetilde{B} are integration constants. The constants actually mean that $v^2 = \text{const}$, i.e., the modulus of the particle's observable velocity remains unchanged in the absence of the electric component of the electromagnetic field.

Then the chr.inv.-vector equations of motion for an our-world particle (3.286) take the form

$$\frac{dv^i}{d\tau} + 2A_{k \cdot}^i v^k + \Delta_{nk}^i v^n v^k = -\frac{e}{mc} \varepsilon^{ikm} v_k H_{*m}, \quad (3.290)$$

while for a mirror-world particle (3.288) we have the same equations, but without the term $2A_{k \cdot}^i v^k$, namely

$$\frac{dv^i}{d\tau} + \Delta_{nk}^i v^n v^k = -\frac{e}{mc} \varepsilon^{ikm} v_k H_{*m}. \quad (3.291)$$

The magnetic strength is determined by the Maxwell equations for a stationary field (3.215, 3.216), which, with zero electric strength and under the constraints assumed in this section, take the form

$$\left. \begin{aligned} \Omega_{*m} H^{*m} &= -2\pi c \rho \\ \varepsilon^{ikm} * \nabla_k (H_{*m} \sqrt{h}) &= \frac{4\pi}{c} j^i \sqrt{h} \end{aligned} \right\} \text{I}, \quad (3.292)$$

$$\left. * \nabla_i H^{*i} = \frac{\partial H^{*i}}{\partial x^i} + \frac{\partial \ln \sqrt{h}}{\partial x^i} H^{*i} = 0 \right\} \text{II}. \quad (3.293)$$

From the first equation of the 1st group, we see that the scalar product of the space non-holonomy pseudovector and the magnetic strength pseudovector is a function of the charge density. As a result, if the charge density of the stationary magnetic field is $\rho = 0$, then the pseudovectors Ω_{*i} and H^{*i} are orthogonal.

Henceforth, we will consider two possible orientations of the magnetic field strength with respect to the pseudovector of the space non-holonomy (rotation).

3.12.1 The magnetic field is co-directed with the non-holonomy field

Assume that the magnetic strength pseudovector H^{*i} is directed along the z axis, i.e., in the same direction as the pseudovector of the angular velocity $\Omega^{*i} = \frac{1}{2} \varepsilon^{ikm} A_{km}$ with which the space rotates. In this case, the space rotation pseudovector has just one non-zero component $\Omega^{*3} = \Omega$, while the magnetic strength pseudovector has as well one non-zero component that has the form

$$\begin{aligned} H^{*3} &= \frac{1}{2} \varepsilon^{3mn} H_{mn} = \frac{1}{2} \left(\varepsilon^{312} H_{12} + \varepsilon^{321} H_{21} \right) = H_{12} = \\ &= \frac{\varphi}{c} \left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} \right) + \frac{2\varphi}{c} \Omega. \end{aligned} \quad (3.294)$$

The condition $\varphi = const$ is derived based on the condition of the absence of the electric component of the field. Hence, the 1st group of the Maxwell equations (3.292) in this case have the form

$$\left. \begin{aligned} \Omega_{*3} H^{*3} &= \frac{\Omega \varphi}{c} \left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} \right) + \frac{2\varphi \Omega^2}{c} = -2\pi c \rho \\ \frac{\partial}{\partial y} (H_{*3} \sqrt{h}) &= \frac{4\pi}{c} j^1 \sqrt{h} \\ -\frac{\partial}{\partial x} (H_{*3} \sqrt{h}) &= \frac{4\pi}{c} j^2 \sqrt{h} \\ j^3 &= 0 \end{aligned} \right\}. \quad (3.295)$$

The 2nd group of the equations (3.293) turn to the trivial relationship $\frac{\partial H^{*3}}{\partial z} = 0$, so we have $H^{*3} = const$. This means that the stationary magnetic field that we are considering is homogeneous along z . Next, we assume that the stationary magnetic field is strictly homogeneous $H^{*i} = const$. Then, from the first equation of the 1st group (3.295), we see that the field is homogeneous under the following two conditions

$$\left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} \right) = const, \quad \rho = -\frac{\varphi \Omega^2}{\pi c^2} = const. \quad (3.296)$$

Hence, the charge density of the stationary magnetic field under consideration is $\rho > 0$, if the field scalar potential is $\varphi < 0$. In this case,

the other equations of the 1st group (3.295) take the form

$$j^1 = \frac{c}{4\pi} \frac{\partial \ln \sqrt{h}}{\partial y}, \quad j^2 = \frac{c}{4\pi} \frac{\partial \ln \sqrt{h}}{\partial x}, \quad j^3 = 0. \quad (3.297)$$

Since $h = 1 + \frac{\Omega^2(x^2+y^2)}{c^2}$ (3.277), this means: the current vector in the homogeneous stationary magnetic field is non-zero only if the space rotation velocity is comparable to the light velocity. In a weak field of the space non-holonomy, we have $h = 1$, hence $j^1 = j^2 = 0$.

Now, expressing the magnetic strength from the Maxwell equations (3.295) we write down the chr.inv.-vector equations of motion for an our-world particle (3.290, 3.291) in the form

$$\left. \begin{aligned} \ddot{x} + \frac{2\Omega}{h} \left[\frac{\Omega^2 xy \dot{x}}{c^2} + \left(1 + \frac{\Omega^2 x^2}{c^2} \right) \dot{y} \right] + \Delta_{11}^1 \dot{x}^2 + 2\Delta_{12}^1 \dot{x} \dot{y} + \\ + \Delta_{22}^1 \dot{y}^2 = -\frac{eH}{mc} \left[-\frac{\Omega^2 xy \dot{x}}{c^2} + \left(1 + \frac{\Omega^2 x^2}{c^2} \right) \dot{y} \right] \\ \ddot{y} - \frac{2\Omega}{h} \left[\frac{\Omega^2 xy \dot{y}}{c^2} + \left(1 + \frac{\Omega^2 y^2}{c^2} \right) \dot{x} \right] + \Delta_{11}^2 \dot{x}^2 + 2\Delta_{12}^2 \dot{x} \dot{y} + \\ + \Delta_{22}^2 \dot{y}^2 = \frac{eH}{mc} \left[-\frac{\Omega^2 xy \dot{y}}{c^2} + \left(1 + \frac{\Omega^2 y^2}{c^2} \right) \dot{x} \right] \\ \ddot{z} = 0 \end{aligned} \right\}, \quad (3.298)$$

while those for an analogous mirror-world particle are

$$\left. \begin{aligned} \ddot{x} + \Delta_{11}^1 \dot{x}^2 + 2\Delta_{12}^1 \dot{x} \dot{y} + \Delta_{22}^1 \dot{y}^2 = \\ = -\frac{eH}{mc} \left[-\frac{\Omega^2 xy \dot{x}}{c^2} + \left(1 + \frac{\Omega^2 x^2}{c^2} \right) \dot{y} \right] \\ \ddot{y} + \Delta_{11}^2 \dot{x}^2 + 2\Delta_{12}^2 \dot{x} \dot{y} + \Delta_{22}^2 \dot{y}^2 = \\ = \frac{eH}{mc} \left[-\frac{\Omega^2 xy \dot{y}}{c^2} + \left(1 + \frac{\Omega^2 y^2}{c^2} \right) \dot{x} \right] \\ \ddot{z} = 0 \end{aligned} \right\}. \quad (3.299)$$

The terms on the right hand side, which contain $\frac{\Omega^2}{c^2}$, appear, because in a rotating space the observable chr.inv.-metric h_{ik} is not Euclidean.

Hence, in the case under consideration there is a difference between the contravariant form of the observable velocity and its covariant form. The right hand side includes the covariant components

$$v_2 = h_{21}v^1 + h_{22}v^2 = -\frac{\Omega^2 xy}{c^2} \dot{x} + \left(1 + \frac{\Omega^2 x^2}{c^2}\right) \dot{y}, \quad (3.300)$$

$$v_1 = h_{11}v^1 + h_{12}v^2 = -\frac{\Omega^2 xy}{c^2} \dot{y} + \left(1 + \frac{\Omega^2 y^2}{c^2}\right) \dot{x}. \quad (3.301)$$

If the space does not rotate ($\Omega = 0$), then the chr.inv.-equations of motion of the our-world particle (3.298) to within their sign match the equations of motion in a homogeneous stationary magnetic field given by Landau and Lifshitz (see their formula 21.2 in *The Classical Theory of Fields*), which have the form

$$\ddot{x} = \frac{eH}{mc} \dot{y}, \quad \ddot{y} = -\frac{eH}{mc} \dot{x}, \quad \ddot{z} = 0, \quad (3.302)$$

while our equations (3.298) under the same simplification mean

$$\ddot{x} = -\frac{eH}{mc} \dot{y}, \quad \ddot{y} = \frac{eH}{mc} \dot{x}, \quad \ddot{z} = 0. \quad (3.303)$$

The difference is derived from the fact that Landau and Lifshitz assumed the magnetic strength in the Lorentz force to have a “plus” sign, while in our equations it has a “minus” sign, which is not that important, because it only depends on the choice of the space signature.

If the space rotates (non-holonomic), then the equations of motion will include the terms containing Ω , $\frac{\Omega^2}{c^2}$ and $\frac{\Omega^4}{c^4}$.

In a strong field of the space non-holonomy, solving the equations that we have obtained is a non-trivial task, which is likely to be tackled in the future with computer-aided numerical methods. Hopefully, the results will be quite interesting.

Let us now find exact solutions to the obtained equations of motion in a weak field of the space non-holonomy, i.e., neglecting second order terms. In this case, the equations of motion that we have obtained (3.298, 3.299) for an our-world particle are simplified

$$\ddot{x} + 2\Omega \dot{y} = -\frac{eH}{mc} \dot{y}, \quad \ddot{y} - 2\Omega \dot{x} = \frac{eH}{mc} \dot{x}, \quad \ddot{z} = 0, \quad (3.304)$$

and for a mirror-world particle they are even simpler

$$\ddot{x} = -\frac{eH}{mc} \dot{y}, \quad \ddot{y} = \frac{eH}{mc} \dot{x}, \quad \ddot{z} = 0. \quad (3.305)$$

First, consider the equations for the our-world particle. The equation along z can be integrated straightaway. The solution is

$$z = \dot{z}_{(0)} \tau + z_{(0)}. \quad (3.306)$$

From here we see that, if at the initial moment of time the particle's velocity along z is zero, then it travels within the xy plane only. Re-write the remaining two equations of (3.304) as follows

$$\frac{d\dot{x}}{d\tau} = -(2\Omega + \omega) \dot{y}, \quad \frac{d\dot{y}}{d\tau} = (2\Omega + \omega) \dot{x}, \quad (3.307)$$

where we denote $\omega = \frac{eH}{mc}$ as in §21 of *The Classical Theory of Fields*. Formulating \dot{x} from the second equation, we derive it then substitute the result into the first equation. Thus, we obtain

$$\frac{d^2 \dot{y}}{d\tau^2} + (2\Omega + \omega)^2 \dot{y} = 0, \quad (3.308)$$

which is an oscillation equation. Its solution is

$$\dot{y} = C_1 \cos(2\Omega + \omega) \tau + C_2 \sin(2\Omega + \omega) \tau, \quad (3.309)$$

where $C_1 = \dot{y}_{(0)}$ and $C_2 = \frac{\ddot{y}_{(0)}}{2\Omega + \omega}$ are integration constants. Substituting \dot{y} (3.309) into the first equation of (3.307), we obtain

$$\frac{d\dot{x}}{d\tau} = -(2\Omega + \omega) \dot{y}_{(0)} \cos(2\Omega + \omega) \tau - \ddot{y}_{(0)} \sin(2\Omega + \omega) \tau, \quad (3.310)$$

or, after integration,

$$\dot{x} = \dot{y}_{(0)} \sin(2\Omega + \omega) \tau - \frac{\ddot{y}_{(0)}}{2\Omega + \omega} \cos(2\Omega + \omega) \tau + C_3, \quad (3.311)$$

where the integration constant is $C_3 = \dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega + \omega}$.

Having all of the constants substituted, the obtained formulae for \dot{x} (3.311) and \dot{y} (3.309) finally transform into

$$\begin{aligned} \dot{x} = \dot{y}_{(0)} \sin(2\Omega + \omega) \tau - \frac{\ddot{y}_{(0)}}{2\Omega + \omega} \cos(2\Omega + \omega) \tau + \\ + \dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega + \omega}, \end{aligned} \quad (3.312)$$

$$\dot{y} = \dot{y}_{(0)} \cos(2\Omega + \omega)\tau + \frac{\ddot{y}_{(0)}}{2\Omega + \omega} \sin(2\Omega + \omega)\tau. \quad (3.313)$$

Hence, the formulae for components of the particle's velocity \dot{x} and \dot{y} in a homogeneous stationary magnetic field are harmonic oscillation equations. The oscillation frequency in a weak field of the space non-holonomy is $2\Omega + \omega = 2\Omega + \frac{eH}{mc}$.

From the live forces integral in the stationary magnetic field (3.289) we see that the square of the particle's velocity is a constant quantity. Calculating $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ for the our-world particle, we obtain that this quantity

$$v^2 = \dot{x}_{(0)}^2 + \dot{y}_{(0)}^2 + \dot{z}_{(0)}^2 + 2\left(\dot{x}_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega + \omega}\right) \times \left[\frac{\dot{y}_{(0)}}{2\Omega + \omega} + \dot{y}_{(0)} \sin(2\Omega + \omega)\tau - \frac{\ddot{y}_{(0)}}{2\Omega + \omega} \cos(2\Omega + \omega)\tau\right] \quad (3.314)$$

is constant $v^2 = \text{const}$, provided that $C_3 = \dot{x}_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega + \omega} = 0$.

Integrating \dot{x} (3.312) and \dot{y} (3.313) by the observable time τ , we obtain the coordinates x and y of the our-world particle travelling in the homogeneous stationary magnetic field

$$x = \left[\frac{\dot{y}_{(0)}}{2\Omega + \omega} \sin(2\Omega + \omega)\tau - \dot{y}_{(0)} \cos(2\Omega + \omega)\tau\right] \times \left[\frac{1}{2\Omega + \omega} + \left(\dot{x}_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega + \omega}\right)\tau + C_4\right], \quad (3.315)$$

$$y = \left[\dot{y}_{(0)} \sin(2\Omega + \omega)\tau + \frac{\ddot{y}_{(0)}}{2\Omega + \omega} \cos(2\Omega + \omega)\tau\right] \times \left[\frac{1}{2\Omega + \omega} + C_5\right], \quad (3.316)$$

where the integration constants are

$$C_4 = x_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega + \omega}, \quad C_5 = y_{(0)} + \frac{\ddot{y}_{(0)}}{(2\Omega + \omega)^2}. \quad (3.317)$$

From (3.315) we see that the particle performs *harmonic oscillations* along x provided that the equation $\dot{x}_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega + \omega} = 0$ is true. This is also the condition for the constant square of the particle's velocity (3.314), which satisfies the live forces integral.

Based on the above solutions, we arrive at the equation of the particle's trajectory within the xy plane

$$\begin{aligned}
 x^2 + y^2 = & \frac{1}{(2\Omega + \omega)^2} \left[\dot{y}_{(0)}^2 + \frac{\ddot{y}_{(0)}^2}{(2\Omega + \omega)^2} \right] - \frac{2C_4}{2\Omega + \omega} \times \\
 & \times \left[\dot{y}_{(0)} \cos(2\Omega + \omega)\tau + \frac{\ddot{y}_{(0)}}{2\Omega + \omega} \sin(2\Omega + \omega)\tau \right] + \\
 & + \left[\dot{y}_{(0)} \sin(2\Omega + \omega)\tau + \frac{\ddot{y}_{(0)}}{2\Omega + \omega} \cos(2\Omega + \omega)\tau \right] \times \\
 & \times \frac{2C_5}{2\Omega + \omega} + C_4^2 + C_5^2.
 \end{aligned} \tag{3.318}$$

Assuming that, at the initial moment of time, $\dot{y}_{(0)} = 0$ and the integration constants C_4 and C_5 are zeroes, we can simplify the obtained formulae (3.315, 3.316), namely

$$x = -\frac{1}{2\Omega + \omega} \dot{y}_{(0)} \cos(2\Omega + \omega)\tau, \tag{3.319}$$

$$y = \frac{1}{2\Omega + \omega} \dot{y}_{(0)} \sin(2\Omega + \omega)\tau. \tag{3.320}$$

With these formulae, our equation of the particle's trajectory (3.318) transforms into the simple equation of a circle

$$x^2 + y^2 = \frac{\dot{y}_{(0)}^2}{(2\Omega + \omega)^2}. \tag{3.321}$$

So, if the initial velocity of an our-world charged particle with respect to the direction of the homogeneous magnetic field (z axis) is zero, then the particle travels within the xy plane along a *circle* of radius

$$r = \frac{\dot{y}_{(0)}}{2\Omega + \omega} = \frac{\dot{y}_{(0)}}{2\Omega + \frac{eH}{mc}}, \tag{3.322}$$

which depends on the magnetic field strength and the angular velocity with which the space rotates.

If the initial velocity of the particle along the magnetic field direction is not zero, then it travels along a *spiral line* of radius r . In general, the particle travels along an *ellipse* within the xy plane (3.318),

the shape of which is different from a circle depending on the initial conditions of the motion.

It is easy to see that our results match those obtained in §21 of *The Classical Theory of Fields*

$$x = -\frac{1}{\omega} \dot{y}_{(0)} \cos \omega \tau, \quad y = \frac{1}{\omega} \dot{y}_{(0)} \sin \omega \tau, \quad (3.323)$$

once we assume $\Omega = 0$, i.e., the space does not rotate. In this particular case, the radius $r = \frac{\dot{y}_{(0)}}{\omega} = \frac{mc}{eH} \dot{y}_{(0)}$ of the particle's trajectory does not depend on the space rotation. If $\Omega \neq 0$, then the non-holonomy field disturbs the particle travelling in the magnetic field by adding up an additional quantity 2Ω to the term $\omega = \frac{eH}{mc}$ in the equations. In a strong field of the space non-holonomy, where Ω cannot be neglected compared to the light velocity, this disturbance is even stronger.

On the other hand, in a non-holonomic space the argument of the trigonometric functions in our equations contains the *sum of two terms*, one of which is derived from the interaction of the particle's charge with the magnetic field strength, and the other is the result of the space rotation (it does not depend on the electric charge of the particle, or even on the presence of the magnetic field at all). This allows us to consider two special cases of the motion of a charged particle in a homogeneous stationary magnetic field that fills a non-holonomic space.

In the first case, where the particle is electrically neutral or the magnetic field is absent, the particle's motion is the same as that under the action of the magnetic component of the Lorentz force, except for the fact that this motion is caused by the space rotation with a velocity 2Ω , comparable to $\omega = \frac{eH}{mc}$.

How real is this case? To answer this question, we need an approximate estimate of the ratio between the angular velocity Ω with which the space rotates and the magnetic field strength H in at least a particular case. The best example would be the atom, because on the scale of the electron orbits, the electromagnetic interactions are several orders of magnitude stronger than others, and the orbital velocities of electrons are relatively large.

Such an estimate can be made on the basis of the second particular case of the motion, where we assume that the condition

$$\frac{eH}{mc} = -2\Omega \quad (3.324)$$

is true and, hence, the argument of the trigonometric functions in the equations of motion becomes zero.

Consider the reference frame of an observer, whose reference space is associated with the nucleus of an atom. Then the ratio in the question (in CGSE and Gaussian systems of units) for an orbiting electron is

$$\begin{aligned} \frac{\Omega}{H} &= -\frac{e}{2m_e c} = -\frac{4.8 \times 10^{-10}}{18.2 \times 10^{-28} \cdot 3.0 \times 10^{10}} = \\ &= -8.8 \times 10^6 \text{ cm}^{1/2} \text{ gram}^{-1/2}, \end{aligned} \quad (3.325)$$

where the “minus” sign is due to the fact that Ω and H in (3.324) are oppositely directed.

Let us now solve the equations of motion of a mirror-world charged particle in the homogeneous stationary magnetic field (3.305), which in a non-holonomic space match the equations

$$\ddot{x} = -\omega \dot{y}, \quad \ddot{y} = \omega \dot{x}, \quad \ddot{z} = 0. \quad (3.326)$$

The solution to the third equation of motion (along z) is simplest and has the form $z = \dot{z}_{(0)}\tau + z_{(0)}$.

The equations of motion along x and y are similar to those for an analogous our-world particle, except that the argument of the trigonometric functions has ω instead of $\omega + 2\Omega$, i.e.

$$\dot{x} = \dot{y}_{(0)} \sin \omega \tau - \frac{\ddot{y}_{(0)}}{\omega} \cos \omega \tau + \dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega}, \quad (3.327)$$

$$\dot{y} = \dot{y}_{(0)} \cos \omega \tau + \frac{\ddot{y}_{(0)}}{\omega} \sin \omega \tau. \quad (3.328)$$

Hence, the formulae for the components of the mirror-world particle's velocity \dot{x} and \dot{y} are the equations of harmonic oscillations at the frequency $\omega = \frac{eH}{mc}$.

Their solutions, i.e., the coordinates of the mirror-world particle travelling in the homogeneous stationary magnetic field have the form

$$x = \frac{1}{\omega} \left(\frac{\ddot{y}_{(0)}}{\omega} \sin \omega \tau - \dot{y}_{(0)} \cos \omega \tau \right) + \left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega} \right) \tau + C_4, \quad (3.329)$$

$$y = \frac{1}{\omega} \left(\dot{y}_{(0)} \sin \omega \tau + \frac{\ddot{y}_{(0)}}{\omega} \cos \omega \tau \right) + C_5, \quad (3.330)$$

where the integration constants are

$$C_4 = x_{(0)} + \frac{\dot{y}_{(0)}}{\omega}, \quad C_5 = y_{(0)} + \frac{\dot{y}_{(0)}}{\omega^2}. \quad (3.331)$$

As we have already mentioned, the live forces integral in a stationary magnetic field (3.289) means the constant relativistic mass of the travelling particle and, hence, the constant square of its observable velocity. Then, using the solutions for the velocities of the mirror-world particle, i.e., the squared quantities \dot{x} , \dot{y} , \dot{z} , we obtain that

$$\begin{aligned} v^2 = & \dot{x}_{(0)}^2 + \dot{y}_{(0)}^2 + \dot{z}_{(0)}^2 + \\ & + 2 \left(\dot{x}_{(0)} + \frac{\dot{y}_{(0)}}{\omega} \right) \left(\frac{\ddot{y}_{(0)}}{\omega} + \dot{y}_{(0)} \sin \omega \tau - \frac{\dot{y}_{(0)}}{\omega} \cos \omega \tau \right) \end{aligned} \quad (3.332)$$

is constant $v^2 = \text{const}$ provided that

$$\dot{x}_{(0)} + \frac{\dot{y}_{(0)}}{\omega} = 0. \quad (3.333)$$

From the formula for x (3.329), we see that the particle performs strictly harmonic oscillations along x under the same condition (3.333). Taking this fact into account, squaring and adding up x (3.329) and y (3.330) for the mirror-world particle in the homogeneous stationary magnetic field, we obtain its trajectory within the xy plane

$$\begin{aligned} x^2 + y^2 = & \\ = & \frac{1}{\omega^2} \left(\dot{y}_{(0)}^2 + \frac{\ddot{y}_{(0)}^2}{\omega^2} \right) - \frac{2C_4}{\omega} \left(\dot{y}_{(0)} \cos \omega \tau + \frac{\dot{y}_{(0)}}{\omega} \sin \omega \tau \right) + \\ & + \left(\dot{y}_{(0)} \sin \omega \tau + \frac{\dot{y}_{(0)}}{\omega} \cos \omega \tau \right) \frac{2C_5}{\omega} + C_4^2 + C_5^2, \end{aligned} \quad (3.334)$$

which differs from the our-world particle trajectory (3.318) by $\omega + 2\Omega$ replaced with ω and by the numerical values of the integration constants (3.331). Therefore, a mirror-world charged particle having zero initial velocity along the z axis (direction of the magnetic field strength) travels along an *ellipse* within the xy plane.

Once we assume $\dot{y}_{(0)}$ as well as the constants C_4 and C_5 to be zeroes, the obtained solutions become much simpler

$$x = -\frac{1}{\omega} \dot{y}_{(0)} \cos \omega \tau, \quad y = \frac{1}{\omega} \dot{y}_{(0)} \sin \omega \tau. \quad (3.335)$$

In such a simplified case, the mirror-world particle which is at rest with respect to the field direction makes a *circle*

$$x^2 + y^2 = \frac{\dot{y}_{(0)}^2}{\omega^2} \quad (3.336)$$

within the xy plane with the radius $r = \frac{\dot{y}_{(0)}}{\omega} = \frac{mc}{eH} \dot{y}_{(0)}$.

Consequently, if the initial velocity of the particle along the magnetic field direction (z axis) is not zero, then the particle travels along a *spiral line* around the magnetic field direction. Hence, the motion of mirror-world charged particles in a homogeneous stationary magnetic field is the same as that of our-world charged particles in the absence of the space non-holonomy.

3.12.2 The magnetic field is orthogonal to the non-holonomy field

We are going to consider the case, where the magnetic strength pseudovector H^{*i} is orthogonal to the pseudovector $\Omega^{*i} = \frac{1}{2} \varepsilon^{ikm} A_{km}$ of the space non-holonomy field. In this case, the first equation of the 1st group of the chr.inv.-Maxwell equations that we have obtained for a stationary magnetic field (3.292) means that the charge density of the field is zero $\rho = 0$.

Assume that the magnetic strength is directed along y (only the component $H^{*2} = H$ is non-zero), while the non-holonomy field is directed along z (only the component $\Omega^{*3} = \Omega$ is non-zero). We also assume that the magnetic field is stationary and homogeneous. Under the above assumptions, the non-zero component of the magnetic strength is

$$H^{*2} = H_{31} = \frac{\varphi}{c} \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) = \text{const}. \quad (3.337)$$

If the non-holonomy field is weak, then the equations of motion of an our-world particle take the form

$$\ddot{x} + 2\Omega\dot{y} = \frac{eH}{mc} \dot{z}, \quad \ddot{y} - 2\Omega\dot{x} = 0, \quad \ddot{z} = -\frac{eH}{mc} \dot{x}, \quad (3.338)$$

which, denoting $\omega = \frac{eH}{mc}$, become even simpler

$$\ddot{x} + 2\Omega\dot{y} = \omega\dot{z}, \quad \ddot{y} - 2\Omega\dot{x} = 0, \quad \ddot{z} = -\omega\dot{x}. \quad (3.339)$$

Differentiating the first equation with respect to τ and substituting \dot{y} and \dot{z} into it from the second and the third equations, we have

$$\ddot{x} + (4\Omega^2 + \omega^2)\dot{x} = 0. \quad (3.340)$$

Setting $\dot{x} = p$, we arrive at the oscillation equation

$$\dot{p} + \tilde{\omega}^2 p = 0, \quad \tilde{\omega} = \sqrt{4\Omega^2 + \omega^2} = \sqrt{4\Omega^2 + \left(\frac{eH}{mc}\right)^2}, \quad (3.341)$$

which solves as follows

$$p = C_1 \cos \tilde{\omega}\tau + C_2 \sin \tilde{\omega}\tau, \quad (3.342)$$

where $C_1 = \dot{x}_{(0)}$ and $C_2 = \frac{\ddot{x}_{(0)}}{\tilde{\omega}^2}$ are integration constants. Integrating $\dot{x} = p$ with respect to τ , we obtain the formula for x

$$x = \frac{\dot{x}_{(0)}}{\tilde{\omega}} \sin \tilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\tilde{\omega}^2} \cos \tilde{\omega}\tau + x_{(0)} + \frac{\dot{x}_{(0)}}{\tilde{\omega}^2}, \quad (3.343)$$

where $x_{(0)} + \frac{\dot{x}_{(0)}}{\tilde{\omega}^2} = C_3$ is an integration constant.

Substituting $\dot{x} = p$ (3.342) into the equations of motion in terms of y and z (3.339) and integrating them, we obtain

$$\dot{y} = \frac{2\Omega}{\tilde{\omega}} \dot{x}_{(0)} \sin \tilde{\omega}\tau - \frac{2\Omega}{\tilde{\omega}^2} \ddot{x}_{(0)} \cos \tilde{\omega}\tau + \dot{y}_{(0)} + \frac{2\Omega}{\tilde{\omega}^2} \dot{x}_{(0)}, \quad (3.344)$$

$$\dot{z} = \frac{\omega}{\tilde{\omega}^2} \ddot{x}_{(0)} \cos \tilde{\omega}\tau - \frac{\omega}{\tilde{\omega}} \dot{x}_{(0)} \sin \tilde{\omega}\tau + \dot{z}_{(0)} - \frac{\omega}{\tilde{\omega}^2} \dot{x}_{(0)}, \quad (3.345)$$

where $\dot{y}_{(0)} + \frac{2\Omega\dot{x}_{(0)}}{\tilde{\omega}^2} = C_4$ and $\dot{z}_{(0)} - \frac{\omega\dot{x}_{(0)}}{\tilde{\omega}^2} = C_5$ are new integration constants. Then integrating the obtained equations (3.344, 3.345) with respect to τ , we obtain the final formulae for y and z

$$y = -\frac{2\Omega}{\tilde{\omega}^2} \left(\dot{x}_{(0)} \cos \tilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\tilde{\omega}} \sin \tilde{\omega}\tau \right) + \dot{y}_{(0)}\tau + \frac{2\Omega}{\tilde{\omega}^2} \ddot{x}_{(0)}\tau + y_{(0)} + \frac{2\Omega}{\tilde{\omega}^2} \dot{x}_{(0)}, \quad (3.346)$$

$$z = \frac{\omega}{\tilde{\omega}^2} \left(\dot{x}_{(0)} \cos \tilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\tilde{\omega}} \sin \tilde{\omega}\tau \right) + \dot{z}_{(0)}\tau - \frac{\omega}{\tilde{\omega}^2} \dot{x}_{(0)}\tau + z_{(0)} - \frac{\omega}{\tilde{\omega}^2} \dot{x}_{(0)}, \quad (3.347)$$

where $y_{(0)} + \frac{2\Omega\dot{x}_{(0)}}{\omega^2} = C_6$ and $z_{(0)} - \frac{\omega\dot{x}_{(0)}}{\omega^2} = C_7$.

If $\Omega = 0$ (the space does not rotate) and some integration constants are zeroes, then the above equations completely match the well-known formulae of relativistic electrodynamics in the case, where the stationary magnetic field is directed along the z axis

$$x = \frac{\dot{x}_{(0)}}{\omega} \sin \bar{\omega} \tau, \quad y = y_{(0)} + \dot{y}_{(0)} \tau, \quad z = \frac{\dot{x}_{(0)}}{\omega} \cos \bar{\omega} \tau. \quad (3.348)$$

So forth, since the live forces integral means that the square of the observable velocity of a charged particle in a stationary magnetic field is constant, we can calculate $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$. Substituting the obtained formulae for the velocity components, we obtain

$$\begin{aligned} v^2 = \dot{x}_{(0)}^2 + \dot{y}_{(0)}^2 + \dot{z}_{(0)}^2 + \frac{2}{\omega} (\ddot{x}_{(0)} + 2\Omega\dot{y}_{(0)} - \omega\dot{z}_{(0)}) \times \\ \times \left(\frac{\dot{x}_{(0)}}{\omega} + \dot{x}_{(0)} \sin \bar{\omega} \tau - \frac{\dot{x}_{(0)}}{\omega} \cos \bar{\omega} \tau \right), \end{aligned} \quad (3.349)$$

therefore $v^2 = const$, provided that

$$\ddot{x}_{(0)} + 2\Omega\dot{y}_{(0)} - \omega\dot{z}_{(0)} = 0. \quad (3.350)$$

The three-dimensional trajectory of the particle can be found by calculating $x^2 + y^2 + z^2$. Thus, we obtain the equation

$$\begin{aligned} x^2 + y^2 + z^2 = \frac{1}{\omega^2} \left(\dot{x}_{(0)}^2 + \frac{\dot{x}_{(0)}^2}{\omega^2} \right) + C_3^2 + C_6^2 + C_7^2 + \\ + (C_4^2 + C_5^2) \tau^2 + 2(C_4 C_6 + C_5 C_7) \tau + \left[(\omega C_7 - 2\Omega C_6) + \right. \\ \left. + 2(\omega C_5 - 2\Omega C_6) \tau \right] \left(\dot{x}_{(0)} \cos \bar{\omega} \tau + \frac{\dot{x}_{(0)}}{\omega} \sin \bar{\omega} \tau \right) \frac{1}{\omega^2} + \\ + \frac{2C_3}{\omega^2} \left(\dot{x}_{(0)} \cos \bar{\omega} \tau - \frac{\dot{x}_{(0)}}{\omega} \sin \bar{\omega} \tau \right), \end{aligned} \quad (3.351)$$

which includes a linear term and a square term with respect to time, as well as a parametric term and two harmonic terms. In a particular case, where the integration constants are zeroes, the obtained formula (3.351) takes the form of the equation of a *sphere*

$$x^2 + y^2 + z^2 = \frac{1}{\omega^2} \left(\dot{x}_{(0)}^2 + \frac{\dot{x}_{(0)}^2}{\omega^2} \right), \quad (3.352)$$

the radius of which is

$$r = \frac{1}{\tilde{\omega}} \sqrt{\dot{x}_{(0)}^2 + \frac{\dot{x}_{(0)}^2}{\tilde{\omega}^2}}, \quad (3.353)$$

where $\tilde{\omega} = \sqrt{4\Omega^2 + \omega^2} = \sqrt{4\Omega^2 + \left(\frac{eH}{mc}\right)^2}$.

So, an our-world charged particle in a homogeneous stationary magnetic field, orthogonal to the space non-holonomy field, travels over a surface of a *sphere*, the radius of which depends on the magnetic field strength and the angular velocity with which the space rotates.

In a particular case, where the non-holonomy field is absent and the initial acceleration of the particle is zero, the obtained trajectory equation simplifies significantly to the equation of a sphere

$$x^2 + y^2 + z^2 = \frac{1}{\omega^2} \dot{x}_{(0)}^2, \quad r = \frac{1}{\omega} \dot{x}_{(0)} = \frac{mc}{eH} \dot{x}_{(0)} \quad (3.354)$$

with the radius depending only on the interaction of the particle's charge with the magnetic field — this is the result, well-known in electrodynamics (see §21 in *The Classical Theory of Fields*).

For a mirror-world charged particle that travels in a homogeneous stationary magnetic field, orthogonal to the non-holonomy field, the equations of motion take the form

$$\ddot{x} = \frac{eH}{mc} \dot{z}, \quad \ddot{y} = 0, \quad \ddot{z} = -\frac{eH}{mc} \dot{x}. \quad (3.355)$$

They are only different from the equations for the our-world particle (3.338) by the absence of the terms that include the angular velocity Ω of the space rotation.

3.13 Motion in a stationary electromagnetic field

In this section, we are going to focus on the motion of a charged particle under the action of both the magnetic and electric components of a stationary electromagnetic field. As a “background” we will consider a non-holonomic space that rotates around the z axis with a constant angular velocity $\Omega_{12} = -\Omega_{21} = \Omega$, so the space has the metric (3.275). In such a space, $F_i = 0$ and $D_{ik} = 0$.

We will solve this problem, assuming that the non-holonomy field is *weak* and, hence, the three-dimensional space has the Euclidean met-

ric. In this case, the Maxwell equations for a stationary electromagnetic field (3.215, 3.216) take the form

$$\left. \begin{aligned} \Omega_{*m} H^{*m} &= -2\pi c\rho \\ \varepsilon^{ikm} \nabla_k (H_{*m} \sqrt{h}) &= \frac{4\pi}{c} j^i \sqrt{h} = 0 \end{aligned} \right\} \text{I}, \quad (3.356)$$

$$\left. \begin{aligned} \Omega_{*m} E^m &= 0 \\ \varepsilon^{ikm} \nabla_k (E_m \sqrt{h}) &= 0 \end{aligned} \right\} \text{II}, \quad (3.357)$$

since the observable homogeneity of a field means the equality to zero of its chr.inv.-derivative [9, 11–13], while in the particular case under consideration the chr.inv.-Christoffel symbols are equal to zero (the metric is Galilean) so the chr.inv.-derivative is the ordinary derivative. Hence, the Maxwell equations mean that, in this case, the following conditions are satisfied:

- a) The space non-holonomy pseudovector and the electric field strength are orthogonal, $\Omega_{*m} E^m = 0$;
- b) The space non-holonomy pseudovector and the magnetic field strength are orthogonal, $\Omega_{*m} H^{*m} = 0$. Consequently, the charge density is zero $\rho = 0$;
- c) The electromagnetic field current is absent, $j^i = 0$.

The latter condition means that the presence of the electromagnetic field currents $j^i \neq 0$ is due to the inhomogeneity of the magnetic strength of the acting electromagnetic field.

Given that the non-holonomy pseudovector is orthogonal to the electric field strength, we can consider the motion of the particle in the two cases of the mutual orientation of the fields:

- 1) $\vec{H} \perp \vec{E}$ and $\vec{H} \parallel \vec{\Omega}$;
- 2) $\vec{H} \parallel \vec{E}$ and $\vec{H} \perp \vec{\Omega}$.

In either case, we assume that the electric strength is co-directed with the x axis. According to the background metric (3.275), the space rotation pseudovector is co-directed with z . Hence, in the first case, the magnetic strength is co-directed with z , and in the second case it is co-directed with x .

The chr.inv.-equations of motion of a charged particle in the stationary electromagnetic field, where the electric strength is co-directed with

x have the following form in our world

$$\frac{dm}{d\tau} = -\frac{eE_1}{c^2} \frac{dx}{d\tau}, \quad (3.358)$$

$$\frac{d}{d\tau}(mv^i) + 2mA_k^i v^k = -e \left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m} \right), \quad (3.359)$$

and in the mirror world the equations take the form

$$\frac{dm}{d\tau} = \frac{eE_1}{c^2} \frac{dx}{d\tau}, \quad (3.360)$$

$$\frac{d}{d\tau}(mv^i) = -e \left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m} \right). \quad (3.361)$$

As before, consider a charged particle *repulsed* by the electromagnetic field. In this case, the components of the electric strength E_i , co-directed with x , in a Galilean reference frame (where the covariant and contravariant components of a tensor quantity are the same) are

$$E_1 = E_x = \frac{\partial\varphi}{\partial x} = \text{const} = -E, \quad E_2 = E_3 = 0. \quad (3.362)$$

Integrating the live forces theorem we obtain the live forces integral for our world and the mirror world, respectively,

$$m = \frac{eE}{c^2} x + B, \quad m = -\frac{eE}{c^2} x + \widetilde{B}, \quad (3.363)$$

where B is an integration constant for our world, and \widetilde{B} is an integration constant for the mirror world. Calculating these constants from the initial conditions at the moment of time $\tau = 0$, we obtain

$$B = m_{(0)} - \frac{eE}{c^2} x_{(0)}, \quad \widetilde{B} = m_{(0)} + \frac{eE}{c^2} x_{(0)}, \quad (3.364)$$

where $m_{(0)}$ is the relativistic mass of the particle, and $x_{(0)}$ is its displacement at the initial moment of time.

From the obtained live forces integrals (3.363), we see that the difference between the two cases under this study is due to the different orientation of the magnetic strength \vec{H} to the electric strength \vec{E} and to the angular velocity $\vec{\Omega}$ with which the space rotates (orientation of the space non-holonomy field). This difference reveals itself only in the

chr.inv.-vector equations of motion, while the chr.inv.-scalar equations of motion (3.358, 3.360) and their solutions (3.363) remain the same.

Note that the vector \vec{E} can also be directed along y , but cannot be directed along z . This is because in the space with such a metric the non-holonomy pseudovector $\vec{\Omega}$ is co-directed with z , while the 2nd group of the Maxwell equations require \vec{E} to be orthogonal to $\vec{\Omega}$.

Now, taking into account the integration results from the live forces theorem (3.363), we will write down the chr.inv.-vector equations for the two cases that are conceivable.

Case 1. Assume that $\vec{H} \perp \vec{E}$ and $\vec{H} \parallel \vec{\Omega}$, so the magnetic strength \vec{H} is directed along z (parallel to the non-holonomy field).

Then, out of all components of the magnetic strength, only the following component is non-zero

$$H^{*3} = H_{12} = \frac{\varphi}{c} \left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x} \right) + \frac{2\varphi}{c} A_{12} = \text{const} = H. \quad (3.365)$$

Consequently, the chr.inv.-vector equations of motion for an our-world particle have the form

$$\left. \begin{aligned} \frac{eE}{c^2} \dot{x}^2 + \left(B + \frac{eE}{c^2} x \right) (\ddot{x} + 2\Omega \dot{y}) &= eE - \frac{eH}{c} \dot{y} \\ \frac{eE}{c^2} \dot{x} \dot{y} + \left(B + \frac{eE}{c^2} x \right) (\dot{y} - 2\Omega \dot{x}) &= \frac{eH}{c} \dot{x} \\ \frac{eE}{c^2} \dot{x} \dot{z} + \left(B + \frac{eE}{c^2} x \right) \ddot{z} &= 0 \end{aligned} \right\}, \quad (3.366)$$

while for a mirror-world particle we have

$$\left. \begin{aligned} \frac{eE}{c^2} \dot{x}^2 + \left(\tilde{B} - \frac{eE}{c^2} x \right) \ddot{x} &= eE - \frac{eH}{c} \dot{y} \\ \frac{eE}{c^2} \dot{x} \dot{y} + \left(\tilde{B} - \frac{eE}{c^2} x \right) \dot{y} &= \frac{eH}{c} \dot{x} \\ \frac{eE}{c^2} \dot{x} \dot{z} + \left(\tilde{B} - \frac{eE}{c^2} x \right) \ddot{z} &= 0 \end{aligned} \right\}. \quad (3.367)$$

Besides, the 1st group of the Maxwell equations require that in the case under study the following condition must be true

$$\Omega_{*3} H^{*3} = -2\pi c \rho, \quad (3.368)$$

where $\Omega_{*3} = \Omega = \text{const}$ and $H^{*3} = H = \text{const}$.

Based on the obtained formula (3.368) we arrive at the obvious conclusion: the above mutual orientation of the space non-holonomy pseudovector and the magnetic field strength is only possible in the case, where electric charges are present in the space, so the charge density is non-zero $\rho \neq 0$.

Case 2. Assume that $\vec{H} \parallel \vec{E}$, $\vec{H} \perp \vec{\Omega}$ and $\vec{E} \perp \vec{\Omega}$, so the magnetic and electric strengths are co-directed with x , while the non-holonomy field is still directed along z .

In this case, out of all components of the magnetic strength only the first component is non-zero

$$H^{*1} = H_{23} = \frac{\varphi}{c} \left(\frac{\partial v_2}{\partial z} - \frac{\partial v_3}{\partial y} \right) = \text{const} = H. \quad (3.369)$$

With this formula, we obtain the chr.inv.-vector equations of motion for an our-world particle and those for a mirror-world particle. For the our-world particle the equations have the form

$$\left. \begin{aligned} \frac{eE}{c^2} \dot{x}^2 + \left(B + \frac{eE}{c^2} x \right) (\ddot{x} + 2\Omega \dot{y}) &= eE \\ \frac{eE}{c^2} \dot{x} \dot{y} + \left(B + \frac{eE}{c^2} x \right) (\dot{y} - 2\Omega \dot{x}) &= -\frac{eH}{c} \dot{z} \\ \frac{eE}{c^2} \dot{x} \dot{z} + \left(B + \frac{eE}{c^2} x \right) \ddot{z} &= \frac{eH}{c} \dot{y} \end{aligned} \right\}, \quad (3.370)$$

while the equations for the mirror-world particle have the form

$$\left. \begin{aligned} \frac{eE}{c^2} \dot{x}^2 + \left(\widetilde{B} - \frac{eE}{c^2} x \right) \ddot{x} &= eE \\ \frac{eE}{c^2} \dot{x} \dot{y} + \left(\widetilde{B} - \frac{eE}{c^2} x \right) \dot{y} &= -\frac{eH}{c} \dot{z} \\ \frac{eE}{c^2} \dot{x} \dot{z} + \left(\widetilde{B} - \frac{eE}{c^2} x \right) \ddot{z} &= \frac{eH}{c} \dot{y} \end{aligned} \right\}. \quad (3.371)$$

Now, having the equations of motion of a charged particle in a stationary electromagnetic field of the above two mutual orientations with respect to the pseudovector of the space non-holonomy field (pseudovector of the space rotation), we can start solving them.

3.13.1 The magnetic field is orthogonal to the electric field and is parallel to the non-holonomy field

Let us solve the chr.inv.-vector equations of motion of the charged particle (3.366, 3.367) in the non-relativistic approximation, i.e., assuming the absolute value of the particle's observable velocity negligible compared to the velocity of light. Hence, we assume that the particle's mass at the initial moment of time is equal to its rest-mass

$$m_{(0)} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \cong m_0. \quad (3.372)$$

Assume that the electric strength E is negligible, so the term $\frac{eEx}{c^2}$ can be withheld. Under these conditions, the chr.inv.-vector equations of motion for an our-world particle take the form

$$\left. \begin{aligned} m_0 (\ddot{x} + 2\Omega\dot{y}) &= eE - \frac{eH}{c} \dot{y} \\ m_0 (\ddot{y} - 2\Omega\dot{x}) &= \frac{eH}{c} \dot{x} \\ m_0 \ddot{z} &= 0 \end{aligned} \right\}, \quad (3.373)$$

while for a mirror-world particle we have

$$m_0 \ddot{x} = eE - \frac{eH}{c} \dot{y}, \quad m_0 \ddot{y} = \frac{eH}{c} \dot{x}, \quad m_0 \ddot{z} = 0. \quad (3.374)$$

These equations match those obtained in §22 in *The Classical Theory of Fields* [10] in the case, where the space non-holonomy is absent ($\Omega = 0$) and the electric strength is directed along the x axis.

The obtained equations for the mirror-world particle are a particular case of the our-world equations at $\Omega = 0$. Therefore, we can only integrate the our-world equations, while the mirror-world solutions are obtained automatically by assuming $\Omega = 0$. Integrating the equation of motion along z we obtain

$$z = \dot{z}_{(0)} \tau + z_{(0)}. \quad (3.375)$$

Integrating the equation along y we arrive at

$$\dot{y} = \left(2\Omega + \frac{eH}{m_0 c} \right) x + C_1, \quad (3.376)$$

where the integration constant is $C_1 = \dot{y}_{(0)} - \left(2\Omega + \frac{eH}{m_0c}\right)x_{(0)}$.

Substituting the obtained solution for \dot{y} into the first equation of (3.373), we obtain a second-order differential equation with respect to x , which has the form

$$\ddot{x} + \omega^2 x = \frac{eE}{m_0} + \omega^2 x_{(0)} - \omega \dot{y}_{(0)}, \quad (3.377)$$

where $\omega = 2\Omega + \frac{eH}{m_0c}$. Introducing a new variable

$$u = x - \frac{A}{\omega^2}, \quad A = \frac{eE}{m_0} + \omega^2 x_{(0)} - \omega \dot{y}_{(0)}, \quad (3.378)$$

we obtain the harmonic oscillation equation

$$\ddot{u} + \omega^2 u = 0, \quad (3.379)$$

which has the following solution

$$u = C_2 \cos \omega \tau + C_3 \sin \omega \tau, \quad (3.380)$$

where the integration constants are $C_2 = u_{(0)}$ and $C_3 = \frac{\dot{u}_{(0)}}{\omega}$. Returning back to the initial variable x by the reverse substitution of the variables, we finally obtain a solution for x , which is

$$x = \frac{1}{\omega} \left(\dot{y}_{(0)} - \frac{eE}{m_0 \omega} \right) \cos \omega \tau + \frac{\dot{x}_{(0)}}{\omega} \sin \omega \tau + \frac{eE}{m_0 \omega^2} + x_{(0)} - \frac{\dot{y}_{(0)}}{\omega}. \quad (3.381)$$

Substituting this formula into the obtained equation for \dot{y} (3.376), then integrating it, we obtain a solution for y , which is

$$y = \frac{1}{\omega} \left(\dot{y}_{(0)} - \frac{eE}{m_0 \omega} \right) \sin \omega \tau - \frac{\dot{x}_{(0)}}{\omega} \cos \omega \tau + \frac{eE}{m_0 \omega^2} + y_{(0)} + \frac{\dot{x}_{(0)}}{\omega}. \quad (3.382)$$

The chr.inv.-vector equations in the mirror world have the same solutions, but because $\Omega = 0$ for them, the frequency is $\omega = \frac{eH}{m_0c}$.

The energy of the our-world and mirror-world particles are $E = mc^2$ and $E = -mc^2$, respectively.

Finally, we obtain solutions for the three-dimensional momentum of the our-world particle

$$\left. \begin{aligned} p^1 &= m_0 \dot{x} = \left(\frac{eE}{\omega} - m_0 \dot{y}_{(0)} \right) \sin \omega \tau + m_0 \dot{x}_{(0)} \cos \omega \tau \\ p^2 &= m_0 \dot{y} = \left(\frac{2\Omega m_0}{\omega} + \frac{eH}{\omega c} \right) \left(\frac{eE}{m_0 \omega} - \dot{y}_{(0)} \right) + m_0 \dot{y}_{(0)} + \\ &\quad + \left(\frac{2\Omega m_0}{\omega} + \frac{eH}{\omega c} \right) \left[\left(\dot{y}_{(0)} - \frac{eE}{m_0 \omega} \right) \cos \omega \tau + \dot{x}_{(0)} \sin \omega \tau \right] \\ p^3 &= m_0 \dot{z} = m_0 \dot{z}_{(0)} \end{aligned} \right\}, \quad (3.383)$$

and for the mirror world particle

$$\left. \begin{aligned} p^1 &= \left(\frac{eE}{\omega} - m_0 \dot{y}_{(0)} \right) \sin \omega \tau + m_0 \dot{x}_{(0)} \cos \omega \tau \\ p^2 &= \frac{eE}{\omega} + m_0 \left[\left(\dot{y}_{(0)} - \frac{eE}{m_0 \omega} \right) \cos \omega \tau + \dot{x}_{(0)} \sin \omega \tau \right] \\ p^3 &= m_0 \dot{z}_{(0)} \end{aligned} \right\}, \quad (3.384)$$

where, in contrast to our world, the frequency is $\omega = \frac{eH}{m_0 c}$.

From here we see that the momentum of an our-world charged particle in the given configuration of the acting fields performs harmonic oscillations along x and y , while along z it is a linear function of the observable time τ (if the particle's initial velocity is $\dot{z} \neq 0$). Within the xy plane the oscillation frequency is $\omega = 2\Omega + \frac{eH}{m_0 c}$.

It should be noted that obtaining exact solutions to the equations of motion in the presence of both the electric and magnetic components of the electromagnetic field is very problematic, because we need to solve elliptic integrals. It can be possible to solve them in the future, when the solutions will be obtained on computers, but this problem is obviously out of the scope of this book. Presumably, Landau and Lifshitz faced a similar problem, because in §22 of *The Classical Theory of Fields*, where they considering a similar problem,* they obtained the equations of motion and then solved them assuming the particle's velocity to be non-relativistic and the electric strength to be weak $\frac{eEx}{c^2} \approx 0$.

*But in contrast to our book, Landau and Lifshitz used the general covariant method and, therefore, they did not take the space non-holonomy into account.

3.13.2 The magnetic field is parallel to the electric field and is orthogonal to the non-holonomy field

Let us solve the chr.inv.-vector equations of motion of a charged particle (3.370, 3.371) in the same approximation as in the previous case. In this case, for an our-world particle and for a mirror world particle the vector equation of motion have the form, respectively

$$\ddot{x} + 2\Omega\dot{y} = \frac{eE}{m_0}, \quad \ddot{y} - 2\Omega\dot{x} = -\frac{eH}{m_0c}\dot{z}, \quad \ddot{z} = \frac{eH}{m_0c}\dot{y}, \quad (3.385)$$

$$\ddot{x} = \frac{eE}{m_0}, \quad \ddot{y} = -\frac{eH}{m_0c}\dot{z}, \quad \ddot{z} = \frac{eH}{m_0c}\dot{y}. \quad (3.386)$$

Integrating the first equation of motion in our world (3.385), which means the motion along x , we obtain

$$\dot{x} = \frac{eE}{m_0}\tau - 2\Omega y + C_1, \quad (3.387)$$

where $C_1 = \text{const} = \dot{x}_{(0)} + 2\Omega y_{(0)}$.

Integrating the third equation of motion (along z) we have

$$\dot{z} = \frac{eH}{m_0c}y + C_2, \quad (3.388)$$

where $C_2 = \text{const} = \dot{z}_{(0)} - \frac{eH}{m_0c}y_{(0)}$.

Substituting the obtained formulae for \dot{x} and \dot{z} into the second equation of motion (3.385), we obtain a linear differential equation of the 2nd order with respect to y , which is

$$\ddot{y} + \left(4\Omega^2 + \frac{e^2H^2}{m_0^2c^2}\right)y = \frac{2\Omega eE}{m_0}\tau + 2\Omega C_1 - \frac{eH}{m_0c}C_2. \quad (3.389)$$

We will solve it, using the variable change method. Thus, introducing a new variable u in the form

$$u = y + \frac{1}{\omega^2} \left(\frac{eH}{m_0c} C_2 - 2\Omega C_1 \right), \quad \omega^2 = 4\Omega^2 + \frac{e^2H^2}{m_0^2c^2}, \quad (3.390)$$

we obtain the inhomogeneous equation of forced oscillations

$$\ddot{u} + \omega^2 u = \frac{2\Omega eE}{m_0}\tau, \quad (3.391)$$

the solution of which is the sum of a general solution to the free oscillation equation

$$\ddot{u} + \omega^2 u = 0, \quad (3.392)$$

and a particular solution to the inhomogeneous equation

$$\ddot{u} = M\tau + N, \quad (3.393)$$

where $M = \text{const}$ and $N = \text{const}$ are constants.

Differentiating \tilde{u} twice with respect to τ and substituting the results into the initial inhomogeneous equation with respect to \ddot{u} (3.391), then equating the obtained coefficients for τ to each other, we obtain that the above linear coefficients M and N are

$$M = \frac{2\Omega eE}{m_0\omega^2}, \quad N = 0. \quad (3.394)$$

As a result, the general solution to the initial inhomogeneous equation (3.391) takes the following form

$$u = C_3 \cos \omega\tau + C_4 \sin \omega\tau + \frac{2\Omega eE}{m_0\omega^2} \tau, \quad (3.395)$$

where the integration constants can be obtained by substituting the initial conditions at $\tau=0$ into the obtained solution (3.395). As a result, we obtain $C_3 = u(0)$ and $C_4 = \frac{\dot{u}(0)}{\omega}$.

Returning back to the initial variable y (3.390), we obtain the final solution for this coordinate

$$y = \left[y(0) + \frac{1}{\omega^2} \left(\frac{eH}{m_0c} C_2 + 2\Omega C_1 \right) \right] \cos \omega\tau + \frac{\dot{y}(0)}{\omega} \sin \omega\tau - \frac{1}{\omega^2} \left(\frac{eH}{m_0c} C_2 + 2\Omega C_1 \right) + \frac{2\Omega eE}{m_0\omega^2} \tau. \quad (3.396)$$

Then, substituting this formula into the equations for \dot{x} and \dot{z} , after integrating we arrive at the solutions for x and z

$$x = \frac{eE}{2m_0} \left(1 - \frac{4\Omega^2}{\omega^2} \right) \tau^2 - \frac{2\Omega}{\omega} (y(0) + A) \sin \omega\tau + \frac{2\Omega \dot{y}(0)}{\omega} \cos \omega\tau + (C_1 + 2\Omega A) \tau + C_5, \quad (3.397)$$

$$z = \frac{eH}{m_0 c \omega} \left[(y_{(0)} + A) \sin \omega \tau - \frac{\dot{y}_{(0)}}{\omega} \cos \omega \tau \right] - \left(\frac{eH}{m_0 c} A - C_2 \right) \tau + C_6, \quad (3.398)$$

where (for convenient notation)

$$A = \frac{1}{\omega^2} \left(\frac{eH}{m_0 c} C_2 - 2\Omega C_1 \right), \quad (3.399)$$

while the new integration constants are

$$C_5 = x_0 - \frac{2\Omega \dot{y}_{(0)}}{\omega}, \quad C_6 = z_{(0)} + \frac{eH \dot{y}_{(0)}}{m_0 c \omega^2}. \quad (3.400)$$

If we assume $\Omega = 0$, then, based on the solutions for an our-world charged particle (3.396–3.398), we immediately obtain the solutions for an analogous charged particle in the mirror world

$$x = \frac{eE}{2m_0} \tau^2 + \dot{x}_{(0)} \tau + x_{(0)}, \quad (3.401)$$

$$y = \frac{\dot{z}_{(0)}}{\omega} \cos \omega \tau + \frac{\dot{y}_{(0)}}{\omega} \sin \omega \tau - \frac{\dot{z}_{(0)}}{\omega} + y_{(0)}, \quad (3.402)$$

$$z = \frac{\dot{z}_{(0)}}{\omega} \sin \omega \tau - \frac{\dot{y}_{(0)}}{\omega} \cos \omega \tau + \frac{\dot{y}_{(0)}}{\omega} + z_{(0)}. \quad (3.403)$$

Consequently, the components of the three-dimensional momentum of the our-world particle under the considered configuration of the acting fields take the form

$$\left. \begin{aligned} p^1 &= m_0 \dot{x}_{(0)} + eE \left(1 - \frac{4\Omega^2}{\omega^2} \right) \tau - \\ &\quad - 2m_0 \Omega \left[\frac{\dot{y}_{(0)}}{\omega} \sin \omega \tau + (y_{(0)} + A) \cos \omega \tau - \frac{\dot{y}_{(0)}}{\omega} - A \right] \\ p^2 &= m_0 \left[\dot{y}_{(0)} \cos \omega \tau - \omega (y_{(0)} + A) \sin \omega \tau \right] + \frac{2\Omega eE}{\omega^2} \\ p^3 &= m_0 \dot{z}_{(0)} + \frac{eH}{c} \times \\ &\quad \times \left[(y_{(0)} + A) \cos \omega \tau + \frac{\dot{y}_{(0)}}{\omega} \sin \omega \tau - A + \frac{2\Omega eE}{m_0 \omega^2} \tau - y_{(0)} \right] \end{aligned} \right\}, \quad (3.404)$$

where the frequency ω is

$$\omega = \sqrt{4\Omega^2 + \frac{e^2 H^2}{m_0^2 c}}. \quad (3.405)$$

In the mirror world, given the above configuration of the acting fields, the components of the three-dimensional momentum of an analogous charged particle are

$$\left. \begin{aligned} p^1 &= m_0 \dot{x}_{(0)} + 2eE\tau \\ p^2 &= m_0 (\dot{y}_{(0)} \cos \omega\tau - \dot{z}_{(0)} \sin \omega\tau) \\ p^3 &= m_0 (\dot{z}_{(0)} \cos \omega\tau - \dot{y}_{(0)} \sin \omega\tau) \end{aligned} \right\}, \quad (3.406)$$

where, in contrast to our world, the frequency is $\omega = \frac{eH}{m_{(0)}c}$.

3.14 Conclusions

In fact, the theory that we have created in this Chapter can be more precisely called the *chronometrically invariant representation of electrodynamics in a pseudo-Riemannian space*. In other words, because the mathematical apparatus of physically observable quantities initially assumes the four-dimensional space-time of General Relativity, we can simply refer to it as the *chronometrically invariant electrodynamics* (or CED). Here, we have obtained only the basics of this theory:

- The chr.inv.-components of the electromagnetic field tensor (Maxwell tensor);
- The Maxwell equations in the chr.inv.-form;
- The law of conservation of electric charge in the chr.inv.-form;
- Lorenz' condition in the chr.inv.-form;
- The d'Alembert equations in the chr.inv.-form (wave propagation equations) for the scalar potential and vector-potential of the electromagnetic field;
- The Lorentz force in the chr.inv.-form;
- The electromagnetic field energy-momentum tensor and its chr. inv.-components;
- The chr.inv.-equations of motion of a charged particle;
- The geometric structure of the four-dimensional electromagnetic field potential.

It is obvious that, the whole scope of the chr.inv.-electrodynamics is much wider than the above obtained results. In addition to what has been obtained, we could obtain the chr.inv.-equations of motion of a distributed charge or study the motion of a particle that has its own electromagnetic emission interacting with the electromagnetic field or, at last, deduce the equations of motion for a charged particle travelling at an arbitrary angle to the field strengths (either for an individual particle or a distributed charge), or solve many other interesting problems.

In addition, of course, here we are talking about non-quantum electrodynamics. As is known, the mathematical apparatus of chronometric invariants was created for the four-dimensional pseudo-Riemannian space. In a space with a different geometry, the operators formally defining physical observables, of course, will also be different. However, the creation of the mathematical methods determining physical observable quantities in the space of quantum mechanics and quantum electrodynamics is in principle also possible: we have carried out the necessary preliminary work in this direction, and only the lack of time, as well as the shift in the focus of our scientific interests to other, incommensurably more interesting problems, stopped the creation of a chronometrically invariant quantum mechanics and a chronometrically invariant electrodynamics.

4.1 Problem statement

In this Chapter we are going to obtain the equations of motion of a particle with an internal rotation momentum (*spin*). As we noted in Chapter 1, these are the parallel transport equations of the four-dimensional dynamic vector of the particle Q^α , which is the sum

$$Q^\alpha = P^\alpha + S^\alpha, \quad (4.1)$$

where $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ is the four-dimensional momentum vector of the particle. The four-dimensional vector S^α is an additional momentum which this particle gains from its internal momentum (spin), so this momentum makes the motion of the particle non-geodesic. Therefore, we will refer to S^α as the *spin momentum*. Since we know the components of the momentum vector P^α , to define summary dynamic vector Q^α we only need to obtain the components of the spin momentum vector S^α .

Our first step, in §4.1, will be defining a particle's spin as a geometric quantity in the four-dimensional pseudo-Riemannian space of General Relativity. Then, in §4.2, we will deduce the spin momentum vector S^α itself. In §4.3, our contribution will be to obtain the equations of motion of a spin particle in the pseudo-Riemannian space, as well as the chr.inv.-projections of the equations. Other sections of this Chapter will focus on the motion of elementary particles.

The numerical value of the spin is $\pm n\hbar$, measured in the fractions of Planck's constant, where n is the so-called *spin quantum number*. As of today, it is known that for various kinds of elementary particles this number is $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. The alternating sign \pm stands for the possible right-wise or left-wise internal rotation of the spin particle under consideration. Besides, the Planck constant \hbar has the dimension of

angular momentum [gram cm² sec⁻¹]. This alone hints that the spin tensor, according to its geometric structure, should be similar to the tensor of an angular momentum, i.e., be an antisymmetric tensor of the 2nd rank. We are going to check if another source can prove this.

Bohr's second postulate states that the length of an electron orbit in an atom must be an *integer* number of de Broglie wavelengths $\lambda = \frac{h}{p}$, which stands for the electron in accordance with the wave-particle concept. In other words, the electron orbit length $2\pi r$ consists of k de Broglie wavelengths

$$2\pi r = k\lambda = k \frac{h}{p}, \quad (4.2)$$

where p is the orbital momentum of the electron. Taking into account that Planck's constant is $\hbar = \frac{h}{2\pi}$, the equation (4.2) takes the form

$$rp = k\hbar. \quad (4.3)$$

Because the radius-vector of an electron orbit r^i is always orthogonal to the electron's orbital momentum p^k , this formula in tensor notation is a vector product, namely

$$\left[r^i; p^k \right] = k\hbar^{ik}. \quad (4.4)$$

From here we conclude that Planck's constant deduced from Bohr's second postulate in tensor notation is present with an antisymmetric tensor of the 2nd rank.

This representation of the Planck constant in a tensor form is linked to the orbital model of atoms — the systems more complicated than the electron or any other elementary particle. Nevertheless, the spin is also defined using this constant as an internal property of elementary particles themselves. Therefore, according to Bohr's second postulate, we can consider the geometric structure of Planck's constant proceeding from another experimental relationship which is related to the internal structure of the electron.

We have such an opportunity thanks to the classical experiment performed by Stern and Gerlach in 1921. One of their results is that any electron has an internal magnetic momentum L_m proportional to the electron's internal rotation momentum (spin)

$$\frac{m_e}{e} L_m = n\hbar, \quad (4.5)$$

where e is the charge of the electron, m_e is its mass, and n is the spin quantum number (for the electron, it is $n = \frac{1}{2}$). The magnetic momentum of a contour covering an area $S = \pi r^2$, which conducts a current I , is $L_m = IS$. The current equals to the charge e divided by its circulation period $T = \frac{2\pi r}{u}$ along the contour

$$I = \frac{eu}{2\pi r}, \quad (4.6)$$

where u is the linear velocity of the charge circulation. Hence, the internal magnetic momentum of the electron is

$$L_m = \frac{1}{2} eur, \quad (4.7)$$

or, in tensor notation,*

$$L_m^{ik} = \frac{1}{2} e [r^i; u^k] = \frac{1}{2} [r^i; p_m^k], \quad (4.8)$$

where r^i is the radius-vector of the internal current circulation provided by the electron, and u^k is the vector of the circulation velocity.

From here we see that Planck's constant calculated from the internal magnetic momentum of an electron (4.5) is also the vector product of two vectors. Therefore, it is an antisymmetric tensor of the 2nd rank

$$\frac{m_e}{2e} [r^i; p_m^k] = n \hbar^{ik}, \quad (4.9)$$

which proves a similar conclusion based on the Bohr second postulate.

Thus, considering the electron quantum relationships in the four-dimensional pseudo-Riemannian space of General Relativity, we introduce the *four-dimensional antisymmetric Plank tensor* $\hbar^{\alpha\beta}$, the spatial components of which are the three-dimensional quantities \hbar^{ik} , i.e.

$$\hbar^{\alpha\beta} = \begin{pmatrix} \hbar^{00} & \hbar^{01} & \hbar^{02} & \hbar^{03} \\ \hbar^{10} & \hbar^{11} & \hbar^{12} & \hbar^{13} \\ \hbar^{20} & \hbar^{21} & \hbar^{22} & \hbar^{23} \\ \hbar^{30} & \hbar^{31} & \hbar^{32} & \hbar^{33} \end{pmatrix}. \quad (4.10)$$

*The equations (4.8) and (4.9) are given for the Minkowski space acceptable for the above experiment. In a Riemannian space, the result of integration depends on the integration path. Therefore, the radius-vector of a finite length is not defined in a Riemannian space, because its length depends on the constantly varying direction.

The antisymmetric Planck tensor $\hbar^{\alpha\beta}$ is dual to the *Planck pseudotensor*, which is $\hbar^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} \hbar_{\mu\nu}$. Therefore, the spin of a particle in the four-dimensional pseudo-Riemannian space is characterized by the antisymmetric Planck tensor $n\hbar^{\alpha\beta}$, or by its dual Planck pseudotensor $n\hbar^{*\alpha\beta}$. Note that the physical nature of the spin does not matter here; it is only sufficient that this fundamental property of particles is characterized by a tensor (or a pseudotensor) of a certain kind. Thanks to this approach, we can solve the problem of the motion of spin particles without any preliminary assumption on their internal structure, i.e., using a strictly formal mathematical method.

Hence, from a geometric point of view, the Planck constant is an antisymmetric tensor of the 2nd rank, the dimension of which is that of angular momentum irrespective of the quantities from which it was obtained (mechanical or electromagnetic).

The latter also means that the Planck tensor does not characterize the rotation of masses inside an atom or any masses inside elementary particles; it is derived based on a fundamental quantum rotation of the space itself and sets all “elementary” rotations in the space irrespective of their nature.

The rotation of a space is characterized by the chr.inv.-tensor A_{ik} (1.36), which results from lowering indices $A_{ik} = h_{im} h_{kn} A^{mn}$ in the components A^{mn} of the contravariant four-dimensional tensor

$$A^{\alpha\beta} = c h^{\alpha\mu} h^{\beta\nu} a_{\mu\nu}, \quad a_{\mu\nu} = \frac{1}{2} \left(\frac{\partial b_\nu}{\partial x^\mu} - \frac{\partial b_\mu}{\partial x^\nu} \right). \quad (4.11)$$

In the accompanying reference frame ($b^i = 0$), the auxiliary quantity $a_{\mu\nu}$ has the components

$$a_{00} = 0, \quad a_{0i} = \frac{1}{2c^2} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \quad a_{ik} = \frac{1}{2c} \left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i} \right), \quad (4.12)$$

so we have

$$\left. \begin{aligned} A_{00} &= 0, & A_{0i} &= -A_{i0} = 0, \\ A_{ik} &= \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} (F_i v_k - F_k v_i) \end{aligned} \right\}. \quad (4.13)$$

In the absence of gravitational fields, the tensor of the angular velocity A_{ik} with which the space rotates depends only on the linear velocity

of this rotation v_i . Therefore we denote it as $A_{\alpha\beta} = \Omega_{\alpha\beta}$

$$\Omega_{00} = 0, \quad \Omega_{0i} = -\Omega_{i0} = 0, \quad \Omega_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right). \quad (4.14)$$

On the other hand, according to the wave-particle concept, any particle corresponds to a wave having the energy $E = mc^2 = \hbar\omega$, where m is the relativistic mass of the particle and ω is its characteristic frequency. In other words, from a purely geometric point of view, any particle can be considered as a wave spread and infinitely close to the position of the particle, the characteristic frequency of which is dependent on a certain distribution of the angular velocities $\omega_{\alpha\beta}$ also defined within this vicinity. As a result, the above quantum relationship in tensor notation becomes $mc^2 = \hbar^{\alpha\beta} \omega_{\alpha\beta}$.

Because the Planck tensor is antisymmetric, all of its diagonal components are zeroes. Its space-time (mixed) components in the accompanying reference frame are also zero similar to the corresponding components of the four-dimensional tensor of the space rotation (4.14). The numerical values of the spatial (three-dimensional) components of the Planck tensor, physically observable in experiments, are $\pm\hbar$ depending on the rotation direction and make up the *three-dimensional chr.inv.-Planck tensor* \hbar^{ik} .

In the case of a left-wise internal rotation, the components \hbar^{12} , \hbar^{23} , \hbar^{31} are positive, while the components \hbar^{13} , \hbar^{32} , \hbar^{21} are negative (and vice versa for a right-wise rotation). Then the geometric structure of the four-dimensional Planck tensor, represented as a matrix, is

$$\hbar^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \hbar & -\hbar \\ 0 & -\hbar & 0 & \hbar \\ 0 & \hbar & -\hbar & 0 \end{pmatrix}. \quad (4.15)$$

In the case of a right-wise internal rotation, the non-zero components \hbar^{12} , \hbar^{23} , \hbar^{31} change their sign to become negative, while the components \hbar^{13} , \hbar^{32} , \hbar^{21} become positive

$$\hbar^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\hbar & \hbar \\ 0 & \hbar & 0 & -\hbar \\ 0 & -\hbar & \hbar & 0 \end{pmatrix}. \quad (4.16)$$

The square of the four-dimensional Planck tensor is calculated based on the following obvious formula

$$\begin{aligned} \hbar_{\alpha\beta}\hbar^{\alpha\beta} = 2\hbar^2 & \left[(g_{11}g_{22} - g_{12}^2) + (g_{11}g_{33} - g_{13}^2) + \right. \\ & + (g_{22}g_{33} - g_{23}^2) + 2(g_{12}g_{23} - g_{22}g_{13} - \\ & \left. - g_{12}g_{33} + g_{13}g_{23} - g_{11}g_{23} + g_{12}g_{13}) \right], \end{aligned} \quad (4.17)$$

and, in a Galilean reference frame in the Minkowski space, where the metric is diagonal unit (2.70), is $\hbar_{\alpha\beta}\hbar^{\alpha\beta} = 6\hbar^2$. However, in the pseudo-Riemannian space of General Relativity, the value of $\hbar_{\alpha\beta}\hbar^{\alpha\beta}$ is calculated using the spatial components of the fundamental metric tensor expressed from the chr.inv.-metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2}v_iv_k$ dependent on the space rotation velocity. Hence, although the physically observable components \hbar^{ik} of the Planck tensor are constants (having opposite signs for left-wise and right-wise rotations), its square in a general case depends on the angular velocity with which the space rotates.

Now, having the Planck tensor components defined, we can deduce the momentum that a particle gains from its spin, as well as the equations of motion of the spin particle travelling in the pseudo-Riemannian space. This will be the focus of the next section, §4.2.

4.2 A spin particle's momentum in the equations of motion

The additional momentum S^α that a particle gains from its spin can be obtained from considering the *action* for spin particles.

The action S for a particle that has an internal scalar field k , with which an external scalar field A interacts thereby displacing the particle at an elementary interval ds , is

$$S = \alpha_{(kA)} \int_a^b kA ds, \quad (4.18)$$

where $\alpha_{(kA)}$ is a scalar constant that characterizes the particle's properties manifested in the interaction, and also equates dimensions in the equation. If the internal scalar field k of the particle corresponds to an external tensor field of the 1st rank A_α , then the action required to displace the particle by the field is

$$S = \alpha_{(kA_\alpha)} \int_a^b kA_\alpha dx^\alpha. \quad (4.19)$$

In the interaction of the particle's internal scalar field k with an external tensor field of the 2nd rank $A_{\alpha\beta}$, the action to displace the particle by that field is

$$S = \alpha_{(kA_{\alpha\beta})} \int_a^b k A_{\alpha\beta} dx^\alpha dx^\beta, \quad (4.20)$$

and so forth. For instance, if an internal vector potential k^α specific of a particle corresponds to an external vector field A_α , then the action to displace the particle by the field is

$$S = \alpha_{(k^\alpha A_\alpha)} \int_a^b k^\alpha A_\alpha ds. \quad (4.21)$$

Besides, the action can be represented, irrespective of the nature of internal properties of particles and external fields, as follows

$$S = \int_{t_1}^{t_2} L dt, \quad (4.22)$$

where L is the so-called *Lagrange function*. Because the dimension of action is [erg sec = gram cm² sec⁻¹], then the Lagrange function has the dimension of energy [erg = gram cm² sec⁻²]. In addition, the derivative of the Lagrange function with respect to the three-dimensional coordinate velocity $u^i = \frac{dx^i}{dt}$ of the particle

$$\frac{\partial L}{\partial u^i} = p_i \quad (4.23)$$

is the covariant notation of its three-dimensional momentum $p^i = cP^i$ that can be used to restore the complete formula for the four-dimensional momentum vector P^α of the particle.

Therefore, having a formula for the action to displace a spin particle, as well as the Lagrange function differentiated with respect to the coordinate velocity of the particle, it is possible to restore the formula for the additional momentum gained by the particle due to its spin.

As is known, the action to displace a free particle in the pseudo-Riemannian space is*

$$S = \int_a^b m_0 c ds. \quad (4.24)$$

*In *The Classical Theory of Fields* [10], Landau and Lifshitz use “minus” before the action, and we always have “plus” before the integral of the action and also before

In a Galilean reference frame in the Minkowski space, since the non-diagonal terms of the fundamental metric tensor are zeroes, the space-time interval is

$$ds = \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta} = c dt \sqrt{1 - \frac{u^2}{c^2}}, \quad (4.25)$$

hence, the action (4.24) becomes

$$S = \int_a^b m_0 c ds = \int_{t_1}^{t_2} m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}} dt. \quad (4.26)$$

Therefore, the Lagrange function of a free particle in a Galilean reference frame in the Minkowski space is

$$L = m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}}. \quad (4.27)$$

Differentiating it with respect to the particle's coordinate velocity, we arrive at the covariant form of its three-dimensional momentum

$$p_i = \frac{\partial L}{\partial u^i} = m_0 c^2 \frac{\partial \sqrt{1 - \frac{u^2}{c^2}}}{\partial u^i} = -\frac{m_0 u_i}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad (4.28)$$

from which, after lifting the index, we arrive at the four-dimensional momentum vector of the free particle as follows

$$P^\alpha = \frac{m_0}{c} \frac{dx^\alpha}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dx^\alpha}{dt} = m_0 \frac{dx^\alpha}{ds}. \quad (4.29)$$

In the final formula, both of the multipliers m_0 and $\frac{dx^\alpha}{ds}$ are general covariant quantities, so they do not depend on the choice of a particular reference frame. For this reason, this formula obtained in a Galilean reference frame in the Minkowski space is also true in any other arbitrary reference frame in any pseudo-Riemannian space.

the Lagrange function. This is because the sign of an action depends on the signature of the pseudo-Riemannian space. Landau and Lifshitz use the signature $(-+++)$, where time is imaginary, the spatial coordinates are real and the three-dimensional coordinate momentum is positive. On the contrary, we use the signature $(+---)$ as Zel'manov [9, 11–13], because, in this case, time is real and the spatial coordinates are imaginary, so the three-dimensional *observable* momentum is positive.

Let us now consider the motion of a particle having an internal structure that in experiments reveals itself as the *spin*. The inner rotation (spin) $n\hbar^{\alpha\beta}$ of a particle in the four-dimensional pseudo-Riemannian space corresponds to the external field $A_{\alpha\beta}$ of the space rotation. Therefore, the summary action to displace a spin particle is

$$S = \int_a^b \left(m_0 c ds + \alpha_{(s)} \hbar^{\alpha\beta} A_{\alpha\beta} ds \right), \quad (4.30)$$

where $\alpha_{(s)}$ [sec cm⁻¹] is a scalar constant characteristic of the particle in the spin interaction. Since the action constants can include only the particle's properties and fundamental physical constants, the constant $\alpha_{(s)}$ is, obviously, the spin quantum number n (function of the internal properties of the particle), divided by the light velocity $\alpha_{(s)} = \frac{n}{c}$. Then, the action to displace a spin particle, produced by the interaction of the particle's spin with the space non-holonomy field $A_{\alpha\beta}$ is

$$S = \alpha_{(s)} \int_a^b \hbar^{\alpha\beta} A_{\alpha\beta} ds = \frac{n}{c} \int_a^b \hbar^{\alpha\beta} A_{\alpha\beta} ds. \quad (4.31)$$

A remark should be made here. Deducing the four-dimensional momentum vector for a spin particle using the same method as for a free particle is impossible. As was shown above, we first obtained the four-dimensional momentum vector of a free particle in a Galilean reference frame in the Minkowski space, where the formula for ds expressed through dt and substituted into the action has the very simple form (4.25). As was noted, the obtained formula for the momentum vector (4.29), due to its property of general covariance, is true in any reference frame in the pseudo-Riemannian space. But as we can see from the above formula for the action for a spin particle, the spin affects the motion of the particle only if the space is non-holonomic $A_{\alpha\beta} \neq 0$, i.e., where the non-diagonal terms g_{0i} of the fundamental metric tensor are non-zeroes. In a Galilean reference frame, by definition, the non-diagonal terms in the metric tensor are zeroes, hence, zeroes are the components of the linear velocity with which the space rotates $v_i = -c \frac{g_{0i}}{\sqrt{g_{00}}}$ and, hence, all components of the non-holonomy tensor $A_{\alpha\beta}$. Therefore, this is worthless and cannot be used to first deduce the formula for the momentum of a spin particle in a Galilean reference frame in the Minkowski space (where it is zero by definition). Instead

we must deduce the momentum of a spin particle directly in the pseudo-Riemannian space.

The space-time interval ds travelled by a particle in the pseudo-Riemannian space, written in terms of the reference frame accompanying an arbitrary observer, is

$$\begin{aligned} ds &= c d\tau \sqrt{1 - \frac{v^2}{c^2}} = \\ &= c dt \left(1 - \frac{w + v_i u^i}{c^2}\right) \sqrt{1 - \frac{u^2}{c^2 \left(1 - \frac{w + v_i u^i}{c^2}\right)^2}}, \end{aligned} \quad (4.32)$$

where the coordinate velocity of the particle $u^i = \frac{dx^i}{dt}$ can be expressed through its observable velocity $v^i = \frac{dx^i}{d\tau}$ as follows

$$v^i = \frac{u^i}{1 - \frac{w + v_i u^i}{c^2}}, \quad v^2 = \frac{h_{ik} u^i u^k}{\left(1 - \frac{w + v_i u^i}{c^2}\right)^2}. \quad (4.33)$$

Then, the additional action (4.31), produced by the interaction of the particle's spin with the space non-holonomy field, becomes

$$S = n \int_{t_1}^{t_2} \hbar^{\alpha\beta} A_{\alpha\beta} \sqrt{\left(1 - \frac{w + v_i u^i}{c^2}\right)^2 - \frac{u^2}{c^2}} dt. \quad (4.34)$$

Therefore, the Lagrange function for this action is

$$L = n \hbar^{\alpha\beta} A_{\alpha\beta} \sqrt{\left(1 - \frac{w + v_i u^i}{c^2}\right)^2 - \frac{u^2}{c^2}}. \quad (4.35)$$

Now to deduce the spin momentum we only need to differentiate the Lagrange function (4.35) with respect to the coordinate velocity of the particle. Taking into account that the internal rotation field tensor $\hbar^{\alpha\beta}$ of the particle and the space rotation field tensor $A_{\alpha\beta}$ (4.13) are not functions of the particle's velocity, after differentiating we obtain

$$\begin{aligned} p_i &= \frac{\partial L}{\partial u^i} = n \hbar^{mn} A_{mn} \frac{\partial}{\partial u^i} \sqrt{\left(1 - \frac{w + v_i u^i}{c^2}\right)^2 - \frac{u^2}{c^2}} = \\ &= - \frac{n \hbar^{mn} A_{mn}}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} (v_i + v_i), \end{aligned} \quad (4.36)$$

where $v_i = h_{ik} v^k$ according to the chronometrically invariant formalism.

Compare (4.36) with the covariant spatial component $p_i = cP_i$ of the four-dimensional momentum vector $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ of a particle in the pseudo-Riemannian space*. If the particle is located in our world, so it travels from the past to the future with respect to us, then its three-dimensional covariant momentum is

$$p_i = cP_i = c g_{i\alpha} P^\alpha = -m(v_i + v_i) = -\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}(v_i + v_i). \quad (4.37)$$

From here we see that the four-dimensional momentum S^α that the particle gains due to its spin (its internal spin momentum) is

$$S^\alpha = \frac{1}{c^2} n \hbar^{\mu\nu} A_{\mu\nu} \frac{dx^\alpha}{ds}, \quad (4.38)$$

or, denoting $\eta_0 = n \hbar^{\mu\nu} A_{\mu\nu} = n \hbar^{mn} A_{mn}$, to make the formula simpler, we obtain

$$S^\alpha = \frac{1}{c^2} \eta_0 \frac{dx^\alpha}{ds}. \quad (4.39)$$

Then the summary vector Q^α (4.1) that characterizes the motion of the spin particle is formulated as follows

$$Q^\alpha = P^\alpha + S^\alpha = m_0 \frac{dx^\alpha}{ds} + \frac{1}{c^2} n \hbar^{\mu\nu} A_{\mu\nu} \frac{dx^\alpha}{ds}. \quad (4.40)$$

So, any spin particle travelling in a non-holonomic space ($A_{\mu\nu} \neq 0$) actually gains an additional momentum that deviates the particle from a geodesic line and thereby makes its motion non-geodesic. In the absence of the space rotation, i.e., in a holonomic space, we have $A_{\mu\nu} = 0$, so the spin of a particle does not affect its motion. However, it is difficult to find (if at all possible) such a sub-atomic region, where the background space does not rotate. Therefore, the spin affects the motion of particles on the scale of atomic physics everywhere in the Universe.

4.3 The equations of motion of a spin particle

The equations of motion of a spin particle are the parallel transport equations of the summary vector $Q^\alpha = P^\alpha + S^\alpha$ (4.40) along the trajectory of

*In this comparison we mean a mass-bearing particle.

the particle, namely

$$\frac{d}{ds} (P^\alpha + S^\alpha) + \Gamma_{\mu\nu}^\alpha (P^\mu + S^\mu) \frac{dx^\nu}{ds} = 0, \quad (4.41)$$

where the square of the vector remains unchanged $Q_\alpha Q^\alpha = \text{const}$ in the Levi-Civita parallel transport along the particle's trajectory.

Let us deduce the chr.inv.-projections of the general covariant equations of motion (4.41). The projections in their general notation, obtained in Chapter 2, have the form

$$\frac{d\varphi}{ds} - \frac{1}{c} F_i q^i \frac{d\tau}{ds} + \frac{1}{c} D_{ik} q^i \frac{dx^k}{ds} = 0, \quad (4.42)$$

$$\begin{aligned} \frac{dq^i}{ds} + \left(\frac{\varphi}{c} \frac{dx^k}{ds} + q^k \frac{d\tau}{ds} \right) (D_k^i + A_k^i) - \\ - \frac{\varphi}{c} F^i \frac{d\tau}{ds} + \Delta_{mk}^i q^m \frac{dx^k}{ds} = 0, \end{aligned} \quad (4.43)$$

where φ is the projection of the summary vector Q^α onto the observer's time line and q^i is its projection onto his spatial section

$$\varphi = b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}} = \frac{P_0}{\sqrt{g_{00}}} + \frac{S_0}{\sqrt{g_{00}}}, \quad (4.44)$$

$$q^i = h_\alpha^i Q^\alpha = Q^i = P^i + S^i. \quad (4.45)$$

Therefore, to solve the problem, it is necessary to derive specific formulae for the φ and q^i , then substitute them into (4.42, 4.43). The chr.inv.-projections of the momentum vector $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m, \quad P^i = \frac{1}{c} m v^i, \quad (4.46)$$

and now we have to deduce the chr.inv.-projections of the spin momentum vector S^α . Taking into account in the formula for S^α (4.39) that the space-time interval, formulated with physically observable quantities, is $ds = c d\tau \sqrt{1 - v^2/c^2}$, we obtain the S^α components

$$S^0 = \frac{1}{c^2} \frac{n \hbar^{mn} A_{mn} (v_i v^i \pm c^2)}{\sqrt{1 - \frac{v^2}{c^2}} c^2 \left(1 - \frac{w}{c^2}\right)}, \quad (4.47)$$

$$S^i = \frac{1}{c^3} \frac{n\hbar^{mn} A_{mn}}{\sqrt{1 - \frac{v^2}{c^2}}} v^i, \quad (4.48)$$

$$S_0 = \pm \frac{1}{c^2} \left(1 - \frac{w}{c^2}\right) \frac{n\hbar^{mn} A_{mn}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (4.49)$$

$$S_i = -\frac{1}{c^3} \frac{n\hbar^{mn} A_{mn}}{\sqrt{1 - \frac{v^2}{c^2}}} (v_i \pm v_i), \quad (4.50)$$

which are formulated with physically observable quantities. Thus, we obtain the chr.inv.-projections of the particle's spin momentum vector

$$\frac{S_0}{\sqrt{g_{00}}} = \pm \frac{1}{c^2} \eta, \quad S^i = \frac{1}{c^3} \eta v^i, \quad (4.51)$$

where the quantity η is

$$\eta = \frac{n\hbar^{mn} A_{mn}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (4.52)$$

while the alternating sign resulting from the time function $\frac{dt}{d\tau}$ (1.63) indicates the particle's motion to the future (upper sign) or to the past (lower sign). Then, the square of the spin momentum vector is

$$S_\alpha S^\alpha = g_{\alpha\beta} S^\alpha S^\beta = \frac{1}{c^4} \eta_0^2 g_{\alpha\beta} \frac{dx^\alpha dx^\beta}{ds^2} = \frac{1}{c^4} \eta_0^2, \quad (4.53)$$

and the square of the summary vector Q^α is

$$Q_\alpha Q^\alpha = g_{\alpha\beta} Q^\alpha Q^\beta = m_0^2 + \frac{2}{c^2} m_0 \eta_0 + \frac{1}{c^4} \eta_0^2. \quad (4.54)$$

Therefore, the square of the summary vector of any spin particle separates into the following three parts:

- a) The square of the momentum vector of the particle $P_\alpha P^\alpha = m_0^2$;
- b) The square of its spin momentum vector $S_\alpha S^\alpha = \frac{1}{c^4} \eta_0^2$;
- c) The term $\frac{2}{c^2} m_0 \eta_0$ describing the spin-gravitational interaction.

To implement parallel transport, it is necessary that the square of the transported summary vector remains unchanged throughout the entire path. But the obtained formula (4.54) means that (because $m_0 = const$)

the square of the summary vector Q^α of a spin particle remains unchanged if only $\eta_0 = const$, i.e., the increment of η_0 is zero

$$d\eta_0 = \frac{\partial\eta_0}{\partial x^\alpha} dx^\alpha = 0 \quad (4.55)$$

along the trajectory of the spin particle.

Dividing both sides of the equation by $d\tau$, which is always possible because any time interval registered by an observer is greater than zero*, we obtain the chr.inv.-conservation condition for the square of the spin particle's summary vector

$$\frac{d\eta_0}{d\tau} = \frac{* \partial\eta_0}{\partial t} + v^k \frac{* \partial\eta_0}{\partial x^k} = 0. \quad (4.56)$$

Substituting $\eta_0 = n \hbar^{mn} A_{mn}$, we have

$$n \hbar^{mn} \left(\frac{* \partial A_{mn}}{\partial t} + v^k \frac{* \partial A_{mn}}{\partial x^k} \right) = 0. \quad (4.57)$$

To illustrate the result, we replace the space non-holonomy tensor A_{ik} , which is actually the tensor of the angular velocity with which the space rotates, with the angular velocity pseudovector

$$\Omega^{*i} = \frac{1}{2} \varepsilon^{imn} A_{mn}, \quad (4.58)$$

which is also a chr.inv.-quantity. Multiplying Ω^{*i} by ε_{ipq}

$$\Omega^{*i} \varepsilon_{ipq} = \frac{1}{2} \varepsilon^{imn} \varepsilon_{ipq} A_{mn} = \frac{1}{2} (\delta_p^m \delta_q^n - \delta_p^n \delta_q^m) A_{mn} = A_{pq}, \quad (4.59)$$

we transform the formula (4.57) to the following form

$$\begin{aligned} n \hbar^{mn} \left[\frac{* \partial}{\partial t} (\varepsilon_{imn} \Omega^{*i}) + v^k \frac{* \partial}{\partial x^k} (\varepsilon_{imn} \Omega^{*i}) \right] = \\ = n \hbar^{mn} \varepsilon_{imn} \left[\frac{1}{\sqrt{h}} \frac{* \partial}{\partial t} (\sqrt{h} \Omega^{*i}) + v^k \frac{1}{\sqrt{h}} \frac{* \partial}{\partial x^k} (\sqrt{h} \Omega^{*i}) \right] = 0. \end{aligned} \quad (4.60)$$

*The condition $d\tau = 0$ makes sense only in a generalized space-time, where the fundamental metric tensor $g_{\alpha\beta}$ can be completely degenerate. In this case, the above condition determined a completely degenerate region (called zero-space), in which there are zero-particles capable of instant displacement, and, hence, they are the carriers of long-range action.

The gravitational inertial force vector and the space non-holonomy tensor are related by the Zelmanov identities, one of which (see formula 13.20 in [9]) has the following form

$$\frac{2}{\sqrt{h}} \frac{* \partial}{\partial t} (\sqrt{h} \Omega^{*i}) + \varepsilon^{ijk} * \nabla_j F_k = 0, \quad (4.61)$$

or, in the other notation

$$\frac{* \partial A_{ik}}{\partial t} + \frac{1}{2} (* \nabla_k F_i - * \nabla_i F_k) = \frac{* \partial A_{ik}}{\partial t} + \frac{1}{2} \left(\frac{* \partial F_k}{\partial x^i} - \frac{* \partial F_i}{\partial x^k} \right) = 0, \quad (4.62)$$

where $\varepsilon^{ijk} * \nabla_j F_k$ is the chr.inv.-curl of the gravitational inertial force field F_k . From here we see that the non-stationarity of the space rotation field A_{ik} is due to the presence of a *curl* of the acting field of the gravitational inertial force F_i .

As a result, taking the Zelmanov identity (4.61) into account, our formula (4.60) transform into

$$-n \hbar^{mn} * \nabla_m F_n + n \hbar^{mn} \varepsilon_{imn} v^k \frac{1}{\sqrt{h}} \frac{* \partial}{\partial x^k} (\sqrt{h} \Omega^{*i}) = 0, \quad (4.63)$$

which can be re-written in the two-side form

$$n \hbar^{mn} * \nabla_m F_n = n \hbar^{mn} \varepsilon_{imn} v^k \left(\Omega^{*i} \frac{\partial \ln \sqrt{h}}{\partial x^k} + \frac{* \partial \Omega^{*i}}{\partial x^k} \right). \quad (4.64)$$

Let us now recall that the above formula is nothing but only the expanded chr.inv.-notation of the conservation condition for the summary vector (4.57). The left hand side of (4.64) is

$$\pm 2n \hbar (* \nabla_1 F_2 - * \nabla_2 F_1 + * \nabla_1 F_3 - * \nabla_3 F_1 + * \nabla_2 F_3 - * \nabla_3 F_2), \quad (4.65)$$

where “plus” and “minus” stand for the right-wise and left-wise rotating reference space of the observer, respectively. Therefore, the left hand side of the equation (4.64) is the chr.inv.-curl of the gravitational inertial force. The right hand side of (4.64) depends on the spatial orientation of the space rotation pseudovector Ω^{*i} .

Therefore, to preserve the square of the momentum vector of a spin particle, transported parallel to itself along the trajectory of the particle, it is necessary that the right hand side and the left hand side of the equation (4.64) be equal to each other along the trajectory.

In a general case, without additional assumptions about the geometric structure of the background space, the above condition requires a balance between the vortical field of the acting gravitational inertial force and the spatial distribution of the space rotation pseudovector.

If the field of the gravitational inertial force is vortexless, then the left hand side of the conservation condition (4.64) is zero and, therefore, this condition becomes

$$n\hbar^{mn}\varepsilon_{imn}v^k\frac{1}{\sqrt{h}}\frac{*}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)=0. \quad (4.66)$$

Using the chr.inv.-derivative $\frac{*}{\partial x^k}=\frac{\partial}{\partial x^k}+\frac{1}{c^2}v_k\frac{*}{\partial t}$, we have

$$n\hbar^{mn}\varepsilon_{imn}v^k\frac{1}{\sqrt{h}}\left[\frac{\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)-\frac{1}{c^2}v_k\frac{*}{\partial t}\left(\sqrt{h}\Omega^{*i}\right)\right]=0. \quad (4.67)$$

Since the force field F_i is vortexless, then, because of (4.66), the second term in this formula is zero. Therefore, the square of the summary vector of a spin particle remains unchanged in the vortexless force field F_i , provided that the chr.inv.-formula (4.66) and the formula containing the ordinary derivative are zeroes

$$n\hbar^{mn}\varepsilon_{imn}v^k\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)=0. \quad (4.68)$$

For example, for mass-bearing particles, this can be in the case where $v^k=0$, so this is when they are at rest with respect to the observer and his reference body. In this case, the vanishing of the derivatives in (4.68) is not essential. In contrast, massless particles travel with the velocity of light. Hence, for them, in the vortexless force field F_i , the derivatives $\frac{\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)$ and $\frac{*}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)$ must be zeroes in any case.

Let us now deduce the chr.inv.-equations of motion of a spin particle in the pseudo-Riemannian space. Substituting (4.46) and (4.51) into (4.44) and (4.45), we obtain that the chr.inv.-projections of the summary vector of the spin particle are

$$\varphi=\pm\left(m+\frac{1}{c^2}\eta\right), \quad q^i=\frac{1}{c}m v^i+\frac{1}{c^3}\eta v^i. \quad (4.69)$$

Having these quantities with $\varphi>0$ substituted into (4.42, 4.43), we obtain the chr.inv.-equations of motion for a mass-bearing spin particle

travelling in our world (it travels from the past to the future)

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = -\frac{1}{c^2} \frac{d\eta}{d\tau} + \frac{\eta}{c^4} F_i v^i - \frac{\eta}{c^4} D_{ik} v^i v^k, \quad (4.70)$$

$$\begin{aligned} \frac{d}{d\tau} (m v^i) + 2m (D_k^i + A_k^i) v^k - m F^i + m \Delta_{nk}^i v^n v^k = \\ = -\frac{1}{c^2} \frac{d}{d\tau} (\eta v^i) - \frac{2\eta}{c^2} (D_k^i + A_k^i) v^k + \frac{\eta}{c^2} F^i - \frac{\eta}{c^2} \Delta_{nk}^i v^n v^k, \end{aligned} \quad (4.71)$$

while for a mass-bearing spin particle travelling in the mirror world (it travels to the past), having the quantities (4.69) for $\varphi < 0$ substituted into (4.42, 4.43), we obtain

$$-\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = \frac{1}{c^2} \frac{d\eta}{d\tau} + \frac{\eta}{c^4} F_i v^i - \frac{\eta}{c^4} D_{ik} v^i v^k, \quad (4.72)$$

$$\begin{aligned} \frac{d}{d\tau} (m v^i) + m F^i + m \Delta_{nk}^i v^n v^k = \\ = -\frac{1}{c^2} \frac{d}{d\tau} (\eta v^i) - \frac{\eta}{c^2} F^i - \frac{\eta}{c^2} \Delta_{nk}^i v^n v^k. \end{aligned} \quad (4.73)$$

The obtained equations are written so that their left hand side has a *geodesic part* characteristic of free (geodesic) motion of the particle, and the right hand side has the terms produced due to the spin of the particle, which makes its motion non-geodesic (*non-geodesic part*). Hence, the right hand side is zero for a spinless particle, and the obtained equations transform into the chr.inv.-equations of free motion. The above form of the equations will facilitate their analysis.

In the framework of the wave-particle concept, a massless particle is described by the four-dimensional wave vector $K^\alpha = \frac{\omega}{c} \frac{dx^\alpha}{d\sigma}$, where $d\sigma^2 = h_{ik} dx^i dx^k$ is the square of the physically observable spatial interval (it is not equal to zero along isotropic trajectories). Because massless particles travel along isotropic trajectories (light propagation trajectories), the vector K^α is also isotropic, i.e., its square is zero. But, because the dimension of K^α is $[\text{cm}^{-1}]$, the equations have the dimension different from the dimension of the equations of motion of mass-bearing particles. Besides, this fact does not allow us to create a joint formula for the action for both massless and mass-bearing particles [9].

On the other hand, the spin is a physical property, possessed by both mass-bearing and massless particles. Therefore, when deducing

the equations of motion of spin particles, we need to use a uniform vector applicable to both kinds of particles. Such a vector can be obtained by applying the physical conditions along isotropic trajectories

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = 0, \quad c d\tau = d\sigma \neq 0 \quad (4.74)$$

to the four-dimensional momentum vector of a mass-bearing particle

$$P^\alpha = m_0 \frac{dx^\alpha}{ds} = \frac{m}{c} \frac{dx^\alpha}{d\tau} = m \frac{dx^\alpha}{d\sigma}. \quad (4.75)$$

As a result the observable spatial interval, not equal to zero along isotropic trajectories, becomes a derivation parameter for mass-bearing particles, while the dimension of the above vector, in contrast to the wave vector K^α [cm^{-1}], matches the dimension of the momentum vector P^α [gram]. The relativistic mass m , not equal to zero for massless particles, can be obtained from the energy equivalent using the $E = mc^2$ formula. For instance, the energy $E = 1 \text{ MeV} = 1.6 \times 10^{-6} \text{ erg}$ of a photon corresponds to a relativistic mass of $m = 1.8 \times 10^{-28} \text{ gram}$.

Therefore, the four-dimensional momentum vector (4.75) can describe the motion of either mass-bearing particles (non-isotropic trajectories) or massless particles (isotropic trajectories). Note that $m_0 = 0$ and $ds = 0$ for massless particles, therefore their ratio in (4.75) is a $\frac{0}{0}$ indeterminacy. However the transition from $\frac{m_0}{ds}$ to $\frac{m}{d\sigma}$ in (4.75) solves this indeterminacy, because the relativistic mass of any massless particle is $m \neq 0$ and also $d\sigma \neq 0$ along its trajectory.

It is obvious that along isotropic trajectories (massless particles) the square of the momentum vector P^α (4.75) is zero

$$P_\alpha P^\alpha = g_{\alpha\beta} P^\alpha P^\beta = m^2 g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = m^2 \frac{ds^2}{d\sigma^2} = 0, \quad (4.76)$$

and the chr.inv.-projections of the vector have the form

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m, \quad P^i = \frac{1}{c} m c^i, \quad (4.77)$$

where c^i is the chr.inv.-vector of the light velocity. In this case, the spin momentum vector of the particle (4.39) is as well isotropic

$$S^\alpha = \frac{1}{c^2} \eta_0 \frac{dx^\alpha}{ds} = \frac{1}{c^2} \eta \frac{dx^\alpha}{c d\tau} = \frac{1}{c^2} \eta \frac{dx^\alpha}{d\sigma}, \quad (4.78)$$

since its square is equal to zero

$$S_\alpha S^\alpha = g_{\alpha\beta} S^\alpha S^\beta = \frac{1}{c^4} \eta^2 g_{\alpha\beta} \frac{dx^\alpha dx^\beta}{d\sigma^2} = \frac{1}{c^4} \eta^2 \frac{ds^2}{d\sigma^2} = 0, \quad (4.79)$$

hence, the square of the summary vector $Q^\alpha = P^\alpha + S^\alpha$ of a massless spin particle is also zero. The chr.inv.-projections of the isotropic spin momentum (4.78) have the form

$$\frac{S_0}{\sqrt{g_{00}}} = \pm \frac{1}{c^2} \eta, \quad S^i = \frac{1}{c^3} \eta c^i, \quad (4.80)$$

so its spatial observable projection matches that for a mass-bearing particle (4.51), where the particle's observable velocity v^i (4.51) is used instead of the chr.inv.-vector c^i of the light velocity. Thus, the chr.inv.-projections of the summary vector of a massless spin particle are

$$\varphi = \pm \left(m + \frac{1}{c^2} \eta \right), \quad q^i = \frac{1}{c} m c^i + \frac{1}{c^3} \eta c^i. \quad (4.81)$$

Substituting them with $\varphi > 0$ into the formulae (4.42, 4.43), we obtain the chr.inv.-equations of motion of a massless spin particle that travels in our world (it travels from the past to the future)

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i c^i + \frac{m}{c^2} D_{ik} c^i c^k = -\frac{1}{c^2} \frac{d\eta}{d\tau} + \frac{\eta}{c^4} F_i c^i - \frac{\eta}{c^4} D_{ik} c^i c^k, \quad (4.82)$$

$$\begin{aligned} \frac{d}{d\tau} (m c^i) + 2m (D_k^i + A_k^i) c^k - m F^i + m \Delta_{nk}^i c^n c^k = \\ = -\frac{1}{c^2} \frac{d}{d\tau} (\eta c^i) - \frac{2\eta}{c^2} (D_k^i + A_k^i) c^k + \frac{\eta}{c^2} F^i - \frac{\eta}{c^2} \Delta_{nk}^i c^n c^k, \end{aligned} \quad (4.83)$$

while for a massless spin particle in the mirror world (it travels from the future to the past), having the quantities (4.81) with $\varphi < 0$ substituted into (4.42, 4.43), the chr.inv.-equations of motion have the form

$$-\frac{dm}{d\tau} - \frac{m}{c^2} F_i c^i + \frac{m}{c^2} D_{ik} c^i c^k = \frac{1}{c^2} \frac{d\eta}{d\tau} + \frac{\eta}{c^4} F_i c^i - \frac{\eta}{c^4} D_{ik} c^i c^k, \quad (4.84)$$

$$\begin{aligned} \frac{d}{d\tau} (m c^i) + m F^i + m \Delta_{nk}^i c^n c^k = \\ = -\frac{1}{c^2} \frac{d}{d\tau} (\eta c^i) - \frac{\eta}{c^2} F^i - \frac{\eta}{c^2} \Delta_{nk}^i c^n c^k. \end{aligned} \quad (4.85)$$

4.4 The physical conditions of spin interaction

We have shown that the spin of a particle (its internal rotation momentum) interacts with an external field of the space rotation, determined by the space non-holonomy tensor $A^{\alpha\beta} = \frac{1}{2} c h^{\alpha\mu} h^{\beta\nu} \left(\frac{\partial b_\nu}{\partial x^\mu} - \frac{\partial b_\mu}{\partial x^\nu} \right)$, which is the curl of the four-dimensional velocity vector b^α of the observer with respect to his reference body. In electromagnetic phenomena, the charge of a particle interacts with an external electromagnetic field — the field of Maxwell’s tensor $F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}$. Therefore, it seems natural to compare the chr.inv.-projections of Maxwell’s tensor $F_{\alpha\beta}$ with the chr.inv.-projections of the space non-holonomy tensor $A_{\alpha\beta}$.

In Chapter 3, we showed that the electromagnetic field tensor $F_{\alpha\beta}$ (Maxwell’s tensor) has two groups of the chr.inv.-projections, produced by the tensor itself and by its dual pseudotensor* $F^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} F_{\mu\nu}$

$$\left. \begin{aligned} \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} &= E^i, & F^{ik} &= H^{ik} \\ \frac{F_{0\cdot}^{*i}}{\sqrt{g_{00}}} &= H^{*i}, & F^{*ik} &= E^{*ik} \end{aligned} \right\}. \quad (4.86)$$

The chr.inv.-projections of the space non-holonomy tensor $A_{\alpha\beta}$ (4.11) and of its dual pseudotensor $A^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} A_{\mu\nu}$ are

$$\left. \begin{aligned} \frac{A_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} &= 0, & A^{ik} &= h^{im} h^{kn} A_{mn} \\ \frac{A_{0\cdot}^{*i}}{\sqrt{g_{00}}} &= 0, & A^{*ik} &= 0 \end{aligned} \right\}. \quad (4.87)$$

Comparing the above formulae, we see that the spin interaction has an analogy in only the “magnetic” component $\mathcal{H}^{ik} = A^{ik} = h^{im} h^{kn} A_{mn}$ of the space non-holonomy field, and the “electric” component of the non-holonomy field is equal to zero, $\mathcal{E}^i = \frac{A_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = 0$. This is no surprise, because the internal rotation field of a particle (its spin) interacts with the space non-holonomy field as with an external field, and both of the fields are produced by motion, like a magnetic field.

*Here $E^{\alpha\beta\mu\nu}$ is the four-dimensional completely antisymmetric discriminant tensor, using which we can make pseudotensors in the four-dimensional pseudo-Riemannian space. See §2.3 in Chapter 2 for details.

Besides the said, the “magnetic” component of the non-holonomy field, which is non-zero $\mathcal{H}^{ik} = A^{ik} \neq 0$, cannot be dual to the zero quantity $\mathcal{H}^{*i} = \frac{A_0^{*i}}{\sqrt{g_{00}}} = 0$. Therefore, the similarity with an electromagnetic field is incomplete. A complete coincidence could not even be expected, because the space non-holonomy tensor and the electromagnetic field tensor have a different structure: the Maxwell tensor $F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}$ is a “pure curl”, and the non-holonomy tensor $A^{\alpha\beta} = \frac{1}{2} c h^{\alpha\mu} h^{\beta\nu} \left(\frac{\partial b_\nu}{\partial x^\mu} - \frac{\partial b_\mu}{\partial x^\nu} \right)$ is not. On the other hand, we have no doubt that in the future a comparative analysis of these fields will lead to a theory of the spin interaction similar to electrodynamics.

The incomplete similarity of the space non-holonomy field to an electromagnetic field leads also to another result. If we define the spin interaction force like the Lorentz force $\Phi^\alpha = \frac{e}{c} F_{\cdot\sigma}^\alpha U^\sigma$, then the obtained formula $\Phi^\alpha = \frac{\eta_0}{c^2} A_{\cdot\sigma}^\alpha U^\sigma$ on the right hand side of the equations of motion of a spin particle will not include all the same terms. Meanwhile, an external force acting on the particle, by definition, must include all the factors that deviate its motion from a geodesic line, i.e., all terms on the right hand side of the equations of motion. This is why the four-dimensional force of the spin interaction, Φ^α [gram/sec], is

$$\Phi^\alpha = \frac{DS^\alpha}{ds} = \frac{dS^\alpha}{ds} + \Gamma_{\mu\nu}^\alpha S^\mu \frac{dx^\nu}{ds}, \quad (4.88)$$

the chr.inv.-projection of which onto the spatial section, after dividing by c , gives the three-dimensional observable force of the spin interaction, Φ^i [gram cm sec⁻²]. For instance, for a mass-bearing particle travelling in our world, using (4.71), we obtain

$$\Phi^i = -\frac{1}{c^2} \frac{d}{d\tau} (\eta v^i) - \frac{2\eta}{c^2} (D_k^i + A_{k\cdot}^i) v^k + \frac{\eta}{c^2} F^i - \frac{\eta}{c^2} \Delta_{nk}^i v^n v^k. \quad (4.89)$$

So forth, by analogy with the electromagnetic field invariants (3.25, 3.26), we obtain the space non-holonomy field invariants

$$J_1 = A_{\alpha\beta} A^{\alpha\beta} = A_{ik} A^{ik} = \varepsilon_{ikm} \varepsilon^{ikn} \Omega^{*m} \Omega_{*n} = 2 \Omega_{*i} \Omega^{*i}, \quad (4.90)$$

$$J_2 = A_{\alpha\beta} A^{*\alpha\beta} = 0, \quad (4.91)$$

where the invariant $J_1 = 2 \Omega_{*i} \Omega^{*i}$ is always different from zero, otherwise the space would be holonomic (not rotating).

Now we are approaching the physical conditions specific of the motion of elementary spin particles. Re-writing the definition of the chr. inv.-vector of the gravitational inertial force (1.38) as

$$F_i = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right) = -c^2 \frac{\partial \ln \left(1 - \frac{w}{c^2} \right)}{\partial x^i} - \frac{* \partial v_i}{\partial t}, \quad (4.92)$$

we formulate the non-holonomy tensor A_{ik} as

$$A_{ik} = \frac{1}{2} \left(\frac{* \partial v_k}{\partial x^i} - \frac{* \partial v_i}{\partial x^k} \right) + v_i \frac{\partial \ln \sqrt{1 - \frac{w}{c^2}}}{\partial x^k} - v_k \frac{\partial \ln \sqrt{1 - \frac{w}{c^2}}}{\partial x^i}. \quad (4.93)$$

From here we see that the non-holonomy tensor A_{ik} is the three-dimensional observable curl of the linear velocity with which the space rotates plus two additional terms formed jointly by the gravitational potential w and the space rotation.

Because of the tiny numerical value of the Planck constant, the spin interaction only affects elementary particles. On the scale of such small masses and distances, the gravitational interaction is negligibly weak. Therefore, we can assume $w \rightarrow 0$. As a result, on the scale of elementary particles the tensor A_{ik} is a “pure” physically observable curl

$$A_{ik} = \frac{1}{2} \left(\frac{* \partial v_k}{\partial x^i} - \frac{* \partial v_i}{\partial x^k} \right), \quad (4.94)$$

the gravitational inertial force (4.92) has only its inertial part

$$F_i = -\frac{* \partial v_i}{\partial t} = -\frac{1}{1 - \frac{w}{c^2}} \frac{\partial v_i}{\partial t} = -\frac{\partial v_i}{\partial t}, \quad (4.95)$$

and the Zelmanov identities

$$\frac{2}{\sqrt{h}} \frac{* \partial}{\partial t} (\sqrt{h} \Omega^{*i}) + \varepsilon^{ijk} * \nabla_j F_k = 0, \quad * \nabla_k \Omega^{*k} + \frac{1}{c^2} F_k \Omega^{*k} = 0, \quad (4.96)$$

take the form

$$\left. \begin{aligned} \frac{1}{\sqrt{h}} \frac{\partial}{\partial t} (\sqrt{h} \Omega^{*i}) + \frac{1}{2} \varepsilon^{ijk} \left(\frac{* \partial^2 v_k}{\partial x^j \partial t} - \frac{* \partial^2 v_j}{\partial x^k \partial t} \right) = 0 \\ * \nabla_k \Omega^{*k} - \frac{1}{c^2} \frac{* \partial v_k}{\partial t} \Omega^{*k} = 0 \end{aligned} \right\}. \quad (4.97)$$

If we substitute $\frac{\partial v_k}{\partial t} = 0$, thereby assuming that the observable rotation of the space is stationary, we obtain ${}^*\nabla_k \Omega^{*k} = 0$, i.e., the space rotation pseudovector remains unchanged. Then the Zelmanov 1st identity becomes

$$\Omega^{*i} D + \frac{\partial \Omega^{*i}}{\partial t} = 0, \quad (4.98)$$

from which we see that $D = \det \|D_n^n\| = \frac{\partial \ln \sqrt{h}}{\partial t} = 0$, i.e., the relative deformation rate of an elementary volume of the space is zero.

So, we have obtained that on the scale of elementary particles, the field of the angular velocities with which the space rotates remains unchanged (${}^*\nabla_k \Omega^{*k} = 0$), and the space does not deform ($D = 0$).

Therefore, it is possible that the stationary state of the space non-holonomy field (it is the external field in the spin interaction) is the necessary condition of the stability of elementary particles. Hence, we can conclude that long-living spin particles have stable internal rotations, while short-living particles are unstable spatial vortexes.

The study of the motion of short-living particles is rather problematic, because we do not have experimental data on the structure of the unstable vortexes that generate them. On the contrary, by studying long-living particles, i.e., their motion in the stationary field of the space rotation, we can obtain exact solutions to the equations of motion. We will focus on this task in the next section §4.5.

4.5 Motion of elementary spin particles

As we have mentioned, the Planck constant, being a tiny absolute value, only “works” for elementary particles, where gravitational interactions is a few orders of magnitude weaker than electromagnetic, weak and strong ones. Hence, assuming $w \rightarrow 0$ in the chr.inv.-equations of motion of spin particles (4.70–4.73) and (4.82–4.85), we will arrive at the chr.inv.-equations of motion of elementary particles.

Besides, as we have obtained in the previous section, §4.4, under a stationary rotation of the space, on the scale of elementary particles the trace of the space deformations tensor is zero $D = 0$. Of course, zero trace of a tensor does not necessarily mean that the tensor itself is zero. On the other hand, a deforming space is a very rare phenomenon. Therefore, when studying the motion of elementary particles, we will assume $D_{ik} = 0$.

In §4.3, we have showed that under a stationary rotation of the space, the conservation condition for the spin momentum vector S^α of a spin particle takes the form (4.68), so that

$$n\hbar^{mn} \varepsilon_{imn} v^k \frac{1}{\sqrt{\hbar}} \frac{\partial}{\partial x^k} (\sqrt{\hbar} \Omega^{*i}) = 0. \quad (4.99)$$

On the other hand, under $\frac{* \partial v_k}{\partial t} = 0$ the Zelmanov 2nd identity applied to elementary particles (4.97) means that

$$* \nabla_k \Omega^{*k} = \frac{\partial \Omega^{*k}}{\partial x^k} + \frac{\partial \sqrt{\hbar}}{\partial x^k} \Omega^{*k} = \frac{1}{\sqrt{\hbar}} \frac{\partial}{\partial x^k} (\sqrt{\hbar} \Omega^{*k}) = 0, \quad (4.100)$$

hence, the first condition of (4.97) is true provided that $\frac{\partial}{\partial x^k} (\sqrt{\hbar} \Omega^{*k}) = 0$, and the space rotation pseudovector is

$$\Omega^{*i} = \frac{\Omega_{(0)}^{*i}}{\sqrt{\hbar}}, \quad \Omega_{(0)}^{*i} = \text{const}. \quad (4.101)$$

Taking all that has been said above into account, and based on the general chr.inv.-equations of motion of a mass-bearing spin particle (4.70, 4.71), we obtain the chr.inv.-equations of motion of an elementary particle. For an our-world particle (it travels to the future with respect to an ordinary observer), the equations have the form

$$\frac{dm}{d\tau} = -\frac{1}{c^2} \frac{d\eta}{d\tau}, \quad (4.102)$$

$$\begin{aligned} \frac{d}{d\tau} (m v^i) + 2m A_{k \cdot}^i v^k + m \Delta_{nk}^i v^n v^k = \\ = -\frac{1}{c^2} \frac{d}{d\tau} (\eta v^i) - \frac{2\eta}{c^2} A_{k \cdot}^i v^k - \frac{\eta}{c^2} \Delta_{nk}^i v^n v^k, \end{aligned} \quad (4.103)$$

while for an elementary spin particle that is located in the mirror world (so it travels to the past), we obtain

$$-\frac{dm}{d\tau} = \frac{1}{c^2} \frac{d\eta}{d\tau}, \quad (4.104)$$

$$\frac{d}{d\tau} (m v^i) + m \Delta_{nk}^i v^n v^k = -\frac{1}{c^2} \frac{d}{d\tau} (\eta v^i) - \frac{\eta}{c^2} \Delta_{nk}^i v^n v^k. \quad (4.105)$$

The chr.inv.-scalar equation of motion is the same for both our-world particles and mirror-world spin particles. Integrating it for an our-world

particle, namely — taking the integral

$$\int_{\tau_1=0}^{\tau_2} \frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) d\tau = 0, \quad (4.106)$$

we obtain

$$m + \frac{\eta}{c^2} = \text{const} = B, \quad (4.107)$$

where B is an integration constant that can be calculated from the initial conditions.

To illustrate the physical sense of the obtained live forces integral, consider an analogy between then chr.inv.-projections

$$\left. \begin{aligned} \frac{P_0}{\sqrt{g_{00}}} &= \pm m, & P^i &= \frac{1}{c} m v^i = \frac{1}{c} p^i \\ \frac{S_0}{\sqrt{g_{00}}} &= \pm \frac{1}{c^2} \eta, & S^i &= \frac{1}{c^3} \eta v^i \end{aligned} \right\} \quad (4.108)$$

of the particle's four-dimensional momentum vector and those of its spin momentum vector, which are $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ and $S^\alpha = \frac{\eta_0}{c^2} \frac{dx^\alpha}{ds}$. Based on the analogy with the relativistic mass $\pm m$, we will refer to the quantity $\pm \frac{1}{c^2} \eta$ as the relativistic *spin mass*, so the quantity $\frac{1}{c^2} \eta_0$ is the rest spin mass. Hence, the live forces theorem for an elementary spin particle (4.107) means that the sum of the particle's relativistic mass and its spin mass remains unchanged along its trajectory.

Now, using the live forces integral*, we consider the chr.inv.-vector equations of motion of a mass-bearing elementary particle, located in our world, i.e., the equations (4.103). Substituting the live force integral (4.107) into (4.103), then having the constant cancelled, we obtain the kinematic equations of motion

$$\frac{dv^i}{d\tau} + 2A_{k \cdot}^i v^k + \Delta_{nk}^i v^n v^k = 0, \quad (4.109)$$

which, in this case, are non-geodesic. The term $\Delta_{nk}^i v^n v^k$, which is the contraction of the chr.inv.-Christoffel symbols with the particle's observable velocity, is relativistic in the sense that it is the square function of the velocity. This term can be neglected, because the observable

*The solution to the chr.inv.-scalar equation of motion.

metric $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ along the trajectory approaches the Euclidean metric. Such a case is possible, if the linear velocity of the space rotation is much lower than the light velocity, and, therefore, the three-dimensional coordinate metric g_{ik} is Euclidean as well. Then, the diagonal components of the chr.inv.-metric tensor are

$$h_{11} = h_{22} = h_{33} = +1, \quad (4.110)$$

while the other components are $h_{ik} = 0$, if $i \neq k$.

Noteworthy that the four-dimensional metric cannot be Galilean in this case, since the spatial section rotates with respect to the time lines that pierce it. In other words, although the observable three-dimensional space (the spatial section) in this case is a flat Euclidean space, the four-dimensional space-time is not the Minkowski space, but a pseudo-Riemannian space with the metric

$$\begin{aligned} ds^2 &= g_{00} dx^0 dx^0 + 2g_{0i} dx^0 dx^i + g_{ik} dx^i dx^k = \\ &= c^2 dt^2 + 2g_{0i} c dt dx^i - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \end{aligned} \quad (4.111)$$

Assume, for instance, that the space rotates with a constant angular velocity $\Omega = \text{const}$ around the x^3 axis. Then the linear velocity $v_i = \Omega_{ik} x^k$ with which the space rotates is

$$v_1 = \Omega_{12} x^2 = \Omega y, \quad v_2 = \Omega_{21} x^1 = -\Omega x, \quad (4.112)$$

where $A_{ik} = \Omega_{ik}$. Then, the space non-holonomy tensor A_{ik} has only the two non-zero components

$$A_{12} = -A_{21} = -\Omega, \quad (4.113)$$

and the chr.inv.-vector equations of motion (4.109) become

$$\frac{dv^1}{d\tau} + 2\Omega v^2 = 0, \quad \frac{dv^2}{d\tau} - 2\Omega v^1 = 0, \quad \frac{dv^3}{d\tau} = 0, \quad (4.114)$$

where the third equation can be solved immediately as

$$v^3 = v_{(0)}^3 = \text{const}. \quad (4.115)$$

Taking into account that $v^3 = \frac{dx^3}{d\tau}$, we represent x^3 as follows

$$x^3 = v_{(0)}^3 \tau + x_{(0)}^3, \quad (4.116)$$

where $x_{(0)}^3$ is the numerical value of the x^3 coordinate at the initial moment of the observable time $\tau = 0$.

So forth, we formulate v^2 from the first equation of (4.114)

$$v^2 = -\frac{1}{2\Omega} \frac{dv^1}{d\tau}, \quad (4.117)$$

then, differentiating (4.117) with respect to $d\tau$, we obtain

$$\frac{dv^2}{d\tau} = -\frac{1}{2\Omega} \frac{d^2v^1}{d\tau^2}, \quad (4.118)$$

and substituting the result (4.118) into the second equation of (4.114) we obtain

$$\frac{d^2v^1}{d\tau^2} + 4\Omega^2 v^1 = 0, \quad (4.119)$$

which is a free oscillation equation. Its solution is

$$v^1 = C_1 \cos(2\Omega\tau) + C_2 \sin(2\Omega\tau), \quad (4.120)$$

where C_1 and C_2 are integration constants, which can be calculated from the conditions at the initial moment of the observable time $\tau = 0$

$$\left. \begin{aligned} v_{(0)}^1 &= C_1 \\ \frac{dv^1}{d\tau} \Big|_{\tau=0} &= -2\Omega C_1 \sin(2\Omega\tau) \Big|_{\tau=0} + 2\Omega C_2 \cos(2\Omega\tau) \Big|_{\tau=0} \end{aligned} \right\}. \quad (4.121)$$

Thus, we obtain $C_1 = v_{(0)}^1$, $C_2 = \frac{\dot{v}_{(0)}^1}{2\Omega}$, $\dot{v}_{(0)}^1 = \frac{dv^1}{d\tau} \Big|_{\tau=0}$. Then, we finally obtain the equation for v^1

$$v^1 = v_{(0)}^1 \cos(2\Omega\tau) + \frac{\dot{v}_{(0)}^1}{2\Omega} \sin(2\Omega\tau), \quad (4.122)$$

so the velocity of the mass-bearing elementary spin particle along x^1 performs sinusoidal oscillations at the frequency equal to the double angular velocity of the space rotation.

Taking into account that $v^1 = \frac{dx^1}{d\tau}$, we integrate the obtained formula (4.122) with respect to $d\tau$. We obtain

$$x^1 = \frac{v_{(0)}^1}{2\Omega} \sin(2\Omega\tau) - \frac{\dot{v}_{(0)}^1}{4\Omega^2} \cos(2\Omega\tau) + C_3. \quad (4.123)$$

Assuming that $x^1 = x_{(0)}^1$ at the initial moment of time $\tau = 0$, we obtain the integration constant $C_3 = x_{(0)}^1 + \frac{\dot{v}_{(0)}^1}{4\Omega^2}$. Then, we have

$$x^1 = \frac{v_{(0)}^1}{2\Omega} \sin(2\Omega\tau) - \frac{\dot{v}_{(0)}^1}{4\Omega^2} \cos(2\Omega\tau) + x_0^1 + \frac{\dot{v}_{(0)}^1}{4\Omega^2}, \quad (4.124)$$

so the x^1 coordinate of the elementary particle also performs free oscillations at the frequency 2Ω .

Now, having the obtained v^1 (4.122) substituted into the second equation (4.114), we arrive at the equation

$$\frac{dv^2}{d\tau} = 2\Omega v_{(0)}^1 \cos(2\Omega\tau) + \dot{v}_{(0)}^1 \sin(2\Omega\tau), \quad (4.125)$$

which, after integration, gives v^2

$$v^2 = v_{(0)}^1 \sin(2\Omega\tau) - \frac{\dot{v}_{(0)}^1}{2\Omega} \cos(2\Omega\tau) + C_4. \quad (4.126)$$

Assuming that $v^2 = v_{(0)}^2$ at the moment of time $\tau = 0$, we obtain the constant $C_4 = v_{(0)}^2 + \frac{\dot{v}_{(0)}^1}{2\Omega}$. Then

$$v^2 = v_{(0)}^1 \sin(2\Omega\tau) - \frac{\dot{v}_{(0)}^1}{2\Omega} \cos(2\Omega\tau) + v_{(0)}^2 + \frac{\dot{v}_{(0)}^1}{2\Omega}. \quad (4.127)$$

Taking into account that $v^2 = \frac{dx^2}{d\tau}$, we integrate the above formula with respect to $d\tau$. As a result, we obtain the formula for the coordinate x^2 of the particle

$$x^2 = -\frac{\dot{v}_{(0)}^1}{4\Omega^2} \sin(2\Omega\tau) - \frac{v_{(0)}^1}{2\Omega} \cos(2\Omega\tau) + v_{(0)}^2 \tau + \frac{\dot{v}_{(0)}^1 \tau}{2\Omega} + C_5. \quad (4.128)$$

The integration constant C_5 can be calculated from the condition $x^2 = x_{(0)}^2$ at $\tau = 0$. It is $C_5 = x_{(0)}^2 + \frac{v_{(0)}^1}{2\Omega}$. Then, finally, the x^2 coordinate of the particle is expressed as

$$x^2 = v_{(0)}^2 \tau + \frac{\dot{v}_{(0)}^1 \tau}{2\Omega} - \frac{\dot{v}_{(0)}^1}{4\Omega^2} \sin(2\Omega\tau) - \frac{v_{(0)}^1}{2\Omega} \cos(2\Omega\tau) + x_{(0)}^2 + \frac{v_{(0)}^1}{2\Omega}. \quad (4.129)$$

From this formula we see that, if at the initial moment of the observable time $\tau = 0$ an elementary spin particle had a velocity $v_{(0)}^2$ along the axis x^2 and an acceleration $\dot{v}_{(0)}^1$ along x^1 , then the particle, in addition to free oscillations along the x^2 axis at the frequency, equal to the double angular velocity of the space rotation Ω , is subjected to a linear displacement at $\Delta x^2 = v_{(0)}^2 \tau + \frac{\dot{v}_{(0)}^1 \tau}{2\Omega}$.

Considering the live forces integral (solution to the chr.inv.-scalar equation of motion) for an elementary spin particle, $m + \frac{\eta}{c^2} = B = const$ (4.107), we can find the integration constant B . Re-writing (4.107) as

$$m_0 + \frac{\eta_0}{c^2} = B \sqrt{1 - \frac{v^2}{c^2}}, \quad (4.130)$$

we conclude that the square of the observable velocity of the particle is $v^2 = const$. Because the velocity components have already been found and since the three-dimensional metric in question is Euclidean, we can represent v^2 as follows

$$\begin{aligned} (v^1)^2 + (v^2)^2 + (v^3)^2 &= \\ &= (v_{(0)}^1)^2 + (v_{(0)}^2)^2 + (v_{(0)}^3)^2 + \frac{(\dot{v}_{(0)}^1)^2}{2\Omega^2} + \frac{\dot{v}_{(0)}^1 \dot{v}_{(0)}^2}{\Omega} + \\ &+ 2 \left(v_{(0)}^2 + \frac{\dot{v}_{(0)}^1}{2\Omega} \right) \left(v_{(0)}^1 \sin(2\Omega\tau) - \frac{\dot{v}_{(0)}^1}{2\Omega} \cos(2\Omega\tau) \right). \end{aligned} \quad (4.131)$$

The square of the velocity is conserved, if $\dot{v}_{(0)}^2 = 0$ and $\dot{v}_{(0)}^1 = 0$. Then the integration constant B of the live forces integral is

$$B = \frac{m_0 + \frac{\eta_0}{c^2}}{\sqrt{1 - \frac{v_{(0)}^2}{c^2}}}, \quad (v_{(0)}^2)^2 = (v_{(0)}^1)^2 + (v_{(0)}^3)^2 = const, \quad (4.132)$$

while the live forces integral itself (4.107) becomes

$$m + \frac{\eta}{c^2} = \frac{m_0 + \frac{\eta_0}{c^2}}{\sqrt{1 - \frac{v_{(0)}^2}{c^2}}}, \quad (4.133)$$

so it is the conservation condition for the sum of the particle's relativistic mass m and its spin mass $\frac{\eta}{c^2}$.

Here we should make a remark about what has been said above about elementary particles. Taking into account $A_{mn} = \varepsilon_{mnk} \Omega^{*k}$ in the definition $\eta_0 = n \hbar^{mn} A_{mn}$, we obtain

$$\eta_0 = n \hbar^{mn} A_{mn} = 2n \hbar_{*k} \Omega^{*k}, \quad (4.134)$$

where $\hbar_{*k} = \frac{1}{2} \varepsilon_{nmk} \hbar^{mn}$. Here, \hbar_{*k} is the three-dimensional pseudovector of the internal momentum of an elementary particle. Hence, η_0 is the scalar product of the three-dimensional pseudovectors: the internal momentum \hbar_{*k} of the particle and the angular velocity Ω^{*k} with which the space rotates. Therefore, we conclude that the spin interaction is absent if the particle's internal rotation pseudovector and the external space rotation pseudovector are observed orthogonal.

Let us return back to the equations of motion of elementary spin particles. Taking into account the integration constants that we have obtained, the chr.inv.-vector equations of motion of an elementary spin particle, located in our world, have the following solutions

$$\left. \begin{aligned} v^1 &= v_{(0)}^1 \cos(2\Omega\tau), & x^1 &= \frac{v_{(0)}^1}{2\Omega} \sin(2\Omega\tau) + x_{(0)}^1 \\ v^2 &= v_{(0)}^2 \sin(2\Omega\tau), & x^2 &= -\frac{v_{(0)}^1}{2\Omega} \cos(2\Omega\tau) + \frac{v_{(0)}^1}{2\Omega} + x_{(0)}^2 \\ v^3 &= v_{(0)}^3, & x^3 &= v_{(0)}^3 \tau + x_{(0)}^3 \end{aligned} \right\}. \quad (4.135)$$

Let us now find the shape of the three-dimensional spatial trajectory along which the elementary particle travels. Let the reference frame of the observer be such that the observed initial displacement of the particle is zero $x_{(0)}^1 = x_{(0)}^2 = x_{(0)}^3 = 0$. Then, its spatial coordinates at an arbitrary moment of time are

$$\left. \begin{aligned} x^1 &= x = a \sin(2\Omega\tau) \\ x^2 &= y = a [1 - \cos(2\Omega\tau)] \\ x^3 &= z = b\tau \end{aligned} \right\}, \quad (4.136)$$

where $a = \frac{v_{(0)}^1}{2\Omega}$ and $b = v_{(0)}^3$. The obtained coordinate solutions are parametric equations of a surface, along which the particle travels. To illustrate what kind of surface it is, we switch from the parametric notation

to the coordinate notation by removing the parameter τ from the equations. Then, calculating $x^2 + y^2$, we obtain

$$\begin{aligned} x^2 + y^2 &= 2a^2 [1 - \cos(2\Omega\tau)] = \\ &= 4a^2 \sin^2(\Omega\tau) = 4a^2 \sin^2 \frac{z\Omega}{b}. \end{aligned} \quad (4.137)$$

At first glance, the obtained result looks like a spiral line equation $x^2 + y^2 = a^2$, $z = b\tau$. However, the similarity is not complete. According to the trajectory equation that we have obtained, an elementary spin particle travels along a spiral wound on the surface of a cylinder so that the particle has a constant velocity $b = v_{(0)}^3$ along the axis of the cylinder (z axis), and the radius of the particle's trajectory (the radius of the cylinder) oscillates with a frequency Ω in the range* from zero up to the maximum $2a = \frac{v_{(0)}^1}{\Omega}$ at $z = \frac{\pi kb}{2\Omega}$.

So, the trajectory of an elementary spin particle in our world looks like a spiral line “wound” on an oscillating cylinder. The lifetime of the particle is equal to the length of the cylinder divided by the velocity of the particle along the axis of the cylinder (z axis). Pulsations of this cylinder are energy “breath ins” and “breath outs” of the particle.

This means that the cylinder that we have mathematically deduced above is the *event cylinder* of an elementary particle from its birth in our world (its materialization) to its death (dematerialization). But even after death the particle's event cylinder does not disappear, but *splits* into the event cylinders of other particles born by this decay either in our world or in the mirror world.

Therefore, the analysis of the births and decays of elementary particles in terms of the General Theory of Relativity means the analysis of the branching points of the event cylinders of these particles, taking into account possible branches leading to the mirror world.

If we consider the motion of two bound spin particles that rotate around a common centre of mass, for example, a positronium atom (a dumbbell-shaped system consisting of an electron and a positron), then we get a DNA-like double spiral — a twisted “rope ladder” with a number of steps (links connecting particles) wound on the pulsating cylinder of their events.

*Where $k = 0, 1, 2, 3, \dots$. If $v_{(0)}^3 = 0$, then the particle simply oscillates within the xy plane (the plane of the cylinder's section).

Let us now solve the chr.inv.-equations of motion of an elementary spin particle travelling in the mirror world (a world with the reverse flow of time). The mentioned equations (4.104, 4.105) under the physical conditions specific of elementary particles* take the form

$$-\frac{dm}{d\tau} = \frac{1}{c^2} \frac{d\eta}{d\tau}, \quad (4.138)$$

$$\frac{d}{d\tau}(m v^i) = -\frac{1}{c^2} \frac{d}{d\tau}(\eta v^i). \quad (4.139)$$

The solution to the chr.inv.-scalar equation is the live forces integral $m + \frac{\eta}{c^2} = B = \text{const}$, as in the case of an analogous our-world particle (4.107). Substituting it into the chr.inv.-vector equations of (4.139), we obtain their solution

$$\frac{dv^i}{d\tau} = 0, \quad (4.140)$$

which means $v^i = v_{(0)}^i = \text{const}$. According to the solution, from the viewpoint of an ordinary observer, an elementary spin particle travels in the mirror world linearly at a constant velocity. This is in contrast to the observable motion of an analogous our-world particle, because it travels along an oscillating “spiral” line.

On the other hand, from the viewpoint of an observer, whose home is the mirror world, the motion of elementary spin particles in our world will be linear and uniform, and in his world elementary spin particles will travel along oscillating “spiral” lines.

We could also get an analysis of the motion of massless (light-like) spin particles in a similar way. But we do not know how adequate our assumption that the linear velocity with which the space rotates is much smaller compared to the light velocity would be. Although, in general, the methods for solving the equations of motion are the same for mass and massless particles.

4.6 A spin particle in an electromagnetic field

In this section, we are going to deduce the chr.inv.-equations of motion for a particle that has both spin and electric charge, and travels in an

*We assumed that the space rotates stationarily at a low speed and does not deform, and the three-dimensional metric is Euclidean.

external electromagnetic field that fills the four-dimensional pseudo-Riemannian space.

So, the summary vector characteristic of such a particle is

$$Q^\alpha = P^\alpha + \frac{e}{c^2} A^\alpha + S^\alpha, \quad (4.141)$$

where P^α is the four-dimensional momentum vector of the particle. The other two four-dimensional vectors are, respectively, an additional momentum gained by the particle from the interaction of its charge with the electromagnetic field, and also an additional momentum gained from the interaction of the particle's spin with the space non-holonomy field.

Since the vectors P^α and S^α are tangential to the four-dimensional trajectory of the particle, we assume that the electromagnetic field potential A^α is also tangential to the trajectory. In this case, it has the form $A^\alpha = \varphi_0 \frac{dx^\alpha}{ds}$, and the formula $q^i = \frac{\varphi}{c} v^i$ (see §3.8) sets the relationship between the scalar potential φ and the vector potential q^i of the electromagnetic field.

Then chr.inv.-projections $\tilde{\varphi}$ and \tilde{q}^i of the particle's summary vector Q^α (4.141) under consideration are

$$\tilde{\varphi} = \pm \left(m + \frac{e\varphi}{c^2} + \frac{\eta}{c^2} \right), \quad \tilde{q}^i = \frac{1}{c^3} m v^i + \frac{1}{c^3} (\eta + e\varphi) v^i, \quad (4.142)$$

where m is the relativistic mass of the particle, φ is the scalar potential of the acting electromagnetic field, while η describes the interaction of the particle's spin with the space non-holonomy field

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \varphi = \frac{\varphi_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \eta = \frac{\eta_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (4.143)$$

The desired equations are deduced in the same way as those for a charge-free spin particle, except for the fact that we have to project the absolute derivative of the sum of the above three vectors (4.141). Using the formulae for $\tilde{\varphi}$ and \tilde{q}^i (4.142), we obtain the chr.inv.-equations of motion of a charged mass-bearing spin particle located in our world (it travels from the past to the future)

$$\begin{aligned} \frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = \\ = -\frac{1}{c^2} \frac{d}{d\tau} (\eta + e\varphi) + \frac{\eta + e\varphi}{c^4} F_i v^i - \frac{\eta + e\varphi}{c^4} D_{ik} v^i v^k, \end{aligned} \quad (4.144)$$

$$\begin{aligned}
\frac{d}{d\tau}(mv^i) + 2m(D_k^i + A_k^i)v^k - mF^i + m\Delta_{nk}^i v^n v^k &= \\
= -\frac{1}{c^2} \frac{d}{d\tau} [(\eta + e\varphi)v^i] - \frac{2(\eta + e\varphi)}{c^2} (D_k^i + A_k^i)v^k + &(4.145) \\
+ \frac{\eta + e\varphi}{c^2} F^i - \frac{\eta + e\varphi}{c^2} \Delta_{nk}^i v^n v^k, &
\end{aligned}$$

while for an analogous particle located in the mirror world (it travels from the future to the past) the equations are

$$\begin{aligned}
-\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k &= \\
= \frac{1}{c^2} \frac{d}{d\tau} (\eta + e\varphi) + \frac{\eta + e\varphi}{c^4} F_i v^i - \frac{\eta + e\varphi}{c^4} D_{ik} v^i v^k, &(4.146)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\tau}(mv^i) + mF^i + m\Delta_{nk}^i v^n v^k &= \\
= -\frac{1}{c^2} \frac{d}{d\tau} [(\eta + e\varphi)v^i] - \frac{\eta + e\varphi}{c^2} F^i - \frac{\eta + e\varphi}{c^2} \Delta_{nk}^i v^n v^k. &(4.147)
\end{aligned}$$

The Levi-Civita parallel transport in a Riemannian space leaves the length of any transported vector unchanged. Hence, its square is invariant in any reference frame. In particular, the square of the transported vector Q^α (4.141) characteristic of a spin particle in an electromagnetic field remains unchanged in the accompanying reference frame

$$\begin{aligned}
Q_\alpha Q^\alpha &= g_{\alpha\beta} \left(P^\alpha + \frac{e}{c^2} A^\alpha + S^\alpha \right) \left(P^\beta + \frac{e}{c^2} A^\beta + S^\beta \right) = \\
&= g_{\alpha\beta} \left(m_0 + \frac{e\varphi_0}{c^2} + \frac{\eta_0}{c^2} \right)^2 \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \left(m_0 + \frac{e\varphi_0}{c^2} + \frac{\eta_0}{c^2} \right)^2. &(4.148)
\end{aligned}$$

In §3.9, we have already shown that when the four-dimensional electromagnetic potential A^α is oriented along the world-line of a charged particle, the right hand side of the chr.inv.-equations of motion of the particle are significantly simplified: the right hand side of the chr.inv.-vector equations of motion takes the form of the chr.inv.-Lorentz force $\Phi^i = -e \left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m} \right)$, and the right hand side of the chr.inv.-scalar equation is the scalar product of the electric strength vector E_i and the observable velocity of the particle.

Keeping the said in mind, we can represent the obtained chr.inv.-equations of motion of a charged spin particle (4.144–4.147) in a more

specific form. Thus, moving the spin interaction terms of the equations to the left hand side and introducing the chr.inv.-Lorentz force, for a charged spin particle that travels in our world we obtain

$$\begin{aligned} \frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) - \frac{1}{c^2} \left(m + \frac{\eta}{c^2} \right) F_i v^i + \\ + \frac{1}{c^2} \left(m + \frac{\eta}{c^2} \right) D_{ik} v^i v^k = - \frac{e}{c^2} E_i v^i, \end{aligned} \quad (4.149)$$

$$\begin{aligned} \frac{d}{d\tau} \left[\left(m + \frac{\eta}{c^2} \right) v^i \right] + 2 \left(m + \frac{\eta}{c^2} \right) (D_k^i + A_k^i) v^k - \left(m + \frac{\eta}{c^2} \right) F^i + \\ + \left(m + \frac{\eta}{c^2} \right) \Delta_{nk}^i v^n v^k = - e \left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m} \right), \end{aligned} \quad (4.150)$$

and for an analogous particle in the mirror world we have

$$\begin{aligned} - \frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) - \frac{1}{c^2} \left(m + \frac{\eta}{c^2} \right) F_i v^i + \\ + \frac{1}{c^2} \left(m + \frac{\eta}{c^2} \right) D_{ik} v^i v^k = - \frac{e}{c^2} E_i v^i, \end{aligned} \quad (4.151)$$

$$\begin{aligned} \frac{d}{d\tau} \left[\left(m + \frac{\eta}{c^2} \right) v^i \right] + \left(m + \frac{\eta}{c^2} \right) F^i + \left(m + \frac{\eta}{c^2} \right) \Delta_{nk}^i v^n v^k = \\ = - e \left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m} \right). \end{aligned} \quad (4.152)$$

To make conclusions on the motion of charged spin particles in the pseudo-Riemannian space, we have to set a specific geometric structure of the space. As in the previous section, §4.5, where we analysed the motion of charge-free spin particles, we now assume that:

- a) Since the gravitational interaction on the scales of elementary particles is negligible, we can assume that $w \rightarrow 0$;
- b) The space rotation is stationary, i.e., $\frac{* \partial v_k}{\partial t} = 0$;
- c) The space does not deform, i.e., $D_{ik} = 0$;
- d) The three-dimensional coordinate metric $g_{ik} dx^i dx^k$ is Euclidean, i.e., $g_{ik} = \begin{cases} -1, & i=k, \\ 0, & i \neq k, \end{cases}$
- e) The space rotates with a constant angular velocity Ω around the axis $x^3 = z$, hence the components of the linear velocity with which the space rotates are $v_1 = \Omega_{12} x^2 = \Omega y$ and $v_2 = \Omega_{21} x^1 = -\Omega x$.

Taking the above constraints into account, we obtain the formula for the space-time interval ds^2 on the scale of elementary particles

$$ds^2 = c^2 dt^2 - 2\Omega y dt dx + 2\Omega x dt dy - dx^2 - dy^2 - dz^2, \quad (4.153)$$

while the physically observable characteristics of the space are

$$F_i = 0, \quad D_{ik} = 0, \quad A_{12} = -A_{21} = -\Omega, \quad A_{23} = A_{31} = 0. \quad (4.154)$$

As in the previous section, §4.5, we assume that the space rotates with a velocity much slower than the velocity of light. In such a case, the metric chr.inv.-tensor h_{ik} is Euclidean, and the chr.inv.-Christoffel symbols Δ_{jk}^i are zeroes, which simplifies the algebra. Then the chr.inv.-equations of motion of a charged spin particle in our world give

$$\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = -\frac{e}{c^2} E_i \frac{dx^i}{d\tau}, \quad (4.155)$$

$$\left. \begin{aligned} \frac{d \left(m + \frac{\eta}{c^2} \right) v^1}{d\tau} + 2 \left(m + \frac{\eta}{c^2} \right) \Omega v^2 &= \\ &= -e \left(E^1 + \frac{1}{c} \varepsilon^{1km} v_k H_{*m} \right) \\ \frac{d \left(m + \frac{\eta}{c^2} \right) v^2}{d\tau} - 2 \left(m + \frac{\eta}{c^2} \right) \Omega v^1 &= \\ &= -e \left(E^2 + \frac{1}{c} \varepsilon^{2km} v_k H_{*m} \right) \\ \frac{d \left(m + \frac{\eta}{c^2} \right) v^3}{d\tau} &= -e \left(E^3 + \frac{1}{c} \varepsilon^{3km} v_k H_{*m} \right) \end{aligned} \right\}, \quad (4.156)$$

while the equations for such a particle in the mirror world give

$$\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = \frac{e}{c^2} E_i \frac{dx^i}{d\tau}, \quad (4.157)$$

$$\left. \begin{aligned} \frac{d \left(m + \frac{\eta}{c^2} \right) v^1}{d\tau} &= -e \left(E^1 + \frac{1}{c} \varepsilon^{1km} v_k H_{*m} \right) \\ \frac{d \left(m + \frac{\eta}{c^2} \right) v^2}{d\tau} &= -e \left(E^2 + \frac{1}{c} \varepsilon^{2km} v_k H_{*m} \right) \\ \frac{d \left(m + \frac{\eta}{c^2} \right) v^3}{d\tau} &= -e \left(E^3 + \frac{1}{c} \varepsilon^{3km} v_k H_{*m} \right) \end{aligned} \right\}. \quad (4.158)$$

Let us look at the chr.inv.-scalar equation of motion in our world (4.155) and those in the mirror world (4.157). We see that the sum of the relativistic mass of the charged spin particle and its spin mass equalizes the work done by the electric component of the acting electromagnetic field in displacing the particle by the elementary interval dx^n . It can be seen from the chr.inv.-vector equations of motion that both in our world (4.156) and in the mirror world (4.158) the sum of the spatial momentum vector of the particle and its spin momentum vector along $x^3 = z$ is determined only by the component of the Lorentz force along the same axis.

Now our task is to calculate the trajectory of a charged spin particle in an electromagnetic field with given properties. As in Chapter 3, we assume that the field is constant, so its electric and magnetic strengths E_i and H^{*i} are

$$E_i = \frac{\partial\varphi}{dx^i}, \quad (4.159)$$

$$H^{*i} = \frac{1}{2} \varepsilon^{imn} H_{mn} = \frac{1}{2c} \varepsilon^{imn} \left[\frac{\partial(\varphi v_m)}{dx^n} - \frac{\partial(\varphi v_n)}{dx^m} - 2\varphi A_{mn} \right]. \quad (4.160)$$

In Chapter 3, we already considered a similar problem. Namely, — we solved the chr.inv.-equations of motion for a charged mass-bearing particle, but without taking its spin into account. Obviously, in the particular case of a charged spinless particle (where the spin of the particle is zero), the solutions for a charged spin particle must coincide with the solutions that we have obtained in Chapter 3 in the framework of “pure” electrodynamics.

To compare our results with those obtained in the framework of electrodynamics, it would be reasonable to analyse the motion of a charged spin particle in the three typical kinds of electromagnetic fields, which were under study in Chapter 3 as well as in *The Classical Theory of Fields* by Landau and Lifshitz [10]:

- a) A homogeneous stationary electric field (the magnetic strength of the field is zero, $H^{*i} = 0$);
- b) A homogeneous stationary magnetic field (the electric strength of the field is zero, $E_i = 0$);
- c) A homogeneous stationary electromagnetic field (both of the field components H^{*i} and E_i are non-zeroes).

On the other hand, in electrodynamics we consider the motion of ordinary macro-particles. It is not obvious that all the three above cases are applicable to the micro-world of elementary particles, given the metric constraints. Here is why.

First, the spin of an elementary particle affects its motion only in the field of the space non-holonomy. Hence, the non-holonomy tensor is $A_{ik} \neq 0$. But from the formulae for the electric strength E_i (4.159) and the magnetic strength H^{*i} (4.160) we see that the space non-holonomy only affects the magnetic field component. Hence, we will focus on the motion of elementary spin particles in an electromagnetic field of the strictly magnetic kind.

Second, the chr.inv.-scalar equation of motion of a charged spin particle (4.155)

$$\left(m_0 + \frac{\eta_0}{c^2}\right) \frac{d}{d\tau} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{e}{c^2} E_i v^i \quad (4.161)$$

in a non-relativistic case, where the particle is much slower than the velocity of light, takes the form

$$E_i v^i = 0, \quad (4.162)$$

so the electric field component *does not perform work* to displace the particle under the constraints specific of the world of elementary particles. Since we are considering a stationary field, the obtained condition (4.162) can be represented as follows

$$E_i v^i = \frac{\partial \varphi}{\partial x^i} v^i = \frac{\partial \varphi}{\partial x^i} \frac{dx^i}{d\tau} = \frac{d\varphi}{d\tau} = 0, \quad (4.163)$$

which obviously means that the scalar electromagnetic potential is constant ($\varphi = \text{const}$), therefore,

$$H^{*i} = \frac{\varphi}{2c} \varepsilon^{imn} \left[\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m} - 2 \left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m} \right) \right]. \quad (4.164)$$

In a relativistic case, the electric component reveals itself (it performs work to displace the particle), provided that the absolute value of the particle's velocity is non-stationary

$$\frac{1}{2c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \left(m_0 + \frac{\eta_0}{c^2}\right) \frac{dv^2}{d\tau} = -\frac{e}{c^2} E_i v^i \neq 0. \quad (4.165)$$

Hence, the electric component of the acting electromagnetic field, given the constraints specific of elementary particles, reveals itself only on those relativistic charged particles, the velocity of which is not constant. All “slow-moving” particles fall out of our consideration in an electromagnetic field of the strictly electric kind.

Therefore, the general case* should be studied only in a stationary electromagnetic field of the *strictly magnetic kind*, where the electric field component is absent. This is what will be done in §4.7.

4.7 Motion in a stationary magnetic field

In this section, we are going to consider the motion of a charged spin particle in a homogeneous stationary magnetic field.

As in the previous section, §4.6, we assume that the space-time has the metric (4.153), where $F_i = 0$ and $D_{ik} = 0$. The space rotates around the z axis (within the xy plane) with a constant angular velocity Ω . Hence, the space non-holonomy tensor has the only non-zero components $A_{12} = -A_{21} = -\Omega = \text{const}$, and the quantity $\eta_0 = n\hbar^{mn}A_{mn}$ that describes the interaction of the particle’s spin (its internal rotation) with the external field of the space non-holonomy is

$$\eta_0 = n(\hbar^{12}A_{12} + \hbar^{21}A_{21}) = 2n\hbar^{12}A_{12} = \pm 2n\hbar\Omega, \quad (4.166)$$

where “plus” stands for the co-directed \hbar and Ω (with $A_{12} = -\Omega$, the numerical value of \hbar^{12} is also negative, $\hbar^{12} = -\hbar$), and “minus” means that they are oppositely directed (with $A_{12} = -\Omega$, we have $\hbar^{12} = +\hbar$).

In this case,† the chr.inv.-equations of motion of a charged elementary spin particle located in our world take the form

$$\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = 0, \quad (4.167)$$

$$\begin{aligned} \frac{d}{d\tau} \left[\left(m + \frac{\eta}{c^2} \right) v^i \right] + 2 \left(m + \frac{\eta}{c^2} \right) A_{k \cdot}^i v^k + \left(m + \frac{\eta}{c^2} \right) \Delta_{nk}^i v^n v^k = \\ = -\frac{e}{c} \varepsilon^{ikm} v_k H_{*m}, \end{aligned} \quad (4.168)$$

*A charged elementary spin particle travelling with an arbitrary velocity, either low or relativistic.

†Provided that the electromagnetic field potential A^α is directed along the four-dimensional trajectory of the particle.

and for an analogous particle located in the mirror world we have

$$-\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = 0, \quad (4.169)$$

$$\frac{d}{d\tau} \left[\left(m + \frac{\eta}{c^2} \right) v^i \right] + \left(m + \frac{\eta}{c^2} \right) \Delta_{nk}^i v^n v^k = -\frac{e}{c} \varepsilon^{ikm} v_k H_{*m}. \quad (4.170)$$

Having the live forces theorem (chr.inv.-scalar equation of motion) integrated, we obtain the live forces integral. In our world and in the mirror world it is, respectively

$$m + \frac{\eta}{c^2} = B = const, \quad m + \frac{\eta}{c^2} = -\widetilde{B} = const, \quad (4.171)$$

where B and \widetilde{B} are integration constants in our world and in the mirror world, respectively. We can obtain these constants by substituting the initial conditions at $\tau = 0$ into (4.171). As a result, we have

$$B = m_0 + \frac{\eta_0}{c^2} = m_0 + \frac{n\hbar^{mn} A_{mn}}{c^2}, \quad (4.172)$$

$$\widetilde{B} = -m_0 - \frac{\eta_0}{c^2} = -m_0 - \frac{n\hbar^{mn} A_{mn}}{c^2}. \quad (4.173)$$

The formulae for the live forces integrals (4.171) mean that, in the absence of the electric field component, the square of the velocity of a charged elementary spin particle is constant, $v^2 = h_{ik} v^i v^k = const$.

Having the formulae for the live forces integrals substituted into (4.168, 4.170), we arrive at the chr.inv.-vector equations of motion in our world and in the mirror world, respectively

$$\frac{dv^i}{d\tau} + 2A_k^i v^k + \Delta_{nk}^i v^n v^k = -\frac{e}{cB} \varepsilon^{ikm} v_k H_{*m}, \quad (4.174)$$

$$\frac{dv^i}{d\tau} + \Delta_{nk}^i v^n v^k = -\frac{e}{c\widetilde{B}} \varepsilon^{ikm} v_k H_{*m}, \quad (4.175)$$

which are similar to the chr.inv.-equations of motion of a charged macro-particle (charged spinless particle) in a homogeneous stationary magnetic field (3.290, 3.291), except that here the integration constant from the living forces integral is not equal to the relativistic mass m of the particle, as it was in electrodynamics (3.290, 3.291), but to the formula

(4.171) that takes into account the interaction of the particle's spin with the space non-holonomy field.

For the readers, who have a special interest in the chronometrically invariant formalism, we make a remark concerning the notations in the chr.inv.-equations of motion.

When obtaining the components of the term $A_k^i v^k$, found only in the our-world equations, we have, for instance, for the index $i = 1$

$$A_k^1 v^k = A_1^1 v^1 + A_2^1 v^2 = h^{12} A_{12} v^1 + h^{11} A_{21} v^2, \quad (4.176)$$

where $A_{12} = -A_{21} = -\Omega$. Then obtaining A_1^1 et A_2^1 , we have

$$A_1^1 = h^{1m} A_{1m} = h^{11} A_{11} + h^{12} A_{12} = h^{12} A_{12}, \quad (4.177)$$

$$A_2^1 = h^{1m} A_{2m} = h^{11} A_{21} + h^{12} A_{22} = h^{11} A_{21}, \quad (4.178)$$

where h^{ik} are the elements of a matrix reciprocal to the matrix h_{ik} , so the required components of h^{ik} are calculated as

$$h^{11} = \frac{h_{22}}{h}, \quad h^{12} = -\frac{h_{12}}{h}. \quad (4.179)$$

Then, since the determinant of the chr.inv.-metric tensor in the case under consideration (see §3.12 for detail) is

$$h = \det \|h_{ik}\| = 1 + \frac{\Omega^2 (x^2 + y^2)}{c^2}, \quad (4.180)$$

the unknown quantity $A_k^1 v^k$ (4.176) is

$$A_k^1 v^k = \frac{\Omega}{h} \left[\frac{\Omega^2}{c^2} xy \dot{x} + \left(1 + \frac{\Omega^2 x^2}{c^2} \right) \dot{y} \right]. \quad (4.181)$$

The component $A_k^2 v^k$, found in the equation of motion along y , can be calculated in the same way.

Let us get back to the chr.inv.-vector equations of motion of the charged spin particle in the homogeneous stationary magnetic field. We approach them in two possible cases of mutual orientation of the magnetic strength and the space non-holonomy pseudovector, when they are co-directed and are orthogonal to each other.

4.7.1 The magnetic field is co-directed with the non-holonomy field

Assume that the space non-holonomy field pseudovector is directed along the z axis, and the space non-holonomy field is weak. Then the chr.inv.-vector equations of motion of a charged elementary spin particle located in our world take the form

$$\dot{x} + 2\Omega\dot{y} = -\frac{eH}{cB}\dot{y}, \quad \dot{y} - 2\Omega\dot{x} = -\frac{eH}{cB}\dot{x}, \quad \dot{z} = 0, \quad (4.182)$$

while for an analogous particle located in the mirror world we have

$$\ddot{x} = -\frac{eH}{cB}\dot{y}, \quad \ddot{y} = -\frac{eH}{cB}\dot{x}, \quad \dot{z} = 0. \quad (4.183)$$

These equations are different from those for a charged spinless particle (3.104, 3.305), deduced under the same assumptions, only by the integration constant B from the live forces integral, which, instead of the relativistic mass of the particle, takes into account here the interaction of the particle's spin with the space non-holonomy field.

Using the solutions obtained in §3.12, we can immediately obtain the formulae for the coordinates of the our-world charged spin particle

$$x = -[\dot{y}_{(0)} \cos(2\Omega + \omega)\tau + \dot{x}_{(0)} \sin(2\Omega + \omega)\tau] \times \\ \times \frac{1}{2\Omega + \omega} + x_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega + \omega}, \quad (4.184)$$

$$y = [\dot{y}_{(0)} \sin(2\Omega + \omega)\tau - \dot{x}_{(0)} \cos(2\Omega + \omega)\tau] \times \\ \times \frac{1}{2\Omega + \omega} + y_{(0)} - \frac{\dot{x}_{(0)}}{2\Omega + \omega}, \quad (4.185)$$

and also those for the mirror-world particle

$$x = -\frac{1}{\omega} [\dot{y}_{(0)} \cos \omega\tau + \dot{x}_{(0)} \sin \omega\tau] + x_{(0)} + \frac{\dot{y}_{(0)}}{\omega}, \quad (4.186)$$

$$y = \frac{1}{\omega} [\dot{y}_{(0)} \sin \omega\tau - \dot{x}_{(0)} \cos \omega\tau] + y_{(0)} - \frac{\dot{x}_{(0)}}{\omega}, \quad (4.187)$$

which are different from the solutions obtained in the framework of electrodynamics only by the frequency ω taking into account the interaction of the particle's spin with the space non-holonomy field.

In our world, particles have positive masses, therefore ω is

$$\omega = \frac{eH}{mc + \frac{\eta}{c}} = \frac{eH \sqrt{1 - \frac{v_{(0)}^2}{c^2}}}{m_0 c + \frac{\eta_0}{c}} = \frac{eH \sqrt{1 - \frac{v_{(0)}^2}{c^2}}}{m_0 c \pm \frac{2n\hbar\Omega}{c}}, \quad (4.188)$$

where the alternate sign in the denominator depends on the mutual orientation of \hbar and Ω : “plus” stands for the co-directed \hbar and Ω , and “minus” means that they are oppositely directed, regardless of a right-hand or left-hand reference frame. See the comment to (4.166) for details.

Particles of the mirror world have negative masses (4.143)

$$m = -\frac{m_0}{\sqrt{1 - \frac{v_{(0)}^2}{c^2}}} < 0, \quad \eta = -\frac{\eta_0}{\sqrt{1 - \frac{v_{(0)}^2}{c^2}}} < 0, \quad (4.189)$$

therefore, ω in the mirror world is

$$\omega = \frac{eH}{mc + \frac{\eta}{c}} = \frac{eH \sqrt{1 - \frac{v_{(0)}^2}{c^2}}}{-m_0 c - \frac{\eta_0}{c}} = \frac{eH \sqrt{1 - \frac{v_{(0)}^2}{c^2}}}{-m_0 c \mp \frac{2n\hbar\Omega}{c}}. \quad (4.190)$$

Note that the obtained formulae for the coordinates x and y (4.184–4.187) take into account the fact that the square of the particle’s velocity remains unchanged both in our world and in the mirror world, which is presented with the conditions (respectively)

$$\dot{x}_{(0)} + \frac{\dot{y}_0}{2\Omega + \omega} = 0, \quad \dot{x}_{(0)} + \frac{\dot{y}_0}{\omega} = 0, \quad (4.191)$$

resulting from the live forces integral (see §3.12 for details).

The third equation of motion (along z) is solved as

$$z = \dot{z}_{(0)}\tau + z_{(0)}. \quad (4.192)$$

The obtained formulas for the coordinates x and y (4.184–4.187) indicate that a charged elementary spin particle travelling in a homogeneous stationary magnetic field parallel to a weak field of the space non-holonomy performs *harmonic oscillations* along x and y . In our world the oscillation frequency is

$$\tilde{\omega} = 2\Omega + \omega = 2\Omega + \frac{eH}{m_0 c \pm \frac{2n\hbar\Omega}{c}} \sqrt{1 - \frac{v_{(0)}^2}{c^2}}, \quad (4.193)$$

and in the mirror world an analogous particle performs similar oscillations at the frequency ω (4.190).

In a weak field of the space non-holonomy, the value of $n\hbar\Omega$ is much less than the energy m_0c^2 . Because we have $\frac{1}{1\pm\alpha} \cong 1 \mp \alpha$ for any small value α , at low velocities of motion we have

$$\tilde{\omega} \cong 2\Omega + \frac{eH}{m_0c} \left(1 \mp \frac{2n\hbar\Omega}{m_0c^2} \right). \quad (4.194)$$

If at the initial moment of time the displacement and the velocity of an our-world charged elementary spin particle satisfy the conditions

$$x_{(0)} + \frac{\dot{y}_0}{2\Omega + \omega} = 0, \quad y_{(0)} - \frac{\dot{x}_0}{2\Omega + \omega} = 0, \quad (4.195)$$

it will travel like a charged spinless particle along a *circle* within the xy plane*

$$x^2 + y^2 = \frac{\dot{y}_0^2}{(2\Omega + \omega)^2}, \quad (4.196)$$

but the radius of its circulation in this case is equal to

$$r = \frac{\dot{y}_0}{2\Omega + \omega} = \frac{\dot{y}_0}{2\Omega + \frac{eH}{m_0c \pm \frac{2n\hbar\Omega}{c}} \sqrt{1 - \frac{v_{(0)}^2}{c^2}}}, \quad (4.197)$$

and is dependent on the value and orientation of its spin. If the initial velocity of a charged spin particle, directed along the magnetic field strength (along z), is not zero, then the particle travels along the magnetic strength along a *spiral line* with the same radius r .

An analogous mirror-world particle, provided that its displacement and velocity at the initial moment of time satisfy the conditions

$$x_{(0)} + \frac{\dot{y}_0}{\omega} = 0, \quad y_{(0)} - \frac{\dot{x}_0}{\omega} = 0, \quad (4.198)$$

will also travel along a circle

$$x^2 + y^2 = \frac{\dot{y}_0^2}{\omega^2}, \quad (4.199)$$

*We set the y axis along the initial momentum of the particle, which is always possible. Then all formulae for the coordinates will have zero initial velocity of the particle along x .

with the radius

$$r = \frac{\dot{y}_0}{\omega} = \frac{\dot{y}_0}{\frac{eH}{-m_0c \mp \frac{2n\hbar\Omega}{c}} \sqrt{1 - \frac{v_{(0)}^2}{c^2}}}. \quad (4.200)$$

In a general case, where there is no additional conditions (4.195, 4.198), the trajectory of a charged elementary spin particle within the xy plane will not be circular.

Let us obtain a formula for the energy and momentum of the particle. Using the formulae for the live forces integral, we obtain the quantity η_0 , which is $\eta_0 = n\hbar^{mn}A_{mn} = n(\hbar^{12}A_{12} + \hbar^{21}A_{21}) = \pm 2n\hbar\Omega$. For details, see (4.166). Then for the particle located in our world we have

$$E_{\text{tot}} = Bc^2 = \frac{m_0c^2 \pm 2n\hbar\Omega}{\sqrt{1 - \frac{v_{(0)}^2}{c^2}}} = \text{const}, \quad (4.201)$$

while in the mirror world we have

$$E_{\text{tot}} = \widetilde{B}c^2 = \frac{-m_0c^2 \mp 2n\hbar\Omega}{\sqrt{1 - \frac{v_{(0)}^2}{c^2}}} = \text{const}. \quad (4.202)$$

Since in this section, §4.7, we have assumed that the electric component of the acting electromagnetic field is absent, the field does not contribute to the total energy of the particle (as it is known, the magnetic component of an electromagnetic field does not perform work to displace electric charges).

From the obtained formulae (4.201, 4.202) we see that the total energy of the particle remains unchanged along the trajectory, while its value depends on the mutual orientation of the particle's internal momentum \hbar and the angular velocity Ω with which the space rotates.

The latter statement requires some comments to be made. By definition the scalar quantity n (value of the spin in \hbar units) is always positive, while \hbar and Ω are the numerical values of the components of the anti-symmetric tensors h^{ik} and Ω_{ik} , which take opposite signs in right-hand and left-hand reference frames. But since we are dealing with the product of the quantities, only their mutual orientation matters, which does not depend on a right-hand or left-hand reference frame.

If \hbar and Ω are co-directed (their scalar product is positive), then the total energy of an our-world spin particle E_{tot} (4.201) is the sum of its relativistic energy $E = mc^2$ and its “spin energy”

$$E_s = \frac{2n\hbar\Omega}{\sqrt{1 - \frac{v_{(0)}^2}{c^2}}}, \quad (4.203)$$

so the total energy of the particle is greater than $E = mc^2$.

If \hbar and Ω are oppositely directed, then E_{tot} is the difference between the relativistic energy and the spin energy of the particle. This mutual orientation permits a specific case, where $m_0c^2 = 2n\hbar\Omega$ and, therefore, the total energy of the particle becomes zero (this case will be discussed in §4.8, concerning elementary particles).

For charged spin particles having negative masses, which inhabit the mirror world, the total energy E_{tot} (4.202) is negative, but its absolute value is as well greater than the relativistic energy $E = -mc^2$, provided that \hbar and Ω are co-directed.

So forth, for the observable total spatial momentum of the our-world particle we have

$$p_{\text{tot}}^i = \frac{m_0c^2 \pm 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} v^i = mv^i \pm \frac{2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} v^i, \quad (4.204)$$

so it is an algebraic sum of the particle’s relativistic observable momentum $p^i = mv^i$ and the spin momentum that the particle gains from the space non-holonomy field. The particle’s total momentum is greater than its relativistic momentum, if \hbar and Ω are co-directed, and it is less otherwise. In the case of the opposite mutual orientation of \hbar and Ω , the total momentum becomes zero (so does the total energy), provided that the condition $m_0c^2 = 2n\hbar\Omega$ is true.

For the mirror-world particle the quantity p_{tot}^i is

$$p_{\text{tot}}^i = \frac{-m_0c^2 \mp 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} v^i = -mv^i \mp \frac{2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} v^i, \quad (4.205)$$

so the particle moves faster (in the mirror world), if \hbar and Ω are co-directed, and it is slower otherwise.

The velocity components of a charged spin particle in the magnetic field co-directed with the space non-holonomy field, taking into account the conditions (4.191), in our world are

$$\dot{x} = \dot{y}_{(0)} \sin(2\Omega + \omega)\tau - \dot{x}_{(0)} \cos(2\Omega + \omega)\tau, \quad (4.206)$$

$$\dot{y} = \dot{y}_{(0)} \cos(2\Omega + \omega)\tau + \dot{x}_{(0)} \sin(2\Omega + \omega)\tau, \quad (4.207)$$

while for an analogous particle located in the mirror world we have

$$\dot{x} = \dot{y}_{(0)} \sin \omega\tau - \dot{x}_{(0)} \cos \omega\tau, \quad (4.208)$$

$$\dot{y} = \dot{y}_{(0)} \cos \omega\tau + \dot{x}_{(0)} \sin \omega\tau. \quad (4.209)$$

Then the total momentum of the particle* in our world is

$$p_{\text{tot}}^1 = \frac{m_0 c^2 \pm 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{y}_{(0)} \sin(2\Omega + \omega)\tau, \quad (4.210)$$

$$p_{\text{tot}}^2 = \frac{m_0 c^2 \pm 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{y}_{(0)} \cos(2\Omega + \omega)\tau, \quad (4.211)$$

$$p_{\text{tot}}^3 = \frac{m_0 c^2 \pm 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{z}_{(0)}, \quad (4.212)$$

where ω is as in (4.188). In the mirror world we have

$$p_{\text{tot}}^1 = \frac{-m_0 c^2 \mp 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{y}_{(0)} \sin \omega\tau, \quad (4.213)$$

$$p_{\text{tot}}^2 = \frac{-m_0 c^2 \mp 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{y}_{(0)} \cos \omega\tau, \quad (4.214)$$

$$p_{\text{tot}}^3 = \frac{-m_0 c^2 \mp 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{z}_{(0)}, \quad (4.215)$$

*The initial momentum of the particle within the xy plane is directed along y .

where ω is as in (4.190). Noteworthy, although the magnetic strength does not appear in the total energy E_{tot} , it appears in the total momentum as a term of the formula for ω (4.190).

4.7.2 The magnetic field is orthogonal to the non-holonomy field

Let us now consider the motion of a mass-bearing charged spin particle in a homogeneous stationary magnetic field, which is orthogonal to the space non-holonomy field. Let the non-holonomy field be weak and directed along z (so, the magnetic field is directed along y). Then the chr.inv.-vector equations of its motion

$$\ddot{x} + 2\Omega\dot{y} = \frac{eH}{cB} \dot{z}, \quad \ddot{y} - 2\Omega\dot{x} = 0, \quad \ddot{z} = -\frac{eH}{cB} \dot{x} \quad (4.216)$$

are similar to those for a charged spinless particle (3.338). The difference from (3.338) is that here the denominator of the right hand side contains, instead of the relativistic mass of the charged particle, the integration constant from the live forces integral, which takes into account the interaction between the particle's spin and the non-holonomy field. After integration, the equations give

$$x = \frac{\dot{x}_{(0)}}{\omega} \sin \tilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\omega^2} \cos \tilde{\omega}\tau + x_{(0)} + \frac{\dot{x}_{(0)}}{\omega^2}, \quad (4.217)$$

$$y = -\frac{2\Omega}{\omega^2} \left(\dot{x}_{(0)} \cos \tilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\omega} \sin \tilde{\omega}\tau \right) + \dot{y}_{(0)}\tau + \frac{2\Omega}{\omega^2} \dot{x}_{(0)}\tau + y_{(0)} + \frac{2\Omega}{\omega^2} \dot{x}_{(0)}, \quad (4.218)$$

$$z = \frac{\omega}{\omega^2} \left(\dot{x}_{(0)} \cos \tilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\omega} \sin \tilde{\omega}\tau \right) + \dot{z}_{(0)}\tau - \frac{\omega}{\omega^2} \dot{x}_{(0)}\tau + z_{(0)} - \frac{\omega}{\omega^2} \dot{x}_{(0)}, \quad (4.219)$$

which are different from the corresponding solutions for a charged spinless particle by the frequency $\tilde{\omega}$ that is dependent on the spin and its mutual orientation with the non-holonomy field

$$\tilde{\omega} = \sqrt{4\Omega^2 + \omega^2} = \sqrt{4\Omega^2 + \frac{e^2 H^2 \left(1 - \frac{v_{(0)}^2}{c^2}\right)^2}{\left(m_0 c^2 \pm \frac{2n\hbar\Omega}{c}\right)^2}}. \quad (4.220)$$

Subsequently, an equation of the trajectory of the charged spin particle is similar to that of the spinless particle. In a particular case, namely — under certain initial conditions, the trajectory equation is the equation of a *sphere*

$$x^2 + y^2 + z^2 = \frac{1}{\omega^2} \dot{x}_{(0)}^2, \quad (4.221)$$

whose radius, in contrast to the radius of the trajectory of the spinless particle, depends on the particle's orientation with respect to the non-holonomy field

$$r = \frac{1}{\sqrt{4\Omega^2 + \frac{e^2 H^2 \left(1 - \frac{v_{(0)}^2}{c^2}\right)^2}{\left(m_0 c^2 \pm \frac{2n\hbar\Omega}{c}\right)^2}}} \dot{x}_{(0)}. \quad (4.222)$$

Let us look at an analogous particle, located in the mirror world, moving in a weak field of the space non-holonomy, directed along y and orthogonal to the magnetic field. For the particle, the chr.inv.-vector equations of motion are

$$\ddot{x} = \frac{eH}{cB} \dot{z}, \quad \ddot{y} = 0, \quad \ddot{z} = -\frac{eH}{cB} \dot{x}, \quad (4.223)$$

so they are different from the equations for the our-world particle (4.216) by the absence of the terms which contain the angular velocity of the space rotation Ω . As a result their solutions can be obtained from the solutions for our world (4.217–4.219), if we assume $\bar{\omega} = \omega$. Subsequently, an equation of the trajectory of the charged spin particle located in the mirror world is

$$x^2 + y^2 + z^2 = \frac{1}{\omega^2} \dot{x}_{(0)}^2, \quad r = \frac{-m_0 c^2 \mp \frac{2n\hbar\Omega}{c}}{eH \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{x}_{(0)}. \quad (4.224)$$

The total energy of the particle E_{tot} in this case, where the magnetic field is orthogonal to the space non-holonomy field, is the same as it was for the case of parallel orientation of the fields. But the formulae for components of the total momentum (4.201, 4.205) are different, because

they include the particle's velocity which depends on mutual orientation of the magnetic field and the non-holonomy field. In the particular case, where the fields are orthogonal to each other, components of the particle's velocity (obtained by derivation of the formulae for 4.217–4.219) in our world are

$$\dot{x} = \dot{x}_{(0)} \cos \tilde{\omega} \tau + \frac{\ddot{x}_{(0)}}{\tilde{\omega}} \sin \tilde{\omega} \tau, \quad (4.225)$$

$$\dot{y} = \frac{2\Omega}{\tilde{\omega}} \dot{x}_{(0)} \sin \tilde{\omega} \tau - \frac{2\Omega}{\tilde{\omega}^2} \ddot{x}_{(0)} \cos \tilde{\omega} \tau + \dot{y}_{(0)} + \frac{2\Omega}{\tilde{\omega}^2} \ddot{x}_{(0)}, \quad (4.226)$$

$$\dot{z} = \frac{\omega}{\tilde{\omega}^2} \ddot{x}_{(0)} \cos \tilde{\omega} \tau - \frac{\omega}{\tilde{\omega}} \dot{x}_{(0)} \sin \tilde{\omega} \tau + \dot{z}_{(0)} - \frac{\omega}{\tilde{\omega}^2} \ddot{x}_{(0)}, \quad (4.227)$$

while in the mirror world we obtain

$$\dot{x} = \dot{x}_{(0)} \cos \omega \tau + \frac{\ddot{x}_{(0)}}{\omega} \sin \omega \tau, \quad (4.228)$$

$$\dot{y} = \dot{y}_{(0)}, \quad (4.229)$$

$$\dot{z} = \frac{1}{\omega} \ddot{x}_{(0)} \cos \tilde{\omega} \tau - \dot{x}_{(0)} \sin \omega \tau + \dot{z}_{(0)} - \frac{1}{\omega} \ddot{x}_{(0)}. \quad (4.230)$$

Now we assume that the initial acceleration of the particle and the integration constants are zeroes, which simplifies the algebra. We also set the x axis along the initial momentum of the particle. In the framework of this consideration we obtain the components of the total momentum for the particle located in our world

$$p_{\text{tot}}^1 = \frac{m_0 c^2 \pm 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{x}_{(0)} \cos \tilde{\omega} \tau, \quad (4.231)$$

$$p_{\text{tot}}^2 = \frac{m_0 c^2 \pm 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \frac{2\Omega}{\tilde{\omega}} \dot{x}_{(0)} \sin \tilde{\omega} \tau, \quad (4.232)$$

$$p_{\text{tot}}^3 = \frac{m_0 c^2 \pm 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \frac{\omega}{\tilde{\omega}} \dot{x}_{(0)} \sin \tilde{\omega} \tau, \quad (4.233)$$

and for the analogous particle located in the mirror world

$$p_{\text{tot}}^1 = \frac{-m_0 c^2 \mp 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{x}_{(0)} \cos \tilde{\omega}\tau, \quad (4.234)$$

$$p_{\text{tot}}^2 = \frac{-m_0 c^2 \mp 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{y}_{(0)} = 0, \quad (4.235)$$

$$p_{\text{tot}}^3 = \frac{-m_0 c^2 \mp 2n\hbar\Omega}{c^2 \sqrt{1 - \frac{v_{(0)}^2}{c^2}}} \dot{x}_{(0)} \sin \tilde{\omega}\tau. \quad (4.236)$$

As it easy to see, the obtained solutions can be transformed into corresponding ones from electrodynamics (§3.12) by assuming $\hbar \rightarrow 0$.

4.8 Quantization of the masses of elementary particles

As obtained before, the chr.inv.-scalar equations of motion of a charged spin particle in an electromagnetic field, located in our world and in the mirror world, respectively, have the form

$$\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = -\frac{e}{c^2} E_i v^i, \quad -\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = -\frac{e}{c^2} E_i v^i. \quad (4.237)$$

Integrating the equations, we obtain the live forces integrals

$$m + \frac{\eta}{c^2} = B, \quad -\left(m + \frac{\eta}{c^2} \right) = \tilde{B}, \quad (4.238)$$

where B is an integration constant in our world and \tilde{B} is that in the mirror world. Integration constants depend on the initial conditions, therefore, it is possible to choose the above constants so as to make them zeroes.

Under what initial conditions are the integration constants equal to zero? For charged spin particles, located in our world and in the mirror world (4.238), we obtain, respectively

$$m + \frac{\eta}{c^2} = 0, \quad -\left(m + \frac{\eta}{c^2} \right) = 0, \quad (4.239)$$

while the right hand side of the chr.inv.-vector equations of motion (4.150, 4.152), which contain the chr.inv.-Lorentz force, also become

zeroes. In other words, with zero integration constants in the scalar chr.inv.-equations, the acting electromagnetic field does not produce a work to displace charged particles.

Having the relativistic square root cancelled in (4.239), which is always possible for any particle having non-zero rest-masses, we represent these formulae in a form that does not depend on the particle's velocity. Then, for mass-bearing particles located in our world we have

$$m_0 c^2 = -n \hbar^{mn} A_{mn}, \quad (4.240)$$

and for mirror-world particles of non-zero masses we have

$$m_0 c^2 = -n \hbar^{mn} A_{mn}. \quad (4.241)$$

We will refer to the formulae (4.240, 4.241) as the *law of quantization of the masses of elementary particles*:

The rest-mass of any mass-bearing spin particle is proportional to the energy of its spin interaction with the space non-holonomy field, taken with the opposite sign.

Or, in other words:

The rest-energy of any mass-bearing spin particle is equal to the energy of its spin interaction with the space non-holonomy field, taken with the opposite sign.

Because in the mirror world the relativistic energy and spin-energy of any particle are negative in (4.239), the “minus” sign stands on the right hand side of (4.241) in the mirror world. So, this law is the same as (4.240), which we have obtained for a spin particle in our world.

Obviously, the above quantum formulae are not applicable to spinless particles.

Let us make some quantitative estimates, based on the obtained quantization law. Considering an elementary particle, we will calculate the numerical value of the quantity* $\eta_0 = n \hbar^{mn} A_{mn}$ as follows. First, we formulate the tensor of the space angular velocity A_{mn} with the pseudovector $\Omega^{*i} = \frac{1}{2} \varepsilon^{imn} A_{mn}$

$$\Omega^{*i} \varepsilon_{imn} = \frac{1}{2} \varepsilon^{ipq} \varepsilon_{imn} A_{pq} = \frac{1}{2} (\delta_m^p \delta_n^q - \delta_n^p \delta_m^q) A_{pq} = A_{mn}, \quad (4.242)$$

*This quantity characterizes the interaction energy of the particle's spin with the space non-holonomy field — the “spin energy”, in other words.

so we have $A_{mn} = \varepsilon_{imn} \Omega^{*i}$. Then, because

$$\frac{1}{2} \varepsilon_{imn} \hbar^{mn} = \hbar_{*i} \quad (4.243)$$

is the Planck pseudovector, the quantity $\eta_0 = n \hbar^{mn} \varepsilon_{imn} \Omega^{*i}$ is

$$\eta_0 = 2n \hbar_{*i} \Omega^{*i}, \quad (4.244)$$

so it is the double scalar product of the Planck three-dimensional pseudovector and the three-dimensional pseudovector of the angular velocity with which the space rotates, multiplied by the particle's spin number. If \hbar_{*i} and Ω^{*i} are co-directed, then the cosine is positive, hence

$$\eta_0 = 2n \hbar_{*i} \Omega^{*i} = 2n \hbar \Omega \cos(\vec{\hbar}; \vec{\Omega}) > 0, \quad (4.245)$$

while if they are oppositely directed, then

$$\eta_0 = 2n \hbar_{*i} \Omega^{*i} = 2n \hbar \Omega \cos(\vec{\hbar}; \vec{\Omega}) < 0. \quad (4.246)$$

Therefore, for any mass-bearing elementary spin particle, the integration constant from the live forces integral becomes zero, provided that the pseudovectors \hbar_{*i} and Ω^{*i} are oppositely directed.

This means that, if the interaction energy of a mass-bearing elementary spin particle with the space non-holonomy field becomes equal to its rest-energy $E = m_0 c^2$, then the momentum of the particle neither manifests itself in our world nor in the mirror world.

Assume that the z axis is co-directed with the angular velocity pseudovector of the space rotation Ω^{*i} . Then out of all three components of the Ω^{*i} the only non-zero component is

$$\begin{aligned} \Omega^{*3} &= \frac{1}{2} \varepsilon^{3mn} A_{mn} = \frac{1}{2} (\varepsilon^{312} A_{12} + \varepsilon^{321} A_{21}) = \\ &= \varepsilon^{312} A_{12} = \frac{e^{312}}{\sqrt{h}} A_{12}. \end{aligned} \quad (4.247)$$

To simplify the algebra we assume that the three-dimensional coordinate metric g_{ik} is Euclidean and the space rotates at a constant angular velocity Ω . Then components of the linear velocity of the space rotation are $v_1 = \Omega x$, $v_2 = -\Omega y$, and $A_{12} = -\Omega$. Hence

$$\Omega^{*3} = \frac{e^{312}}{\sqrt{h}} A_{12} = \frac{A_{12}}{\sqrt{h}} = -\frac{\Omega}{\sqrt{h}}. \quad (4.248)$$

The square root of the determinant of the chr.inv.-metric tensor, as defined in (4.180), is

$$\sqrt{h} = \sqrt{\det \|h_{ik}\|} = \sqrt{1 + \frac{\Omega^2(x^2 + y^2)}{c^2}}. \quad (4.249)$$

Because we are dealing with very small coordinate values on the scales of elementary particles, we can assume $\sqrt{h} \approx 1$ and, according to (4.248), also $\Omega^{*3} = -\Omega = \text{const}$. Then the law of quantization of the masses of elementary particles (4.240), considered in our world and in the mirror world, becomes

$$m_0 = \frac{2n\hbar\Omega}{c^2}. \quad (4.250)$$

Hence, for any elementary mass-bearing particle, located in our world, the following relationship between its rest-mass m_0 and the angular velocity Ω with which the space rotates is obvious

$$\Omega = \frac{m_0 c^2}{2n\hbar}. \quad (4.251)$$

This means that the rest-mass (true mass) of an observable object, under ordinary conditions does not depend on the properties of the observer's reference space. On the contrary, for elementary particles it becomes strictly dependent on these properties, in particular — it depends on the angular velocity of the space rotation.

Hence, proceeding from the quantization law, we can calculate the rotation frequencies of the observer's reference space, which correspond to the rest-masses of elementary particles.

The results, proceeding from the calculations for elementary particles of known kinds, are given in Table 4.1.

These results show that on the scale of elementary particles, the observer's space is always non-holonomic. So forth for instance, in observation of an electron $r_e = 2.8 \times 10^{-13}$ cm the linear velocity of rotation of the observer's space is $v = \Omega r = 2200$ km/sec*. Because other elemen-

*This value of v equals the velocity of an electron in the Bohr 1st orbit, although when calculating the velocity of the space rotation (see Table 4.1) we considered a *free electron*, i.e., the one not related to an atomic nucleus and quantization of orbits in an atom of hydrogen. The reason is that the "genetic" quantum non-holonomy of the space seems not only to define rest-masses of elementary particles, but to be the reason of rotation of electrons in atoms.

Elementary particles	Rest-mass	Spin	Ω , sec^{-1}
Leptons			
electron e^- , positron e^+	1	1/2	7.782×10^{20}
electron neutrino ν_e and electron anti-neutrino $\bar{\nu}_e$	$< 4 \times 10^{-4}$	1/2	$< 3 \times 10^{17}$
μ -meson neutrino ν_μ and μ -meson anti-neutrino $\bar{\nu}_\mu$	< 8	1/2	$< 6 \times 10^{21}$
μ^- -meson, μ^+ -meson	206.766	1/2	1.609×10^{23}
Baryons			
<i>nucleons</i>			
proton p, anti-proton \bar{p}	1836.09	1/2	1.429×10^{24}
neutron n, anti-neutron \bar{n}	1838.63	1/2	1.431×10^{24}
<i>hyperons</i>			
Λ^0 -hyperon, anti- Λ^0 -hyperon	2182.75	1/2	1.699×10^{24}
Σ^+ -hyperon, anti- Σ^+ -hyperon	2327.6	1/2	1.811×10^{24}
Σ^- -hyperon, anti- Σ^- -hyperon	2342.6	1/2	1.823×10^{24}
Σ^0 -hyperon, anti- Σ^0 -hyperon	2333.4	1/2	1.816×10^{24}
Ξ^- -hyperon, anti- Ξ^- -hyperon	2584.7	1/2	2.011×10^{24}
Ξ^0 -hyperon, anti- Ξ^0 -hyperon	2572	1/2	2.00×10^{24}
Ω^- -hyperon, anti- Ω^- -hyperon	3278	3/2	8.50×10^{23}

Table 4.1: Frequencies of rotation of the observer's reference space, which correspond to elementary mass-bearing particles.

tary particles are even smaller, this linear velocity seems to be the upper limit*.

So, what did we get? Generally, the observer compares the results of his measurements with the standards located in his reference body. But the body and himself are not related to the observed object and do not affect it during observations. Hence, in the macro-world there is no dependence of the true properties of the observed bodies (rest-mass, rest-energy, etc.) on the properties of the reference body and reference space — their properties are *not related* to each other.

In other words, although observed images are distorted by the influence from the physical properties of the observer's reference frame,

*It is interesting that the angular velocities of the space rotation in baryons (see Table 4.1) up within the order of the magnitude match the frequency $\sim 10^{23} \text{ sec}^{-1}$ which characterizes elementary particles as oscillators [27].

the observer himself and his reference body in the macro-world do not affect the measured objects anyhow.

But the world of elementary particles presents a big difference. In this section, we have seen that once we reach the scale of elementary particles, where the spin, a quantum property of the particles, significantly affects their motion, while the physical properties of the reference body (reference space) and those of the particles become tightly linked to each other, so the reference body *affects* the observed particles. In other words, the observer does not just compare the properties of the observed particles to those of his references any longer, but instead directly *affects* the observed particles. The observer shapes their properties in a tight quantum relationship with the properties of his references.

We can explain the above in other words as follows. When looking at the world of elementary particles, *there is no border* between the observer (his reference body and reference space) and the observed particles. Hence, we have an opportunity to define a *relationship* between the space non-holonomy field, linked to the observer, and the rest-masses of the observed particles (objects of his observations), which in the macro-world are not related to the reference body. So, the obtained law of quantization of the masses is only true for elementary particles.

Please note that we have obtained the above result using only the geometric methods of the General Theory of Relativity, and not the probabilistic methods used in Quantum Mechanics. In the future, this result can possibly become a “bridge” between these two theories.

4.9 The Compton wavelength

So, we have obtained that, in observations of an mass-bearing elementary particle, the rotation frequency of the observer’s space is $\Omega = \frac{m_0 c^2}{2n\hbar}$ (4.251). Let us find the wavelength corresponding to that frequency. Assuming that this wave, i.e., the wave of the space non-holonomy, propagates with the light velocity $\lambda\Omega = c$, we have

$$\lambda = \frac{c}{\Omega} = 2n \frac{\hbar}{m_0 c} . \quad (4.252)$$

In other words, when we observe a mass-bearing particle with the spin $n = \frac{1}{2}$, the length of the space non-holonomy wave is equal to Compton’s wavelength of this particle $\lambda_c = \frac{\hbar}{m_0 c}$.

What does this mean? The Compton effect, named after Compton who discovered it in 1922, is the “diffraction” of a photon on a free electron, which results in the decrease of its own frequency

$$\Delta\lambda = \lambda_2 - \lambda_1 = \frac{h}{m_e c} (1 - \cos \vartheta) = \lambda_c^e (1 - \cos \vartheta), \quad (4.253)$$

where λ_1 and λ_2 are the photon’s wavelengths before and after the encounter, and ϑ is the “diffraction” angle. The multiplier λ_c^e , specific to the electron, at first was called the *Compton wavelength* of the electron. Later it was discovered that other elementary particles during the “diffraction” of photons reveal as well the specific wavelengths $\lambda_c = \frac{h}{m_0 c}$, or $\lambda_c = \frac{\hbar}{m_0 c}$. That is, elementary particles of every kind (electrons, protons, neutrons etc.) have their own Compton wavelengths. The physical sense behind the quantity will be explained later. As obtained, within an area smaller than λ_c , any elementary particle is no longer a point object and its interaction with other particles (and with the observer) is described by Quantum Mechanics. Hence, the λ_c -sized area is sometimes interpreted as the “size” of the elementary particle.

As for the results that we have obtained in the previous section, §4.8, they can be interpreted as follows. In the observation of a mass-bearing elementary particle, the observer’s space rotates so fast that the angular velocity of its rotation makes a specific wavelength equal to the Compton wavelength specific of the observed particle (the “size”, inside which the particle is no longer a point object). In other words, it is the angular velocity of the space rotation (the wavelength in the space non-holonomy field), which determines the Compton observable wavelength (specific “size”) of the elementary particle.

4.10 Massless spin particles

Because massless particles do not have an electric charge, the chr.inv.-scalar equation of their motion in our world and in the mirror world are as follows, respectively,

$$\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = 0, \quad -\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = 0. \quad (4.254)$$

Their integration always gives a constant equal to zero, hence we always obtain the formulae (4.239). Therefore, for massless particles in

our world and in the mirror world, we have

$$mc^2 = -\eta. \quad (4.255)$$

On the other hand, it is obvious that the term “rest-mass” is not applicable to massless particles — they are always on the move. Their relativistic masses are defined from the energy equivalent $E = mc^2$, measured in electron-volts. Consequently, massless particles have no rest spin energy $\eta_0 = n\hbar^{mn}A_{mn}$.

Nevertheless, the Planck tensor found in the spin energy η enables the quantization of the relativistic masses of massless particles and of the angular velocities of the space rotation. Hence, to obtain the angular velocities of the space rotation for massless particles, we need an expanded formula of their relativistic spin energy η , which would not contain the relativistic square root.

Quantum Mechanics speaks of the “helicity” of massless particles — the projection of the spin of a massless particle onto the direction of its momentum. The reason for introducing this term is the fact that massless particles cannot be at rest with respect to any ordinary observer, since they always travel with the velocity of light with respect to him. Therefore, we can always assume that the spin of any massless particle is tangential to its light-like trajectory (either co-directed or oppositely directed to it).

Keeping in mind that the spin quantum number n of any massless particle is 1, we assume that for a massless particle

$$\eta = \hbar^{mn}\tilde{A}_{mn}, \quad (4.256)$$

where \tilde{A}_{mn} is the angular velocity chr.inv.-tensor of rotation of its light-like space.

Hence, to obtain the relativistic spin energy of a massless particle (4.256) we need to find the components of the angular velocity chr.inv.-tensor of the light-like space rotation. We are going to create the tensor similar to the space rotation four-dimensional tensor $A^{\alpha\beta}$ (4.11), which describes the rotation of the reference space, travelling with respect to the observer with an arbitrary velocity (this means a non-accompanying reference frame). As a result we obtain

$$\tilde{A}^{\alpha\beta} = \frac{1}{2} c \tilde{h}^{\alpha\mu} \tilde{h}^{\beta\mu} \tilde{a}_{\mu\nu}, \quad \tilde{a}_{\mu\nu} = \frac{\partial \tilde{b}_\nu}{\partial x^\mu} - \frac{\partial \tilde{b}_\mu}{\partial x^\nu}, \quad (4.257)$$

where \tilde{b}^α is the four-dimensional velocity of the light-like reference frame with respect to the observer, and

$$\tilde{h}^{\alpha\mu} = -g^{\alpha\mu} + \tilde{b}^\alpha \tilde{b}^\mu \quad (4.258)$$

is the four-dimensional generalization of the chr.inv.-metric tensor for the light-like reference space.

The home space of massless particles is a space-time region corresponding to the four-dimensional light-like (isotropic) cone given by the equation $g_{\alpha\beta} dx^\alpha dx^\beta = 0$. This cone exists at any point of the four-dimensional pseudo-Riemannian space with the signature (+---).

The four-dimensional velocity vector of the light-like reference frame of massless particles is

$$\tilde{b}^\alpha = \frac{dx^\alpha}{d\sigma} = \frac{dx^\alpha}{cd\tau}, \quad \tilde{b}_\alpha \tilde{b}^\alpha = 0, \quad (4.259)$$

so its chr.inv.-projections in the reference frame of an ordinary “subluminal” observer are

$$\frac{\tilde{b}_0}{\sqrt{g_{00}}} = \pm 1, \quad \tilde{b}^i = \frac{1}{c} \frac{dx^i}{d\tau} = \frac{1}{c} c^i, \quad (4.260)$$

while the other components of the isotropic vector (4.259) are

$$\tilde{b}^0 = \frac{1}{\sqrt{g_{00}}} \left(\frac{1}{c^2} v_i c^i \pm 1 \right), \quad \tilde{b}_i = -\frac{1}{c} (c_i \pm v_i). \quad (4.261)$$

The isotropic condition of a massless particle’s four-dimensional velocity, $b_\alpha b^\alpha = 0$, in the chr.inv.-form has the form

$$h_{ik} c^i c^k = c^2 = const, \quad (4.262)$$

where h_{ik} is the chr.inv.-metric tensor of an ordinary “subluminal” observer’s reference space. The components of the four-dimensional light-like metric tensor $\tilde{h}^{\alpha\beta}$ (4.258), the three-dimensional components of which make up the light-like space chr.inv.-metric tensor \tilde{h}^{ik} , are

$$\left. \begin{aligned} \tilde{h}^{00} &= \frac{v_k v^k \pm 2v_k c^k + \frac{1}{c^2} v_k v_n c^k c^n}{c^2 \left(1 - \frac{w}{c^2}\right)^2} \\ \tilde{h}^{0i} &= \frac{v^i \pm c^i + \frac{1}{c^2} v_k c^k c^i}{c \left(1 - \frac{w}{c^2}\right)}, \quad \tilde{h}^{ik} = h^{ik} + \frac{1}{c^2} c^i c^k \end{aligned} \right\}, \quad (4.263)$$

where c^i is the chr.inv.-vector of the light velocity, “plus” stands for the light-like space with the direct flow of time (our world), and “minus” stands for the reverse-time (mirror) world.

Deduce the components of the curl of the four-dimensional velocity vector of a massless particle, which is found in the formula (4.257). After some algebra we obtain

$$\left. \begin{aligned} \tilde{a}_{00} = 0, \quad \tilde{a}_{0i} = \frac{1}{2c^2} \left(1 - \frac{w}{c^2} \right) \left(\pm F_i - \frac{* \partial c_i}{\partial t} \right) \\ \tilde{a}_{ik} = \frac{1}{2c} \left(\frac{\partial c_i}{\partial x^k} - \frac{\partial c_k}{\partial x^i} \right) \pm \frac{1}{2c} \left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i} \right) \end{aligned} \right\}. \quad (4.264)$$

Generally, to define the spin energy of a massless particle (4.256), we need the covariant spatial components of the space rotation tensor, namely — the lower-index components \tilde{A}_{ik} . To deduce them, we take the formula for the contravariant components \tilde{A}^{ik} and lower their indices similar to any chr.inv.-quantity, using the chr.inv.-metric tensor of the observer’s reference space.

Substituting the obtained components $\tilde{h}^{\alpha\beta}$ and $\tilde{a}_{\alpha\beta}$ into

$$\tilde{A}^{ik} = c \left(\tilde{h}^{i0} \tilde{h}^{k0} \tilde{a}_{00} + \tilde{h}^{i0} \tilde{h}^{km} \tilde{a}_{0m} + \tilde{h}^{im} \tilde{h}^{k0} \tilde{a}_{m0} + \tilde{h}^{im} \tilde{h}^{kn} \tilde{a}_{mn} \right) \quad (4.265)$$

we arrive at the general formula

$$\begin{aligned} \tilde{A}^{ik} = & h^{im} h^{kn} \left[\frac{1}{2} \left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m} \right) + \frac{1}{2c^2} (F_n c_m - F_m c_n) \right] \pm \\ & \pm h^{im} h^{kn} \left[\frac{1}{2} \left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m} \right) + \frac{1}{2c^2} (F_n v_m - F_m v_n) \right] + \\ & + \left(\frac{1}{c^2} v_n c^n \pm 1 \right) \left(c^k h^{im} - c^i h^{km} \right) \frac{* \partial c_m}{\partial t} - \\ & - \left(v^k h^{im} - v^i h^{km} \right) \frac{* \partial c_m}{\partial t} + \frac{1}{2c^2} c^m \left(c^i h^{kn} - c^k h^{in} \right) \times \\ & \times \left[\left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m} \right) \pm \left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m} \right) \right]. \end{aligned} \quad (4.266)$$

In this formula, the quantity $\frac{1}{2} \left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m} \right) + \frac{1}{2c^2} (F_n v_m - F_m v_n)$, by definition, is the chr.inv.-tensor of the angular velocity of the observer’s

space rotation A_{mn} , which is the non-holonomy tensor of the *non-isotropic space* * at the same time.

The quantity $\frac{1}{2} \left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m} \right) + \frac{1}{2c^2} (F_n c_m - F_m c_n)$ by its structure is similar to the tensor A_{mn} , but instead of the linear velocity v_i with which the non-isotropic space rotates, it has the components of the covariant chr.inv.-vector of the light velocity $c_m = h_{mn} c^n$. The vector c_m is a physically observable quantity, because it is obtained by lowering indices in the chr.inv.-vector c^n using the chr.inv.-metric tensor h_{mn} . We denote that tensor as \check{A}_{mn} , where the inward curved cap (croissant) means that the quantity belongs to the *isotropic space* † with the direct flow of time — the “upper” part of the light cone, which in a curved space-time gets a “round” shape. Then we obtain

$$\check{A}_{mn} = \frac{1}{2} \left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m} \right) + \frac{1}{2c^2} (F_n c_m - F_m c_n). \quad (4.267)$$

In the particular case, where the gravitational potential is negligible (i.e., where $w \approx 0$) the tensor becomes

$$\check{A}_{mn} = \frac{1}{2} \left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m} \right), \quad (4.268)$$

so it is the chr.inv.-curl of the light velocity. Therefore, we will refer to \check{A}_{mn} as the *isotropic space curl*.

The following example gives a geometric illustration of the isotropic space curl. As is known, the necessary and sufficient condition of the equality $A_{mn} = 0$ (the space holonomy condition) is the equality to zero of all components $v_i = -c \frac{g_{0i}}{\sqrt{g_{00}}}$, i.e., the absence of the space rotation. The tensor \check{A}_{mn} is defined only in the isotropic space, inhabited by massless particles. Outside the isotropic space it is nonsense, because the

*We refer to a region of the four-dimensional space-time, where particles having non-zero rest-masses exist, as the *non-isotropic space*. This is the region of the world-trajectories, along which $ds \neq 0$. Subsequently, if the interval ds is real, then the particles travel with subluminal velocities (ordinary particles); if it is imaginary, then the particles travel with superluminal velocities (tachyons). So, the space of both subluminal particles and superluminal tachyons is non-isotropic by definition.

†We refer to a region of the four-dimensional space-time, inhabited by massless (light-like) particles, as the *isotropic space*. This region can also be called the *light membrane*. From a geometric point of view, the light membrane is the four-dimensional surface of the isotropic cone, i.e., the set of its four-dimensional elements that are the world-lines of the propagation of light.

“interior” of the light cone is inhabited by subluminal particles, while tachyons inhabit its “exterior”.

Our subject here is massless particles (photons). From (4.268) it is seen that the non-holonomy of the isotropic space is linked to the curl nature of the linear velocity of massless particles c_m . Hence, any photon is a spatial curl of the isotropic space, and the photon’s spin results from the interaction between its internal curl field with the external tensor field \check{A}_{mn} .

To make the explanations even more illustrative, we depict the home regions of different kinds of particles. The light cone exists in every point of space (space-time). The light cone equation $g_{\alpha\beta} dx^\alpha dx^\beta = 0$ in the chr.inv.-notation takes the form

$$c^2 \tau^2 - h_{ik} x^i x^k = 0, \quad h_{ik} x^i x^k = \sigma^2. \quad (4.269)$$

On Minkowski’s diagram, the light cone “interior” is filled with the non-isotropic space, where subluminal particles exist. Outside it, there is also a region of the non-isotropic space, inhabited by superluminal tachyons. The specific space of massless particles is a *space-time membrane* between these two non-isotropic regions. The picture is mirror-symmetric: in the upper part of the cone, there is the subluminal space with the direct flow of time (our world), separated in the observer’s spatial section from the lower part — the subluminal space with the reverse flow of time (the mirror world). In other words, the upper part of the cone is inhabited by real particles (they have positive masses and energies), while the lower part is inhabited by their mirror “counterparts” (masses and energies of which are negative from our point of view).

Therefore, the rotation of the subluminal non-isotropic space that is “inside” the light cone involves the surrounding light membrane (isotropic space). As a result, the light cone begins a rotation described by the tensor \check{A}_{mn} (isotropic space curl). Of course, we can assume a reverse order of events, where the light cone rotation involves the “content” of its internal part. But, since particles “inside” the cone have non-zero rest-masses, they are “heavier” than massless particles on the light membrane. Hence, the internal “content” of the light cone is also an inertial media.

Now we come back to the formula for the relativistic spin energy of a massless particle $\eta = \check{h}^{mn} \check{A}_{mn}$ (4.256). Lowering indices in the isotropic

space non-holonomy tensor \tilde{A}^{ik} (4.266), we obtain

$$\begin{aligned} \tilde{A}_{ik} = & \pm A_{ik} + \check{A}_{ik} + \frac{1}{2c^2} c^m \left\{ c_i \left[\frac{\partial(c_m \pm v_m)}{\partial x^k} - \frac{\partial(c_k \pm v_k)}{\partial x^m} \right] - \right. \\ & \left. - c_k \left[\frac{\partial(c_m \pm v_m)}{\partial x^i} - \frac{\partial(c_i \pm v_i)}{\partial x^m} \right] \right\} + \left(v_i \frac{* \partial c_k}{\partial t} - v_k \frac{* \partial c_i}{\partial t} \right) + \\ & + \left(\frac{1}{c^2} v_n v^n \pm 1 \right) \left(c_k \frac{* \partial c_i}{\partial t} - c_i \frac{* \partial c_k}{\partial t} \right). \end{aligned} \quad (4.270)$$

Having \tilde{A}_{ik} contracted with the Planck tensor \hbar^{ik} , we obtain

$$\begin{aligned} \eta = & \eta_0 + n \hbar^{ik} \check{A}_{ik} + \left[\left(\frac{1}{c^2} v_n v^n \pm 1 \right) \left(c_k \frac{* \partial c_i}{\partial t} - c_i \frac{* \partial c_k}{\partial t} \right) + \right. \\ & \left. + \left(v_i \frac{* \partial c_k}{\partial t} - v_k \frac{* \partial c_i}{\partial t} \right) \right] n \hbar^{ik} + \frac{1}{2c^2} n \hbar^{ik} c^m \left\{ c_i \left[\frac{\partial(c_m \pm v_m)}{\partial x^k} - \right. \right. \\ & \left. \left. - \frac{\partial(c_k \pm v_k)}{\partial x^m} \right] - c_k \left[\frac{\partial(c_m \pm v_m)}{\partial x^i} - \frac{\partial(c_i \pm v_i)}{\partial x^m} \right] \right\}, \end{aligned} \quad (4.271)$$

where “plus” stands for our world and “minus” — for the mirror world.

The quantity $\eta_0 = \eta \sqrt{1 - v^2/c^2} = 0$ for massless particles is zero, because they travel with the light velocity. Hence, keeping in mind that $\eta_0 = n \hbar^{mn} A_{mn}$, we obtain an additional condition imposed on the non-holonomy tensor of the isotropic space \tilde{A}_{ik} : at any point of the trajectory of any massless particle, the condition

$$\hbar^{mn} A_{mn} = 2\hbar (A_{12} + A_{23} + A_{31}) = 0 \quad (4.272)$$

must be true. Or, in the other notation, $\Omega^1 + \Omega^2 + \Omega^3 = 0$.

Therefore, in a region, where the observer “sees” a massless particle, the angular velocity with which the observer’s non-isotropic space rotates is equal to zero.

The other terms that make up the relativistic spin energy of the massless particle (4.271) are due to possible non-stationary state of the light velocity $\frac{* \partial c_i}{\partial t}$ as well as other dependencies that include the squared velocity of light.

So forth, we will analyse the obtained formula (4.271) under the following two simplification assumptions:

- a) The gravitational potential is negligible ($w \approx 0$);
 b) The three-dimensional chr.inv.-velocity of light is stationary.

In this case, the quantities A_{ik} and \check{A}_{ik} , which are the observer's space non-holonomy tensor and the isotropic space curl, become

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right), \quad \check{A}_{ik} = \frac{1}{2} \left(\frac{\partial c_k}{\partial x^i} - \frac{\partial c_i}{\partial x^k} \right), \quad (4.273)$$

and, therefore, the formula (4.271) for the relativistic spin energy of a massless particle takes the following form

$$\eta = n \left(\hbar^{ik} \check{A}_{ik} + \frac{1}{c^2} c_i c^m \hbar^{ik} \check{A}_{km} \right). \quad (4.274)$$

Therefore, the quantity η that describes the action of the spin of a massless particle, is determined (in addition to the particle's spin) only by the isotropic spatial curl and does not depend in any way on the non-holonomy (rotation) of the observer's space.

To make further deductions simpler, we transform η (4.274) as follows. Similar to the pseudovector $\check{\Omega}^{*i} = \frac{1}{2} \varepsilon^{ikm} A_{km}$ of the rotation angular velocity of the observer's space, we introduce a pseudovector

$$\check{\check{\Omega}}^{*i} = \frac{1}{2} \varepsilon^{ikm} \check{A}_{km}, \quad (4.275)$$

which can be interpreted as the pseudovector of the rotation angular velocity of the isotropic space.

Subsequently, $\check{A}_{km} = \varepsilon_{kmn} \check{\check{\Omega}}^{*n}$. Then the formula for η (4.274) can be represented as follows

$$\eta = n \left(\hbar_{*i} \check{\check{\Omega}}^{*i} + \frac{1}{c^2} c_i c^m \hbar^{ik} \varepsilon_{kmn} \check{\check{\Omega}}^{*n} \right). \quad (4.276)$$

This means that the internal mechanical curl (spin) of a massless particle only reveals itself in the interaction with the isotropic space curl. The result of the interaction is the scalar product $\hbar_{*i} \check{\check{\Omega}}^{*i}$, to which the massless particle's spin is attributed. Hence, massless particles are elementary light-like curls of the isotropic space itself.

Let us estimate the rotations of the isotropic space for massless particles having different energies. At present, we know for sure that among massless particles are photons — the quanta of an electromagnetic field.

Kind of photons	Frequency $\check{\Omega}$, sec ⁻¹
Radiowaves	$10^3 - 10^{11}$
Infra-red rays	$10^{11} - 1.2 \times 10^{15}$
Visible light	$1.2 \times 10^{15} - 2.4 \times 10^{15}$
Ultraviolet rays	$2.4 \times 10^{15} - 10^{17}$
X-rays	$10^{17} - 10^{19}$
Gamma rays	$10^{19} - 10^{23}$ and above

Table 4.2: The rotation frequencies of the isotropic space, which correspond to photons.

The spin quantum number of any photon is 1, and the energy $E = \hbar \omega$ of a photon is positive in our world. Hence, taking into account the live forces integral for massless particles (4.255), for photons we have

$$\hbar \omega = \hbar_{*i} \check{\Omega}^{*i} + \frac{1}{c^2} c_i c^m \hbar^{ik} \varepsilon_{kmn} \check{\Omega}^{*n}. \quad (4.277)$$

Assume that the rotation pseudovector $\check{\Omega}^{*i}$ of the isotropic space is directed along the z axis, while the light velocity is directed along y . Then, the relationship (4.277) obtained for photons becomes $\hbar \omega = 2\hbar \check{\Omega}$, or, after having the Planck constant cancelled,

$$\check{\Omega} = \frac{\omega}{2} = \frac{2\pi\nu}{2} = \pi\nu, \quad (4.278)$$

so the isotropic space rotation frequency $\check{\Omega}$ for a massless particle is constant and coincides the particle's own frequency ν . Thanks to this formula, resulting from the quantization law of the relativistic masses of massless particles, we can estimate the isotropic space angular velocities that correspond to photons of different energy levels. Table 4.2 gives the results of our calculation.

From Table 4.2, we see that the angular velocity of rotation of the isotropic space in photons of the gamma rays range is of the order of the ordinary space rotation frequencies in electrons and other elementary particles (see Table 4.1).

4.11 Conclusions

Here is what we have obtained in this Chapter. Firstly, we have obtained that the spin of any particle is characterized by the four-dimensional an-

tisymmetric tensor of the 2nd rank called the Planck tensor. Its diagonal and space-time components are zeroes, while the non-diagonal spatial components are $\pm\hbar$ depending on the spatial direction of the spin and our choice of a right or left-handed reference frame.

The spin (internal vortical field) of a particle interacts with an external field of the space non-holonomy. As a result, the particle gains an additional momentum, which deviates the particle's motion from a geodesic line. This interaction energy is found from the chr.inv.-scalar equation of motion of the particle (live forces theorem), so the equation must be taken into account when solving the chr.inv.-vector equations of motion.

A particular solution to the chr.inv.-scalar equation of motion is the law of quantization of the masses of elementary spin particles, which unambiguously links:

- The rest-masses of mass-bearing elementary particles with the angular velocity of the observer's space rotation;
- The relativistic masses of photons with the angular velocity of rotation of their internal light-like space.

Because the region, where light-like particles exist, is home to four-dimensional isotropic trajectories, such terms as the "isotropic space" and the "light-like space" can be used as synonyms.

Please note that we have obtained the results using only the geometric methods of the General Theory of Relativity, not the probabilistic methods of Quantum Mechanics. In the future, this result can possibly become a "bridge" between these two theories.

5.1 Introduction

According to the recent data, the average density of matter in the Universe is about $5\text{--}10 \times 10^{-30}$ gram/cm³. The average density of substance in galaxies is greater, $\sim 10^{-24}$ gram/cm³ in our Galaxy. Astronomical observations show that most part of the cosmic mass is accumulated in compact objects, such as stars, the total volume of which is incomparable to the entire Universe (this is called the “island” distribution of substance). Therefore, our Universe is predominantly empty.

For a long time, the words “emptiness” and “vacuum” were considered synonymous. But since the 1920s, the geometric methods of the General Theory of Relativity have showed that these are two different states of matter.

The distribution of matter in the Universe is characterized by the energy-momentum tensor, which is linked to the geometric structure of the space-time (expressed with the fundamental metric tensor) by the *field equations*. In Einstein’s theory of gravitation, which is an application of Riemannian geometry, these are *Einstein’s equations**

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa T_{\alpha\beta} + \lambda g_{\alpha\beta}. \quad (5.1)$$

These equations, in addition to the energy-momentum tensor and the fundamental metric tensor, include:

- 1) $R_{\alpha\sigma} = R_{\alpha\beta\sigma}^{\dots\beta}$ is Ricci’s tensor[†], which is the contraction result of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ by two indices;
- 2) $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature;

*The left hand side of the field equations (5.1) is often referred to as the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$, in the notation $G_{\alpha\beta} = -\kappa T_{\alpha\beta} + \lambda g_{\alpha\beta}$.

[†]Gregorio Ricci-Curbastro (1853–1925), an Italian mathematician who was the teacher of Tullio Levi-Civita in Padua in the 1890s.

- 3) $\kappa = \frac{8\pi G}{c^2} = 1.862 \times 10^{-27}$ cm/gram is Einstein's gravitational constant, while $G = 6.672 \times 10^{-8}$ cm³/gram sec² is Gauss' gravitational constant. Note that Landau and Lifshitz [10] use $\kappa = \frac{8\pi G}{c^4}$ instead of $\kappa = \frac{8\pi G}{c^2}$ as used by Zelmanov. To understand the reason, why $\kappa = \frac{8\pi G}{c^4}$ is not in our study, consider the chr.inv.-projections of the energy-momentum tensor $T_{\alpha\beta}$: the chr.inv.-scalar of the observable mass density $\frac{T_{00}}{g_{00}} = \rho$, the chr.inv.-vector of the observable momentum density $\frac{cT_0^i}{\sqrt{g_{00}}} = J^i$ and the chr.inv.-tensor of the observable momentum flux density $c^2 T^{ik} = U^{ik}$ [9]. Therefore, the scalar chr.inv.-projection of the Einstein equations is $\frac{G_{00}}{g_{00}} = -\frac{\kappa T_{00}}{g_{00}} + \lambda$. As is known, the dimension of the Ricci tensor is [cm⁻²], hence the Einstein tensor $G_{\alpha\beta}$ and the quantity $\frac{\kappa T_{00}}{g_{00}} = \frac{8\pi G\rho}{c^2}$ have the same dimension. Consequently, it is obvious that the dimension of the energy-momentum tensor $T_{\alpha\beta}$ is that of mass density [gram/cm³]. This means that, when we use $\frac{8\pi G}{c^4}$ on the right hand side of the Einstein equations, we actually use not the energy-momentum tensor itself, but the quantity $c^2 T_{\alpha\beta}$, the chr.inv.-scalar projection and chr.inv.-vector projection of which are the observable energy density $\frac{c^2 T_{00}}{g_{00}} = \rho c^2$ and the observable energy flux $\frac{c^3 T_0^i}{\sqrt{g_{00}}} = c^2 J^i$, respectively;
- 4) λ [cm⁻²] is the so-called *cosmological term*, which describes non-Newtonian forces of attraction or repulsion, depending on the sign before λ ($\lambda > 0$ stands for repulsion, $\lambda < 0$ stands for attraction). The term is referred to as "cosmological", because it is assumed that the forces described by λ grow up proportionally with distance and, therefore, reveal themselves in full scale at "cosmological" distances comparable to the size of the entire Universe. Because the non-Newtonian gravitational field (λ -field) has never been observed, the cosmological term in our Universe is $|\lambda| < 10^{-56}$ cm⁻² (as of today's measurement accuracy).

Looking at the Einstein equations (5.1), we see that the energy-momentum tensor describing the distribution of matter is linked to both the fundamental metric tensor and the Ricci tensor, and, therefore, to the Riemann-Christoffel curvature tensor. The equality of the Riemann-Christoffel tensor to zero in a space is the necessary and sufficient condition for the space to be flat. The Riemann-Christoffel tensor is non-zero

in a curved space only. It reveals itself as an increment of an arbitrary vector V^α in its parallel transport along a closed contour

$$\Delta V^\mu = -\frac{1}{2} R_{\alpha\beta\gamma}^{\dots\mu} V^\alpha \Delta\sigma^{\beta\gamma}, \quad (5.2)$$

where $\Delta\sigma^{\beta\gamma}$ is the area within this contour. As a result, the initial vector V^α and the vector $V^\alpha + \Delta V^\alpha$ have different directions. From a quantitative point of view, the difference is described by a quantity K called the *four-dimensional curvature* of the pseudo-Riemannian space along the given parallel transport (see [9] for details)

$$K = \lim_{\Delta\sigma \rightarrow 0} \frac{\tan \varphi}{\Delta\sigma}, \quad (5.3)$$

where $\tan \varphi$ is the tangent of the angle between the vector V^α and the projection of the vector $V^\alpha + \Delta V^\alpha$ onto the area constructed by the transport contour. For instance, we consider a surface and a “geodesic” triangle on it, produced by crossing three geodesic lines. We transport a vector, defined in any arbitrary point of that triangle, parallel to itself along the sides of the triangle. The summary rotation angle φ after the vector returns to the initial point is $\varphi = \Sigma - \pi$ (where Σ is the sum of the internal angles of the triangle). Assume the surface curvature K is equal at all of its points. Then

$$K = \lim_{\Delta\sigma \rightarrow 0} \frac{\tan \varphi}{\Delta\sigma} = \frac{\varphi}{\sigma} = \text{const}, \quad (5.4)$$

where σ is the triangle’s area, and $\varphi = K\sigma$ is called the *spherical excess*. If $\varphi = 0$, then the curvature is $K = 0$, so the surface is flat. In this case the sum of all internal angles of the geodesic triangle is π (a flat space). If $\Sigma > \pi$ (the transported vector is rotated towards the circuit), then there is a positive spherical excess, so the curvature $K > 0$. An example of such a space is the surface of a sphere: a triangle on the surface is convex. If $\Sigma < \pi$ (the transported vector is rotated counter the circuit), then the spherical excess is negative and the curvature is $K < 0$.

Einstein had postulated that gravitation is caused by the space-time curvature. He understood the space (space-time) curvature as the inequality to zero of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta} \neq 0$ (as assumed in Riemannian geometry). This concept completely includes Newtonian gravitational concept, so Einstein’s four-dimensional gravitation-curvature for an ordinary physical observer can reveal itself as follows:

- a) Newtonian gravitation;
- b) Rotation of the three-dimensional space (three-dimensional spatial section);
- c) Deformation of the three-dimensional space;
- d) The three-dimensional curvature, so that there are non-zero first derivatives of the Christoffel symbols.

According to Mach's Principle, on the basis of which Einstein's theory of gravitation rests, "... the property of inertia is completely determined by the interaction of matter" [28], so the space-time curvature is produced by the matter that fills the space-time. Proceeding from the above and from the Einstein equations (5.1), we can give the *mathematical definitions to the emptiness and the physical vacuum*:

The emptiness is the state of a space-time, for which the Ricci tensor is $R_{\alpha\beta} = 0$, which means the absence of any substance ($T_{\alpha\beta} = 0$) and the non-Newtonian gravitational fields ($\lambda = 0$). The field equations (5.1) in the emptiness* are as simple as $R_{\alpha\beta} = 0$;

The physical vacuum (or, simply, the vacuum) is the state of a space-time, where there is no substance $T_{\alpha\beta} = 0$, but $\lambda \neq 0$ and, hence, $R_{\alpha\beta} \neq 0$. The emptiness is a particular case of the vacuum in the absence of the λ -field. The field equations in the physical vacuum have the form

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \lambda g_{\alpha\beta}. \quad (5.5)$$

The Einstein equations are applicable to the most varied cases of distributed matter, except for the cases, where the density is close to that of the substance inside atomic nuclei. It is hard to give an accurate mathematical description to all of the cases of distributed matter, because such a problem is so general and it cannot be approached *per se*. On the other hand, the average density of substance in our Universe is so small, $5-10 \times 10^{-30}$ gram/cm³, that we can assume it near the vacuum. The Einstein equations say that the energy-momentum tensor is functionally dependent on the metric tensor and the Ricci tensor (curvature tensor, contracted by two indices). At such small numerical values of

*If we put down the Einstein equations for an empty space $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0$ in the mixed form $R_{\alpha}^{\beta} - \frac{1}{2} g_{\alpha}^{\beta} R = 0$, then after contraction ($R_{\alpha}^{\alpha} - \frac{1}{2} g_{\alpha}^{\alpha} R = 0$) we obtain $R - \frac{1}{2} 4R = 0$. So the scalar curvature in the emptiness is $R = 0$. Hence, the field equations (Einstein equations) in an empty space are $R_{\alpha\beta} = 0$.

density, we can assume the energy-momentum tensor to be proportional to the metric tensor $T_{\alpha\beta} \sim g_{\alpha\beta}$ and, hence, proportional to the Ricci tensor. Therefore, besides the field equations in the vacuum (5.5), we can consider the field equations

$$R_{\alpha\beta} = k g_{\alpha\beta}, \quad k = \text{const}, \quad (5.6)$$

where the energy-momentum tensor is different from the metric tensor only by a constant. This case, including the absence of masses (i.e., in the vacuum) as well as some other conditions close to it, related to our Universe, were studied in detail by Petrov [29, 30]. He called the spaces, for which the energy-momentum tensor is proportional to the metric tensor (and, hence, to the Ricci tensor) *Einstein spaces*.

A space with $R_{\alpha\beta} = k g_{\alpha\beta}$ (Einstein space) is homogeneous at every of its points, has no mass fluxes, while the density of the matter that fills the space (including any substances) is everywhere constant. In this case,

$$R = g^{\alpha\beta} R_{\alpha\beta} = k g_{\alpha\beta} g^{\alpha\beta} = 4k, \quad (5.7)$$

while the Einstein tensor takes the form

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -k g_{\alpha\beta}, \quad (5.8)$$

where $k g_{\alpha\beta}$ is the analogue of the energy-momentum tensor of the matter that fills the Einstein space.

To find out what kinds of matter fill Einstein spaces, Petrov studied the algebraic structure of the energy-momentum tensor. This is what he did: the tensor $T_{\alpha\beta}$ is compared to the metric tensor in an arbitrary point; for this point the difference $T_{\alpha\beta} - \xi g_{\alpha\beta}$ is calculated, where ξ are the so-called eigenvalues of the matrix $T_{\alpha\beta}$; the difference is equated to zero to find the values of ξ , which make the equality true. This problem is also called the matrix eigenvalues problem*. The matrix eigenvalues set allows us to define the algebraic kind of this matrix. For a sign-constant metric, this problem had been solved already, but Petrov proposed a method to bring any matrix to a canonical form in the space of a sign-alternating metric, which allowed using it in the pseudo-Riemannian space, in particular, to study the algebraic structure of the

*Generally, the problem should be solved at a given point, but the obtained result is applicable to any point of the space.

energy-momentum tensor. This can be illustrated as follows. The eigenvalues of the matrix elements $T_{\alpha\beta}$ are similar to the basis vectors of the metric tensor matrix, so the eigenvalues define a kind of “skeleton” of the tensor $T_{\alpha\beta}$ (the skeleton of matter); but even if we know what the skeleton is, we cannot know exactly what the muscles are. Nevertheless, the structure of such a skeleton (the length and mutual direction of the basis vectors) can be depicted based on the properties of matter, such as homogeneity or isotropy, and their relation to the space curvature.

As a result, Petrov had shown that all Einstein spaces have three basic algebraic kinds of the energy-momentum tensor and a few subtypes. According to his algebraic classification of the energy-momentum tensor and the curvature tensor, all Einstein spaces are sub-divided into three basic kinds, which is called *Petrov’s classification**.

The Einstein spaces of the kind I are best understood, because the field of gravitation in such a space is produced by a massive island (“island” distribution of substance), while the space itself can be empty or filled with the vacuum. The curvature of such a space is created by the island mass and by the vacuum. At the infinite distance from the island mass, in the absence of the vacuum, this space remains flat. Devoid of any island masses but filled with the vacuum, the space of the kind I has a curvature (e.g. de Sitter spaces). An empty space of the kind I, i.e., the one devoid of any island masses or the vacuum, is flat.

The Einstein spaces of the kind II and of the kind III are more exotic, because they are curved by themselves. Their curvature is neither related to the island distribution of masses, nor the presence of the vacuum. The kind II and the kind III are generally attributed to radiation fields, for instance, to gravitational waves.

A few years later, Gliner [32–34] in his study of the algebraic structure of the energy-momentum tensor of the vacuum-like states of matter ($T_{\alpha\beta} \sim g_{\alpha\beta}$, $R_{\alpha\beta} = k g_{\alpha\beta}$) outlined its special kind, for which all four eigenvalues are the same, so the three spatial vectors and the time vector of the “ortho-reference” of the tensor $T_{\alpha\beta}$ are equal to each other†.

*A chr.inv.-interpretation of the algebraic classification of Einstein spaces (or, in other words, of Petrov’s gravitational fields) was introduced in 1970 by a co-author of this book, Borissova, née Grigoreva [31].

†If we introduce a local flat space, tangential to the given Riemannian space at a given point, then the eigenvalues ξ of the tensor $T_{\alpha\beta}$ are the quantities in an ortho-reference, corresponding to this tensor, in contrast to the eigenvalues of the metric ten-

The matter that corresponds to the energy-momentum tensor of such a structure has a constant density $\mu = \text{const}$, equal to the coinciding eigenvalues of the energy-momentum tensor $\mu = \xi$ (the dimension of μ is the same as that of $T_{\alpha\beta}$ [gram/cm³]). The energy-momentum tensor in this case is*

$$T_{\alpha\beta} = \mu g_{\alpha\beta}, \quad (5.9)$$

and the field equations with $\lambda = 0$ have the form

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa \mu g_{\alpha\beta}, \quad (5.10)$$

while with the cosmological term $\lambda \neq 0$, they are

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa \mu g_{\alpha\beta} + \lambda g_{\alpha\beta}. \quad (5.11)$$

Gliner called this state of matter the μ -vacuum [32–34], because it is related to a vacuum-like state of substance ($T_{\alpha\beta} \sim g_{\alpha\beta}$, $R_{\alpha\beta} = k g_{\alpha\beta}$), which is not exactly the vacuum (in the vacuum, $T_{\alpha\beta} = 0$). At the same time, Gliner showed that spaces filled with the μ -vacuum are Einstein spaces, so three basic kinds of the μ -vacuum exist, which correspond to the three basic algebraic kinds of the energy-momentum tensor (and of the curvature tensor). In other words, an Einstein space of each kind (I, II, and III), provided that matter is present in it, is filled with the μ -vacuum of the corresponding kind (I, II, or III).

Actually, because when taken in the “ortho-reference” of the energy-momentum tensor of the μ -vacuum, all the three spatial vectors and the time vector are the same (all the four directions have the same significance), the μ -vacuum is the highest degree of isotropic matter. Besides, since Einstein spaces are homogeneous, i.e., the matter density is there everywhere equal [29, 30], the μ -vacuum that fills such a space does not only have a constant density, but is homogeneous as well.

As we have seen, Einstein spaces can be filled with the μ -vacuum, with the physical vacuum ($T_{\alpha\beta} = 0$) or with the emptiness. Besides, there

or $g_{\alpha\beta}$ in an ortho-reference, defined in this tangential space.

*Gliner used the signature $(-+++)$. Therefore, he had $T_{\alpha\beta} = -\mu g_{\alpha\beta}$. So, since the observable density of matter is positive, $\rho = \frac{T_{00}}{g_{00}} = -\mu > 0$, he had negative numerical values of the μ . In our book, we use the signature $(+---)$, because in this case the three-dimensional observable interval is positive. Therefore, we have $\mu > 0$ and $T_{\alpha\beta} = \mu g_{\alpha\beta}$.

can exist isolated “islands” of mass, which also produce the space curvature. Therefore, the Einstein spaces of the kind I are the best illustration of our knowledge of the Universe as a whole. And, thus, to study the geometry of the Universe and the physical states of the matter that fills it, it is the same as studying the Einstein spaces of the kind I.

Petrov had proposed and proved a theorem (see §13 in [29]), which we call *Petrov’s theorem*:

Petrov’s theorem

Any space of a constant curvature is an Einstein space... <so that > ... the Einstein spaces of the kind II and of the kind III cannot be constant curvature spaces.

Hence, a constant curvature space is an Einstein space of the kind I, according to the Petrov classification. If $K = 0$, then an Einstein space of the kind I is flat. This makes our study of the vacuum and vacuum-like states of matter in the Universe even simpler, because by today we have well studied constant curvature spaces. For example, these are *de Sitter spaces*, or, in other words, the spaces with the de Sitter metric.

Any de Sitter space has $T_{\alpha\beta} = 0$ and $\lambda \neq 0$, so it is filled with the ordinary vacuum and does not contain “islands” of substance. On the other hand, we know that the average density of matter in the Universe is rather low. Looking at it in general, we can neglect the presence of occasional “islands” and inhomogeneities of substance, which locally distort it. Hence, our space can be generally assumed as a de Sitter space with the constant curvature radius equal to the observable radius of the Universe.

Theoretically a de Sitter space can have either a positive curvature $K > 0$ or a negative curvature $K < 0$. Analysis (see Synge’s book) shows that in de Sitter worlds with $K < 0$ time-like geodesic lines are closed: a test-particle repeats its motion again and again along the same trajectory. This brings to mind some ideas, which seem to be too “revolutionary” from the point of view of today’s physics [35]. For this reason, most physicists (Synge, Gliner, Petrov, and others) have left negative curvature de Sitter spaces beyond the scope of their consideration.

As is known, positive curvature Riemannian spaces are the generalization of an ordinary sphere, while the negative curvature ones are the generalization of the Lobachewski-Bolyai space (an imaginary-radius sphere). According to Poincaré’s interpretation, negative curva-

ture spaces lie on the internal surface of a sphere. Using the methods of chronometric invariants, Zelmanov showed that in the four-dimensional pseudo-Riemannian space the three-dimensional observable curvature is negative to the Riemannian four-dimensional curvature. Since we perceive our planet as a sphere, the observable curvature is positive in our world. If any hypothetical beings inhabited the “internal” surface of the Earth, they would perceive it as concave, and their world would be of a negative curvature.

This illustration inspired some researchers for the idea of the possible existence of our mirror twin, the *mirror Universe* inhabited by antipodes. Initially it was assumed that, once our world has a positive curvature, the mirror Universe must be a negative curvature space. But Synge showed (see [35, Chapter VII]) that space-like geodesic trajectories are open in a positive curvature de Sitter space, and in a negative curvature de Sitter space they are closed. In other words, a negative curvature de Sitter space is not a mirror reflection of its positive curvature counterpart.

On the other hand, in our study [19] (see also §1.3 herein) we found another approach to the concept of the mirror Universe. We considered the motion of free particles with the reverse time flow. As a result, it was obtained that the observable scalar component of the four-dimensional momentum vector of a particle is its negative relativistic mass. Noteworthy, particles having “mirror” masses were obtained as a formal result of projecting the four-dimensional momentum of a particle onto the time line associated with an ordinary observer, and the projection result was not related to changing the space curvature sign, i.e., particles with either the direct or reverse flow of time can either exist in positive or negative curvature spaces.

These results obtained by the geometric methods of the General Theory of Relativity inevitably affect our views of matter and cosmology of our Universe.

In §5.2, we are going to obtain the energy-momentum tensor of the vacuum and, at the same time, a formula for its observable density. We will also introduce a classification of matter according to the obtained formula of the energy-momentum tensor (*T-classification* of matter). In §5.3, we are going to consider the physical properties of the vacuum in the Einstein spaces of the kind I; in particular, we will discuss the physical properties of the vacuum in a de Sitter space and make conclu-

sions on the global structure of the Universe. Following this approach, in §5.4, we will set forth the concept of the origin and evolution of the Universe as a result of the *Inversion Explosion* from a pre-particle that possessed some specific properties. In §5.5, we will obtain a formula for the non-Newtonian gravitational inertial force, which is proportional to distance, and §5.6 and §5.7 will focus on the gravitational collapse in a Schwarzschild space (a *gravitational collapsar*) and in a de Sitter space (an inflationary collapse and an *inflanton*). In Chapter 6, we will show that our Universe and the mirror Universe are the mirror time flow worlds, which co-exist in a de Sitter space with a four-dimensional negative curvature. Also we will find the physical conditions, which allow a transition through the membrane that separates our world and the mirror Universe.

5.2 The observable density of the vacuum. Non-Newtonian gravity. The T-classification of matter

The Einstein equations (i.e., the field equations in Einstein's theory of gravitation) are the functions that link the space curvature to the distribution of matter. Their general form is $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa T_{\alpha\beta} + \lambda g_{\alpha\beta}$. The left hand side, as is known, describes the geometric structure of the space, while the right hand side describes the matter that fills the space. The sign of the second term on the right hand side depends on the sign of λ . As we will see below, the sign of λ and so the type of Newtonian gravitation (attraction or repulsion) is directly linked to the sign of the vacuum density.

Einstein spaces are defined by the condition $T_{\alpha\beta} \sim g_{\alpha\beta}$, and the field equations for them have the form $R_{\alpha\beta} = k g_{\alpha\beta}$. Such field equations can exist in the two cases: a) in a space, where $T_{\alpha\beta} \neq 0$, i.e., in a substance; b) in a space, where $T_{\alpha\beta} = 0$, i.e., in the vacuum. But, since in Einstein spaces, filled with the vacuum, the energy-momentum tensor is equal to zero, it cannot be proportional to the metric tensor; this fact contradicts the definition of Einstein spaces ($T_{\alpha\beta} \sim g_{\alpha\beta}$).

So what is the problem here? In the absence of any substance, but in the vacuum, the field equations are $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \lambda g_{\alpha\beta}$ and, hence, the space curvature is produced by the λ -field (non-Newtonian fields of gravitation), and not by a substance. In the absence of both a substance and the λ -field, we have $R_{\alpha\beta} = 0$, so the space is empty but generally it is not flat.

We see that the λ -field and the vacuum are actually the same thing, therefore, the *vacuum is a non-Newtonian field of gravitation*. We will call this point of the theory the *physical definition of the physical vacuum*. Hence, the λ -field is the action of the vacuum potential.

This means that the term $\lambda g_{\alpha\beta}$ of the field equations cannot be omitted in the vacuum, no matter how small it is, since it describes the vacuum, which is one of the causes that make the space curved. Then the field equations $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa T_{\alpha\beta} + \lambda g_{\alpha\beta}$ take the form

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa \widetilde{T}_{\alpha\beta}, \quad (5.12)$$

on the right hand side of which the tensor

$$\widetilde{T}_{\alpha\beta} = T_{\alpha\beta} + \check{T}_{\alpha\beta} = T_{\alpha\beta} - \frac{\lambda}{\kappa} g_{\alpha\beta} \quad (5.13)$$

is the energy-momentum tensor that describes matter in general (both substance and the vacuum). The first term here is the energy-momentum tensor of a distributed substance. The second term

$$\check{T}_{\alpha\beta} = -\frac{\lambda}{\kappa} g_{\alpha\beta} \quad (5.14)$$

is the energy-momentum tensor of the vacuum.

Therefore, because Einstein spaces can be filled with the vacuum, their mathematical definition is better to be set forth in a more general form $\widetilde{T}_{\alpha\beta} \sim g_{\alpha\beta}$, which takes the presence of both substance and the vacuum (λ -field) into account. In particular, doing this helps to avoid contradictions when considering Einstein empty spaces.

Note that the energy-momentum tensor of the vacuum (5.14) is the direct consequence of the field equations in a general form.

If $\lambda > 0$ (the non-Newtonian forces of gravitation repulsion), then the observable density of the vacuum is negative

$$\check{\rho} = \frac{\check{T}_{00}}{g_{00}} = -\frac{\lambda}{\kappa} = -\frac{|\lambda|}{\kappa} < 0, \quad (5.15)$$

and if $\lambda < 0$ (the non-Newtonian forces of gravitation attraction), then the observable density of the vacuum is, on the contrary, positive

$$\check{\rho} = \frac{\check{T}_{00}}{g_{00}} = -\frac{\lambda}{\kappa} = \frac{|\lambda|}{\kappa} > 0. \quad (5.16)$$

The latter fact, as we will see in §5.3, is of great importance, because a de Sitter space with $\lambda < 0$, being a constant-negative curvature space* filled with the vacuum only (no substance present), best fits our observation data on our Universe in general.

Therefore, based on the studies by Petrov and Gliner and taking into account our note on the existence of the energy-momentum tensor of the vacuum (λ -field) and, hence, the physical properties of the vacuum, we can introduce a new “geometric” classification of the states of matter according to the energy-momentum tensor. We will call this classification the *T-classification of matter*:

- I) The emptiness: $T_{\alpha\beta} = 0$ and $\lambda = 0$ (a space-time without matter).
In this case, the field equations are $R_{\alpha\beta} = 0$;
- II) The physical vacuum (or, simply, the vacuum): $T_{\alpha\beta} = 0$ and $\lambda \neq 0$.
In this case, the field equations are $G_{\alpha\beta} = \lambda g_{\alpha\beta}$;
- III) The μ -vacuum: $T_{\alpha\beta} = \mu g_{\alpha\beta}$, $\mu = const$ (a vacuum-like state of substance). In this case, the field equations are $G_{\alpha\beta} = -\kappa \mu g_{\alpha\beta}$;
- IV) Substance: $T_{\alpha\beta} \neq 0$, $T_{\alpha\beta} \neq g_{\alpha\beta}$ (this state comprises both an ordinary substance and electromagnetic fields).

Generally, the energy-momentum tensor of substance (the kind IV according to the T-classification) is not proportional to the metric tensor. On the other hand, there are such states of substance, in which the energy-momentum tensor contains a term proportional to the metric tensor, but since it also contains other terms, these states of substance are not the μ -vacuum. Such, for instance, is an ideal fluid

$$T_{\alpha\beta} = \left(\rho - \frac{p}{c^2} \right) U_\alpha U_\beta - \frac{p}{c^2} g_{\alpha\beta}, \quad (5.17)$$

where p is the fluid pressure, and also electromagnetic fields

$$T_{\alpha\beta} = F_{\rho\sigma} F^{\rho\sigma} g_{\alpha\beta} - F_{\alpha\sigma} F_{\beta}^{\sigma}, \quad (5.18)$$

where $F_{\rho\sigma} F^{\rho\sigma}$ is the first invariant of the electromagnetic field under consideration (3.27), and $F_{\alpha\beta}$ is the Maxwell tensor. If $p = \rho c^2$ (a substance inside atomic nuclei) and $p = const$, the energy-momentum tensor of an ideal fluid seems to be proportional to the metric tensor.

But in the next section, §5.3, we will show that the equation of state of the μ -vacuum has a different form $p = -\rho c^2$, which is the state of inf-

*We mean here the Riemannian four-dimensional curvature.

lation (the expansion of a medium having a positive density). Hence, the pressure and density in atomic nuclei should not be constant as to prevent the transition of their internal substance into a vacuum-like state.

Note that the introduced T-classification of matter, just like the field equations, is only about a *distributed matter* that affects the space curvature, but not about test-particles (material points, the masses of which are so small that their effect on the space curvature can be neglected). Therefore, the energy-momentum tensor is not defined for particles; they must be considered beyond the T-classification of matter.

5.3 The physical properties of the vacuum. Cosmology

Einstein spaces are defined by the field equations like $R_{\alpha\beta} = k g_{\alpha\beta}$, where $k = \text{const.}$ With $\lambda \neq 0$ and $T_{\alpha\beta} = \mu g_{\alpha\beta}$ the space is filled with a matter, the energy-momentum tensor which is proportional to the fundamental metric tensor, so this kind of matter is the μ -vacuum. As we saw in the previous section, §5.2, the energy-momentum tensor of the vacuum is also proportional to the metric tensor. This means that the physical properties of the vacuum and those of the μ -vacuum are mostly the same, except for a scalar coefficient that determines the composition of the matter (a substance or the λ -field) as well as the absolute value of the acting forces. Therefore, we will consider an Einstein space filled with the vacuum or the μ -vacuum. In this case, the field equations take the form

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -(\kappa\mu - \lambda) g_{\alpha\beta}. \quad (5.19)$$

Writing them in the mixed form and then contracting indices, we obtain the scalar curvature

$$R = 4(\kappa\mu - \lambda), \quad (5.20)$$

then substituting it into the initial equations (5.19), we obtain the field equations in their final form

$$R_{\alpha\beta} = (\kappa\mu - \lambda) g_{\alpha\beta}, \quad (5.21)$$

where $\kappa\mu - \lambda = \text{const} = k$.

Let us consider the physical properties of the vacuum and the μ -vacuum. The physically observable properties of a medium are expressed with the chr.inv.-projections of its energy-momentum tensor: the ob-

servable density $\rho = \frac{T_{00}}{g_{00}}$, the observable momentum density $J^i = \frac{c T_0^i}{\sqrt{g_{00}}}$ and the observable stress tensor $U^{ik} = c^2 T^{ik}$.

For the energy-momentum tensor of the μ -vacuum, $T_{\alpha\beta} = \mu g_{\alpha\beta}$, the chr.inv.-projections have the following form

$$\rho = \frac{T_{00}}{g_{00}} = \mu, \quad (5.22)$$

$$J^i = \frac{c T_0^i}{\sqrt{g_{00}}} = 0, \quad (5.23)$$

$$U^{ik} = c^2 T^{ik} = -\mu c^2 h^{ik} = -\rho c^2 h^{ik}. \quad (5.24)$$

For the energy-momentum tensor $\check{T}_{\alpha\beta} = -\frac{\lambda}{\varkappa} g_{\alpha\beta}$ (5.14), which describes the vacuum, the chr.inv.-projections are

$$\check{\rho} = \frac{\check{T}_{00}}{g_{00}} = -\frac{\lambda}{\varkappa}, \quad (5.25)$$

$$\check{J}^i = \frac{c \check{T}_0^i}{\sqrt{g_{00}}} = 0, \quad (5.26)$$

$$\check{U}^{ik} = c^2 \check{T}^{ik} = \frac{\lambda}{\varkappa} c^2 h^{ik} = -\check{\rho} c^2 h^{ik}. \quad (5.27)$$

We see that the μ -vacuum and the vacuum (λ -field) have a constant density, so these are the kinds of *uniformly distributed matter*. They are also *non-emitting media*, since the energy flux $c^2 J^i$ in them is zero

$$c^2 \check{J}^i = \frac{c^3 \check{T}_0^i}{\sqrt{g_{00}}} = 0, \quad c^2 J^i = \frac{c^3 T_0^i}{\sqrt{g_{00}}} = 0. \quad (5.28)$$

In the reference frame that accompanies the medium, the stress tensor is equal to (see Zelmanov's book [9])

$$U_{ik} = p_0 h_{ik} - \alpha_{ik} = p h_{ik} - \beta_{ik}, \quad (5.29)$$

where p_0 is the equilibrium pressure, defined from the state equation, p is the true pressure, α_{ik} is the *viscosity of the 2nd kind* (the viscous stress tensor), $\alpha = \alpha_i^i$ is the trace of the tensor α_{ik} , and $\beta_{ik} = \alpha_{ik} - \frac{1}{3} \alpha h_{ik}$ is the anisotropic part of the tensor α_{ik} , which is called the *viscosity of the 1st kind* (it reveals itself in anisotropic deformations).

Expressing the μ -vacuum stress tensor (5.24) in the reference frame accompanying the μ -vacuum itself, we obtain

$$U_{ik} = p h_{ik} = -\rho c^2 h_{ik}, \quad (5.30)$$

and, similarly, for the stress tensor of the vacuum (5.27), we have

$$\check{U}_{ik} = \check{p} h_{ik} = -\check{\rho} c^2 h_{ik}. \quad (5.31)$$

This means that the μ -vacuum and the vacuum are non-viscous media ($\alpha_{ik} = 0, \beta_{ik} = 0$), the equation of state of which* is the same

$$\check{p} = -\check{\rho} c^2, \quad p = -\rho c^2. \quad (5.32)$$

Such a state is known as *inflation*, because at the positive density of a medium the pressure inside it is negative, so the medium expands.

So, these are the physical properties of the μ -vacuum and the vacuum: these are homogeneous $\rho = const$, non-viscous $\alpha_{ik} = \beta_{ik} = 0$ and non-emitting $J^i = 0$ media that are in the state of inflation.

Let us now consider the vacuum that fills constant curvature spaces, in particular, a de Sitter space — the approximation of our Universe.

In constant curvature spaces, the Riemann-Christoffel tensor is (see Chapter VII in Synge's book [35])

$$R_{\alpha\beta\gamma\delta} = K (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), \quad K = const. \quad (5.33)$$

Having the tensor contracted by two indices, we obtain a formula for the Ricci tensor, which on subsequent contraction allows us to deduce the scalar curvature. As a result we have

$$R_{\alpha\beta} = -3K g_{\alpha\beta}, \quad R = -12K. \quad (5.34)$$

Assuming our Universe to be a constant curvature space, we obtain the field equations formulated with the curvature

$$3K g_{\alpha\beta} = -\kappa T_{\alpha\beta} + \lambda g_{\alpha\beta}. \quad (5.35)$$

*The equation of state of a distributed matter is the relationship between the pressure and density in the medium. For instance, $p = 0$ is the equation of state of a dust medium, $p = \rho c^2$ is the equation of state of a matter inside atomic nuclei, $p = \frac{1}{3} \rho c^2$ is the equation of state of an ultra-relativistic gas.

Re-write this formula as $(\lambda - 3K) g_{\alpha\beta} = \kappa T_{\alpha\beta}$. Then, the energy-momentum tensor of a substance filling a constant curvature space is

$$T_{\alpha\beta} = \frac{\lambda - 3K}{\kappa} g_{\alpha\beta}. \quad (5.36)$$

We see that, in a constant curvature space, the problem of geometri- zation of matter solves by itself: the energy-momentum tensor (5.36) contains only the metric tensor and fundamental constants.

A de Sitter space is a constant curvature space, where $T_{\alpha\beta} = 0$ and $\lambda \neq 0$, hence, it is filled with the vacuum (any substance is absent). Then, equating the energy-momentum tensor of substance (5.36) to zero, we obtain the same result as that of Synge: $\lambda = 3K$ in any de Sitter space.

Taking into account this relationship, the formula for the observable density of the vacuum in a de Sitter world becomes

$$\check{\rho} = -\frac{\lambda}{\kappa} = -\frac{3K}{\kappa} = -\frac{3Kc^2}{8\pi G}. \quad (5.37)$$

Now we are arriving at the key question about the sign of the four- dimensional curvature in our Universe. The reason to ask this question is not only curiosity. Depending on the answer, the de Sitter world cos- mology can fit the available observational data or can lead to a result totally alien to the commonly accepted astronomical facts.

Given that the four-dimensional curvature is positive $K > 0$, the vac- uum density is negative and, hence, the inflationary pressure is greater than zero: the vacuum contracts. Then, since $\lambda > 0$, the non-Newtonian gravitational forces are the forces of repulsion. At the positive inflatio- nary pressure of the vacuum, which tends to compress the entire space, we should observe the repulsing forces of non-Newtonian gravitation. First, since the λ -forces are proportional to distance, their expanding ef- fect would grow along with the growth of the Universe's radius, there- fore the expansion would accelerate. Second, if the Universe were ever less than the distance at which the compressive pressure of the vacuum is equal to the expanding action of the λ -forces, the expansion would become impossible.

If, on the contrary, the four-dimensional curvature is negative $K < 0$, the inflationary pressure is less than zero — the vacuum expands. Be- sides, since $\lambda < 0$ in this case, the non-Newtonian forces of gravitation are the forces of attraction. Then, the Universe can still be expanding

from nearly a point until the moment of time, when the vacuum density becomes so low that its expanding force becomes equal to the compressing force of the non-Newtonian λ -forces.

As seen, the question of the curvature sign is the most crucial question for cosmology of our Universe.

But human perception is three-dimensional and, therefore, an ordinary observer cannot judge anything on the sign of the four-dimensional curvature by means of his direct observations. What can be done then? The way out of the situation is in the theory of chronometric invariants, which determine physical observable quantities.

Among the goals that Zelmanov set for himself was to build the curvature tensor of the three-dimensional spatial section associated with an observer — his observable three-dimensional space, which is inhomogeneous, non-holonomic (rotating), deforming, and curved, in a general case. The Zelmanov curvature tensor (see formula 5.40 for the tensor itself, and 5.41 for its contractions) has all the properties of the Riemann-Christoffel tensor in the three-dimensional space of the observer and, at the same time, has the property of chronometric invariance.

Zelmanov had deduced this tensor based on the similarity with the Riemann-Christoffel curvature tensor, which is the result of the non-commutativity of the second derivatives from an arbitrary vector in a Riemannian space. Deducing the difference of the second chr.inv.-derivatives from an arbitrary vector, he obtained the equation

$${}^* \nabla_i {}^* \nabla_k Q_l - {}^* \nabla_k {}^* \nabla_i Q_l = \frac{2A_{ik}}{c^2} \frac{{}^* \partial Q_l}{\partial t} + H_{lki}{}^{..j} Q_j, \quad (5.38)$$

where the chr.inv.-tensor

$$H_{lki}{}^{..j} = \frac{{}^* \partial \Delta_{il}^j}{\partial x^k} - \frac{{}^* \partial \Delta_{kl}^j}{\partial x^i} + \Delta_{il}^m \Delta_{km}^j - \Delta_{kl}^m \Delta_{im}^j \quad (5.39)$$

is similar to Schouten's tensor from the theory of non-holonomic manifolds*. But in a general case, where the space rotates ($A_{ik} \neq 0$), the tensor $H_{lki}{}^{..j}$ is algebraically different from the Riemann-Christoffel tensor.

*Schouten had created the theory of non-holonomic manifolds for an arbitrary dimension space by considering an m -dimensional sub-space of an n -dimensional space, where $m < n$ [36]. In the theory of chronometric invariants, we actually consider an observer associated with an ($m = 3$)-dimensional sub-space of the ($n = 4$)-dimensional pseudo-Riemannian space. At the same time, the theory of chronometric invariants is applicable to any metric space in general. See [9].

Therefore, Zelmanov had introduced a new tensor

$$C_{lkij} = \frac{1}{4} (H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}), \quad (5.40)$$

which was not only a chr.inv.-quantity, but it also has all algebraic properties of the Riemann-Christoffel tensor. Therefore, C_{lkij} is the physically observable curvature tensor of the three-dimensional observable space of an observer, who accompanies his reference body. Having it contracted, we obtain the chr.inv.-quantities

$$C_{kj} = C_{kij}^{\dots i} = h^{im} C_{kimj}, \quad C = C_j^j = h^{lj} C_{lj}, \quad (5.41)$$

which also characterize the observable three-dimensional space curvature. Because C_{lkij} , C_{kj} and C are chr.inv.-quantities, they are physically observables for the observer. In particular, the C is the *three-dimensional observable scalar curvature* [9].

Concerning the physical properties of the vacuum applied to cosmology, we need to know how the observable three-dimensional curvature C is linked to the four-dimensional curvature K in a general case and in a de Sitter space in particular. We are going to consider this problem step-by-step.

The Riemann-Christoffel four-dimensional curvature tensor is a tensor of the 4th-rank, hence it has $n^4 = 256$ components, out of which only 20 are significant. The remaining components are either zeroes or identical to each other, because the Riemann-Christoffel tensor is:

- a) Symmetric by each pair of its indices $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$;
- b) Antisymmetric with respect to the transposition of indices inside each of the pairs $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$, $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$;
- c) It satisfies the property $R_{\alpha(\beta\gamma\delta)} = 0$, where round brackets stand for the (β, γ, δ) -transpositions.

The significant components of the Riemann-Christoffel tensor produce the three chr.inv.-tensors

$$X^{ik} = -c^2 \frac{R_{0 \cdot 0}^{\cdot i \cdot k}}{g_{00}}, \quad Y^{ijk} = -c \frac{R_{0 \dots}^{\cdot ijk}}{\sqrt{g_{00}}}, \quad Z^{ijkl} = c^2 R^{ijkl}. \quad (5.42)$$

The tensor X^{ik} has 6 components, Y^{ijk} has 9 components, while Z^{ijkl} has only 9 due to its symmetry. The Y^{ijk} components are constructed based on the property $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$. Substituting the com-

ponents of the Riemann-Christoffel tensor, and having indices lowered, Zelmanov had obtained [9]

$$X_{ij} = \frac{*\partial D_{ij}}{\partial t} - (D_i^l + A_i^l)(D_{jl} + A_{jl}) + (*\nabla_i F_j + *\nabla_j F_i) - \frac{1}{c^2} F_i F_j, \quad (5.43)$$

$$Y_{ijk} = *\nabla_i (D_{jk} + A_{jk}) - *\nabla_j (D_{ik} + A_{ik}) + \frac{2}{c^2} A_{ij} F_k, \quad (5.44)$$

$$Z_{iklj} = D_{ik} D_{lj} - D_{il} D_{kj} + A_{ik} A_{lj} - A_{il} A_{kj} + 2A_{ij} A_{kl} - c^2 C_{iklj}. \quad (5.45)$$

From the above formulae we see that the spatial chr.inv.-components of the Riemann-Christoffel tensor Z_{iklj} (5.45) are linked to the chr.inv.-tensor of the three-dimensional observable curvature C_{iklj} .

Let us now deduce a formula for the three-dimensional observable curvature in a constant curvature space. In such a space the Riemann-Christoffel tensor has the form (5.33). Then

$$R_{0i0k} = -K h_{ik} g_{00}, \quad (5.46)$$

$$R_{0ijk} = \frac{K}{c} \sqrt{g_{00}} (v_j h_{ik} - v_k h_{ij}), \quad (5.47)$$

$$R_{ijkl} = K \left[h_{ik} h_{jl} - h_{il} h_{kj} + \frac{1}{c^2} v_i (v_l h_{kj} - v_k h_{jl}) + \frac{1}{c^2} v_j (v_k h_{il} - v_l h_{ik}) \right]. \quad (5.48)$$

Deducing its chr.inv.-projections (5.42), we obtain

$$X^{ik} = c^2 K h^{ik}, \quad Y^{ijk} = 0, \quad Z^{ijkl} = c^2 K (h^{ik} h^{jl} - h^{il} h^{jk}), \quad (5.49)$$

hence,

$$Z_{ijkl} = c^2 K (h_{ik} h_{jl} - h_{il} h_{jk}), \quad (5.50)$$

$$Z_{jl} = Z_{.jil}^{i\dots} = 2c^2 K h_{jl}, \quad Z = Z_j^j = 6c^2 K. \quad (5.51)$$

On the other hand, we know the formula for Z_{ijkl} in an arbitrary curvature space (5.45), which is linked to the three-dimensional observable

curvature. Obviously, it is as well true for $K = \text{const}$. Then, having the general formula (5.45) contracted, we obtain

$$Z_{il} = D_{ik}D_l^k - D_{il}D + A_{ik}A_l^k + 2A_{ik}A_l^k - c^2C_{il}, \quad (5.52)$$

$$Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2C. \quad (5.53)$$

In a constant curvature space, we have $Z = 6c^2K$ (5.51). Hence, in such a space the relationship between the four-dimensional scalar curvature K and the three-dimensional observable scalar curvature C is

$$6c^2K = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2C. \quad (5.54)$$

We see that in a constant curvature space that does not rotate or deform, the four-dimensional curvature has the opposite sign to the three-dimensional observable curvature. In a de Sitter space (since there is no rotation or deformation), we have

$$K = -\frac{1}{6}C, \quad (5.55)$$

so there the three-dimensional observable curvature is $C = -6K$.

Taking the results that we have obtained above, we are now going to create a cosmological model of our Universe based on only the two experimental facts: a) the sign of the observable density of matter in the Universe, and b) the sign of the observable three-dimensional curvature of the background space of the Universe.

At first, our everyday experience shows that the density of matter in our Universe is positive, no matter how rarefied it may be. Then to ensure that the vacuum density (5.37) is positive, the cosmological term is negative $\lambda < 0$ (in this case, the non-Newtonian gravitational forces are the forces of attraction) and, hence, the four-dimensional curvature is negative $K < 0$.

Secondly, as Ivanenko referred to McVittie's presentation [37] in his Preface to the 1962 edition of Weber's book [28]:

“Though the data of cosmological observations are obviously not exact, but, for instance, McVittie maintains that the best results of the observation of the Hubble redshift to $H \approx 75$ km/sec Mpc and of the average density of matter $\rho \approx 10^{-31}$ gram/cm³ support the idea of the non-vanishing cosmological term $\lambda < 0$.”

Therefore, we assume that the vacuum density in our Universe is positive and the three-dimensional observable curvature is $C > 0$. As a result, the four-dimensional curvature is $K < 0$ and, hence, the cosmological term is $\lambda < 0$. Then, from (5.37) we obtain the observable density of the vacuum in our Universe

$$\check{\rho} = -\frac{\lambda}{\varkappa} = -\frac{3K}{\varkappa} = \frac{C}{2\varkappa} > 0, \quad (5.56)$$

so the inflationary pressure in the vacuum is negative $\check{p} = -\check{\rho}c^2$ (the vacuum expands). Since the homogeneous distribution is among the physical properties of the vacuum, the negative inflationary pressure in the vacuum also means the expansion of the Universe as a whole.

Therefore, the observable three-dimensional space of our Universe (its curvature is $C > 0$) is a three-dimensional expanding sphere, which is a sub-space of the four-dimensional space-time with the curvature $K < 0$ (a space of the Lobachewski-Bolyai geometry).

Of course a de Sitter space is merely an approximation of our Universe. Astronomical data say that although “islands” of masses are occasional and do not affect the global curvature, their effect on the space curvature near them is significant (a deviation of light rays and similar effects). But in our study of the Universe as a whole we can neglect the occasional “islands” of substance and the local non-uniformities in the curvature. In this case, the background space of our Universe can be considered as a de Sitter space with a negative four-dimensional curvature (hence, the observed three-dimensional curvature is positive).

5.4 The concept of the Inversion Explosion of the Universe

From the previous section, §5.3, we know that $\lambda = 3K$ in a de Sitter space. According to its physical sense, the λ -term is approximately the same as the curvature. For a three-dimensional spherical sub-space of the de Sitter space, the observable curvature $C = -6K$ is

$$C = \frac{1}{R^2}, \quad (5.57)$$

where R is the observable curvature radius (radius of the sphere). Then, the four-dimensional curvature of the de Sitter space is

$$K = -\frac{1}{6R^2}, \quad (5.58)$$

i.e., the larger the sphere's radius, the smaller the space curvature K . According to astronomical estimates, our Universe originated 10–20 billion years ago. Hence, the distance covered by a photon since it was born at the dawn of the Universe is $R_H \approx 10^{27} - 10^{28}$ cm. This distance is referred to as the *radius of the event horizon*. Assuming that our Universe as a whole is a de Sitter space with $K < 0$, for the four-dimensional curvature and, hence, for the λ -term $\lambda = 3K$, we have the estimate

$$K = -\frac{1}{6R_H^2} \approx -10^{-56} \text{ cm}^{-2}. \quad (5.59)$$

On the other hand, similar figures for the event horizon, space curvature radius and λ -term are available from di Bartini [38,39], who studied the relationships between fundamental physical constants based on the methods of combinatorial topology. In his works, the Universe's radius is interpreted as the largest distance, determined from the topological context. According to the *di Bartini inversion relationship*

$$\frac{R\rho}{r^2} = 1, \quad (5.60)$$

the space radius R (which is the largest distance in the Universe) is the result of the spherical inversion of the gravitational radius of the electron $\rho = 1.347 \times 10^{-55}$ cm into the space outside the electron with respect to its classical radius $r = 2.818 \times 10^{-13}$ cm (which is the radius of the spherical inversion, according to di Bartini). The space radius (event horizon radius) is equal to

$$R = 5.895 \times 10^{29} \text{ cm}. \quad (5.61)$$

Following this way, di Bartini had defined the *space mass* (which is the mass within the space radius) and the *space density* as

$$M = 3.986 \times 10^{57} \text{ gram}, \quad \rho = 9.87 \times 10^{-34} \text{ gram/cm}^3. \quad (5.62)$$

As a matter of fact, the theoretical results that di Bartini had obtained say that the space of the Universe (ranged from the classical radius of the electron to the event horizon) is the external inversion image of the internal space of a certain particle with the size of the electron (ranged from the gravitational radius of the electron to its classical radius). From other points of view, the particle is different from the electron: its mass

is equal to the space mass $M = 3.986 \times 10^{57}$ gram, while the mass of the electron is only $m = 9.11 \times 10^{-28}$ gram.

The space inside such a particle cannot be represented as a de Sitter space: the vacuum density in a de Sitter space with $K < 0$ and the curvature observable radius $r = 2.818 \times 10^{-13}$ cm is

$$\check{\rho} = -\frac{3K}{\varkappa} = -\frac{1}{2\varkappa} r^2 = 3.39 \times 10^{51} \text{ gram/cm}^3, \quad (5.63)$$

while that inside the di Bartini particle is

$$\rho = \frac{M}{2\pi^2 r^3} = 9.03 \times 10^{93} \text{ gram/cm}^3. \quad (5.64)$$

On the other hand, the external space of such a particle, which is the inversion image of its internal space, can be assumed as a de Sitter space in accordance with its properties. Let us assume that a space with the curvature radius, equal to the di Bartini radius $R = 5.895 \times 10^{29}$ cm, is a de Sitter space with $K < 0$. Then the four-dimensional curvature K and the λ -term of the space are

$$K = -\frac{1}{6R^2} = -4.8 \times 10^{-61} \text{ cm}^{-2}, \quad (5.65)$$

$$\lambda = 3K = -\frac{1}{2R^2} = -14.4 \times 10^{-61} \text{ cm}^{-2}, \quad (5.66)$$

so they are five orders of magnitude less than the observed estimate, which is $|\lambda| < 10^{-56}$. This can be explained by the fact that the Universe continues to expand and, in the distant future, the numerical values of the space curvature and the cosmological term will decrease, approaching the numbers in (5.65, 5.66), calculated for the largest distance (space radius, according to di Bartini).

The estimated value of the vacuum density in a de Sitter space of the di Bartini space radius is

$$\check{\rho} = -\frac{3K}{\varkappa} = -\frac{3Kc^2}{8\pi G} \approx 7.7 \times 10^{-34} \text{ gram/cm}^3, \quad (5.67)$$

which is also less than the observed average density of matter in the Universe ($5\text{--}10 \times 10^{-30}$ gram/cm³), but is very close to the matter density in the space of the di Bartini radius 9.87×10^{-34} gram/cm³.

To calculate how long our Universe will continue to expand, we must calculate the difference between the observed event horizon radius R_H and the curvature radius R . Assume that the maximum event horizon radius of the Universe $R_{H(\max)}$ is equal to the di Bartini space radius $R = R_{H(\max)} = 5.895 \times 10^{29}$ cm (5.61), which is the outer inversion distance. Then, comparing this value with the observed event horizon radius $R_H \approx 10^{27} - 10^{28}$ cm, we obtain $\Delta R = R_{H(\max)} - R_H \approx 5.8 \times 10^{29}$ cm. Hence, the time left for the expansion of our Universe is

$$t = \frac{\Delta R}{c} \approx 600 \text{ billion years.} \quad (5.68)$$

These calculations of the vacuum density and other properties of the de Sitter space pave the way for conclusions on the origin and evolution of our Universe and allow the only interpretation of the di Bartini inversion relationship. We will call this interpretation the *cosmological concept of the Inversion Explosion of the Universe*. This cosmological concept is based on our analysis of the properties of the de Sitter space using the geometric methods of the General Theory of Relativity and taking into account the di Bartini inversion relation, which is the result of modern knowledge of fundamental physical constants. We can formulate this concept as follows:

At the very beginning, there was a single pre-particle with a radius equal to the classical radius of the electron, and with a mass equal to the mass of the entire Universe.

Then the inversion explosion occurred: a topological transition inverted the matter from within the pre-particle with respect to its surface into the outer world, which gave birth to our expanding Universe. At present, 10–20 billion years since the explosion, the Universe is at the early stage of its evolution. The expansion will continue for almost 600 billion years.

At the end of this period, the expanding Universe will reach its curvature radius, at which the non-Newtonian forces of gravitation, proportional to distance, will be equal to the inflationary expanding pressure of the vacuum. The expansion will discontinue and stability will be reached, which will last until the next inversion topological transition occurs.

The calculated parameters of matter at different stages of the evolution of the Universe are presented in Table 5.1. The evolution stages are

Evolution stage	Age, years	Space radius, cm	Density, gram/cm ³	λ -term, cm ⁻²
Pre-particle	0	2.82×10^{-13}	9.03×10^{93}	?
Present time	$10-20 \times 10^9$	$10^{27} - 10^{28}$	$5-10 \times 10^{-30}$	$< 10^{-56}$
After expansion	623×10^9	5.89×10^{29}	9.87×10^{-34}	1.44×10^{-60}

Table 5.1: Parameters of matter and space at different stages of the evolution of the Universe.

the pre-particle before the inversion explosion, the stage of the inversion expansion at the present time, and the stage after the expansion.

The reasons for this topological transition, which led to the spherical inversion of the matter from within the pre-particle (after its Inversion Explosion), remain unknown... but so do the reasons for the “emergence” of the Universe in some other contemporary cosmological concepts, for instance, in the Big Bang concept (the explosion of the Universe from a singular point).

5.5 Non-Newtonian gravitational forces

The Einstein spaces of the kind I, including constant curvature spaces, besides those that have occasional “islands of matter” can be either empty or filled with a homogeneously distributed matter. But an empty Einstein space of the kind I (its curvature is $K = 0$) is dramatically different from non-empty spaces ($K = const \neq 0$).

To make our discourse more concrete, let us consider the field of gravitation in the most typical empty and non-empty Einstein spaces of the kind I.

If an island of mass is a ball (the spherically symmetric distribution of mass in the island) located in emptiness, then the curvature of such a space is derived from the Newtonian field of gravitation, produced by the island, and such a space is not a constant curvature space. At an infinite large distance from the island, the space becomes flat, i.e., a constant curvature space with $K = 0$. A typical example of the field of gravitation, produced by a spherically symmetric island of mass in emptiness is the field determined by the Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (5.69)$$

where r is the distance from the island, and r_g is the gravitational radius of the island.

A Schwarzschild space neither rotates nor deforms. The components of the chr.inv.-vector of the gravitational inertial force (1.38) in such a space can be deduced as follows. According to the metric (5.69), the component g_{00} is

$$g_{00} = 1 - \frac{r_g}{r}, \quad (5.70)$$

then, differentiating the gravitational potential $w = c^2(1 - \sqrt{g_{00}})$ with respect to x^i , we obtain

$$\frac{\partial w}{\partial x^i} = -\frac{c^2}{2\sqrt{g_{00}}} \frac{\partial g_{00}}{\partial x^i}. \quad (5.71)$$

Substituting it into the formula for the gravitational inertial force (1.38), and taking into account the fact that such a space does not rotate (this follows from the metric, where all $g_{0i} = 0$), we obtain

$$F_1 = -\frac{c^2 r_g}{2r^2} \frac{1}{1 - \frac{r_g}{r}}, \quad F^1 = -\frac{c^2 r_g}{2r^2}. \quad (5.72)$$

Therefore, the vector F^i in a Schwarzschild space describes a Newtonian gravitational force, which is reciprocal to the square of the distance r from the gravitating mass.

If a space is *filled* with the spherically symmetric distribution of the physical vacuum (λ -field) and does not include any island of mass, its curvature is everywhere the same. An example of such a field is that described by the de Sitter metric*

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \quad (5.73)$$

Note that although any de Sitter space has no islands of mass, which create ordinary Newtonian fields of gravitation, we can always consider

*According to the latest theoretical studies [40], the de Sitter space metric (5.73) satisfies to the condition of the spherical symmetry in only a limiting case, where $\lambda = 0$. In a general case of $\lambda \neq 0$, a de Sitter space can be spherically symmetric only if it has zero volume (i.e., only if the de Sitter space degenerates into a point). This means that an actual de Sitter space (wherein $\lambda \neq 0$, i.e., a space filled by the vacuum) should not have the property of the spherical symmetry.

the motion of small test-particles, since their own Newtonian fields are so weak that they can be neglected.

Any de Sitter space is a constant curvature space that becomes flat only in the absence of the λ -fields. A de Sitter space neither rotates nor deforms, and the non-zero components of the chr.inv.-vector of the gravitational inertial force in such a space take the form

$$F_1 = \frac{\lambda c^2}{3} \frac{r}{1 - \frac{\lambda r^2}{3}}, \quad F^1 = \frac{\lambda c^2}{3} r. \quad (5.74)$$

As is seen, the vector F^i in a de Sitter space describes a kind of non-Newtonian gravitational forces, which are proportional to r : if $\lambda < 0$, then these are the forces of attraction, if $\lambda > 0$, then these are the forces of repulsion. The forces of non-Newtonian gravitation (we will call them the λ -forces) increase with the distance at which they act.

Therefore, we can see the principal difference between empty and non-empty Einstein spaces of the kind I: in empty spaces with an island of mass only Newtonian forces exist, while in the spaces filled with the vacuum and without islands of mass there are only non-Newtonian gravitation forces. An example of a “mixed” space of the kind I is that of the Kottler metric [41]

$$\left. \begin{aligned} ds^2 = & \left(1 + \frac{ar^2}{3} + \frac{b}{r} \right) c^2 dt^2 - \frac{dr^2}{1 + \frac{ar^2}{3} + \frac{b}{r}} - \\ & r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \\ F_1 = & -c^2 \frac{\frac{ar}{3} - \frac{b}{2r^2}}{1 + \frac{ar^2}{3} + \frac{b}{r}}, \quad F^1 = -c^2 \left(\frac{ar}{3} - \frac{b}{2r^2} \right) \end{aligned} \right\}, \quad (5.75)$$

where both Newtonian forces and the λ -forces exist: a Kottler space is filled with the vacuum and also includes islands of mass, the latter which produce Newtonian gravitational forces.

On the other hand, Kottler had proposed this metric with two unknown constants a and b to define which additional constraints are required. Hence, despite some attractive features of the Kottler metric, only two of its “limiting” cases are of practical interest. These are the Schwarzschild metric (Newtonian gravitational forces) and the de Sitter metric (non-Newtonian gravitational forces, i.e., the λ -forces).

5.6 Gravitational collapse

Obviously, it is a certain approximating assumption to represent our Universe as a de Sitter space (filled with the vacuum without islands of mass) or a Schwarzschild space (an island of mass in emptiness). The real metric of our world is “something in between”. However, in problems related to non-Newtonian gravity (caused by the physical vacuum), where the influence of concentrated masses can be neglected, the de Sitter metric is optimal. And, in problems with the gravitation caused by massive islands, the Schwarzschild metric is reasonable. An illustrative example of such a “split” of the models is collapse — the state of a space (space-time) region, where $g_{00} = 0$.

The formula for the gravitational potential w deduced for an arbitrary space metric is (1.38). Then

$$g_{00} = \left(1 - \frac{w}{c^2}\right)^2 = 1 - \frac{2w}{c^2} + \frac{w^2}{c^4}, \quad (5.76)$$

hence, the state of collapse ($g_{00} = 0$) occurs at $w = c^2$.

Scientists usually consider *gravitational collapse* — the compressed state of an island of mass as a result of the action of Newtonian gravity, which compressed the island to a very small size, equal to the gravitational radius of the mass. Hence, “strict” gravitational collapse occurs in a space of the Schwarzschild metric (5.69), because only the Newtonian field of a spherically symmetric island of mass in emptiness is present in such a space.

At a large distance from a massive island, the gravitational field becomes weak and the gravitational field potential becomes

$$w = \frac{GM}{r}, \quad (5.77)$$

where G is the Gauss gravitational constant, M is the island’s mass that produced the gravitational field. Since the third term in (5.76) is so small in a weak field that it can be neglected, the formula for g_{00} becomes

$$g_{00} = 1 - \frac{2GM}{c^2 r}, \quad (5.78)$$

so gravitational collapse in a Schwarzschild space occurs if

$$\frac{2GM}{c^2 r} = 1, \quad (5.79)$$

where the quantity

$$r_g = \frac{2GM}{c^2}, \quad (5.80)$$

which has the dimension of length, is called the *gravitational radius* of the island of mass. Then g_{00} can be presented as follows

$$g_{00} = 1 - \frac{r_g}{r}. \quad (5.81)$$

From here we see that gravitational collapse occurs in a Schwarzschild space at the distance $r = r_g$ from the centre of mass.

If the entire mass of the spherically symmetric island (which is the source of the Newtonian field) is concentrated under the gravitational radius of the mass, the surface of such an island of mass is referred to as the *Schwarzschild sphere*. Such objects are also called *gravitational collapsars*, because under the gravitational radius an escape velocity is higher than the velocity of light, so light cannot leave such objects from within.

It is easy to see from formula (5.69) that, in a Schwarzschild field of gravitation, the three-dimensional space does not rotate ($g_{0i} = 0$) and, hence, the interval of the physically observable time (1.25) is

$$d\tau = \sqrt{g_{00}} dt = \sqrt{1 - \frac{r_g}{r}} dt. \quad (5.82)$$

So, at the distance $r = r_g$ the observable time interval is equal to zero $d\tau = 0$: from the point of view of an external observer, the observable time on the surface of a Schwarzschild sphere stops*. Inside the Schwarzschild sphere the observable time interval becomes imaginary.

*At $g_{00} = 0$ (the state of gravitational collapse) the observable time interval (1.25) is $d\tau = -\frac{1}{c^2} v_i dx^i$, where $v_i = -c \frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity with which the space rotates (1.37). Only assuming $g_{0i} = 0$ and $v_i = 0$, the collapse condition can be defined correctly: for an external observer the observable time flow on the surface of a collapsar stops $d\tau = 0$, while the four-dimensional interval is $ds^2 = -d\sigma^2 = g_{ik} dx^i dx^k$. From here a conclusion can be made: the space is holonomic on the surface of a collapsar, so the collapsar does not rotate.

As we had showed in our first book [19], a completely degenerate space-time region (called the zero-space), where $ds = 0$, $d\tau = 0$ and $d\sigma = 0$, collapses if it does not rotate. Here we proved a more general theorem: if $g_{00} = 0$, then the space is holonomic irrespective of whether it is degenerate ($g = 0$) or not ($g < 0$, the ordinary space-time of General Relativity).

We can also be sure that an ordinary observer who is located on the Earth surface, apparently stays outside the Schwarzschild sphere of the Earth, the radius of which is 0.443 cm, and he can only look at the process of gravitational collapse from “outside”.

If $r = r_g$, then the quantity

$$g_{11} = -\frac{1}{1 - \frac{r_g}{r}} \quad (5.83)$$

grows up to infinity. But the determinant of the metric tensor $g_{\alpha\beta}$ is

$$g = -r^4 \sin^2\theta < 0, \quad (5.84)$$

so a space-time region inside a gravitational collapsar is generally not degenerate, although collapse is also possible in the zero-space.

At this point a note concerning photometric distance and metric observable distance should be made. The quantity r is not a metric distance along the axis $x^1 = r$, because the metric (5.69) has dr^2 with the coefficient $(1 - \frac{r_g}{r})^{-1}$. The quantity r is a *photometric distance* defined as the function of an illumination, produced by a stable source of light and reciprocal to the square of the distance from the source. In other words, r is the radius of a non-Euclidean sphere of the surface area $4\pi r^2$ [9].

According to the theory of chronometric invariants, the elementary observable metric distance between any two infinitely close points in a Schwarzschild space is

$$d\sigma = \sqrt{\frac{dr^2}{1 - \frac{r_g}{r}} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)}. \quad (5.85)$$

At $\theta = const$ and $\varphi = const$, it is

$$\sigma = \int_{r_1}^{r_2} \sqrt{h_{11}} dr = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{r_g}{r}}}, \quad (5.86)$$

and it is not the same as the photometric distance r .

To define the space-time metric inside a Schwarzschild sphere, we formulate the external metric (5.69) for a radius $r < r_g$. We obtain

$$ds^2 = -\left(\frac{r_g}{r} - 1\right) c^2 dt^2 + \frac{dr^2}{\frac{r_g}{r} - 1} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \quad (5.87)$$

Introducing the notations $r = c\tilde{t}$ and $ct = \tilde{r}$, we obtain

$$ds^2 = \frac{c^2 d\tilde{t}^2}{\frac{r_g}{c\tilde{t}} - 1} - \left(\frac{r_g}{c\tilde{t}} - 1\right) d\tilde{r}^2 - c^2 d\tilde{t}^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (5.88)$$

so the space-time metric inside the Schwarzschild sphere is similar to the external metric, provided that the time coordinate and the spatial coordinate r swap their rôles: the photometric distance r outside the collapsar is the coordinate time $c\tilde{t}$ inside it, while outside the collapsar the coordinate time ct is the photometric distance \tilde{r} inside it.

From the first term of the Schwarzschild internal metric (5.88) we see that it is not stationary, but exists within a limited period of time

$$\tilde{t} = \frac{r_g}{c}. \quad (5.89)$$

For the Sun, the gravitational radius of which is about 3 km, the life span of such a space is approximately $< 10^{-5}$ sec. For the Earth, the gravitational radius of which is as small as 0.443 cm, the life span of the internal Schwarzschild space is even lesser, 1.5×10^{-11} sec.

Comparing the metrics inside a gravitational collapsar (5.88) and outside it (5.69), we conclude the following:

- a) The space of both metrics is holonomic, i.e., it does not rotate ($A_{ik} = 0$);
- b) The external metric is stationary, and the vector of the observable gravitational inertial force is $F^1 = -\frac{GM}{r^2}$;
- c) The internal metric is non-stationary, and the observable gravitational inertial force is zero.

Let us give more detailed analysis of the external and internal metrics. To make the analysis simpler, we assume $\theta = const$ and $\varphi = const$, so that out of the possible three spatial directions we limit our study to the radial direction only. Then the external metric is

$$ds^2 = -\left(\frac{r_g}{r} - 1\right) c^2 dt^2 + \frac{dr^2}{\frac{r_g}{r} - 1}, \quad (5.90)$$

while for the internal metric we have

$$ds^2 = \frac{c^2 d\tilde{t}^2}{\frac{r_g}{c\tilde{t}} - 1} - \left(\frac{r_g}{c\tilde{t}} - 1\right) d\tilde{r}^2. \quad (5.91)$$

Calculating the physically observable distance (5.86) to the attracting centre of the collapsar along the radial direction r , we obtain

$$\sigma = \int \frac{dr}{\sqrt{1 - \frac{r_g}{r}}} = \sqrt{r(r - r_g)} + r_g \ln(\sqrt{r} + \sqrt{r - r_g}) + \text{const.} \quad (5.92)$$

We see that at $r = r_g$ the observable distance is a constant value

$$\sigma_g = r_g \ln \sqrt{r_g} + \text{const.} \quad (5.93)$$

This means that a Schwarzschild sphere, defined by a photometric radius r_g , for an external observer is a sphere with the observable radius $\sigma_g = r_g \ln \sqrt{r_g} + \text{const}$ (5.93). Therefore, for an external observer any gravitational collapsar is a sphere with a constant observable radius, on the surface of which the observable time stops.

Let us look within a collapsar. For an external observer, the observable time interval (5.82) inside a Schwarzschild sphere is imaginary

$$d\tau = i \sqrt{\frac{r_g}{r} - 1} dt, \quad (5.94)$$

or, in the “internal” coordinates $r = c\tilde{r}$ and $ct = \tilde{r}$ (from the point of view of an “internal” observer),

$$d\tilde{\tau} = \frac{1}{\sqrt{\frac{r_g}{c\tilde{r}} - 1}} d\tilde{r}. \quad (5.95)$$

Hence, for an external observer, the “imaginary” time inside a collapsar (5.94) stops on its surface, while the “internal” observer sees the flow of his observable time on the surface growing infinitely.

So, when looking at a collapsar from outside, the physically observable distance inside it, according to the metric (5.87), is

$$\sigma = \int \frac{dr}{\sqrt{\frac{r_g}{r} - 1}} = -\sqrt{r(r - r_g)} + r_g \arctan \sqrt{\frac{r_g}{r} - 1} + \text{const.}, \quad (5.96)$$

and, from the point of view of an “internal” observer, it is

$$\tilde{\sigma} = \int \sqrt{\frac{r_g}{c\tilde{r}} - 1} d\tilde{r}. \quad (5.97)$$

From here we see that at $r = c\tilde{t} = r_g$ for an external observer the observed distance between any two points tends to a constant, and for an “internal” observer the observed distance decreases to zero.

In conclusion, let us touch upon the question of what happens to particles falling “from the outside” onto the Schwarzschild sphere along its radial direction. For the outer metric of a collapsar, we have

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \quad d\tau = \left(1 - \frac{r_g}{r}\right) dt, \quad d\sigma = \frac{dr}{1 - \frac{r_g}{r}}. \quad (5.98)$$

For real-mass particles $ds^2 > 0$, for light-like particles $ds^2 = 0$, for superluminal tachyons $ds^2 < 0$ (their masses are imaginary). In radial motion towards a gravitational collapsar, these conditions are:

- 1) Real-mass particles: $\left(\frac{d\tau}{dt}\right)^2 < c^2 \left(1 - \frac{r_g}{r}\right)^2$;
- 2) Light-like particles: $\left(\frac{d\tau}{dt}\right)^2 = c^2 \left(1 - \frac{r_g}{r}\right)^2$;
- 3) Imaginary particles-tachyons: $\left(\frac{d\tau}{dt}\right)^2 > c^2 \left(1 - \frac{r_g}{r}\right)^2$.

Since $r = r_g$ on the surface of a Schwarzschild sphere, then $\frac{d\tau}{dt} = 0$. Hence, any particle, including a light-like one, will stop there. A four-dimensional interval on the surface of the sphere is

$$ds^2 = -d\sigma^2 < 0, \quad (5.99)$$

which means that the surface of a Schwarzschild spheres (gravitational collapsar) is home to particles having imaginary rest-masses.

5.7 Inflationary collapse

A de Sitter space has no islands of mass, hence, Newtonian gravitational fields are absent there. Therefore, gravitational collapse is impossible in a de Sitter space. Nevertheless, the condition $g_{00} = 0$ is a strictly geometric definition of collapse, not necessarily related to Newtonian fields. Therefore, we can consider collapse in any arbitrary space.

Consider the de Sitter metric (5.73). It describes a non-Newtonian gravitational field in a constant curvature space without islands of mass. In a de Sitter space, collapse can occur due to non-Newtonian gravitational forces. From the de Sitter metric (5.73), we see that

$$g_{00} = 1 - \frac{\lambda r^2}{3}, \quad (5.100)$$

so the gravitational potential $w = c^2(1 - \sqrt{g_{00}})$ is

$$w = c^2 \left(1 - \sqrt{1 - \frac{\lambda r^2}{3}} \right). \quad (5.101)$$

Because it is the potential of a non-Newtonian gravitational field, produced by the vacuum (λ -field), we will call it the λ -potential.

Since $\lambda = 3K$ in any de Sitter space, hence

- 1) $g_{00} = 1 - Kr^2 > 0$ at distances $r < \frac{1}{\sqrt{K}}$;
- 2) $g_{00} = 1 - Kr^2 < 0$ at distances $r > \frac{1}{\sqrt{K}}$;
- 3) $g_{00} = 1 - Kr^2 = 0$ (collapse) at the distance $r = \frac{1}{\sqrt{K}}$.

If the four-dimensional curvature is $K < 0$, then the numerical value of $g_{00} = 1 - Kr^2$ is always greater than zero. Hence, collapse is only possible in a de Sitter space with $K > 0$.

In §5.3, we showed that the basic space of our Universe as a whole has $K < 0$. But we can assume the presence of local inhomogeneities with $K > 0$, which do not affect the space curvature in general. In particular, collapse can occur in such inhomogeneities. Therefore, it is reasonable to consider a de Sitter space with $K > 0$ as a local space in the vicinities of some compact objects.

In de Sitter spaces the three-dimensional observable curvature C is linked to the four-dimensional curvature with the relationship $C = -6K$ (5.55). Then, assuming the observable three-dimensional space to be a sphere, we obtain $C = \frac{1}{R^2}$ (5.57) and, hence $K = -\frac{1}{6R^2}$ (5.58), where R is the observable curvature radius. In the case of $K < 0$, the numerical value of R is real, but at $K > 0$ it becomes imaginary.

So, collapse in a de Sitter space is only possible at $K > 0$. In this case, the observable curvature radius is imaginary. Denote $R = iR^*$, where R^* is its absolute value. Then, in a de Sitter space with $K > 0$ we have

$$K = \frac{1}{6R^{*2}}, \quad (5.102)$$

and the collapse condition $g_{00} = 1 - Kr^2$ can be written as follows

$$r = R^* \sqrt{6}, \quad (5.103)$$

so, at the distance $r = R^* \sqrt{6}$ in a de Sitter space with $K > 0$ the condition $g_{00} = 0$ is true: the observable time flow stops and collapse occurs.

That is, the region of a de Sitter space under the radius $r = R^* \sqrt{6}$ stays in collapse. Since the vacuum (it fills any de Sitter space) stays in inflation, we will call such a collapse *inflationary collapse* to differentiate it from gravitational collapse (it occurs in a Schwarzschild space), while $r = R^* \sqrt{6}$ (5.77) will be referred to as the *inflationary radius*, r_{inf} . Then the collapsed region of a de Sitter space, which is under the inflationary radius, will be referred to as an *inflationary collapsar*, or, simply, an *inflanton*.

Inside an inflanton we have $K > 0$, so the three-dimensional observable curvature is $C < 0$. In this case, the vacuum density is negative (the inflationary pressure is positive, hence, the vacuum compresses) and $\lambda > 0$, so there are non-Newtonian forces of repulsion. This means that an inflationary collapsar (inflanton) is filled with the vacuum having a negative density, which is in the state of fragile balance between the compressing pressure of the vacuum and the expanding forces of non-Newtonian gravitation.

In a de Sitter space with $K > 0$, we have

$$d\tau = \sqrt{g_{00}} dt = \sqrt{1 - Kr^2} dt = \sqrt{1 - \frac{r^2}{r_{\text{inf}}^2}} dt, \quad (5.104)$$

so on the surface of an inflationary sphere the observable time flow stops $d\tau = 0$. Besides, the assumed space-time signature (+---), i.e., the condition $g_{00} > 0$, is true at $r < r_{\text{inf}}$.

Using the term “inflationary radius” we represent the de Sitter metric with $K > 0$ as follows

$$ds^2 = \left(1 - \frac{r^2}{r_{\text{inf}}^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r^2}{r_{\text{inf}}^2}} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (5.105)$$

then the chr.inv.-vector of the gravitational inertial force (5.74) has the non-zero components

$$F_1 = \frac{c^2}{1 - \frac{r^2}{r_{\text{inf}}^2}} \frac{r}{r_{\text{inf}}^2}, \quad F^1 = c^2 \frac{r}{r_{\text{inf}}^2}. \quad (5.106)$$

Let us deduce formulae for the observable distances and the observable inflationary radius in an inflanton. To make our calculations simpler, we assume $\theta = \text{const}$ and $\varphi = \text{const}$, i.e., out of all three spatial

directions only the radial direction will be considered. Then the observable three-dimensional interval is

$$\sigma = \int \sqrt{h_{11}} dr = \int \frac{dr}{\sqrt{1 - Kr^2}} = r_{\text{inf}} \arcsin \frac{r}{r_{\text{inf}}} + \text{const}, \quad (5.107)$$

so the observable inflationary radius is constant

$$\sigma_{\text{inf}} = \int_0^{r_{\text{inf}}} \frac{dr}{\sqrt{1 - Kr^2}} = \frac{\pi}{2} r_{\text{inf}}. \quad (5.108)$$

In a space with the Schwarzschild metric, which we considered in the previous section, §5.6, a collapsar is a collapsed compact mass, which produces the curvature of the space as a whole. An ordinary observer, whose home is a Schwarzschild space, stays always outside gravitational collapsars.

In a de Sitter space, a collapsar is the vacuum that fills the entire space, and the surface of the collapsar has a radius equal to the space curvature radius. Therefore, an ordinary observer, whose home is a de Sitter space, stays always under the surface of an inflationary collapsar and, therefore, he “watches” the inflationary collapsar from within.

To look beyond an inflationary collapsar, we consider the de Sitter metric with $K > 0$ (5.105) for distances $r > r_{\text{inf}}$. Considering the radial direction, in the coordinates associated with an ordinary observer (the “internal” coordinates of the collapsar), we obtain

$$ds^2 = - \left(\frac{r^2}{r_{\text{inf}}^2} - 1 \right) c^2 dt^2 + \frac{dr^2}{\frac{r^2}{r_{\text{inf}}^2} - 1}, \quad (5.109)$$

or, from the point of view of an observer, who is located outside the collapsar (in the “external” coordinates $r = c\tilde{r}$ and $ct = \tilde{r}$), we have

$$ds^2 = \frac{c^2 d\tilde{t}^2}{\frac{c^2 \tilde{t}^2}{r_{\text{inf}}^2} - 1} - \left(\frac{c^2 \tilde{t}^2}{r_{\text{inf}}^2} - 1 \right) d\tilde{r}^2. \quad (5.110)$$

5.8 Conclusions

At a low density of matter (as observed, $5\text{--}10 \times 10^{-30}$ gram/cm³ in the Metagalaxy, i.e., the space is nearly empty), we can assume that the energy-momentum tensor is $T_{\alpha\beta} \sim g_{\alpha\beta}$. In this case the Einstein equa-

tions take the form $R_{\alpha\beta} = k g_{\alpha\beta}$, where $k = \text{const}$. This case was studied in detail by Petrov [29, 30]. He called this kind of spaces *Einstein spaces*. According to Gliner [32, 33], who studied the algebraic structure of the energy-momentum tensor, a special type of the tensor can be outlined: $T_{\alpha\beta} = \mu g_{\alpha\beta}$, where $\mu = \text{const}$ is the density of matter. It characterizes a vacuum-like state of matter. He called this state of matter the μ -vacuum. Gliner had also showed that a space filled with the μ -vacuum is an Einstein space.

We have disclosed the physical sense of the energy-momentum tensor of the vacuum $\check{T}_{\alpha\beta} = \lambda g_{\alpha\beta}$ and that of the μ -vacuum $T_{\alpha\beta} = \mu g_{\alpha\beta}$. We have also deduced the formulae for the physically observable properties of the vacuum and the μ -vacuum, such as their density, momentum density and stress-tensor. We have also showed that the vacuum is a homogeneous, non-viscous, non-emitting and inflating (expanding at a positive density) medium. Proceeding from Petrov's studies and Gliner's studies and taking into account the deduced energy-momentum tensor of the vacuum (and, hence, the physical properties of the vacuum) we have suggested a "geometrical" classification of matter according to the energy-momentum tensor. We called it the *T-classification of matter*:

- The emptiness — the state, in which the energy-momentum tensor of matter is zero ($T_{\alpha\beta} = 0$), and non-Newtonian gravitation fields are absent ($\lambda = 0$);
- The physical vacuum (or, simply, the vacuum) — the state, in which substance is absent ($T_{\alpha\beta} = 0$), but there are non-Newtonian gravitational fields ($\lambda \neq 0$);
- The μ -vacuum $T_{\alpha\beta} = \mu g_{\alpha\beta}$, $\mu = \text{const}$ — a vacuum-like state of matter;
- Substance $T_{\alpha\beta} \neq 0$, $T_{\alpha\beta} \neq g_{\alpha\beta}$ — the state that includes ordinary matter and electromagnetic field.

Routine experience shows that: the density of matter in our Universe is positive. With a positive density of the vacuum, the cosmological term is $\lambda < 0$, i.e., non-Newtonian gravitational forces are the forces of attraction, and its inflation pressure is negative (the vacuum expands).

Considering spaces filled exclusively with the vacuum (and no substance inside), such as a de Sitter space, we have found that the collapse condition ($g_{00} = 0$) is realized therein in the form of a collapsed region that we called an *inflationary collapsar*, or an *inflanton*. Inside an

inflanton, there is $\lambda > 0$, i.e., the vacuum density is negative, the pressure is positive, and non-Newtonian gravitational forces are the forces of repulsion that cause the inflanton to exist in an equilibrium of the compressing pressure of the vacuum and the expanding forces of non-Newtonian gravitation.

6.1 Introducing the concept of the mirror world

As we mentioned in §5.1, the attempts to represent our Universe and the mirror Universe as two spaces with positive and negative curvature failed: even when considering a space of the de Sitter metric, which is one of the simplest space-time metrics, trajectories in a positive curvature de Sitter space are substantially different from those in its negative curvature twin (see Chapter VII in Synge's book [35]).

On the other hand, many researchers, beginning with Dirac, intuitively predicted that the mirror Universe (as the antipode of our Universe) should be sought not in a space with the opposite sign of the space curvature, but in a space where particles have masses and energies with the opposite sign. That is, since the masses of particles in our Universe are positive, then particles in the mirror Universe must be obviously negative.

Joseph Weber [28] wrote that neither Newton's law of gravitation nor the relativistic theory of gravitation ruled out the existence of negative masses; rather, our empirical experience says that they have never been observed. Both Newton's theory of gravitation and Einstein's General Theory of Relativity predicted negative mass behaviour quite different from what electrodynamics prescribes for negative charges. For two bodies, one with a positive mass and the other with a negative mass, but equal in magnitude, one would expect the positive mass to attract the negative mass and the negative mass to repulse the positive mass, so that one would chase the other! If the motion occurs along a line connecting the centres of two bodies, then such a system will move with a constant acceleration. This problem was studied by Bondi [42]. Assuming that the gravitational mass of the positron is negative (observations show that its inertial mass is positive) and using the methods of Quantum Electrodynamics, Schiff found that there is a difference between the

inertial mass of the positron and its gravitational mass. The difference turned out to be much larger than the error of the Eötvös experiment, which showed the equality of gravitational and inertial mass [43]. As a result, Schiff arrived at the conclusion that a negative gravitational mass in the positron cannot exist (see Chapter 1, §2 in Weber’s book [28]).

Besides, the “co-habitation” of positive and negative masses in the same space-time region would cause ongoing annihilation. The possible consequences of the “mixed” existence of particles, having both positive and negative masses, were also studied by Terletskii [44, 45].

Therefore, the idea of the mirror Universe as a world of negative masses and energies faced two obstacles:

- a) The *experimentum crucis* that would point directly at the exchange interactions between our Universe and the mirror Universe;
- b) The absence of a theory that would clearly explain the separation of the worlds with positive and negative masses as different space-time regions.

In this section, §6.1, we are going to tackle the second (theoretical) part of the problem.

Let us apply the term “mirror properties” to the space-time metric. To do this, write the square of the space-time interval in the chr.inv.-form

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \quad (6.1)$$

where

$$d\sigma^2 = h_{ik} dx^i dx^k, \quad (6.2)$$

$$d\tau = \left(1 - \frac{w}{c^2}\right) dt - \frac{1}{c^2} v_i dx^i = \left(1 - \frac{w + v_i u^i}{c^2}\right) dt. \quad (6.3)$$

From here we see that the physically observable elementary spatial interval (6.2) is the square function of the elementary spatial increments dx^i . The spatial coordinates x^i are all equal, so there is no principal difference between the translational motion to the right or to the left, up or down. Therefore, we will no longer consider mirror reflections with respect to spatial coordinates.

Time is a different thing. The physically observable time τ for an ordinary observer always flows from the past to the future, so that $d\tau > 0$. But there are two cases, in which the observable time stops. At first, this is possible in the ordinary space-time under the condition of collapse.

Secondly, this happens in the zero-space — the completely degenerate four-dimensional space-time. Therefore, the state of an observer, whose own observable time stops, can be regarded to be transitional, i.e., unavailable under ordinary conditions.

We will consider the problem of the mirror Universe for both $d\tau > 0$ and $d\tau = 0$. In the latter case, the analysis will be done separately for collapsed regions of the ordinary space-time and for those in the zero-space. We start the analysis from the ordinary case, where $d\tau > 0$. From the formula for the physically observable time (6.3), it is obvious that the condition $d\tau > 0$ is true if

$$w + v_i u^i < c^2. \quad (6.4)$$

If the space does not rotate ($v_i = 0$), then the above formula transforms into $w < c^2$, which corresponds to the space-time structure outside the state of collapse.

Then ds^2 (6.1) can be expanded as follows

$$\begin{aligned} ds^2 = & \left(1 - \frac{w}{c^2}\right)^2 c^2 dt^2 - 2\left(1 - \frac{w}{c^2}\right) v_i dx^i dt - \\ & - h_{ik} dx^i dx^k + \frac{1}{c^2} v_i v_k dx^i dx^k, \end{aligned} \quad (6.5)$$

on the other hand

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = c^2 d\tau^2 \left(1 - \frac{v^2}{c^2}\right), \quad v^2 = h_{ik} v^i v^k. \quad (6.6)$$

Let us divide both sides of the formula for ds^2 (6.5) by the next quantities, according to the kind of particle trajectory:

- 1) $c^2 d\tau^2 \left(1 - \frac{v^2}{c^2}\right)$, if the space-time interval is real $ds^2 > 0$;
- 2) $c^2 d\tau^2$, if the space-time interval is equal to zero $ds^2 = 0$;
- 3) $-c^2 d\tau^2 \left(\frac{v^2}{c^2} - 1\right)$, if the interval is imaginary $ds^2 < 0$.

As a result, in all three cases we obtain the same square equation with respect to the function of the “true coordinate time” t dependent on the physically observable time τ registered by the observer. The square equation has the form

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{2v_i v^i}{c^2 \left(1 - \frac{w}{c^2}\right)} \frac{dt}{d\tau} + \frac{1}{\left(1 - \frac{w}{c^2}\right)^2} \left(\frac{1}{c^4} v_i v_k v^i v^k - 1\right) = 0, \quad (6.7)$$

which has two solutions

$$\left(\frac{dt}{d\tau}\right)_1 = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{1}{c^2} v_i v^i + 1\right), \quad (6.8)$$

$$\left(\frac{dt}{d\tau}\right)_2 = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{1}{c^2} v_i v^i - 1\right). \quad (6.9)$$

Integrating t with respect to τ , we obtain

$$t = \frac{1}{c^2} \int \frac{v_i dx^i}{1 - \frac{w}{c^2}} \pm \int \frac{d\tau}{1 - \frac{w}{c^2}} + const, \quad (6.10)$$

which can be easily integrated, if the space does not rotate and the gravitational potential is $w = 0$. Then the integral is $t = \pm \tau + const$. The right choice of the initial conditions can make the integration constant equal to zero. In this case, we obtain the solution

$$t = \pm \tau, \quad \tau > 0, \quad (6.11)$$

which graphically represents two beams, which are the mirror reflections of each other with respect to $\tau > 0$. We can say that here, the observer's own time serves as a mirror membrane, the mirror surface of which separates two worlds: one with the direct flow of the coordinate time* from the past to the future $t = \tau$, and the other, mirror world, with the reverse flow of the coordinate from the future to the past $t = -\tau$.

Noteworthy, the world with the reverse flow of time is not like a videotape being rewound. Both worlds are quite equal, but for an ordinary observer the mentioned time coordinate in the mirror Universe is negative. The mirror surface of the membrane in this case only reflects the time flow, but does not affect it.

Now we assume that the space does not rotate $v_i = 0$, but the gravitational potential is not zero $w \neq 0$. Then we have

$$t = \pm \int \frac{d\tau}{1 - \frac{w}{c^2}} + const. \quad (6.12)$$

*The physically observable time τ registered by any observer everywhere flows from the past to the future, so the condition $d\tau > 0$ is true in the reference frame associated with any observer.

If the gravitational potential is weak ($w \ll c^2$), then the obtained integral (6.12) becomes

$$t = \pm \left(\tau + \frac{1}{c^2} \int w d\tau \right) = \pm (\tau + \Delta t), \quad (6.13)$$

where Δt is a correction that takes into account the potential field w . The quantity w can denote any scalar potential field — either a field of Newtonian potential or a field of non-Newtonian gravitation.

If the gravitational field is strong, then the integral has the form (6.12). The stronger the field potential w , the faster the coordinate time flow (6.12). In the limiting case, where $w = c^2$, we have $t \rightarrow \infty$. On the other hand, at $w = c^2$ collapse occurs $d\tau = 0$. We will consider this case below, but for now we are still assuming that $w < c^2$.

Let us consider the coordinate time in a Schwarzschild space and a de Sitter space. If the potential w describes a Newtonian gravitational field (a Schwarzschild space), then

$$t = \pm \int \frac{d\tau}{1 - \frac{GM}{c^2 r}} = \pm \int \frac{d\tau}{1 - \frac{r_g}{r}}, \quad (6.14)$$

i.e., the closer we approach the gravitational radius associated with the mass M , the greater the difference between the coordinate time and the physically observable time registered by the observer. If the potential w describes a non-Newtonian gravitational field (a de Sitter space), then

$$t = \pm \int \frac{d\tau}{\sqrt{1 - \frac{\lambda r^2}{3}}} = \pm \int \frac{d\tau}{\sqrt{1 - \frac{r^2}{r_{\text{inf}}^2}}}, \quad (6.15)$$

which means that the closer the measured photometric distance r to the inflationary radius in the space, the faster the coordinate time flow. In the limiting case, where $r \rightarrow r_{\text{inf}}$, we have $t \rightarrow \infty$.

Therefore, in a space that does not rotate, but is filled with a gravitational field, the coordinate time flow is faster when the field potential is stronger. This is true both in a Newtonian gravitational field and in a non-Newtonian gravitational field.

Let us now consider a general case, where the space rotates and is filled with a gravitational field. Then the coordinate time t has the form (6.10), which includes:

- a) The “rotational” time determined by the term $v_i dx^i$, which has the dimension of rotation momentum divided by unit mass;
- b) The ordinary coordinate time, linked to the physically observable time registered by the observer.

From the integral for t (6.10), we see that the “rotational” time, produced by the space rotation, exists independently from the observer because it does not depend on τ . For an observer, who rests on the surface of the Earth (anywhere apart from the poles), the “rotational” time can be interpreted as the time determined by the rotation of the planet around its axis. The “rotational” time always exists irrespective of whether the observer records it at this particular location or not. The regular coordinate time is linked to our presence (it depends on the registered time τ) and to the gravitational field at the point of observation (in particular, to the field of the Newtonian potential).

Noteworthy, with $v_i \neq 0$, the time coordinate t at the initial moment of observation (when the physically observable time registered by the observer is $\tau_0 = 0$) is not zero.

Representing the integral for t (6.10) as

$$t = \int \frac{\frac{1}{c^2} v_i dx^i \pm d\tau}{1 - \frac{w}{c^2}}, \quad (6.16)$$

we obtain that the quantity under the integral sign is:

- 1) Positive, if $\frac{1}{c^2} v_i dx^i > \mp d\tau$;
- 2) Zero, if $\frac{1}{c^2} v_i dx^i = \pm d\tau$;
- 3) Negative, if $\frac{1}{c^2} v_i dx^i < \mp d\tau$.

Hence, the coordinate time t of an object that we observe stops, if the scalar product of the linear velocity with which the space rotates and the observable velocity of the object is $v_i v^i = \pm c^2$. This happens, if the numerical values of both velocities equal to the velocity of light, and they are either co-directed or oppositely directed.

A region of the space-time, where the condition $v_i v^i = \pm c^2$ is true and, hence, the coordinate time stops for a real observer, is the *mirror membrane* separating two regions of the space-time — the region with the direct flow of the coordinate time and the region with the reverse coordinate time flow. Obviously, no one observer under ordinary physical conditions in an Earth-bound laboratory can accompany such a space.

We will refer to the space-time region, where coordinate time takes negative numerical values as the *mirror space*.

Let us analyse the properties of particles that inhabit the mirror space, in comparison with the properties of particles located in the ordinary space, where the time coordinate is always positive.

The four-dimensional momentum vector of a mass-bearing particle having a non-zero rest-mass m_0 is

$$P^\alpha = m_0 \frac{dx^\alpha}{ds}, \quad (6.17)$$

the chr.inv.-projections of which are

$$\frac{P_0}{\sqrt{g_{00}}} = m \frac{dt}{d\tau} = \pm m, \quad P^i = \frac{m}{c} v^i, \quad (6.18)$$

where “plus” stands for the direct coordinate time flow, and “minus” stands for the reverse coordinate time flow with respect to the physically observable time registered by the observer. The square of P^α is

$$P_\alpha P^\alpha = g_{\alpha\beta} P^\alpha P^\beta = m_0^2, \quad (6.19)$$

while its length is

$$\left| \sqrt{P_\alpha P^\alpha} \right| = m_0. \quad (6.20)$$

We see that any particle having a non-zero rest-mass, being a four-dimensional object, is projected onto the observer’s time line as a dipole consisting of a positive mass $+m$ and a negative mass $-m$. But when the vector P^α is projected onto the spatial section, we obtain a single projection — the particle’s three-dimensional observable momentum $p^i = m v^i$. In other words, each observable particle with a positive relativistic mass has its own *mirror twin* with the same negative mass: the particle and its mirror twin are only different by the sign of the mass, while the three-dimensional momenta of both particles are positive.

Similarly, for the four-dimensional wave vector

$$K^\alpha = \frac{\omega}{c} \frac{dx^\alpha}{d\sigma} = k \frac{dx^\alpha}{d\sigma}, \quad (6.21)$$

which describes a massless particle, the chr.inv.-projections are

$$\frac{K_0}{\sqrt{g_{00}}} = \pm k, \quad K^i = \frac{k}{c} c^i. \quad (6.22)$$

This means that any massless particle, as a four-dimensional object, exists also in the two states: in our world (the direct flow of time) it is a massless particle with a positive frequency, while in the mirror world (the reverse flow of time) it is a massless particle with the same negative frequency.

Let us define the *material Universe* as the four-dimensional space (space-time) filled with a substance and fields. Then, since any particle is a four-dimensional dipole object, we can consider the material Universe as a four-dimensional dipole object that exists in the two states: *our Universe*, inhabited by particles with positive masses and frequencies, and as its mirror twin — the *mirror Universe*, where masses and frequencies of particles are negative, while the three-dimensional observable momentum remains positive.

On the other hand, our Universe and the mirror Universe are merely two different regions of the same four-dimensional space-time.

For instance, when analysing the properties of the Universe as a whole, we neglect Newtonian fields, produced by occasional islands of substance, so we assume the basic four-dimensional space of our Universe to be a de Sitter space with a negative four-dimensional curvature, while its three-dimensional observable curvature is positive (see §5.5). Hence, we can assume that our Universe as a whole is a region of the de Sitter space with the negative four-dimensional curvature, where the time coordinate is positive as well as the masses and frequencies of particles located in the region. Besides, vice versa, the mirror Universe is a region of the same de Sitter space, where the time coordinate is negative as well as the masses and frequencies of particles located in it.

The space-time membrane that separates our Universe from the mirror Universe in the space-time, does not allow them to “mix”, thereby preventing their total annihilation. This membrane will be discussed below, in the end.

Let us consider the dipole structure of the Universe under the condition $d\tau = 0$, i.e., the collapsed regions of the ordinary space-time as well as the completely degenerate space-time region (zero-space).

First, as we have shown, the condition $d\tau = 0$ is true in collapsed regions of the ordinary (non-degenerate) space-time, provided that the space is holonomic (it does not rotate). In this case,

$$d\tau = \left(1 - \frac{w}{c^2}\right) dt = 0. \quad (6.23)$$

This condition is true for collapse of any kind, i.e., for the gravitational fields of any kind, including the fields of a non-Newtonian gravitational potential. At $d\tau = 0$ (6.23), the four-dimensional metric is

$$ds^2 = -d\sigma^2 = -h_{ik} dx^i dx^k = g_{ik} dx^i dx^k = g_{ik} u^i u^k dt^2, \quad (6.24)$$

hence, in this case the absolute value of the interval ds is

$$|ds| = i d\sigma = i \sqrt{h_{ik} u^i u^k} dt = i u dt, \quad u^2 = h_{ik} u^i u^k, \quad (6.25)$$

therefore, the four-dimensional momentum vector of a particle that is located on the surface of a collapsar is

$$P^\alpha = m_0 \frac{dx^\alpha}{d\sigma}, \quad d\sigma = u dt, \quad (6.26)$$

and its square is

$$P_\alpha P^\alpha = g_{\alpha\beta} P^\alpha P^\beta = -m_0^2, \quad (6.27)$$

hence, the length of the vector P^α (6.26) is imaginary

$$\left| \sqrt{P_\alpha P^\alpha} \right| = i m_0. \quad (6.28)$$

The latter, in particular, means that the surface of a collapsar is inhabited by particles with imaginary rest-masses. But, at the same time, this does not mean that superluminal particles (tachyons) should be found there. This is because the rest-masses of tachyons are real (in that time they are ordinary particles), but their relativistic masses become imaginary only after the particles accelerate to superluminal velocities thus become tachyons.

On the surface of any collapsar the term “observable velocity” is void, because the physically observable time registered by an ordinary observer stops on the surface ($d\tau = 0$).

The components of the four-dimensional momentum vector of a particle on the surface of a collapsar (6.26), can be formally written as

$$P^0 = \frac{m_0 c}{u}, \quad P^i = \frac{m_0}{u} u^i. \quad (6.29)$$

We cannot observe such a particle, because the observable time stops on the surface of a collapsar. On the other hand, the velocity $u^i = \frac{dx^i}{dt}$, found in this formula, is a coordinate velocity; it does not depend on the observer’s measured time that stops there. Hence, we can

interpret the spatial vector $P^i = \frac{m_0}{u} u^i$ as the coordinate momentum of the particle, and the quantity $\frac{m_0}{u} c^3$ can be interpreted as the particle's energy. The energy has only one sign here, hence, the surface of any collapsar as a four-dimensional region is not a dipole four-dimensional object, presented in the form of two mirror twins. This means that the surface of any collapsar, irrespective of its Newtonian or non-Newtonian nature, exists in a single state.

On the other hand, the surface of a collapsar ($g_{00} = 0$) can be regarded as a membrane, which separates the four-dimensional space-time regions outside the state of collapse and under the collapse state. Outside the state of collapse, we have $g_{00} > 0$, so the observer's measured time τ is real. Under the collapse state, we have $g_{00} < 0$, hence τ is imaginary. When an ordinary observer, when entering a collapsar, crosses its surface, his measured time is subjected to a 90° "turn", changing its rôle to that of the measured spatial coordinates.

The term "light-like particle" is nonsense on the surface of a collapsar. This is because $d\sigma = c d\tau$ for light-like particles by definition, hence, on the surface of a collapsar ($d\tau = 0$) for such particles we have

$$u = \sqrt{h_{ik} u^i u^k} = \sqrt{\frac{h_{ik} dx^i dx^k}{dt^2}} = \frac{d\sigma}{dt} = \frac{c d\tau}{dt} = 0. \quad (6.30)$$

Secondly, the physically observable time registered by the observer stops ($d\tau = 0$) in the completely degenerate space-time (zero-space), since there, by definition, $d\tau = 0$ and $d\sigma = 0$. As we have shown in the previous book [19], the above degeneration conditions can be written in the following expanded form as follows

$$w + v_i u^i = c^2, \quad g_{ik} u^i u^k = c^2 \left(1 - \frac{w}{c^2}\right)^2. \quad (6.31)$$

Particles inherent in the completely degenerate space-time (called *zero-particles*) have zero relativistic mass $m = 0$, but non-zero masses M (1.79) and non-zero corresponding momenta p^i

$$M = \frac{m}{1 - \frac{1}{c^2} (w + v_i u^i)}, \quad p^i = M u^i, \quad (6.32)$$

which are not sign-alternate quantities.

Therefore, mirror twins are only found in the ordinary matter — these are massless and mass-bearing particles, which are not in the state

of collapse. Collapsed objects in the ordinary space-time (gravitational collapsars and inflationary collapsars) do not have the property of mirror dipoles, they therefore are common objects for our Universe and the mirror Universe. Zero space objects, since they also do not have the property of mirror dipoles, lie outside the basic space-time due to the complete degeneration of the metric. It is possible to enter such “neutral zones”, which are on the surface of collapsed objects in the ordinary space-time and in the zero-space, from either our Universe (where the coordinate time is positive) or the mirror Universe (where the coordinate time is negative).

6.2 The conditions to move through the membrane, to the mirror world

Now we are going to discuss the question of the membrane that separates our Universe from the mirror Universe in the space-time, thereby preventing the total annihilation of all particles with negative and positive masses.

In our Universe, we have $dt > 0$, and $dt < 0$ is true in the mirror Universe. Hence, the membrane is a region of the space-time, where $dt = 0$ (the coordinate time stops). Mathematically, this means

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{1}{c^2} v_i v^i \pm 1 \right) = 0, \quad (6.33)$$

which can also be presented as the physical condition

$$dt = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{1}{c^2} v_i dx^i \pm d\tau \right) = 0. \quad (6.34)$$

The latter notation is more versatile, because it is applicable not only to the space-time of General Relativity, but also to a generalized space-time, where the total degeneration of the metric is possible.

Inside the membrane, the conditions $t = \text{const}$ and $dt = 0$, in accordance with (6.34), are determined by the formula

$$v_i dx^i \pm c^2 d\tau = 0, \quad (6.35)$$

which can be also written in the form

$$v_i v^i = \pm c^2. \quad (6.36)$$

The above condition can be represented as follows

$$v_i v^i = |v_i| |v^i| \cos(v_i; v^i) = \pm c^2. \quad (6.37)$$

This condition is satisfied, if the numerical values of the velocities v_i and v^i are equal to the velocity of light and are either co-directed (“plus”) or oppositely directed (“minus”).

Therefore, from a physical point of view, the membrane that we are talking about is a space that is in motion with the velocity of light and, at the same time, rotates with the velocity of light. So, the membrane is a world of light-like spiral trajectories. It is possible that such a space is inhabited by particles having the helicity property (e.g., massless light-like particles — photons).

Having $dt = 0$ substituted into the formula for ds^2 , we obtain the space metric inside the membrane

$$ds^2 = g_{ik} dx^i dx^k, \quad (6.38)$$

which is the same as the metric on the surface of a collapsar. The above metric is a particular case of a space-time metric with the signature (+---). Therefore, ds^2 inside the membrane is always negative.

This means that, in a region of the four-dimensional space-time, which serves the membrane between our Universe and the mirror Universe, the four-dimensional interval is space-like.

The difference from the space-like metric on the surface of a collapsar (6.24) is that the space on the surface of a collapsar does not rotate, so there is $g_{ik} = -h_{ik}$, while in the internal space of the membrane we have $g_{ik} = -h_{ik} + \frac{1}{c^2} v_i v_k$ (1.18). Or, in other words, inside the membrane we have the metric

$$ds^2 = g_{ik} dx^i dx^k = -h_{ik} dx^i dx^k + \frac{1}{c^2} v_i v_k dx^i dx^k, \quad (6.39)$$

which is space-like due to the space rotation (there is $v_i dx^i = \pm c^2 d\tau$).

As a result, an ordinary mass-bearing particle (irrespective of the sign of its mass) in its “natural” form cannot pass through the membrane: this region of the space-time is inhabited by light-like particles travelling along light-like spirals.

On the other hand, the limiting case of particles with $m > 0$ or $m < 0$ are particles with zero relativistic mass $m = 0$. From a geometric point

of view, the region, in which such particles exist, is tangential to the regions inhabited by particles with either $m > 0$ or $m < 0$. This means that zero-mass particles can have exchange interactions with either our-world particles $m > 0$ or mirror-world particles $m < 0$.

Particles with zero relativistic mass, by definition, exist in a region of the space-time, where $ds^2 = 0$ and $c^2 d\tau^2 = d\sigma^2 = 0$. Equating ds^2 inside the membrane (6.38) to zero, we obtain

$$ds^2 = g_{ik} dx^i dx^k = 0, \quad (6.40)$$

so this condition satisfies in the two cases:

- 1) All numerical values dx^i are zeroes, so $dx^i = 0$;
- 2) The three-dimensional metric is degenerate $\tilde{g} = \det \|g_{ik}\| = 0$.

The first case takes place in the ordinary space-time under the limiting condition on the surface of a collapsar, where all its surface shrinks into a point ($dx^i = 0$). In this case, the metric on the collapsar's surface, according to $ds^2 = -h_{ik} dx^i dx^k = g_{ik} dx^i dx^k$ (6.24), becomes zero.

The second case takes place on the surface of a collapsar located in the zero-space: since the condition $g_{ik} dx^i dx^k = \left(1 - \frac{w}{c^2}\right)^2 c^2 dt^2$ is true there, then at $w = c^2$ we have always $g_{ik} dx^i dx^k = 0$.

The first case is asymptotic, so it never takes place in reality. Therefore, we can expect that “middlemen” in the exchange between our Universe and the mirror Universe are those particles with zero relativistic mass, which inhabit the surface of the collapsars located in the completely degenerate space-time. In other words, the mentioned “middlemen” are those zero-particles that are inherent in the surface of zero-space collapsars.

6.3 Conclusions

So we have shown that our Universe is the observable region of the space-time, where time coordinate is positive, thus all particles have positive masses and energies. The mirror Universe is the region of the space-time, in which, from the viewpoint of an ordinary observer, time coordinate is negative and all particles have negative masses and energies. From the viewpoint of an our-world observer, the mirror Universe is the world with the reverse flow of time, where particles travel from the future to the past with respect to us.

These two worlds, our Universe and the mirror Universe, are separated by the membrane — the region of the space-time, inhabited by light-like particles that travel along light-like spirals. On the scale of elementary particles, such a space can be home to light-like particles that have helicity (e.g., photons). The mentioned membrane prevents mixing positive-mass and negative-mass particles, so it prevents their total annihilation. The “middlemen” in the exchange interaction between our world and the mirror world can be particles with zero relativistic masses (zero-particles) under the physical conditions on the surface of the collapsars located in the completely degenerate space-time (zero-space collapsars).

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About the authors

Larissa Borissova (b. 1944, Moscow, Russia) is a graduate of the Sternberg Astronomical Institute (Faculty of Physics, Moscow University). Since 1964 she was trained by Dr. Abraham Zelmanov (1913–1987), a famous cosmologist and researcher in General Relativity. Since 1968 she worked with Prof. Kyril Stanyukovich (1916–1989), a prominent scientist in gas dynamics and General Relativity. In 1975, Larissa Borissova got a “Candidate of Science” degree (Soviet PhD) in gravitational waves. She has published about 50 scientific papers and 3 books on General Relativity. In 2005, Larissa Borissova became a co-founder and Editor of *Progress in Physics*, and in 2008, an Editor of *The Abraham Zelmanov Journal*. She has since continued her scientific studies as an independent researcher.

Dmitri Rabounski (b. 1965, Moscow, Russia). Since 1983 he was trained by Prof. Kyril Stanyukovich (1916–1989), a prominent scientist in gas dynamics and General Relativity. He was also trained with Dr. Abraham Zelmanov (1913–1987), a famous cosmologist and researcher in General Relativity. He was also trained by Dr. Vitaly Bronshten (1918–2004), the well-known expert in the physics of destruction of bodies in the atmosphere. Dmitri Rabounski has published about 50 scientific papers and 3 books on General Relativity. In 2005, he started a new journal on physics, *Progress in Physics*, where he is the Editor-in-Chief. In 2008, he started a new journal on General Relativity, *The Abraham Zelmanov Journal*, while continuing his scientific studies as an independent researcher.

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Book format: 60 × 90 1/16
Typeface family: Tempora TLF
Default font size: 11 pt
Typos corrected on: 20.12.2023

Fields, Vacuum and the Mirror Universe

Fields and particles in
the space-time of General Relativity

by **L. Borissova and D. Rabounski**

The 3rd revised edition

New Scientific Frontiers
London, 2023

