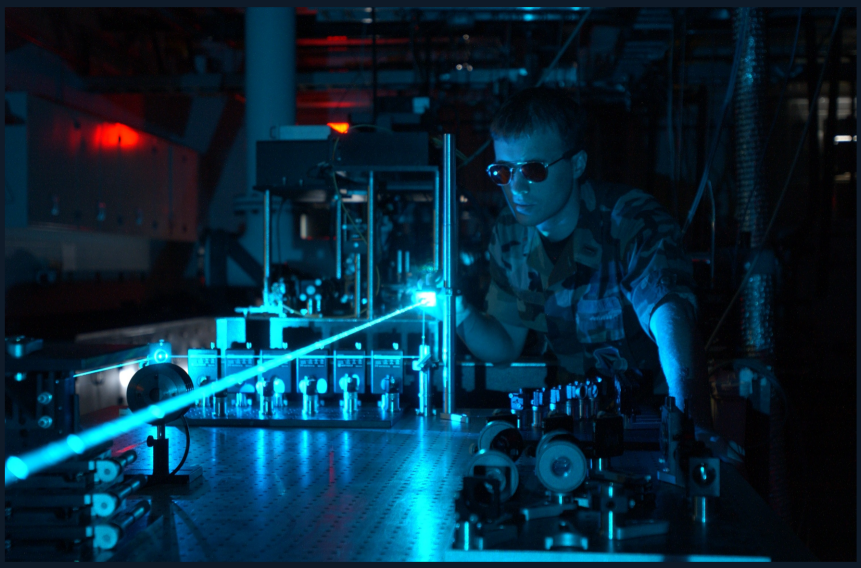
PARTICLES HERE
AND
BEYOND THE MIRROR

The 4th revised edition

D. Rabounski and L. Borissova



Particles Here and Beyond the Mirror

Three kinds of particles inherent
in the space-time of General Relativity

by Dmitri Rabounski and Larissa Borissova

The 4th revised edition

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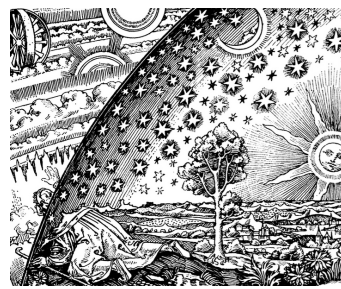

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Summary: — This is a research on all kinds of particles, which could be conceivable in the space-time of General Relativity. In addition to mass-bearing particles and light-like particles, zero-particles are predicted: such particles can exist in a completely degenerate space-time region (zero-space). Zero-particles seem as standing light waves, which travel in an instant (the non-quantum teleportation of photons); they might be observed in a further development of the "stopped light experiment" that was first conducted in 2000 by Lene Hau. The theoretical existence of two separate regions in the space-time is also shown, where the observable time flows to the future and to the past (our world and the mirror world). These regions are separated by a space-time membrane wherein the observable time stops. A few other certain problems are considered. It is shown, using Killing's equations, that geodesic motion of particles is a result of stationary geodesic rotation of the space which hosts them. As for the theory of gravitational wave detectors, it is shown that both free-mass detectors and solid-body detectors may register a gravitational wave only if the detector extremities oscillate relative to each other.

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Preface

The background behind this book is as follows. In 1991 the authors started a research to find out what kinds of particles can theoretically inhabit the space-time of General Relativity. To this end, they used the mathematical apparatus of chronometric invariants (physically observable quantities) introduced in the 1940s by Abraham L. Zelmanov.

The study was completed to reveal that besides mass-bearing and massless (light-like) particles, particles of the third kind may also exist. Their trajectories lie beyond the regular region in space-time. For an ordinary observer, the trajectories have zero four-dimensional length and zero three-dimensional observable length. Besides, along the trajectories the interval of observable time is also zero. Mathematically, this means that such particles inhabit a space-time with a completely degenerate metric (completely degenerate space-time). We have therefore called such a space the "zero-space" and such particles — "zero-particles".

For an ordinary observer, the motion of particles in the zero-space is instantaneous. Therefore, zero-particles do realize the long-range action. Through possible interaction with our-world's mass-bearing or massless particles, zero-particles can instantly transmit signals to any point in our three-dimensional space (a phenomenon that the authors call the "non-quantum teleportation").

Considering zero-particles in the frames of the wave-particle duality, the authors have obtained that for an ordinary observer they are standing waves and the whole zero-space is filled with a system of standing light-like waves (zero-particles), i.e. standing light-holograms. This result corresponds to the well-known "stopped light experiment" that was first conducted in 2000 by Lene Hau (in Harvard, USA).

Using the mathematical method of physically observable quantities, the authors have also showed that two separate regions in inhomogeneous space-time exist, where the physically observable (proper) time

flows to the future and to the past, while such a duality is not found in a homogeneous space-time. These regions are referred to as our world and the mirror world respectively; they are separated by a space-time membrane wherein observable time stops.

A few other certain problems are considered. It is shown, through Killing's equations, that geodesic motion of particles is a result of stationary geodesic rotation of the space which hosts them.

This book includes a chapter on the theory of gravitational wave detectors: it is shown that both free-mass detectors and solid-body detectors can register a gravitational wave only if the detector extremities oscillate relative to each other.

In the 3rd edition, the authors have added a list of chronometrically invariant derivatives, as well as references to their recent publications. We have also fixed typographical errors found in the previous editions.

Calais, January 12, 2023 Patrick Marquet

Editor's Foreword

> "Only through the pure contemplation … which becomes absorbed entirely in the object, are the Ideas comprehended; and the nature of genius consists precisely in the preeminent ability for such contemplation. …This demands a complete forgetting of our own person."
>
> Arthur Schopenhauer

> "Genius does what it must, and Talent does what it can."
>
> Owen Meredith

Einstein's theory of space-time and gravitation, the General Theory of Relativity, has nearly reached its centennial relative adulthood. While this theory has revolutionized our basic understanding of the structure of space-time and its respective dynamical interaction with energy fields and matter in the rather rhapsodic-aesthetic light of differential geometry, after the savory dominance of the classical Newtonian-mechanical and Maxwellian-electromagnetic worldviews, it has become incumbent upon the shoulders of the most capable — and most sincere and passionate — of scientists to shed light on a few still largely mysterious, fundamental features associated with the nature of the theory. Without doubt, these scientists number only a few today, as those capable of filling a pure niche with real object-illumination in the dark, and not a mere spark, and absolutely not mere brilliance. They are the infinitely self-reserved ones who, at once, see the foundational and material aspect — including the philosophical, theoretical and experimental aspects — of the theory beyond everyone else.

The authors of the present book — like their preeminent teacher before them who spear-headed the Soviet general relativistic and cosmological school, Abraham Leonidovich Zelmanov — certainly are such capable, natural, reflexive fillers in the loom of Einstein's theory. With respect to one's possession of fundamental theoretical and experimental

strength and intellectual clarity, and of immense creativity, authenticity, and integrity, both physicists form a vigorous, perpetual dimension of the physico-mathematical school of Zelmanov himself.

Among the seemingly many truly elusive and more moderate problems faced in General Relativity, gravitation, and cosmology, one must further discern the truly most important ones by way of proper scientific-epistemological qualification as to whether or not the problems (as they are) are truly fundamental — in contrast to the rawness of a new plethora of merely fanciful (yet lacking in true in-depth quality) post-modern, solipsistic-toy-models of the universe available (and easily so) nowadays. Of course, such a distinctive weight is the emphasis while keeping in mind possible ways of generalizing Einstein's theory towards a similarly qualified unified field theory — and thus complete geometrization — of not just gravitation, but also of other physical fields, including the constituents of matter.

Notwithstanding the fact that various experimental tests have been carried out to verify the theory within a simple, limited, tangible range of largely earth-bound human experiences and suavities, one crucial reason for the rather lengthy "single theoretical incubation period" of General Relativity in its original form since its very inception has been precisely the profound degree of depth of the philosophical aspect — and further abstract edification — of the theory as related to its existence as a scientific theory of physical reality and as an impetus for philosophical considerations regarding our place in the universe. However, referring back to the aforementioned fundamental problems, there is a great qualitative lacuna between past-time researchers — in a line emanating from Einstein himself and culminating all the way with, among others, Abraham Leonidovich Zelmanov — and many of today's own as regards the fundamental epistemological standard and cognizance, including the critical dimension of human experiment, in the vein of identifying the important problems truly relevant to the theory and the cosmos as a whole.

At least, four of these truly fundamental problems "native to the landscape of General Relativity" are presented and solved here. These are profoundly encompassed by the authors' commanding investigation into the kinds of particles theoretically conceivable in the generally inhomogeneous, anisotropic, non-simply connected space-time structure of General Relativity (including various kinds of degenerate pseudo-

Riemannian manifolds and zero-particles), their respective consideration of the dynamics of particles therein (covering both geodesic and non-geodesic motion), their in-depth study of gravitational waves followed by a substantial modification of the theory of gravitational wave detectors and their formidable creation of a general relativistic theory of frozen light (the first such account in immediate connexion to the experiment of stopped or retarded light, which is peculiar to this book).

It must be emphasized that while swimming extensively through the sky and ocean of these cosmical problems, one must respect the profundity and power of the mathematical apparatus left behind by Zelmanov himself at the apex and zenith of his profound intellectual presence, i.e., the theory of chronometric invariants. This, being more than just a tool for regularly projecting space-time quantities (i.e., mathematical representations — tensor fields) onto the observer's coordinate lines, is not a trivial matter at all: the full creation of the theory of chronometric invariants enables us — a few who truly understand it — to cast General Relativity in an elegant kinemetric semi-three-dimensional (hence "chronometric") form wherein the fundamental observer, seen as a co-moving space-time "patch", generally moves, deforms, and rotates along with the entire universe while occupying an infinitesimal dynamical volume thereof. The fact that such an observer is integral to the theory, as in quantum mechanics, renders him beyond just an immutable abstract kinematic point-like addition to the actual space-time substratum. This forms the basis of the chronometrically invariant formalism of General Relativity.

I am hereby proud and privileged to have edited this insightful masterpiece by Rabounski and Borissova, who also wrote a *magnum opus* on General Relativity, *Fields, Vacuum, and the Mirror Universe*.

Yogyakarta, April 11, 2012 Indranu Suhendro

Acknowledgements

First, we would like to express our sincere gratitude to our mentors of General Relativity in Moscow. These are Abraham L. Zelmanov (1913–1987) and Kyril P. Stanyukovich (1916–1989). Many years of friendly acquaintance, individual training and countless hours of scientific conversations with them had sown the seeds of fundamental ideas, which by now have grown in our minds and are reflected in these pages. We are also very grateful to Kyril I. Dombrowski (1913–1997), a mathematician whose friendly conversations and discussions with us had a great influence on our scientific outlooks.

All preparation of the text of the first English edition of this book in 2001 was undertaken by Gregory V. Semionov (Moscow, Russia), to whom we are very grateful.

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Special thanks go to Anatole V. Belyakov (Tverin Kariela, Russia), who has translated all our books from English into Russian.

This is our first book written in 1997. Then we expanded the book with new results. In the 4th English edition, we have completely revised the entire text of the book and made many necessary corrections.

Puschino, March 7, 2023 D. Rabounski and L. Borissova

Chapter 1 Three Kinds of Particles Inherent in the Space-Time of General Relativity

1.1 Problem statement

The main goal of the theory of motion of particles is to define the three-dimensional (spatial) coordinates of a particle at any given moment of time. In order to do this, one should be aware of three things. First, one should know in what type of space-time the events take place. That is, one should know the geometric structure of space and time, just as one should know the conditions of a road to be able to drive on it. Second, one should know the physical properties of the travelling particle. Third, knowledge of the equations of motion of particles of a certain kind is necessary.

The first problem actually leads to the choice of a space from the spaces known in mathematics, in order to represent just the right geometry for space and time which best fits the geometric representation of the observed world.

The view of the world as a space-time continuum takes its origin from Hermann Minkowski's historical speech *Raum und Zeit*, which he delivered on September 21, 1908, in Köln, Germany, at the 80th Assembly of the Society of German Natural Scientists and Physicians (Die Gesellschaft Deutscher Naturforscher und Ärzte). There he introduced the term "space-time" into physics and gave a geometric interpretation of the principle of invariance of the speed of light and Lorentz' transformations.

A few years later, in 1912, Marcel Grossmann, in his private conversation with Albert Einstein, a close friend of him, proposed Riemannian geometry as the geometry of the observed world. Later Einstein came

to the idea that became the corner-stone of his General Theory of Relativity: the "geometric concept of the world", according to which the geometric structure of space-time determines all properties of the Universe. Thus, Einstein's General Theory of Relativity, completed by him in 1915, is the first geometric theory of space-time and particle motion since the dawn of modern science.

Consideration of the problem in detail had led Einstein to the conclusion that the only way to represent space-time in the way that fits the modern experimental data is given by a *four-dimensional pseudo-Riemannian space* with one time axis and three spatial axes, i.e., with the sign-alternating Minkowski signature $(+---)$ or $(-+++)$. This is a particular case of the family of Riemannian spaces, i.e., spaces where geometry is Riemannian (in such spaces, the squared distance ds^2 between any two infinitely close points is determined by the invariant metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = inv$). In general, in a Riemannian space, coordinate axes can be of any kind. A four-dimensional pseudo-Riemannian space differs in that there is a principal difference between the three-dimensional space, perceived as space, and the fourth axis — time.

From a mathematical point of view, the above means the following: the three spatial axes are real, while the time axis is imaginary (or vice versa), and the choice of such conditions is arbitrary.

A particular case of such spaces is a flat, homogeneous and isotropic four-dimensional pseudo-Riemannian space referred to as *Minkowski's space*. This is the basic space-time of Special Relativity — a particular abstract case, free of gravitational fields, rotation, deformation, and curvature. In a general case, the real pseudo-Riemannian space is curved, inhomogeneous and anisotropic. This is the basic space-time of General Relativity, where we encounter gravitational fields, rotation, deformation, and curvature.

So, Einstein's General Theory of Relativity is based on the view of the world as a four-dimensional space-time, where any and all objects possess not a three-dimensional volume alone, but their "longitude" in time. In other words, any physical body, including ours, is a really existing four-dimensional instance with the shape of a cylinder elongated in time (event cylinder of the body), created by the interweaving of other event cylinders at the moment of its "birth" and decayed into many other ones at the moment of its "death". For example, for a human, the "time length" is the duration of his life from conception until death.

Shortly after Eddington gave the first experimental proof in 1919 that light rays are deviated by the Sun's gravitational field, many researchers faced strong obstacles in fitting together calculations made in the framework of Einstein's theory with existing results of observations and experiments. Successful experiments verifying the theory over the last 100 years have explicitly shown that the four-dimensional pseudo-Riemannian space is the basic space-time of the observed world (as far as the modern measurement precision allows us to judge). And if the inevitable evolution of human civilization and thought, as well as of experimental technology, indicates that the four-dimensional pseudo-Riemannian space can no longer explain the results of new experiments, then this will mean nothing other than the need to assume a more general space, which will include the four-dimensional pseudo-Riemannian space as a particular case.

In this book, the main focus will be on the motion of particles, based on the geometric concept of the world-structure: we will assume that the geometry of our space-time determines all properties of the observed world. Therefore, in contrast to other researchers, we are not going to constrain the geometry of the space-time by any limitations, but solve the problems of physics in the way that the space-time geometry requires their solution.

So, any particle in the four-dimensional space-time corresponds to its own world-line determining the three-dimensional (spatial) coordinates of the particle at any given moment of time. Therefore, our task to determine all possible kinds of particles evolves into considering all allowable types of trajectories of motion in the space-time.

Generally speaking, in terms of the equations of motion of a free particle in a metric space (space-time), one actually refers to the *equations of geodesic lines*, which are the four-dimensional equations of the world-trajectory of a free particle*

$$\frac{d^2x^\alpha}{d\rho^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\rho}\frac{dx^\nu}{d\rho} = 0\,, \tag{1.1}$$

where $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols of the 2nd kind, and ρ is a parameter of derivation along the geodesic line.

*Here and so forth space-time indices are Greek, for instance $\alpha, \beta, = 0, 1, 2, 3$, while spatial indices — Roman, for instance $i, k = 1, 2, 3$.

From a purely geometric point of view, the equations of geodesic lines are the equations of the Levi-Civita parallel transport [1] of the four-dimensional kinematic vector

$$Q^\alpha = \frac{dx^\alpha}{d\rho}\,, \tag{1.2}$$

namely — the following equations

$$\frac{\mathrm{D}Q^\alpha}{d\rho} = \frac{dQ^\alpha}{d\rho} + \Gamma^\alpha_{\mu\nu} Q^\mu \frac{dx^\nu}{d\rho} = 0\,, \tag{1.3}$$

where $\mathrm{D}Q^\alpha = dQ^\alpha + \Gamma^\alpha_{\mu\nu} Q^\mu dx^\nu$ is the absolute differential of the kinematic vector Q^α transported parallel to itself and tangential to the trajectory of transport (a geodesic line).

The Levi-Civita parallel transport means that the length of the transported vector remains unchanged

$$Q_\alpha Q^\alpha = g_{\alpha\beta}\, Q^\alpha Q^\beta = const \tag{1.4}$$

along the entire world-trajectory, where $g_{\alpha\beta}$ is the fundamental metric tensor of the space.

At this point, we note that the equations of geodesic lines are purely kinematic as they do not contain the physical properties of the travelling object. Therefore, to obtain the complete picture of motion of particles we must consider *dynamical equations of motion*, solving which will give us not only the trajectories of the travelling particles, but also their properties such as their energy, frequency, etc.

To do this, we must define: a) the possible types of trajectories in the four-dimensional space-time (pseudo-Riemannian space); b) the dynamical vector for each type of trajectory; c) the derivation parameter for each type of trajectory.

First we consider what types of trajectories are allowable in the four-dimensional pseudo-Riemannian space.

As mentioned above, along a geodesic line in a Riemannian space the condition $g_{\alpha\beta}\, Q^\alpha Q^\beta = const$ is true.

If along geodesic lines $g_{\alpha\beta}\, Q^\alpha Q^\beta \neq 0$, then such lines are called *non-isotropic geodesics*. Along non-isotropic geodesics the square of the four-dimensional interval is non-zero

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \neq 0\,, \tag{1.5}$$

and the interval ds takes the form

$$ds = \sqrt{g_{\alpha\beta}\, dx^\alpha dx^\beta} \quad \text{if} \quad ds^2 > 0\,, \tag{1.6}$$

$$ds = \sqrt{-g_{\alpha\beta}\, dx^\alpha dx^\beta} \quad \text{if} \quad ds^2 < 0\,. \tag{1.7}$$

If along geodesic lines $g_{\alpha\beta}\, Q^\alpha Q^\beta = 0$, then such lines are called *isotropic geodesics*. Along isotropic geodesics the square of the four-dimensional interval is zero

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = c^2 d\tau^2 - d\sigma^2 = 0\,, \tag{1.8}$$

while the physically observable three-dimensional (spatial) interval $d\sigma$ and the interval of the physically observable time interval $d\tau$ are non-zero (therefore isotropic trajectories are particularly degenerate).

This ends the list of types of trajectories in the four-dimensional pseudo-Riemannian space (basic space-time of General Relativity), which are known to scientists until now.

Further, we will show that trajectories of the third type are theoretically allowable in the space, along which the four-dimensional interval, the physically observable time interval and the observable three-dimensional interval are zero. Such trajectories lie beyond the regular four-dimensional pseudo-Riemannian space, in a completely degenerate space-time region. We call them "completely degenerate" because, from the viewpoint of an ordinary observer, all distances and intervals of time in such a region degenerate into zero. Nevertheless, transition into such a degenerate space-time region from the regular space-time is quite possible (upon reaching certain physical conditions). And, perhaps, for an observer, whose home is such a completely degenerate space-time region, such quantities as "time" and "space" are measured in units different from ours.

Therefore, we will consider the regular four-dimensional pseudo-Riemannian space (space-time) together with the completely degenerate space-time as an *extended space-time*, in which both non-degenerate (isotropic and non-isotropic) and degenerate trajectories exist.

Therefore, in such an extended four-dimensional space-time, which is an actual "extension" of the basic space-time of General Relativity, which includes a completely degenerate space-time region, three types of trajectories are allowed:

1) Non-isotropic trajectories (pseudo-Riemannian space). Motion on such trajectories is possible with subluminal and superluminal velocities;
2) Isotropic trajectories (pseudo-Riemannian space). On such trajectories, motion is possible with the velocity of light only;
3) Completely degenerate trajectories (zero-trajectories), which lie in the completely degenerate space-time.

According to these types of trajectories, three kinds of particles can be distinguished, which can exist in the four-dimensional space-time:

1) Mass-bearing particles (their rest-masses are $m_0 \neq 0$). Such particles travel along non-isotropic trajectories ($ds \neq 0$) with subluminal velocities (real mass-bearing particles) and with superluminal velocities (imaginary mass-bearing particles — tachyons);
2) Massless particles (their rest-masses are $m_0 = 0$) travel along isotropic trajectories ($ds = 0$) with the velocity of light. These are light-like particles, e.g., photons;
3) Particles of the 3rd kind travel along trajectories in the completely degenerate space-time.

Besides, from a purely mathematical point of view, the equations of geodesic lines contain the same vector Q^α and the same parameter ρ irrespective of whether the considered trajectories are isotropic or non-isotropic. This indicates that there must exist such equations of motion, which have a common form for mass-bearing and massless particles. We will proceed to search for such generalized equations of motion.

In the next §1.2, we will explain the basics of the mathematical apparatus of physically observable quantities (chronometric invariants), which will be used as our main tool in this book. In §1.3, we prove the existence of a generalized dynamical vector and derivation parameter, which are the same for mass-bearing and massless particles. In §1.4, we focus on the physical conditions of the complete degeneration of a pseudo-Riemannian space. In §1.5, we consider the properties of particles in an extended four-dimensional space-time, which allows the complete degeneration of the space metric. In §1.6–§1.8, the chronometrically invariant dynamical equations of motion valid for all of the three kinds of particles allowed in the extended four-dimensional space-time will be obtained. In §1.9 and §1.10, we show that the equations of geodesic lines and Newton's laws of Classical Mechanics are par-

ticular cases of the above dynamical equations of motion. The next §1.11 and §1.12 will be devoted to two aspects of the obtained equations: 1) the conditions transforming the extended space-time into the regular space-time; 2) the asymmetry of motion to the future (direct flow of time) and to the past (reverse flow of time). In §1.13 and §1.14, we focus on the physical conditions of the direct and reverse flows of time. In §1.15 and §1.16, we discuss certain specific cases such as a superluminal observer and gravitational collapse.

1.2 Chronometrically invariant (observable) quantities

In order to build a descriptive picture of any physical theory, we need to express the obtained results through real physical quantities, which can be measured in experiments (*physically observable quantities*). In the General Theory of Relativity, this problem is not a trivial one, because we consider objects in a four-dimensional space-time; therefore we must determine which components of four-dimensional tensor quantities are physically observable.

Here is the problem in a nutshell. All equations of the General Theory of Relativity are usually expressed in the *general covariant form*, which is independent of our choice of reference frame. Such equations, as well as the variables they contain, are four-dimensional. Which of the four-dimensional variables are observable in real physical experiments, i.e., which components are physically observable quantities?

Intuitively we may assume that the three-dimensional components of a four-dimensional tensor constitute a physically observable quantity. At the same time, we cannot be absolutely sure that what we observe are truly the three-dimensional components *per se*, if not more complicated variables that depend on other factors such as the properties of the physical standards of the observer's reference space.

A four-dimensional vector (1st rank tensor) has as few as 4 components: 1 time component and 3 spatial components. A 2nd rank tensor, e.g., a rotation or deformation tensor, has 16 components: 1 time component, 9 spatial components and 6 mixed (time-space) components. Now, are the mixed components physically observable quantities? This is another question that seemingly has no definite answer. Tensors of higher ranks have even more components; for instance the Riemann-Christoffel curvature tensor has 256 components, so the problem of

the heuristic recognition of its physically observable components becomes far more complicated. Besides, there is an obstacle related to the recognition of physically observable components of covariant tensors (in which indices occupy the lower position) and of mixed type tensors, which have both lower and upper indices.

We see that finding physically observable quantities in the General Theory of Relativity is not a trivial problem. Ideally, we would like to have a mathematical technique for unambiguously calculating physically observable quantities for tensors of any given rank.

Numerous attempts to develop such a mathematical method were made in the 1930s by some researchers of that time. A contribution was done by L. D. Landau and E. M. Lifshitz in their famous *The Classical Theory of Fields*, first published in 1939 [2]. Aside for discussing the problem of physically observable quantities, in §84 of their book, they introduced the physically observable time interval and the observable three-dimensional interval, which depend on the physical properties (physical standards) of the reference space of an observer. But all the attempts made in the 1930s were limited to solving certain particular problems. None of them led to a versatile mathematical apparatus.

A complete mathematical apparatus for calculating physically observable quantities in a four-dimensional pseudo-Riemannian space was first introduced by Abraham Zelmanov and is known as the *theory of chronometric invariants*. Zelmanov's mathematical apparatus was first presented in 1944 in his PhD thesis [3], where it is given in detail, then — in his short papers of 1956–1957 [4, 5].

A similar result was obtained by Carlo Cattaneo [6–9], an Italian mathematician who worked independently of Zelmanov. Cattaneo published his first paper on this subject in 1958 [6]. He highly appreciated Zelmanov's theory of chronometric invariants and referred to it in 1968, in his last publication [9]. On the other hand, his result was very far from a complete theory, because he limited himself to general considerations on this problem and did not emphasize the physical and geometric observable properties of the local physical space associated with an observer (as Zelmanov did).

The essence of Zelmanov's mathematical apparatus of physically observable quantities (chronometric invariants), which he developed specifically for the four-dimensional, curved, inhomogeneous pseudo-Riemannian space (space-time), is as follows.

This mathematical apparatus is very extensive. For this reason, we present here only the necessary basics of this technique*.

At any point of the space-time we can place a three-dimensional *spatial section* $x^0 = ct = const$ (three-dimensional space) orthogonal to a given *time line* $x^i = const$. If a spatial section is everywhere orthogonal to the time lines, which pierce it at each point, then such a space is called *holonomic*. Otherwise, if the spatial section is non-orthogonal everywhere to the aforementioned time lines, then the space is called *non-holonomic*.

The reference frame of a real observer includes a coordinate grid spanned over a real physical body (his reference body located near him) and real clocks installed at each point of the coordinate grid.

The coordinate grid and clocks represent a set of real references, to which the observer compares the results of his observations. Therefore, physically observable quantities, as actually registered by a particular observer, must be the result of a truly fundamental (i.e., "chronometrical") projection of four-dimensional quantities onto the time line and the spatial section (his local three-dimensional space) associated with his reference body.

The operator of projection onto the time line of an observer is the four-dimensional velocity world-vector

$$b^\alpha = \frac{dx^\alpha}{ds} \tag{1.9}$$

of his reference body with respect to him. The world-vector b^α is tangential to the observer's world-line (his four-dimensional trajectory) at each point. Therefore, it is a unit-length vector

$$b_\alpha b^\alpha = g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \frac{g_{\alpha\beta}\, dx^\alpha dx^\beta}{ds^2} = +1\,. \tag{1.10}$$

The operator of projection onto the spatial section of the observer (his local three-dimensional space) is determined as a four-dimensional

*To date, the most complete description (compendium) of the mathematical apparatus of physically observable quantities in General Relativity is given in our recent article. In this article, we have collected everything (or almost everything) that we know on this topic from Zelmanov and what has been obtained over the past decades: Rabounski D. and Borissova L. Physical observables in General Relativity and the Zelmanov chronometric invariants. *Progress in Physics*, 2023, vol. 19, no. 1, 3–29.

symmetric tensor $h_{\alpha\beta}$, which is

$$\left.\begin{aligned} h_{\alpha\beta} &= -g_{\alpha\beta} + b_\alpha b_\beta \\ h^{\alpha\beta} &= -g^{\alpha\beta} + b^\alpha b^\beta \\ h^\beta_\alpha &= -g^\beta_\alpha + b_\alpha b^\beta \end{aligned}\right\}. \tag{1.11}$$

The vector b^α and the tensor $h_{\alpha\beta}$ are orthogonal to each other. Mathematically this means that their common contraction is zero

$$\left.\begin{aligned} h_{\alpha\beta} b^\alpha &= -g_{\alpha\beta} b^\alpha + b_\alpha b^\alpha b_\beta = -b_\beta + b_\beta = 0 \\ h^{\alpha\beta} b_\alpha &= -g^{\alpha\beta} b_\alpha + b^\beta b_\alpha b^\alpha = -b^\beta + b^\beta = 0 \\ h^\alpha_\beta b_\alpha &= -g^\alpha_\beta b_\alpha + b_\beta b^\alpha b_\alpha = -b_\beta + b_\beta = 0 \\ h^\beta_\alpha b^\alpha &= -g^\beta_\alpha b^\alpha + b^\beta b_\alpha b^\alpha = -b^\beta + b^\beta = 0 \end{aligned}\right\}, \tag{1.12}$$

therefore the main properties of the operators of projection are commonly expressed, obviously, as follows

$$b_\alpha b^\alpha = +1, \qquad h^\beta_\alpha b^\alpha = 0. \tag{1.13}$$

If an observer is at rest with respect to his reference body (such a case is known as the *accompanying reference frame*), then $b^i = 0$ in his reference frame. In this case, the coordinate grids of the same spatial section are connected to each other through the transformations

$$\left.\begin{aligned} \tilde{x}^0 &= \tilde{x}^0\left(x^0, x^1, x^2, x^3\right) \\ \tilde{x}^i &= \tilde{x}^i\left(x^1, x^2, x^3\right), \qquad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \end{aligned}\right\} \tag{1.14}$$

where the third equation displays the fact that the spatial coordinates in the tilde-marked grid are independent of the time of the non-tilded grid, which is equivalent to a coordinate grid, at any point of which the time lines piercing it are fixed $x^i = const$. A transformation of the spatial coordinates is nothing but the transition from one coordinate grid to another within the same spatial section. A transformation of time means changing the whole set of clocks, so this is the transition to another spatial section (another three-dimensional reference space). This means replacing one reference body with all of its physical standards

with another reference body that has its own physical standards. But, using different physical standards, the observer gets different results of measurements (other observable quantities). Therefore, the physically observable projections in the accompanying reference frame must be invariant with respect to the transformation of time, i.e., they must be invariant with respect to the transformations (1.14). In other words, such quantities must have the property of *chronometric invariance*.

Therefore, we call physically observable quantities determined in the accompanying reference frame *chronometrically invariant quantities*, or *chronometric invariants* in short.

The projection tensor $h_{\alpha\beta}$, when considered in the reference frame accompanying an observer, has all properties attributed to the fundamental metric tensor, namely

$$h_i^\alpha h_\alpha^k = \delta_i^k - b_i b^k = \delta_i^k\,, \qquad \delta_i^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.15}$$

where δ_i^k is the unit three-dimensional tensor*. Therefore, in the accompanying reference frame, the three-dimensional tensor h_{ik} can lift and lower indices in chronometrically invariant quantities.

So, in the accompanying reference frame, the main properties of the operators of projection are

$$b_\alpha b^\alpha = +1\,, \qquad h_\alpha^i b^\alpha = 0\,, \qquad h_i^\alpha h_\alpha^k = \delta_i^k\,. \tag{1.16}$$

Calculate the components of the projection operators in the accompanying reference frame.

The component b^0 comes from the condition $b_\alpha b^\alpha = g_{\alpha\beta} b^\alpha b^\beta = 1$, which in the accompanying reference frame (where $b^i = 0$) transforms obviously into $b_\alpha b^\alpha = g_{00} b^0 b^0 = 1$. As a result, the component b^0 and the rest components of the b^α are

$$\left.\begin{aligned} b^0 &= \frac{1}{\sqrt{g_{00}}}\,, & b^i &= 0 \\ b_0 &= g_{0\alpha} b^\alpha = \sqrt{g_{00}}\,, & b_i &= g_{i\alpha} b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}} \end{aligned}\right\}, \tag{1.17}$$

*This tensor δ_i^k is the three-dimensional part of the four-dimensional unit tensor δ_β^α, which can be used to replace (lift and lower) indices in four-dimensional quantities.

while the components of $h_{\alpha\beta}$ are

$$\left.\begin{array}{lll} h_{00} = 0\,, & h^{00} = -g^{00} + \dfrac{1}{g_{00}}\,, & h^0_0 = 0 \\ h_{0i} = 0\,, & h^{0i} = -g^{0i}, & h^i_0 = \delta^i_0 = 0 \\ h_{i0} = 0\,, & h^{i0} = -g^{i0}, & h^0_i = \dfrac{g_{i0}}{g_{00}} \\ h_{ik} = -g_{ik} + \dfrac{g_{0i}\,g_{0k}}{g_{00}}\,, & h^{ik} = -g^{ik}, & h^i_k = -g^i_k = \delta^i_k \end{array}\right\}. \quad (1.18)$$

In the framework of the chronometrically invariant formalism, Zelmanov had developed a common mathematical method how to calculate the chronometrically invariant (physically observable) projections of any general covariant (four-dimensional) tensor quantity. He formulated it as a theorem, which we call *Zelmanov's theorem*:

Zelmanov's theorem

Let there be a four-dimensional tensor $Q^{\mu\nu\ldots\rho}_{\alpha\beta\ldots\sigma}$ of the r-th rank, where $Q^{ik\ldots p}_{00\ldots0}$ is the three-dimensional part of $Q^{\mu\nu\ldots\rho}_{00\ldots0}$, in which all upper indices are non-zero, and all m lower indices are zeroes. Then,

$$T^{ik\ldots p} = (g_{00})^{-\frac{m}{2}}\, Q^{ik\ldots p}_{00\ldots0} \quad (1.19)$$

is a chronometrically invariant three-dimensional contravariant tensor of the $(r-m)$-th rank. This means that the chr.inv.-tensor $T^{ik\ldots p}$ is the result of m-fold projection of the initial tensor $Q^{\mu\nu\ldots\rho}_{\alpha\beta\ldots\sigma}$ onto the time line by the indices $\alpha, \beta \ldots \sigma$ and onto the spatial section by $r-m$ indices $\mu, \nu \ldots \rho$.

In particular, according to the theorem, the chronometrically invariant (physically observable) projections of any four-dimensional vector Q^α are the two following quantities

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha Q^\alpha = Q^i, \quad (1.20)$$

while the chr.inv.-projections of any symmetric tensor of the 2nd rank $Q^{\alpha\beta}$ are the three following quantities

$$b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}\,, \quad h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}\,, \quad h^i_\alpha h^k_\beta\, Q^{\alpha\beta} = Q^{ik}. \quad (1.21)$$

The chr.inv.-projections of a four-dimensional coordinate interval dx^α are the interval of the physically observable time

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\,\sqrt{g_{00}}}\, dx^i, \tag{1.22}$$

and the intervals of each of the respective three-dimensional spatial coordinates dx^i. The physically observable velocity of a particle is the three-dimensional chr.inv.-vector

$$\mathrm{v}^i = \frac{dx^i}{d\tau}, \qquad \mathrm{v}_i \mathrm{v}^i = h_{ik}\, \mathrm{v}^i \mathrm{v}^k = \mathrm{v}^2, \tag{1.23}$$

which at isotropic trajectories becomes the three-dimensional chr.inv.-vector of the physically observable velocity of light

$$c^i = \mathrm{v}^i = \frac{dx^i}{d\tau}, \quad c_i c^i = h_{ik}\, c^i c^k = c^2. \tag{1.24}$$

Projecting the covariant and contravariant fundamental metric tensor onto the spatial section of an observer, in the accompanying reference frame ($b^i = 0$), we obtain

$$\left.\begin{aligned} h_i^\alpha h_k^\beta\, g_{\alpha\beta} &= g_{ik} - b_i b_k = -h_{ik} \\ h_\alpha^i h_\beta^k\, g^{\alpha\beta} &= g^{ik} - b^i b^k = g^{ik} = -h^{ik} \end{aligned}\right\}, \tag{1.25}$$

which means that the chr.inv.-quantity

$$h_{ik} = -g_{ik} + b_i b_k \tag{1.26}$$

is the *chr.inv.-metric tensor* (i.e., the *observable metric tensor*), using which we can lift and lower indices in any three-dimensional chr.inv.-quantity in the accompanying reference frame. The contravariant and mixed components of the observable metric tensor are, obviously,

$$h^{ik} = -g^{ik}, \qquad h_k^i = -g_k^i = \delta_k^i\,. \tag{1.27}$$

Expressing $g_{\alpha\beta}$ through the definition $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$, we obtain the formula for the four-dimensional interval

$$ds^2 = b_\alpha b_\beta\, dx^\alpha dx^\beta - h_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{1.28}$$

expressed through the projection operators b_α and $h_{\alpha\beta}$. In this formula $b_\alpha\, dx^\alpha = c\, d\tau$, so the first term is $b_\alpha\, b_\beta\, dx^\alpha dx^\beta = c^2 d\tau^2$. The second term $h_{\alpha\beta}\, dx^\alpha dx^\beta = d\sigma^2$ in the accompanying reference frame is the square of the physically observable three-dimensional interval*

$$d\sigma^2 = h_{ik}\, dx^i dx^k. \tag{1.29}$$

Thus, the formula, where the four-dimensional interval is expressed through physically observable quantities, is

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{1.30}$$

The main physically observable properties attributed to the accompanying reference space had been deduced by Zelmanov in the framework of the theory. In particular, he proceeded from the property of non-commutativity (non-zero difference between the mixed 2nd derivatives with respect to the coordinates)

$$\frac{{}^*\partial^2}{\partial x^i \partial t} - \frac{{}^*\partial^2}{\partial t\, \partial x^i} = \frac{1}{c^2} F_i \frac{{}^*\partial}{\partial t}\,, \tag{1.31}$$

$$\frac{{}^*\partial^2}{\partial x^i \partial x^k} - \frac{{}^*\partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{{}^*\partial}{\partial t} \tag{1.32}$$

of the chr.inv.-derivation operators that he had introduced

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial x^0}\,. \tag{1.33}$$

The first two physically observable properties are characterized by the chr.inv.-vector of the gravitational inertial force F_i and the antisymmetric chr.inv.-tensor A_{ik} of the angular velocity with which the reference space rotates. They are

$$F_i = \frac{1}{1 - \frac{\mathrm{w}}{c^2}} \left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \tag{1.34}$$

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} \left(F_i v_k - F_k v_i \right). \tag{1.35}$$

*This is due to the fact that the $h_{\alpha\beta}$ in the accompanying reference frame has all properties of the fundamental metric tensor $g_{\alpha\beta}$.

The quantities w and v_i characterize the observer's reference space. These are the gravitational potential

$$\mathrm{w} = c^2\,(1 - \sqrt{g_{00}})\,, \qquad 1 - \frac{\mathrm{w}}{c^2} = \sqrt{g_{00}}\,, \tag{1.36}$$

and the linear velocity with which the space rotates

$$\left.\begin{aligned} v_i &= -c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, & v^i &= -c\,g^{0i}\sqrt{g_{00}} \\ v_i &= h_{ik}\,v^k, & v^2 &= v_k\,v^k = h_{ik}\,v^i v^k \end{aligned}\right\}. \tag{1.37}$$

We note that w and v_i do not have the property of chronometric invariance, despite the fact that $v_i = h_{ik}\,v^k$ is obtained as for a chr.inv.-quantity, through lowering the index by the chr.inv.-metric tensor h_{ik}.

Zelmanov had also found that the chr.inv.-quantities F_i and A_{ik} are related by two identities that we call *Zelmanov's identities*

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{1.38}$$

$$\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\,(F_i A_{km} + F_k A_{mi} + F_m A_{ik}) = 0\,. \tag{1.39}$$

In the framework of quasi-Newtonian approximation, i.e., in a weak gravitational field at velocities much lower than the velocity of light and in the absence of rotation of the space, the F_i becomes an ordinary non-relativistic gravitational force $F_i = \frac{\partial \mathrm{w}}{\partial x^i}$.

Zelmanov had also proved a theorem setting up the condition for a space to be non-holonomic:

Zelmanov's theorem on the space holonomity condition

> For a four-dimensional region of a space (space-time), the identical equality to zero of the tensor A_{ik} is the necessary and sufficient condition for the orthogonality of the spatial sections to the time lines everywhere in this region.

In other words, the necessary and sufficient condition for a space to be holonomic is achieved by equating to zero the tensor A_{ik}. Naturally, if the spatial sections are everywhere orthogonal to the time lines (in such a case the space is holonomic), then the quantities g_{0i} are zero. Since $g_{0i} = 0$, we have $v_i = 0$ and $A_{ik} = 0$. Therefore, we also refer to the tensor A_{ik} as the *space non-holonomity tensor*.

If somewhere the conditions $F_i = 0$ and $A_{ik} = 0$ are satisfied together, then there the conditions $g_{00} = 1$ and $g_{0i} = 0$ are satisfied (i.e., the conditions $g_{00} = 1$ and $g_{0i} = 0$ can be satisfied through the transformation of time in such a region). In such a region, according to (1.22), we have $d\tau = dt$: the difference between the coordinate time t and the physically observable time τ disappears in the absence of gravitational fields and rotation of the space. In other words, according to the theory of chronometric invariants, the difference between the coordinate time t and the physically observable time τ is due to both gravitation and rotation attributed to the reference space of the observer (the Earth for an Earth-bound observer), or due to each of these factors separately.

On the other hand, it is doubtful to find such a region in the Universe, wherein gravitational fields or rotation of the background space would be absent in clear. Therefore, in the real world, the physically observable time τ and the coordinate time t differ from each other. This means that the real space of our Universe is non-holonomic and is filled with gravitational fields, and a holonomic space free of gravitational fields can only be a local approximation to it.

The space holonomity condition is linked directly to the problem of integrability of time. The formula for the physically observable time interval (1.22) has no integrating multiplier. In other words, this formula cannot be reduced to the form

$$d\tau = A dt\,, \tag{1.40}$$

where the multiplier A depends on only t and x^i: in a non-holonomic space, the formula (1.22) has a non-zero second term depending on the coordinate interval dx^i and g_{0i}. In a holonomic space $A_{ik} = 0$, so $g_{0i} = 0$. In this case, the second term of (1.22) is zero, while the first term is the elementary coordinate time interval dt with an integrating multiplier

$$A = \sqrt{g_{00}} = f(x^0, x^i), \tag{1.41}$$

so we are allowed to write the integral

$$d\tau = \int \sqrt{g_{00}}\, dt\,. \tag{1.42}$$

Hence, time is integrable in a holonomic space ($A_{ik} = 0$), but cannot be integrated in a non-holonomic space ($A_{ik} \neq 0$). In the case, where

time is integrable (a holonomic space), we can synchronize clocks at two distant points in the space by moving a control clock along the path between these two points. In the case, where time cannot be integrated (a non-holonomic space), the clock synchronization at two distant points is impossible in principle: the greater the distance between these two points, the greater the deviation of time on these clocks.

The Earth's space is non-holonomic due to the daily rotation of the Earth around its axis. Hence, two clocks located at different points on the Earth's surface should show a deviation between the intervals of time registered on each of them. The greater the distance between these clocks, the greater the expected deviation of the physically observable time registered on them. This effect was surely verified by the well-known Hafele-Keating experiments [10–15] on moving a set of standard atomic clocks on board a civil jet airplane around the globe. When flying along the Earth's rotation, the observer's space on board the airplane had more rotation than the space of another observer, who remained motionless on the ground at the departure/arrival point. When flying against the Earth's rotation it was vice versa. As a result, the atomic clocks on board the airplane showed a significant deviation of the registered time depending on the velocity with which the space on board the airplane rotated.

Synchronization of clocks at different points on the Earth's surface is the most important task of maritime navigation and aviation. Therefore, in the past, navigators introduced desynchronization corrections based on tables containing empirically obtained values that take the Earth's rotation into account. Now, thanks to the theory of chronometric invariants, we know the origin of the corrections and can calculate them on the basis of General Relativity.

In addition to gravitation and rotation, a reference body can deform, changing its coordinate grid with time. This fact must also be taken into account in measurements. This can be done by selecting in the equations the three-dimensional symmetric chr.inv.-tensor of the deformation rate of the observer's reference space

$$\left.\begin{aligned} &D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}, \qquad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t} \\ &D = h^{ik}D_{ik} = D^n_n = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}, \qquad h = \det\|h_{ik}\| \end{aligned}\right\}. \tag{1.43}$$

In addition to the above, when Zelmanov tried to develop a four-dimensional (general covariant) generalization of the chronometrically invariant formalism called the *orthometric monad formalism* [16], he had deduced formulae for the four-dimensional quantities

$$F_\alpha = -2c^2 b^\beta a_{\beta\alpha}\,, \tag{1.44}$$

$$A_{\alpha\beta} = c\,h^\mu_\alpha\, h^\nu_\beta\, a_{\mu\nu}\,, \tag{1.45}$$

$$D_{\alpha\beta} = c\,h^\mu_\alpha\, h^\nu_\beta\, d_{\mu\nu}\,, \tag{1.46}$$

which are the general covariant generalization of the chr.inv.-quantities F_i, A_{ik}, D_{ik}. The auxiliary quantities $a_{\alpha\beta}$ and $d_{\alpha\beta}$ here are

$$a_{\alpha\beta} = \frac{1}{2}\Big(\nabla_\alpha b_\beta - \nabla_\beta b_\alpha\Big), \quad d_{\alpha\beta} = \frac{1}{2}\Big(\nabla_\alpha b_\beta + \nabla_\beta b_\alpha\Big). \tag{1.47}$$

The regular Christoffel symbols of the 2nd rank $\Gamma^\alpha_{\mu\nu}$ and the regular Christoffel symbols of the 1st rank $\Gamma_{\mu\nu,\sigma}$

$$\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\,\Gamma_{\mu\nu,\sigma} = \frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) \tag{1.48}$$

are related to the corresponding chr.inv.-Christoffel symbols

$$\Delta^i_{jk} = h^{im}\Delta_{jk,m} = \frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^j} - \frac{{}^*\partial h_{jk}}{\partial x^m}\right), \tag{1.49}$$

which Zelmanov had determined similarly to $\Gamma^\alpha_{\mu\nu}$. The only difference is that here instead of the fundamental metric tensor $g_{\alpha\beta}$ the chr.inv.-metric tensor h_{ik} is used.

Zelmanov had found that the regular Christoffel symbols are connected with the other chr.inv.-characteristics of the accompanying reference space by the following relations

$$D^i_k + A^{\cdot i}_{k\cdot} = \frac{c}{\sqrt{g_{00}}}\left(\Gamma^i_{0k} - \frac{g_{0k}\Gamma^i_{00}}{g_{00}}\right), \tag{1.50}$$

$$F^k = -\frac{c^2\,\Gamma^k_{00}}{g_{00}}\,, \tag{1.51}$$

$$g^{i\alpha}g^{k\beta}\,\Gamma^m_{\alpha\beta} = h^{iq}h^{ks}\Delta^m_{qs}\,. \tag{1.52}$$

By analogy with the respective absolute (general covariant) derivatives, Zelmanov had also introduced the chr.inv.-derivatives

$$^*\nabla_i Q_k = \frac{^*\partial Q_k}{dx^i} - \Delta^l_{ik} Q_l, \tag{1.53}$$

$$^*\nabla_i Q^k = \frac{^*\partial Q^k}{dx^i} + \Delta^k_{il} Q^l, \tag{1.54}$$

$$^*\nabla_i Q_{jk} = \frac{^*\partial Q_{jk}}{dx^i} - \Delta^l_{ij} Q_{lk} - \Delta^l_{ik} Q_{jl}, \tag{1.55}$$

$$^*\nabla_i Q^k_j = \frac{^*\partial Q^k_j}{dx^i} - \Delta^l_{ij} Q^k_l + \Delta^k_{il} Q^l_j, \tag{1.56}$$

$$^*\nabla_i Q^{jk} = \frac{^*\partial Q^{jk}}{dx^i} + \Delta^j_{il} Q^{lk} + \Delta^k_{il} Q^{jl}, \tag{1.57}$$

$$^*\nabla_i Q^i = \frac{^*\partial Q^i}{\partial x^i} + \Delta^j_{ji} Q^i, \qquad \Delta^j_{ji} = \frac{^*\partial \ln\sqrt{h}}{\partial x^i}, \tag{1.58}$$

$$^*\nabla_i Q^{ji} = \frac{^*\partial Q^{ji}}{\partial x^i} + \Delta^j_{il} Q^{il} + \Delta^l_{li} Q^{ji}, \quad \Delta^l_{li} = \frac{^*\partial \ln\sqrt{h}}{\partial x^i}. \tag{1.59}$$

Zelmanov had also deduced the chr.inv.-projections of the Riemann-Christoffel tensor. He followed the same way by which the Riemann-Christoffel tensor was built, proceeding from the non-commutativity of the second derivatives of an arbitrary vector taken in the given space. Taking the second chr.inv.-derivatives of an arbitrary vector

$$^*\nabla_i {}^*\nabla_k Q_l - {}^*\nabla_k {}^*\nabla_i Q_l = \frac{2A_{ik}}{c^2}\frac{^*\partial Q_l}{\partial t} + H^{\cdots j}_{lki\cdot} Q_j, \tag{1.60}$$

he obtained the chr.inv.-tensor

$$H^{\cdots j}_{lki\cdot} = \frac{^*\partial \Delta^j_{il}}{\partial x^k} - \frac{^*\partial \Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}, \tag{1.61}$$

which is like Schouten's tensor from the theory of non-holonomic manifolds [17]. The $H^{\cdots j}_{lki\cdot}$ differs algebraically from the Riemann-Christoffel tensor by the presence of the space rotation velocity tensor A_{ik} in the formula (1.60). Nevertheless it gives the chr.inv.-tensor

$$C_{lkij} = \frac{1}{4}\Big(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\Big), \tag{1.62}$$

which has all the algebraic properties of the Riemann-Christoffel tensor in the observer's three-dimensional space. Therefore, Zelmanov called the C_{iklj} the *chr.inv.-curvature tensor*. Its contraction step-by-step gives the chr.inv.-scalar C, which is the observable three-dimensional curvature of the observer's spatial section (his reference space)

$$C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im} C_{kimj}\,, \qquad C = C_j^j = h^{lj} C_{lj}\,, \tag{1.63}$$

Substituting the necessary components of the Riemann-Christoffel tensor into the formulae for its chr.inv.-projections

$$X^{ik} = -c^2 \frac{R_{0\cdot 0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \quad Y^{ijk} = -c\,\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \quad Z^{ijkl} = c^2 R^{ijkl}. \tag{1.64}$$

and by lowering indices, after transformations Zelmanov obtained the above chr.inv.-components in the form

$$X_{ij} = \frac{{}^*\partial D_{ij}}{\partial t} - \left(D_i^l + A_{i\cdot}^{\cdot l}\right)\left(D_{jl} + A_{jl}\right) + \\ + \left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2} F_i F_j\,, \tag{1.65}$$

$$Y_{ijk} = {}^*\nabla_i \left(D_{jk} + A_{jk}\right) - {}^*\nabla_j \left(D_{ik} + A_{ik}\right) + \frac{2}{c^2} A_{ij} F_k\,, \tag{1.66}$$

$$Z_{iklj} = D_{ik} D_{lj} - D_{il} D_{kj} + A_{ik} A_{lj} - \\ - A_{il} A_{kj} + 2 A_{ij} A_{kl} - c^2 C_{iklj}\,, \tag{1.67}$$

where $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$ just like in the Riemann-Christoffel tensor. Contraction of the spatial observable projection Z_{iklj} step-by-step gives the following chr.inv.-quantities

$$Z_{il} = D_{ik} D_l^k - D_{il} D + A_{ik} A_{l\cdot}^{\cdot k} + 2 A_{ik} A_{\cdot l}^{k\cdot} - c^2 C_{il}\,, \tag{1.68}$$

$$Z = h^{il} Z_{il} = D_{ik} D^{ik} - D^2 - A_{ik} A^{ik} - c^2 C\,. \tag{1.69}$$

These are the necessary basics of the mathematical apparatus of physically observable quantities, known also as the Zelmanov chronometric invariants [3–5].

With the above definitions, we can find out how any geometric object of the four-dimensional pseudo-Riemannian space (space-time) looks like from the viewpoint of an observer, whose home is this space.

For instance, having any equation obtained in the general covariant tensor analysis, we can calculate the chr.inv.-projections of it onto the time line and the spatial section associated with any particular observer, then formulate the chr.inv.-projections in terms of the physically observable properties of his reference space. In this way, we will arrive at equations containing only quantities measurable in practice.

1.3 Mass-bearing particles and massless particles

According to modern concepts, a mass-bearing particle is characterized by the four-dimensional momentum vector P^α, while a massless particle is characterized by the four-dimensional wave vector K^α, i.e.

$$P^\alpha = m_0 \frac{dx^\alpha}{ds}\,, \qquad K^\alpha = \frac{\omega}{c}\frac{dx^\alpha}{d\sigma}\,, \tag{1.70}$$

where m_0 is the rest-mass of the mass-bearing particle, while ω is the frequency of the massless particle. The space-time interval ds is used as the derivation parameter for mass-bearing particles (non-isotropic trajectories, $ds \neq 0$). Along isotropic trajectories $ds = 0$ (massless particles), but the observable three-dimensional interval is $d\sigma \neq 0$. Therefore, $d\sigma$ is used as the derivation parameter for massless particles.

The square of the momentum vector P^α along the trajectories of mass-bearing particles is not zero and is constant

$$P_\alpha P^\alpha = g_{\alpha\beta}\, P^\alpha P^\beta = m_0^2 = const \neq 0\,, \tag{1.71}$$

i.e., the P^α is a non-isotropic vector. The square of the wave vector K^α is zero along the trajectories of massless particles

$$K_\alpha K^\alpha = g_{\alpha\beta}\, K^\alpha K^\beta = \frac{\omega^2}{c^2}\frac{g_{\alpha\beta}\, dx^\alpha dx^\beta}{d\sigma^2} = \frac{\omega^2}{c^2}\frac{ds^2}{d\sigma^2} = 0\,, \tag{1.72}$$

therefore the K^α is an isotropic vector.

Since ds^2 in the chr.inv.-form (1.30) expresses itself through the square of the relativistic root as

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = c^2 d\tau^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right), \quad \mathrm{v}^2 = h_{ik}\mathrm{v}^i \mathrm{v}^k, \tag{1.73}$$

we can put the P^α and K^α down as

$$P^\alpha = m_0 \frac{dx^\alpha}{ds} = \frac{m}{c}\frac{dx^\alpha}{d\tau}\,, \qquad K^\alpha = \frac{\omega}{c}\frac{dx^\alpha}{d\sigma} = \frac{k}{c}\frac{dx^\alpha}{d\tau}\,, \tag{1.74}$$

where $k = \frac{\omega}{c}$ is the massless particle's wave number, and m is the mass-bearing particle's relativistic mass.

From the obtained formulae, we see that the physically observable time τ can be used as a universal derivation parameter with respect to both isotropic and non-isotropic trajectories, i.e., as the single derivation parameter for mass-bearing and massless particles.

The contravariant components of the P^α and K^α are

$$P^0 = m\,\frac{dt}{d\tau}\,, \qquad P^i = \frac{m}{c}\frac{dx^i}{d\tau} = \frac{1}{c}\,m\mathrm{v}^i\,, \tag{1.75}$$

$$K^0 = k\,\frac{dt}{d\tau}\,, \qquad K^i = \frac{k}{c}\frac{dx^i}{d\tau} = \frac{1}{c}\,k\mathrm{v}^i\,, \tag{1.76}$$

where $m\mathrm{v}^i$ is the three-dimensional chr.inv.-momentum vector of the mass-bearing particle, and $k\mathrm{v}^i$ is the three-dimensional chr.inv.-wave vector of the massless particle. The observable velocity of massless particles is the observable chr.inv.-velocity of light $\mathrm{v}^i = c^i$ (1.24).

The $\frac{dt}{d\tau}$ can be obtained based on the square of the four-dimensional velocity vector U^α of a particle, which in the case of a subluminal velocity, the light velocity and a superluminal velocity is, respectively,

$$g_{\alpha\beta}\,U^\alpha U^\beta = +1\,, \qquad U^\alpha = \frac{dx^\alpha}{ds}\,, \qquad ds = c\,d\tau\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\,, \tag{1.77}$$

$$g_{\alpha\beta}\,U^\alpha U^\beta = 0\,, \qquad U^\alpha = \frac{dx^\alpha}{d\sigma}\,, \qquad ds = 0\,,\ \ d\sigma = c\,d\tau\,, \tag{1.78}$$

$$g_{\alpha\beta}\,U^\alpha U^\beta = -1\,, \qquad U^\alpha = \frac{dx^\alpha}{|ds|}\,, \qquad ds = c\,d\tau\sqrt{\frac{\mathrm{v}^2}{c^2}-1}\,. \tag{1.79}$$

Using the definitions of h_{ik}, v_i, v^i in the formulae for $g_{\alpha\beta}\,U^\alpha U^\beta$, we arrive at three identical quadratic equations with respect to $\frac{dt}{d\tau}$ for subluminal velocities, the light velocity and superluminal velocities

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{2v_i\mathrm{v}^i}{c^2\left(1-\frac{\mathrm{w}}{c^2}\right)}\frac{dt}{d\tau} + \frac{1}{\left(1-\frac{\mathrm{w}}{c^2}\right)^2}\left(\frac{1}{c^4}\,v_i v_k\,\mathrm{v}^i\mathrm{v}^k - 1\right) = 0\,. \tag{1.80}$$

This quadratic equation has two solutions

$$\left(\frac{dt}{d\tau}\right)_{1,2} = \frac{1}{1-\frac{\mathrm{w}}{c^2}}\left(\frac{1}{c^2}\,v_i\mathrm{v}^i \pm 1\right). \tag{1.81}$$

The function $\frac{dt}{d\tau}$ allows you to find out in which direction in time the particle is travelling. If $\frac{dt}{d\tau} > 0$, then the time coordinate parameter t increases: the particle travels from the past to the future, which means the direct flow of time. If $\frac{dt}{d\tau} < 0$, then the time coordinate parameter decreases: the particle travels from the future to the past (the reverse flow of time).

We assume $1 - \frac{\mathrm{w}}{c^2} = \sqrt{g_{00}} > 0$, because the other cases $\sqrt{g_{00}} = 0$ and $\sqrt{g_{00}} < 0$ contradict the signature condition $(+---)$. Therefore, the coordinate time t stops ($\frac{dt}{d\tau} = 0$) provided that

$$v_i \mathrm{v}^i = -c^2, \qquad v_i \mathrm{v}^i = +c^2. \tag{1.82}$$

The coordinate time t has direct flow $\frac{dt}{d\tau} > 0$, if in the first and second solutions we have, respectively

$$\frac{1}{c^2} v_i \mathrm{v}^i + 1 > 0\,, \qquad \frac{1}{c^2} v_i \mathrm{v}^i - 1 > 0\,. \tag{1.83}$$

The coordinate time t has reverse flow $\frac{dt}{d\tau} < 0$, if

$$\frac{1}{c^2} v_i \mathrm{v}^i + 1 < 0\,, \qquad \frac{1}{c^2} v_i \mathrm{v}^i - 1 < 0\,. \tag{1.84}$$

For subluminal particles, $v_i \mathrm{v}^i < c^2$ is always true. Hence, the direct flow of time for regularly observed mass-bearing particles takes place under the first condition of (1.83), and the reverse flow of time takes place under the second condition of (1.84).

It should be noted that we have considered the problem of the direction of the coordinate time t, assuming that the physically observable time interval is always $d\tau > 0$. This is associated with the perception of any ordinary observer to see the events of his world in order from the past to the future.

Calculate the covariant components P_i and K_i, then — the chr.inv.-projections of the four-dimensional vectors P^α and K^α onto the time line of an observer. Using the formulae (1.75), (1.76), (1.81), we obtain

$$P_i = -\frac{m}{c}\,(\mathrm{v}_i \pm v_i)\,, \qquad K_i = -\frac{k}{c}\,(\mathrm{v}_i \pm v_i)\,, \tag{1.85}$$

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m\,, \qquad \frac{K_0}{\sqrt{g_{00}}} = \pm k\,, \tag{1.86}$$

where the time projections $+m$ and $+k$ take place when observing the motion of these particles to the future (direct flow of time), and $-m$ and $-k$ are registered when observing the motion of these particles to the past (reverse flow of time).

Therefore, the physically observable quantities are as follows. For a mass-bearing particle these are its relativistic mass $\pm m$ and the three-dimensional quantity $\frac{1}{c} m \mathrm{v}^i$, where $m \mathrm{v}^i$ is the observable momentum vector of the particle. For a massless particle these are its wave number $\pm k$ and the three-dimensional quantity $\frac{1}{c} k \mathrm{v}^i$, where $k \mathrm{v}^i$ is the observable wave vector of the particle.

From the obtained formulae (1.85) and (1.86), we see that the observable wave vector $k \mathrm{v}^i$ characterizing a massless particle is the complete analogue of the observable momentum vector $m \mathrm{v}^i$ that characterizes a mass-bearing particle.

Substituting the obtained formulae for P^0, P^i, K^0, K^i, and also the formula for g_{ik} expressed through $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ into the formulae for $P_\alpha P^\alpha$ (1.71) and $K_\alpha K^\alpha$ (1.72), we arrive at the relations between the physically observable energy and the physically observable momentum for a mass-bearing particle

$$\frac{E^2}{c^2} - m^2 \mathrm{v}_i \mathrm{v}^i = \frac{E_0^2}{c^2}, \tag{1.87}$$

and also that for a massless particle

$$h_{ik} \mathrm{v}^i \mathrm{v}^k = c^2, \tag{1.88}$$

where $E = \pm mc^2$ is the relativistic energy of the mass-bearing particle, while $E_0 = m_0 c^2$ is its rest-energy.

Therefore, by comparing the new common formulae for P^α and K^α (1.74), which we have obtained, we arrive at an universal derivation parameter, which is the physically observable time τ and is the same for both mass-bearing and massless particles. This is despite the fact that the four-dimensional dynamical vectors for particles of each of these two kinds, i.e., the vectors P^α and K^α, *differ* from each other.

Now we are going to find a universal dynamical vector, which in particular cases can be represented as the dynamical vector of a mass-bearing particle P^α and that of a massless particle K^α.

We will tackle this problem by assuming that the wave-particle duality, first introduced by Louis de Broglie for massless particles, is peculiar

to particles of all kinds without any exception. That is, we will consider the motion of massless and mass-bearing particles as the propagation of waves in the approximation of geometric optics.

The four-dimensional wave vector of a massless particle K^α in the approximation of geometric optics is [2]

$$K_\alpha = \frac{\partial\psi}{\partial x^\alpha}\,, \tag{1.89}$$

where ψ is the wave phase (*eikonal*). In the same way, we introduce the four-dimensional momentum vector of a mass-bearing particle

$$P_\alpha = \frac{\hbar}{c}\frac{\partial\psi}{\partial x^\alpha}\,, \tag{1.90}$$

where $\hbar$ is Planck's constant, while the coefficient $\frac{\hbar}{c}$ equates the dimensions of both parts of the equation. From these formulae we obtain

$$\frac{K_0}{\sqrt{g_{00}}} = \frac{1}{c}\frac{{}^*\partial\psi}{\partial t}\,, \qquad \frac{P_0}{\sqrt{g_{00}}} = \frac{\hbar}{c^2}\frac{{}^*\partial\psi}{\partial t}\,. \tag{1.91}$$

Equating the quantities (1.91) to (1.86) we obtain

$$\pm\omega = \frac{{}^*\partial\psi}{\partial t}\,, \qquad \pm m = \frac{\hbar}{c^2}\frac{{}^*\partial\psi}{\partial t}\,, \tag{1.92}$$

where $+\omega$ for a massless particle and $+m$ for a mass-bearing particle take place at the wave phase ψ that increases with time, while $-\omega$ and $-m$ take place at the wave phase decreasing with time. Thus, we obtain a formula for the energy of both massless and mass-bearing particles, which takes their dual (wave-particle) nature into account. It is

$$\pm mc^2 = \pm\hbar\omega = \hbar\frac{{}^*\partial\psi}{\partial t} = E\,. \tag{1.93}$$

Now from (1.90) we obtain the dependence between the chr.inv.-momentum p^i of a particle and its wave phase ψ

$$p^i = m\mathrm{v}^i = -\hbar h^{ik}\frac{{}^*\partial\psi}{\partial x^k}\,, \qquad p_i = m\mathrm{v}_i = -\hbar\frac{{}^*\partial\psi}{\partial x^i}\,. \tag{1.94}$$

Furthermore, as is known [2], the condition $K_\alpha K^\alpha = 0$ can be presented in the form

$$g^{\alpha\beta}\frac{\partial\psi}{\partial x^\alpha}\frac{\partial\psi}{\partial x^\beta} = 0\,, \tag{1.95}$$

which is the basic equation of geometric optics, known as the *eikonal equation*. Formulating the ordinary derivation operators through the chr.inv.-derivation operators and taking into account that

$$g^{00} = \frac{1 - \frac{1}{c^2} v_i v^i}{g_{00}}, \quad g^{ik} = -h^{ik}, \quad v^i = -c g^{0i} \sqrt{g_{00}}, \tag{1.96}$$

we arrive at the chr.inv.-eikonal equation for massless particles

$$\frac{1}{c^2} \left(\frac{{}^*\partial\psi}{\partial t} \right)^2 - h^{ik} \frac{{}^*\partial\psi}{\partial x^i} \frac{{}^*\partial\psi}{\partial x^k} = 0. \tag{1.97}$$

Following the same way, we obtain the chr.inv.-eikonal equation for mass-bearing particles

$$\frac{1}{c^2} \left(\frac{{}^*\partial\psi}{\partial t} \right)^2 - h^{ik} \frac{{}^*\partial\psi}{\partial x^i} \frac{{}^*\partial\psi}{\partial x^k} = \frac{m_0^2 c^2}{\hbar^2}, \tag{1.98}$$

which at $m_0 = 0$ becomes the same as the former one.

Substituting the relativistic mass m (1.92) into (1.74), we obtain the dynamical vector P^α that characterizes the motion of both massless and mass-bearing particles in the approximation of geometric optics

$$P^\alpha = \frac{\hbar\omega}{c^3} \frac{dx^\alpha}{\partial\tau}, \qquad P_\alpha P^\alpha = \frac{\hbar^2\omega^2}{c^4} \left(1 - \frac{\mathrm{v}^2}{c^2} \right). \tag{1.99}$$

The length of the vector P^α is real for $\mathrm{v} < c$, is zero for $\mathrm{v} = c$, and is imaginary for $\mathrm{v} > c$. Therefore, the obtained dynamical vector P^α characterizes a particle with any rest-mass (real, zero, or imaginary).

The observable chr.inv.-projections of the universal vector P^α are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm \frac{\hbar\omega}{c^2}, \qquad P^i = \frac{\hbar\omega}{c^3} \mathrm{v}^i, \tag{1.100}$$

where the time chr.inv.-projection has the dimension of mass, and the quantity $p^i = c P^i$ has the dimensions of momentum.

1.4 Completely degenerate space-time. Zero-particles

As is known, along the trajectories of massless particles (isotropic trajectories) the four-dimensional interval is zero

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = 0, \qquad c^2 d\tau^2 = d\sigma^2 \neq 0. \tag{1.101}$$

Note that we have $ds^2 = 0$ not only at $c^2 d\tau^2 = d\sigma^2$, but also when even a stricter condition is true, $c^2 d\tau^2 = d\sigma^2 = 0$. The condition $d\tau^2 = 0$ means that the physically observable time τ has the same numerical value along the entire trajectory. The second condition $d\sigma^2 = 0$ means that all the three-dimensional trajectories have zero length. Taking into account the definitions of $d\tau$ (1.22) and $d\sigma^2$ (1.29), and also the fact that $h_{00} = h_{0i} = 0$ in a reference frame accompanying the observer, we re-write the conditions $d\tau^2 = 0$ and $d\sigma^2 = 0$ in the expanded form

$$c\,d\tau = \left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right] c\,dt = 0\,, \qquad dt \neq 0\,, \tag{1.102}$$

$$d\sigma^2 = h_{ik}\,dx^i dx^k = 0\,, \tag{1.103}$$

where $u^i = \frac{dx^i}{dt}$ is the three-dimensional coordinate velocity of the particle, which is not a physically observable chr.inv.-quantity.

As is known, the necessary and sufficient condition for a metric to be complete degenerate means zero value of the determinant of its metric tensor. For the three-dimensional physically observable metric $d\sigma^2 = h_{ik}\,dx^i dx^k$ this condition is

$$h = \det \|h_{ik}\| = 0\,. \tag{1.104}$$

On the other hand, the determinant of the chr.inv.-metric tensor h_{ik} has the form [3–5]

$$h = -\frac{g}{g_{00}}\,, \qquad g = \det \|g_{\alpha\beta}\|\,. \tag{1.105}$$

The degeneration of the three-dimensional form $d\sigma^2$ (i.e., $h = 0$) means the degeneration of the four-dimensional form ds^2 (i.e., $g = 0$). Hence, a four-dimensional space (space-time), wherein the conditions (1.102) and (1.103) are true, is a *completely degenerate space-time*.

Substituting $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ into (1.103), then dividing it by dt^2, we obtain the (1.102) and (1.103), i.e., the *physical conditions of degeneration of the space*, in the final form

$$\mathrm{w} + v_i u^i = c^2, \qquad g_{ik}\,u^i u^k = c^2\left(1 - \frac{\mathrm{w}}{c^2}\right)^2, \tag{1.106}$$

where $v_i u^i$ is the scalar product of the linear velocity v_i with which the space rotates and the coordinate velocity u^i of the particle.

If all quantities $v_i = 0$ (i.e., the space is holonomic), then $\mathrm{w} = c^2$ and also $\sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} = 0$. This means that the gravitational potential of the reference body w is strong enough at the given point of the space to bring the space to gravitational collapse at this point. This case will not be discussed here.

Consider the degeneration of a four-dimensional space (space-time), which is non-holonomic. In this case, we have $v_i \neq 0$, i.e., the spatial section belonging to the observer rotates.

Using the definition of $d\tau$ (1.22), we obtain the relation between the coordinate velocity u^i and the observable velocity v^i in the space

$$\mathrm{v}^i = \frac{u^i}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)}, \tag{1.107}$$

which takes the first degeneration condition into account.

Thus, we re-write ds^2 in a form, where the first degeneration condition is presented explicitly

$$ds^2 = c^2 d\tau^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right) = c^2 dt^2 \left\{\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)\right]^2 - \frac{u^2}{c^2}\right\}. \tag{1.108}$$

It is obvious that a degenerate space-time can only host the particles for which the physical conditions of degeneration (1.106) are true.

We will refer to such a completely degenerate space-time as the *zero-space*, while the particles allowed in such a completely degenerate space-time (zero-space) will be referred to as *zero-particles*.

1.5 An extended space for particles of all three kinds

When we studied the motion of mass-bearing and massless particles, we considered it in a four-dimensional space-time, the metric of which is strictly non-degenerate ($g < 0$). Now, we are going to consider the motion of particles in such a space-time, the metric of which can be degenerate ($g \leqslant 0$).

We have already obtained the metric of such an extended space-time (see formula 1.108). Hence, the momentum vector of a mass-bearing particle P^α in such an extended space-time ($g \leqslant 0$) has the form

$$P^\alpha = m_0 \frac{dx^\alpha}{ds} = \frac{M}{c}\frac{dx^\alpha}{dt}, \tag{1.109}$$

$$M = \frac{m_0}{\sqrt{\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)\right]^2 - \frac{u^2}{c^2}}}\,, \tag{1.110}$$

where M stands for the *gravitational rotational mass* of the particle. Such a mass M depends not only on the three-dimensional velocity of the particle with respect to the observer, but also on the gravitational potential w (associated with the reference body's field) and on the linear velocity v_i with which the space rotates.

From the obtained formula (1.109) we see that in a four-dimensional space-time, wherein the metric can be degenerate ($g \leqslant 0$), the generalized derivation parameter is the coordinate time t.

Substituting v^2 from (1.107) and $m_0 = m\sqrt{1 - \mathrm{v}^2/c^2}$ into this formula, we arrive at the relation

$$M = \frac{m}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)} \tag{1.111}$$

between the relativistic mass m of any particle in the space and its gravitational rotational mass M.

From the obtained formula we see that the M is a ratio between two quantities, each one is zero in the case where the space metric is degenerate ($g = 0$), but the ratio itself is not zero $M \neq 0$.

This fact is no surprise. The same is true for the relativistic mass m in the case of $\mathrm{v}^2 = c^2$. As soon as $m_0 = 0$ and $\sqrt{1 - \mathrm{v}^2/c^2} = 0$, the ratio of these quantities is still $m \neq 0$.

Therefore, light-like (massless) particles are the limiting case of mass-bearing particles at $\mathrm{v} \to c$. Zero-particles can be regarded the limiting case of light-like ones travelling in a non-holonomic space at the observable velocity v^i (1.107) that depends on the gravitational potential w of the reference body's field and on the direction with respect to the linear velocity v_i with which the space rotates.

The time component of the world-vector P^α (1.109) and the physically observable projection of the vector onto the time line are

$$P^0 = M = \frac{m}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)}\,, \tag{1.112}$$

$$\frac{P_0}{\sqrt{g_{00}}} = M\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right] = m\,, \tag{1.113}$$

while the spatial components of the vector are

$$P^i = \frac{M}{c}\, u^i = \frac{m}{c}\, \mathrm{v}^i, \tag{1.114}$$

$$P_i = -\frac{M}{c}\left[u_i + v_i - \frac{1}{c^2}\, v_i \left(\mathrm{w} + v_k u^k\right)\right]. \tag{1.115}$$

In a completely degenerate region of the extended space-time, i.e., under the physical conditions of degeneration (1.106), the components of the dynamic vector P^α of a particle become

$$P^0 = M \neq 0\,, \qquad \frac{P_0}{\sqrt{g_{00}}} = m = 0\,, \tag{1.116}$$

$$P^i = \frac{M}{c}\, u^i, \qquad P_i = -\frac{M}{c}\, u_i\,, \tag{1.117}$$

i.e., a particle of the degenerate space-time (a zero-particle) has a zero relativistic mass, but its gravitational rotational mass is not zero.

Consider mass-bearing particles in the extended space-time within the wave-particle duality concept. In such a case, the components of the universal dynamical vector $P_\alpha = \frac{\hbar}{c}\frac{{}^*\partial\psi}{\partial x^\alpha}$ (1.90) of a particle are

$$\frac{P_0}{\sqrt{g_{00}}} = m = M\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right] = \frac{\hbar}{c^2}\frac{{}^*\partial\psi}{\partial t}\,, \tag{1.118}$$

$$P_i = \frac{\hbar}{c}\left(\frac{{}^*\partial\psi}{\partial x^i} - \frac{1}{c^2}\, v_i\, \frac{{}^*\partial\psi}{\partial t}\right), \tag{1.119}$$

$$P^i = \frac{m}{c}\, \mathrm{v}^i = \frac{M}{c}\, u^i = -\frac{\hbar}{c}\, h^{ik}\, \frac{{}^*\partial\psi}{\partial x^k}\,, \tag{1.120}$$

$$P^0 = M = \frac{\hbar}{c^2\left(1 - \frac{\mathrm{w}}{c^2}\right)}\left(\frac{{}^*\partial\psi}{\partial t} - v^i\, \frac{{}^*\partial\psi}{\partial x^i}\right). \tag{1.121}$$

From these components, the following two formulae can be obtained

$$Mc^2 = \frac{1}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)}\, \hbar\, \frac{{}^*\partial\psi}{\partial t} = \hbar\,\Omega = E_{\mathrm{tot}}\,, \tag{1.122}$$

$$Mu^i = -\hbar h^{ik}\, \frac{{}^*\partial\psi}{\partial x^k}\,, \tag{1.123}$$

where Ω is the gravitational rotational frequency of the particle, while ω is its regular frequency

$$\Omega = \frac{\omega}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)}, \qquad \omega = \frac{{}^*\partial\psi}{\partial t}. \tag{1.124}$$

The first relation (1.122) links the gravitational rotational mass M of a particle to its corresponding total energy E_{tot}. The second relation (1.123) links the three-dimensional generalized momentum Mu^i of the particle to the gradient of its wave phase ψ.

The condition $P_\alpha P^\alpha = const$ in the approximation of geometric optics is the eikonal equation (1.98). For corpuscular matter in the extended space-time, this condition takes the chr.inv.-form

$$\frac{E^2}{c^2} - M^2 u^2 = \frac{E_0^2}{c^2}, \tag{1.125}$$

where $M^2 u^2$ is the square of the generalized three-dimensional momentum vector, $E = mc^2$, and $E_0 = m_0 c^2$. Using this formula, we obtain the formula for the universal dynamical vector

$$P^\alpha = \frac{\hbar\Omega}{c^3}\frac{dx^\alpha}{dt} = \frac{\hbar\,\frac{{}^*\partial\psi}{\partial t}}{c^3\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right]}\frac{dx^\alpha}{\partial t}, \tag{1.126}$$

$$P_\alpha P^\alpha = \frac{\hbar^2\Omega^2}{c^4}\left\{\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right]^2 - \frac{u^2}{c^2}\right\}, \tag{1.127}$$

where the first degeneration condition has been included.

In a completely degenerate region of the extended space-time, we have $m_0 = 0$, $m = 0$, $\omega = \frac{{}^*\partial\psi}{\partial t} = 0$, and $P_\alpha P^\alpha = 0$. That is, from the viewpoint of an observer, whose home is our world, particles of a degenerate region (zero-particles) have zero rest-mass m_0, zero relativistic mass m, zero relativistic frequency ω (corresponding to the relativistic mass in the framework of the wave-particle duality), while the length of the four-dimensional dynamical vector of any zero-particle is indeed conserved. On the contrary, for zero-particles, the gravitational rotational mass M (1.110), the three-dimensional generalized momentum Mu^i (1.123), and the gravitational rotational frequency Ω (1.124), which corresponds to the mass M according to the wave-particle duality, are not zero.

The zero-space metric $d\mu^2$ is not invariant from the viewpoint of an internal observer in the zero-space. It can be proven based on the second degeneration condition $d\sigma^2 = h_{ik}\, dx^i dx^k = 0$. Using $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$, dividing by dt^2, and then substituting the first degeneration condition $\mathrm{w} + v_i u^i = c^2$, we arrive at the internal zero-space metric

$$d\mu^2 = g_{ik}\, dx^i dx^k = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 \neq inv\,, \tag{1.128}$$

which is not invariant. Hence, from the viewpoint of an observer in the zero-space, the four-dimensional vector of any zero-particle has a length that is not conserved along the trajectory of the particle

$$U_\alpha U^\alpha = g_{ik}\, u^i u^k = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 \neq const. \tag{1.129}$$

The eikonal equation for zero-particles is obtained by substituting the conditions $m = 0$, $\omega = \frac{{}^*\partial\psi}{\partial t} = 0$, $P_\alpha P^\alpha = 0$ into the eikonal equation (1.97) or (1.98) obtained for mass-bearing and massless particles, respectively. Thus, we obtain the eikonal equation for zero-particles in the reference frame of an ordinary observer, whose home is our world,

$$h^{ik} \frac{{}^*\partial\psi}{\partial x^i} \frac{{}^*\partial\psi}{\partial x^k} = 0\,. \tag{1.130}$$

This is a standing wave equation. This means that zero-particles look from our point of view as *standing light waves* — waves of stopped light (*information circles*, or *light-like holograms*).

As a result, our theoretical investigation of the extended space-time, wherein the metric can be completely degenerate, we conclude that in such a space-time there are two ultimate space-time barriers:

1) The *light barrier*, to overcome which a particle must exceed the velocity of light;
2) The *zero-transition*, to overcome which a particle must be in the state of specific rotation depending on a particular distribution of matter (the degeneration conditions).

1.6 The equations of motion: general considerations

Let us obtain the dynamical equations of motion of free particles in the extended space-time ($g \leqslant 0$), i.e., the equations of motion for mass-bearing, massless, and zero particles in a common form.

Geometrically, the equations in question are those of the Levi-Civita parallel transport applied to the universal dynamical vector P^α, i.e.,

$$\mathrm{D}P^\alpha = dP^\alpha + \Gamma^\alpha_{\mu\nu} P^\mu dx^\nu = 0\,. \tag{1.131}$$

The parallel transport equations (1.131) are written in the general covariant form. In order to use the equations to solve real problems of physics, the equations must contain only chronometrically invariant (physically observable) quantities. To transform the equations to the chr.inv.-form, we project them onto the time line and the spatial section of the reference frame accompanying our reference body. We obtain

$$\left.\begin{aligned} b_\alpha \mathrm{D}P^\alpha &= \sqrt{g_{00}}\left(dP^0 + \Gamma^0_{\mu\nu} P^\mu dx^\nu\right) + \\ &\qquad + \frac{g_{0i}}{\sqrt{g_{00}}}\left(dP^i + \Gamma^i_{\mu\nu} P^\mu dx^\nu\right) = 0 \\ h^i_\beta \mathrm{D}P^\beta &= dP^i + \Gamma^i_{\mu\nu} P^\mu dx^\nu = 0 \end{aligned}\right\}. \tag{1.132}$$

The Christoffel symbols found in the chr.inv.-equations (1.132) are not yet expressed in terms of chr.inv.-quantities. Express the Christoffel symbols of the 2nd kind $\Gamma^\alpha_{\mu\nu}$ and those of the 1st kind $\Gamma_{\mu\nu,\sigma}$

$$\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}\,, \quad \Gamma_{\mu\nu,\rho} = \frac{1}{2}\left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right) \tag{1.133}$$

through the chr.inv.-properties of the accompanying reference space. Formulating the $g^{\alpha\beta}$ components and the first derivatives of $g_{\alpha\beta}$ in terms of F_i, A_{ik}, D_{ik}, w, and v_i, after some algebra we obtain

$$\Gamma_{00,0} = -\frac{1}{c^3}\left(1 - \frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial t}\,, \tag{1.134}$$

$$\Gamma_{00,i} = \frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 F_i + \frac{1}{c^4}\, v_i \frac{\partial \mathrm{w}}{\partial t}\,, \tag{1.135}$$

$$\Gamma_{0i,0} = -\frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{1.136}$$

$$\Gamma_{0i,j} = -\frac{1}{c}\left(1 - \frac{\mathrm{w}}{c^2}\right)\left(D_{ij} + A_{ij} + \frac{1}{c^2}\, F_j v_i\right) + \frac{1}{c^3}\, v_j \frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{1.137}$$

$$\Gamma_{ij,0} = \frac{1}{c}\left(1 - \frac{\mathrm{w}}{c^2}\right)\left[D_{ij} - \frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) + \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right)\right], \tag{1.138}$$

$$\Gamma_{ij,k} = -\Delta_{ij,k} + \frac{1}{c^2}\left[v_i A_{jk} + v_j A_{ik} + \frac{1}{2} v_k \left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j} \right) - \right.$$
$$\left. - \frac{1}{2c^2} v_k \left(F_i v_j + F_j v_i \right) \right] + \frac{1}{c^4} F_k v_i v_j \,, \quad (1.139)$$

$$\Gamma^0_{00} = -\frac{1}{c^3}\left[\frac{1}{1-\frac{\mathrm{w}}{c^2}} \frac{\partial \mathrm{w}}{\partial t} + \left(1 - \frac{\mathrm{w}}{c^2}\right) v_k F^k \right], \quad (1.140)$$

$$\Gamma^k_{00} = -\frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 F^k, \quad (1.141)$$

$$\Gamma^0_{0i} = \frac{1}{c^2}\left[-\frac{1}{1-\frac{\mathrm{w}}{c^2}} \frac{\partial \mathrm{w}}{\partial x^i} + v_k \left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2} v_i F^k \right) \right], \quad (1.142)$$

$$\Gamma^k_{0i} = \frac{1}{c}\left(1 - \frac{\mathrm{w}}{c^2}\right)\left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2} v_i F^k \right), \quad (1.143)$$

$$\Gamma^0_{ij} = -\frac{1}{c\left(1-\frac{\mathrm{w}}{c^2}\right)} \left\{ -D_{ij} + \frac{1}{c^2} v_n \times \right.$$
$$\times \left[v_j \left(D^n_i + A^{\cdot n}_{i\cdot} \right) + v_i \left(D^n_j + A^{\cdot n}_{j\cdot} \right) + \frac{1}{c^2} v_i v_j F^n \right] +$$
$$\left. + \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} \right) - \frac{1}{2c^2}\left(F_i v_j + F_j v_i \right) - \Delta^n_{ij} v_n \right\}, \quad (1.144)$$

$$\Gamma^k_{ij} = \Delta^k_{ij} - \frac{1}{c^2}\left[v_i \left(D^k_j + A^{\cdot k}_{j\cdot} \right) + v_j \left(D^k_i + A^{\cdot k}_{i\cdot} \right) + \frac{1}{c^2} v_i v_j F^k \right], \quad (1.145)$$

where Δ^i_{jk} are the chr.inv.-Christoffel symbols (1.49).

Expressing the ordinary derivation operators through the chr.inv.-derivation operators, then writing down $dx^0 = c\,dt$ through $d\tau$ (1.22), we obtain a chr.inv.-formula for the regular differential

$$d = \frac{\partial}{\partial x^\alpha} dx^\alpha = \frac{{}^*\partial}{\partial t} d\tau + \frac{{}^*\partial}{\partial x^i} dx^i. \quad (1.146)$$

Denoting the chr.inv.-projections of the P^α as φ and q^i, we have

$$\frac{P_0}{\sqrt{g_{00}}} = \varphi \,, \qquad P^i = q^i, \quad (1.147)$$

$$P^0 = \frac{1}{\sqrt{g_{00}}}\left(\varphi + \frac{1}{c} v_k q^k \right), \qquad P_i = -\frac{\varphi}{c} v_i - q_i \,. \quad (1.148)$$

Substituting these formulae into (1.132), we arrive at the parallel transport chr.inv.-equations of the vector P^α, which are

$$\left.\begin{aligned} & d\varphi + \frac{1}{c}\left(-F_i q^i d\tau + D_{ik}\, q^i dx^k\right) = 0 \\ & dq^i + \left(\frac{\varphi}{c}\, dx^k + q^k d\tau\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \\ & \qquad\qquad - \frac{\varphi}{c}\, F^i d\tau + \Delta^i_{mk}\, q^m dx^k = 0 \end{aligned}\right\}. \quad (1.149)$$

From the obtained equations (1.149) we can make an easy transition to particular dynamical equations of motion, with φ and q^i for different kinds of particles substituted into (1.149) and divided by dt.

1.7 The equations of motion in the extended space

The corpuscular and wave forms of the universal dynamical vector P^α for this case have been obtained in §1.5.

1.7.1 The equations of motion of real mass-bearing particles

From (1.113) and (1.114) we obtain the chr.inv.-projections of the P^α taken in the corpuscular form for real mass-bearing particles

$$\varphi = M\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)\right], \qquad q^i = M\frac{u^i}{c}, \quad (1.150)$$

where $\dfrac{u^2}{\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right]^2} < c^2$, $d\tau \neq 0$, $dt \neq 0$.

From here we immediately arrive at the corpuscular form of the dynamical equations of motion of real mass-bearing particles

$$\left.\begin{aligned} & \frac{d}{dt}\left\{M\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)\right]\right\} - \\ & \qquad - \frac{M}{c^2}\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)\right] F_i u^i + \frac{M}{c^2}\, D_{ik}\, u^i u^k = 0 \\ & \frac{d}{dt}\left(M u^i\right) + 2M\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)\right]\left(D^i_n + A^{\cdot i}_{n\cdot}\right) u^n - \\ & \qquad - M\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)\right] F^i + M\Delta^i_{nk}\, u^n u^k = 0 \end{aligned}\right\}, \quad (1.151)$$

where $d = \frac{{}^*\partial}{\partial t}\, d\tau + \frac{{}^*\partial}{\partial x^i}\, dx^i$, $\frac{d}{d\tau} = \frac{{}^*\partial}{\partial t} + \mathrm{v}^i \frac{{}^*\partial}{\partial x^i}$, and also

$$\frac{d}{dt} = \frac{{}^*\partial}{\partial t}\frac{d\tau}{dt} + u^i \frac{{}^*\partial}{\partial x^i} = \left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_m u^m\right)\right]\frac{{}^*\partial}{\partial t} + u^i \frac{{}^*\partial}{\partial x^i}\,. \quad (1.152)$$

For the wave form of the universal dynamical vector P^α of real mass-bearing particles we obtain, according to (1.118) and (1.120),

$$\varphi = \frac{\hbar}{c^2}\frac{{}^*\partial\psi}{\partial t}\,, \qquad q^i = -\frac{\hbar}{c}\, h^{ik}\frac{{}^*\partial\psi}{\partial x^k}\,, \quad (1.153)$$

where the physically observable change of the wave phase ψ with time, i.e., the chr.inv.-function $\frac{{}^*\partial\psi}{\partial t}$, is positive for the particles travelling from the past to the future, and is negative for those travelling from the future to the past. From here we arrive at the wave form of (1.151), i.e., at the dynamical equations of wave propagation corresponding to real mass-bearing particles according to the wave-particle duality

$$\left.\begin{aligned}
&\pm\frac{d}{d\tau}\left(\frac{{}^*\partial\psi}{\partial t}\right) + \left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_p u^p\right)\right] F^i \frac{{}^*\partial\psi}{\partial x^i} - D^i_k u^k \frac{{}^*\partial\psi}{\partial x^i} = 0 \\
&\frac{d}{d\tau}\left(h^{ik}\frac{{}^*\partial\psi}{\partial x^k}\right) \pm \frac{1}{c^2}\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_p u^p\right)\right]\frac{{}^*\partial\psi}{\partial t} F^i - \\
&\qquad - \left\{\pm\frac{1}{c^2}\frac{{}^*\partial\psi}{\partial t}\, u^k - h^{km}\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_p u^p\right)\right]\frac{{}^*\partial\psi}{\partial x^m}\right\}\times \\
&\qquad\qquad \times\left(D^i_k + A^{\cdot i}_{k\cdot}\right) + h^{mn}\Delta^i_{mk} u^k \frac{{}^*\partial\psi}{\partial x^n} = 0
\end{aligned}\right\}. \quad (1.154)$$

We see that the first term in the time chr.inv.-equation and two terms in the spatial chr.inv.-equations of (1.154) are positive for the wave-particles travelling from the past to the future, and are negative when travelling from the future to the past.

1.7.2 The equations of motion of imaginary mass-bearing particles

In this case, the chr.inv.-projections φ and q^i of the corpuscular vector P^α differ from those for real particles (1.150) by the factor $i = \sqrt{-1}$

$$\varphi = iM\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_k u^k\right)\right], \qquad q^i = iM\frac{u^i}{c}\,, \quad (1.155)$$

where $\dfrac{u^2}{\left[1-\frac{1}{c^2}\left(\mathrm{w}+v_i u^i\right)\right]^2} > c^2$, $d\tau \neq 0$, $dt \neq 0$.

Respectively, the corpuscular form of the dynamical equations of motion of imaginary mass-bearing particles (superluminal particles — tachyons) differ from the equations obtained for real (subluminal) particles (1.151) by the coefficient i in the mass term M.

The chr.inv.-projections φ and q^i of the dynamical wave vector of imaginary mass-bearing particles are the same as those for real particles (1.153). Hence, the dynamical equations of wave propagation are the same for both imaginary wave-particles and real ones (1.154).

1.7.3 The equations of motion of massless particles

According to (1.107), for massless (light-like) particles in the extended space-time (taking the condition $\mathrm{v} = c$ into account) we have

$$\frac{u^2}{\left[1-\frac{1}{c^2}\left(\mathrm{w}+v_i u^i\right)\right]^2} = c^2, \quad d\tau \neq 0\,, \quad dt \neq 0\,. \tag{1.156}$$

Using this condition in the φ and q^i obtained for real mass-bearing particles considered as corpuscles, i.e., in (1.150), we obtain

$$\varphi = M\frac{u}{c}\,, \qquad q^i = M\frac{u^i}{c}\,. \tag{1.157}$$

Respectively, the corpuscular form of the dynamical equations of motion of massless particles is

$$\left.\begin{aligned} &\frac{d}{dt}(Mu) - \frac{Mu}{c^2}F_i u^i + \frac{M}{c}D_{ik}u^i u^k = 0 \\ &\frac{d}{dt}(Mu^i) + 2M\frac{u}{c}\left(D^i_n + A^{\cdot i}_{n\cdot}\right)u^n - \\ &\qquad\qquad - M\frac{u}{c}F^i + M\Delta^i_{nk}u^n u^k = 0 \end{aligned}\right\}. \tag{1.158}$$

The chr.inv.-projections φ and q^i of the dynamical wave vector of massless particles are the same as the φ and q^i of the dynamical wave vector of mass-bearing particles (1.153). As a result, the dynamical equations of wave propagation associated with massless particles in the framework of the wave-particle duality are the same as those we have obtained for mass-bearing wave-particles (1.154).

1.7.4 The equations of motion of zero-particles

In the degenerate space-time, i.e., under the degeneration conditions, the chr.inv.-projections of the P^α taken in the corpuscular form are

$$\varphi = 0\,, \qquad q^i = M\frac{u^i}{c}\,, \tag{1.159}$$

where $\mathrm{w} + v_k u^k = c^2$, $d\tau = 0$, $dt \neq 0$. Substituting them into the chr.inv.-equations of the Levi-Civita parallel transport (1.149), we obtain the corpuscular form of the dynamical equations of motion of zero-particles

$$\left.\begin{aligned} &\frac{M}{c^2}\,D_{ik}\,u^i u^k = 0 \\ &\frac{d}{dt}(Mu^i) + M\Delta^i_{nk}\,u^n u^k = 0 \end{aligned}\right\}. \tag{1.160}$$

The chr.inv.-projections φ and q^i of the wave form of the generalized dynamical vector P^α in the degenerate space-time are

$$\varphi = 0\,, \qquad q^i = -\frac{\hbar}{c}\,h^{ik}\,\frac{{}^*\partial\psi}{\partial x^k}\,, \tag{1.161}$$

from which we arrive at the equations

$$\left.\begin{aligned} &D^m_k\,u^k\,\frac{{}^*\partial\psi}{\partial x^m} = 0 \\ &\frac{d}{dt}\left(h^{ik}\,\frac{{}^*\partial\psi}{\partial x^k}\right) + h^{mn}\Delta^i_{mk}\,u^k\,\frac{{}^*\partial\psi}{\partial x^n} = 0 \end{aligned}\right\}, \tag{1.162}$$

which are the dynamical equations of wave propagation corresponding to zero-particles in the framework of the wave-particle duality.

1.8 The equations of motion in the regular space

In this case, the corpuscular and the wave forms of the universal dynamical vector P^α have been obtained in §1.3.

1.8.1 The equations of motion of real mass-bearing particles

According to (1.86) and (1.75), the corpuscular form of the P^α characteristic of real mass-bearing particles has the chr.inv.-projections

$$\varphi = \pm m\,, \qquad q^i = \frac{1}{c}\,m\mathrm{v}^i, \tag{1.163}$$

where $\mathrm{v}^2 < c^2$, $d\tau \neq 0$, $dt \neq 0$. After substituting these chr.inv.-quantities into (1.149), we obtain the dynamical equations of motion of the mass-bearing particles that have positive relativistic masses $m > 0$ (they travel from the past to the future)

$$\left.\begin{aligned} &\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = 0 \\ &\frac{d(m\mathrm{v}^i)}{d\tau} + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0 \end{aligned}\right\}, \tag{1.164}$$

and also the equations of motion of the particles that have negative relativistic masses $m < 0$ (they travel from the future to the past)

$$\left.\begin{aligned} &-\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = 0 \\ &\frac{d(m\mathrm{v}^i)}{d\tau} + mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0 \end{aligned}\right\}. \tag{1.165}$$

For the wave form of the P^α, from (1.91) and (1.94) we obtain

$$\varphi = \frac{\hbar}{c^2}\frac{{}^*\partial\psi}{\partial t}, \qquad q^i = -\frac{\hbar}{c} h^{ik}\frac{{}^*\partial\psi}{\partial x^k}, \tag{1.166}$$

which are the same as those we have obtained for the wave form of the P^α in the extended space-time (1.153). We see that the chr.inv.-function $\frac{{}^*\partial\psi}{\partial t}$, i.e., the physically observable change of the wave phase with time, is positive when travelling from the past to the future, and is negative when travelling from the future to the past.

Taking into account the above and the fact that the chr.inv.-Levi-Civita parallel transport equations (1.149) in the strictly non-degenerate space-time must be divided by the physically observable time interval $d\tau$, we obtain the dynamical equations of wave propagation associated with mass-bearing real particles

$$\left.\begin{aligned} &\pm\frac{d}{d\tau}\left(\frac{{}^*\partial\psi}{\partial t}\right) + F^i\frac{{}^*\partial\psi}{\partial x^i} - D^i_k \mathrm{v}^k \frac{{}^*\partial\psi}{\partial x^i} = 0 \\ &\frac{d}{d\tau}\left(h^{ik}\frac{{}^*\partial\psi}{\partial x^k}\right) - \left(D^i_k + A^{\cdot i}_{k\cdot}\right)\left(\pm\frac{1}{c^2}\frac{{}^*\partial\psi}{\partial t}\mathrm{v}^k - h^{km}\frac{{}^*\partial\psi}{\partial x^m}\right) \pm \\ &\qquad\qquad \pm\frac{1}{c^2}\frac{{}^*\partial\psi}{\partial t}F^i + h^{mn}\Delta^i_{mk}\mathrm{v}^k\frac{{}^*\partial\psi}{\partial x^n} = 0 \end{aligned}\right\}. \tag{1.167}$$

From the equations (1.167) we see that the first term of the time chr.inv.-equation and two terms of the spatial chr.inv.-equations are positive for the wave-particles travelling from the past to the future, and are negative when travelling from the future to the past.

1.8.2 The equations of motion of imaginary mass-bearing particles

In this case, the corpuscular form of the φ and q^i differ from that obtained for real mass-bearing particles (1.163) by only factor $i = \sqrt{-1}$

$$\varphi = \pm i m, \qquad q^i = \frac{1}{c} i m \mathrm{v}^i, \tag{1.168}$$

where $\mathrm{v}^2 > c^2$, $d\tau \neq 0$, $dt \neq 0$. Respectively, the corpuscular form of the dynamical equations of motion of imaginary (superluminal) particles differ from those we have obtained for real (subluminal) particles by only the coefficient i in the mass term m.

The wave form of φ and q^i for imaginary mass-bearing particles is the same as that for real mass-bearing particles (1.166). Therefore, the dynamical equations of wave propagation corresponding to imaginary mass-bearing particles, are the same as the dynamical equations of wave propagation corresponding to real mass-bearing particles (1.167).

We now see that in the framework of the wave concept there is no difference with what velocity a mass-bearing particle travels (a wave propagates) — slower than the velocity of light or faster than light. On the contrary, in the framework of the corpuscular concept the equations of motion of superluminal (imaginary) particles differ from those of subluminal (real) particles by the presence of the coefficient i in the mass term m.

1.8.3 The equations of motion of massless particles

In this case, the corpuscular form of the φ and q^i takes the form

$$\varphi = \pm \frac{\omega}{c} = \pm k, \qquad q^i = \frac{1}{c} k \mathrm{v}^i = \frac{1}{c} k c^i, \tag{1.169}$$

where $\mathrm{v}^2 = c^2$, $d\tau \neq 0$, $dt \neq 0$, and also the physically observable chr.inv.-velocity of light c^i (1.24) is attributed to any massless particle

$$\mathrm{v}^i = \frac{dx^i}{d\tau} = c^i, \qquad c_i c^i = h_{ik} c^i c^k = c^2. \tag{1.170}$$

Using the above parameters in the parallel transport equations, we obtain the corpuscular dynamical equations of motion

$$\left.\begin{aligned} &\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i + \frac{\omega}{c^2} D_{ik} c^i c^k = 0 \\ &\frac{d(\omega c^i)}{d\tau} + 2\omega \left(D^i_k + A^{\cdot i}_{k\cdot}\right) c^k - \omega F^i + \omega \Delta^i_{nk} c^n c^k = 0 \end{aligned}\right\} \tag{1.171}$$

for the massless particles that have positive relativistic frequencies $\omega > 0$ (they travel from the past to the future), and also

$$\left.\begin{aligned} &-\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i + \frac{\omega}{c^2} D_{ik} c^i c^k = 0 \\ &\frac{d(\omega c^i)}{d\tau} + \omega F^i + \omega \Delta^i_{nk} c^n c^k = 0 \end{aligned}\right\} \tag{1.172}$$

for those having $\omega < 0$ (they travel from the future to the past).

The wave form of the φ and q^i for massless particles is the same as that for mass-bearing particles (1.166). Therefore, the dynamical equations of wave propagation corresponding to massless (light-like) particles in the framework of the wave-particle duality are identical to those of mass-bearing particles in the framework of this concept (1.167). The only difference is the particles' observable velocity v^i replaced with the chr.inv.-vector of the physically observable light velocity c^i.

1.9 A particular case: the equations of geodesic lines

What are the geodesic equations? As we mentioned in §1.1, these are the kinematic equations of particle motion along the shortest (geodesic) trajectories. From a geometric point of view, the geodesic equations are the Levi-Civita parallel transport equations

$$\frac{\mathrm{D}Q^\alpha}{d\rho} = \frac{dQ^\alpha}{d\rho} + \Gamma^\alpha_{\mu\nu} Q^\mu \frac{dx^\nu}{d\rho} = \frac{d^2x^\alpha}{d\rho^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\rho}\frac{dx^\nu}{d\rho} = 0 \tag{1.173}$$

of the four-dimensional kinematic vector $Q^\alpha = \frac{dx^\alpha}{d\rho}$ characteristic of a particle (it is tangential to the trajectory at its every point). Respectively, the non-isotropic geodesic equations (they determine the trajectories of free mass-bearing particles) have the form

$$\frac{\mathrm{D}Q^\alpha}{ds} = \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,, \qquad Q^\alpha = \frac{dx^\alpha}{ds}\,, \tag{1.174}$$

and the isotropic geodesic equations (determining the trajectories of free massless particles) have the form

$$\frac{\mathrm{D}Q^\alpha}{d\sigma} = \frac{d^2x^\alpha}{d\sigma^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0\,, \qquad Q^\alpha = \frac{dx^\alpha}{d\sigma}\,. \tag{1.175}$$

On the other hand any kinematic vector, similar to the dynamical vector P^α of a mass-bearing particle or to the wave vector K^α of a massless particle, is a particular case of an arbitrary vector Q^α, for which we have obtained the universal chr.inv.-equations of the Levi-Civita parallel transport (1.149).

Hence, with the chr.inv.-projections φ and q^i of the kinematic vector of a mass-bearing particle, substituted into the universal chr.inv.-equations of the Levi-Civita parallel transport (1.149), we should immediately arrive at the non-isotropic geodesic equations in the chr.inv.-form. Similarly, substituting the φ and q^i of the kinematic vector of a massless particle, we should arrive at the chr.inv.-equations of isotropic geodesics. This is what we are going to do now.

For the kinematic vector of mass-bearing particles we have the following chr.inv.-projections

$$\left.\begin{aligned} \varphi &= \frac{Q_0}{\sqrt{g_{00}}} = \frac{g_{0\alpha}Q^\alpha}{\sqrt{g_{00}}} = \pm\frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} \\ q^i &= Q^i = \frac{dx^i}{ds} = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{dx^i}{c\,d\tau} = \frac{1}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,\mathrm{v}^i \end{aligned}\right\}. \tag{1.176}$$

For the kinematic vector of massless particles, taking into account that $d\sigma = c\,d\tau$ on isotropic trajectories, we have

$$\left.\begin{aligned} \varphi &= \sqrt{g_{00}}\,\frac{dx^0}{d\sigma} + \frac{1}{c\sqrt{g_{00}}}\,g_{0k}\,c^k = \pm 1 \\ q^i &= \frac{dx^i}{d\sigma} = \frac{dx^i}{c\,d\tau} = \frac{1}{c}\,c^i \end{aligned}\right\}. \tag{1.177}$$

So forth, we substitute the above φ and q^i into the universal chr.inv.-equations of the Levi-Civita parallel transport (1.149). As a result, we obtain the chr.inv.-non-isotropic geodesic equations (trajectories of

mass-bearing free particles)

$$\left.\begin{aligned}
&\pm\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\right)-\frac{F_i\mathrm{v}^i}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}+\frac{D_{ik}\mathrm{v}^i\mathrm{v}^k}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}=0\\
&\frac{d}{d\tau}\left(\frac{\mathrm{v}^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\right)\mp\frac{F^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}+\frac{\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}+\\
&\qquad\qquad+\frac{(1\pm1)}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k=0
\end{aligned}\right\}\tag{1.178}$$

and also the chr.inv.-isotropic geodesic equations (trajectories of massless free particles)

$$\left.\begin{aligned}
&D_{ik}c^ic^k-F_ic^i=0\\
&\frac{dc^i}{d\tau}\mp F^i+\Delta^i_{nk}c^nc^k+(1\pm1)\left(D^i_k+A^{\cdot i}_{k\cdot}\right)c^k=0
\end{aligned}\right\},\tag{1.179}$$

where the upper sign in the alternating terms of the equations stands for the particles travelling from the past to the future (direct flow of time), and the lower sign stands for the particles travelling from the future to the past (reverse flow of time).

As you can see, we again have an asymmetry of motion along the time axis. The same asymmetry was found in the dynamical equations of motion. We see from the above equations, this asymmetry does not depend on the physical properties of the travelling particles, but rather on the properties of the reference space of the observer (actually, on the properties of his reference body), such as F^i, A_{ik}, D_{ik}. In the absence of gravitational fields, as well as rotation and deformation of the observer's reference space, the mentioned asymmetry vanishes.

1.10 A particular case: Newton's laws

In this paragraph we prove that the dynamical chr.inv.-equations of motion of mass-bearing particles are the four-dimensional generalization of Newton's 1st and 2nd laws in a four-dimensional space (space-time) that is non-holonomic (i.e., is rotating, $A_{ik}\neq0$) and deforming ($D_{ik}\neq0$), and is also filled with a gravitational field ($F^i\neq0$).

At low velocities we have $m = m_0$, so the general covariant dynamical equations of motion take the form

$$\frac{\mathrm{D}P^\alpha}{ds} = m_0 \frac{d^2x^\alpha}{ds^2} + m_0 \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,, \tag{1.180}$$

where having these equations divided by m_0, the dynamical equations turn immediately into kinematic ones, i.e., the regular non-isotropic geodesic equations.

These are the dynamical equations of motion of the so-called "free particles" — the particles that fall freely under the action of a gravitational field.

The motion of particles under the action of the gravitational field and an additional non-gravitational force R^α is non-geodesic

$$m_0 \frac{d^2x^\alpha}{ds^2} + m_0 \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = R^\alpha. \tag{1.181}$$

All these are the dynamical equations of motion of particles in the *four-dimensional space-time*, while Newton's laws are determined for the *three-dimensional space*. In particular, the derivation parameter we use in the above four-dimensional equations is the space-time interval, not applicable to a three-dimensional space.

Let us now look at the dynamical chr.inv.-equations of motion of mass-bearing particles. At low velocities of motion, the equations are

$$\left.\begin{aligned} &\frac{m_0}{c^2}\left(D_{ik}\,\mathrm{v}^i\mathrm{v}^k - F_i\,\mathrm{v}^i\right) = 0 \\ &m_0\frac{d^2x^i}{d\tau^2} - m_0 F^i + m_0 \Delta^i_{nk}\,\mathrm{v}^n\mathrm{v}^k + 2m_0\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k = 0 \end{aligned}\right\}, \tag{1.182}$$

where the spatial chr.inv.-projections are the actual dynamical equations of motion along the three-dimensional spatial section associated with the observer (his three-dimensional space).

In a four-dimensional space (space-time), wherein the spatial sections have the Euclidean metric, we have $h^k_i = \delta^k_i$, and the space deformation tensor is zero $D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} = 0$. In such a case $\Delta^i_{kn} = 0$, therefore $m_0\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0$. If there also $F^i = 0$ and $A_{ik} = 0$, then the spatial chr.inv.-projections of the equations of motion take the form

$$m_0 \frac{d^2x^i}{d\tau^2} = 0\,, \tag{1.183}$$

or, in another form,

$$\mathrm{v}^i = \frac{dx^i}{d\tau} = const. \tag{1.184}$$

As a result of the above, we arrive at the conclusion that the four-dimensional generalization of Newton's 1st law for mass-bearing particles can be formulated as follows:

Newton's 1st law

If a particle is free from the action of gravitational inertial forces (or such acting forces are balanced) and, at the same time, the space does not rotate or deform, the particle travels uniformly and rectilinearly.

Such a condition, as is seen from the formulae for the Christoffel symbols (1.140–1.145), is only possible in the case, where all $\Gamma^{\alpha}_{\mu\nu} = 0$, because any component of the Christoffel symbols is a function of at least one of the quantities F^i, A_{ik}, D_{ik}.

Let us now assume that $F^i \neq 0$, but $A_{ik} = 0$ and $D_{ik} = 0$. In such a case, the spatial chr.inv.-equations of motion take the form

$$\frac{d^2x^i}{d\tau^2} = F^i. \tag{1.185}$$

On the other hand, the gravitational potential and the force F^i as well as the quantities A_{ik} and D_{ik} according to their definitions describe the reference body and the local space associated with it. The quantity F^i is the gravitational inertial force acting on a unit-mass particle. The force acting on a particle, the rest-mass of which is m_0 is

$$\Phi^i = m_0 F^i, \tag{1.186}$$

therefore the spatial chr.inv.-equations of motion become

$$m_0 \frac{d^2x^i}{d\tau^2} = \Phi^i. \tag{1.187}$$

Accordingly, the four-dimensional generalization of Newton's 2nd law for mass-bearing particles can be formulated as follows:

Newton's 2nd law

In the space that does not rotate or deform, the acceleration that a particle gains from a gravitational field is proportional to the gravitational inertial force acting on the particle from this field, and inversely proportional to the particle's mass.

Having any particular value of the gravitational inertial force Φ^i substituted into the spatial chr.inv.-equations of motion of a mass-bearing particle, which are the second equation of (1.182),

$$m_0 \frac{d^2 x^i}{d\tau^2} + m_0 \Delta^i_{nk} \mathrm{v}^n \mathrm{v}^k + 2m_0 \left(D^i_k + A^{\cdot i}_{k\cdot} \right) \mathrm{v}^k = \Phi^i, \qquad (1.188)$$

we can solve the equations in order to obtain the three-dimensional observable coordinates of the particle in the three-dimensional space at any moment of time (from which we find the particle's trajectory).

As is seen from the equations, the presence of the gravitational inertial force is not mandatory to make the motion curvilinear and uneven. This happens if at least one of the quantities F^i, A_{ik}, D_{ik} is different from zero. Hence, theoretically, a particle can travel uniformly and curvilinearly in even the absence of gravitational inertial forces, but in the case where the space rotates or deforms (or both of these factors are present in the space).

If a particle travels under the joint action of the gravitational inertial force Φ^i and another force R^i of a non-gravitational nature, the spatial chr.inv.-equations of its motion take the form

$$m_0 \frac{d^2 x^i}{d\tau^2} + m_0 \Delta^i_{nk} \mathrm{v}^n \mathrm{v}^k + 2m_0 \left(D^i_k + A^{\cdot i}_{k\cdot} \right) \mathrm{v}^k = \Phi^i + R^i. \qquad (1.189)$$

In a flat three-dimensional space, there $\Delta^i_{kn} = 0$ is true, so the second term in the equations vanishes. Due to the fact that such a space does not rotate or deform, the spatial chr.inv.-equations of motion of a mass-bearing particle in the space take the form

$$m_0 \frac{d^2 x^i}{d\tau^2} = \Phi^i, \qquad m_0 \frac{d^2 x^i}{d\tau^2} = \Phi^i + R^i, \qquad (1.190)$$

respectively, in the case of only the gravitational inertial force Φ^i, and in the case together with an additional force R^i of a non-gravitational nature, which deviates the particles from a geodesic line.

So, we have obtained that motion under the action of gravitational inertial forces is possible in either curved or flat space. Why?

As is known, the curvature of a Riemannian space is characterized by the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ consisting of the second derivatives of the fundamental metric tensor $g_{\alpha\beta}$ and its first derivatives. The necessary and sufficient condition for a Riemannian space to

be curved is $R_{\alpha\beta\gamma\delta} \neq 0$. To have a non-zero curvature, it is necessary and sufficient that the second derivatives of $g_{\alpha\beta}$ be non-zero.

On the other hand we also know that the first derivatives of the fundamental metric tensor $g_{\alpha\beta}$ in a flat space may not be zero. Namely, — the chr.inv.-equations of motion contain the quantities Δ^i_{kn}, F^i, A_{ik}, D_{ik}, which depend on the first derivatives of $g_{\alpha\beta}$. Therefore, even at $R_{\alpha\beta\gamma\delta} = 0$ (i.e., in a flat space) the Christoffel symbols Δ^i_{kn}, the gravitational inertial force F^i, the space rotation tensor A_{ik} and the space deformation tensor D_{ik} may not be zero.

1.11 Analysis of the equations: the ultimate transitions between the basic space and the zero-space

At $\mathrm{w} = -v_i u^i$ the quantities of the extended space-time ($g \leqslant 0$) are replaced by those of the strictly non-degenerate space-time ($g < 0$)

$$d\tau = \left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right] dt = dt\,, \tag{1.191}$$

$$u^i = \frac{dx^i}{dt} = \frac{dx^i}{d\tau} = \mathrm{v}^i, \tag{1.192}$$

$$M = \frac{m}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)} = m\,, \tag{1.193}$$

$$P^0 = M = m\,, \qquad P^i = \frac{1}{c} M u^i = \frac{1}{c} m \mathrm{v}^i, \tag{1.194}$$

and the coordinate time t coincides with the physically observable time τ in this transition.

Of course, if $\mathrm{w} \to 0$ (a weak gravitational field) and $v_i = 0$ (the space does not rotate), then the above transformation occurs under a narrower condition $\mathrm{w} = -v_i u^i = 0$. On the other hand, it is doubtful to find a region free of rotation and gravitational fields in the observed part of the Universe. We therefore see that the transition to the regular (strictly non-degenerate) space-time always happens at

$$\mathrm{w} = -v_i u^i = -v_i \mathrm{v}^i. \tag{1.195}$$

Substituting this condition into the equations of motion, which we have obtained in §1.7 and §1.8, we arrive at the following conclusions on the geometric structure of the extended space-time.

The corpuscular equations of motion (*ball-particles*) in the extended space-time transform *completely* into those in the regular (strictly non-degenerate) space-time, i.e., no terms are vanished or new terms are added up, only in the case of motion from the past to the future ($m > 0$, $im > 0$, $\omega > 0$). For ball-particles travelling from the future to the past ($m < 0$, $im < 0$, $\omega < 0$), such a transformation is *incomplete*.

On the other hand, the wave equations of motion (*wave-particles*) in the extended space-time transform *completely* into those in the regular space-time for both the particles with $m > 0$, $im > 0$, $\omega > 0$ (they travel from the past to the future) and the particles with $m < 0$, $im < 0$, $\omega < 0$ (they travel from the future to the past).

In the next §1.12 we will find why this asymmetry takes place in the four-dimensional space (space-time).

In the regular space-time ($g < 0$) we have P^0 (1.75), which after the substitution of $\frac{dt}{d\tau}$ (1.81) and the transition condition $\mathrm{w} = -v_i u^i = -v_i \mathrm{v}^i$ becomes the sign-alternating relativistic mass

$$P^0 = m\frac{dt}{d\tau} = \frac{m}{1 - \frac{\mathrm{w}}{c^2}}\left(\frac{1}{c^2}\, v_i \mathrm{v}^i \pm 1\right) = \pm m\,. \tag{1.196}$$

On the other hand, in the extended space-time ($g \leqslant 0$), we have obtained $P^0 = M$, but using another method (1.112), without the use of $\frac{dt}{d\tau}$ which is the source of the alternating sign in the formula (1.196).

Hence, the component $P^0 = \pm m$ we have obtained in the regular space-time (1.196), which takes two numerical values, cannot be a particular case of the single value $P^0 = M$ we have obtained in the extended space-time.

To understand the reason why, we turn from the sign-alternating formula $P^0 = \pm m$ specific to the regular space-time to the formula $P^0 = M$ specific to the extended space-time. This can be easily done by substituting the already known relation between the physically observable velocity v^i and the coordinate velocity u^i (1.107) into the sign-alternating formula $P^0 = \pm m$ (1.196).

Thus, we obtain the formula for the component P^0 in the extended space-time

$$P^0 = \frac{m}{1 - \frac{\mathrm{w}}{c^2}}\left[\frac{1}{c^2}\,\frac{v_i u^i}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)} \pm 1\right], \tag{1.197}$$

which has the alternating sign.

For the particles travelling in the extended space-time from the past to the future,

$$P^0 = \frac{m}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)} = +M, \tag{1.198}$$

which is the same as (1.112). For the particles travelling from the future to the past, we have

$$P^0 = \frac{m\left[\frac{1}{c^2}\left(2v_i u^i + \mathrm{w}\right) - 1\right]}{\left(1 - \frac{\mathrm{w}}{c^2}\right)\left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right]} = -M. \tag{1.199}$$

In the regular space-time, the first formula $P^0 = +M$ (1.198) unambiguously transforms into $P^0 = +m$, and the second formula $P^0 = -M$ (1.199) transforms into $P^0 = -m$.

It should be noted that the remarks made on the sign-alternating formulae for the P^0 do not affect all the dynamical equations of motion, which we have obtained for the extended space-time. This is because the obtained equations of motion include the gravitational rotational mass in the general notation, M, without any respect to a particular composition of it. Substituting these two values of the M into the equations of motion, we arrive at merely the equations of two kinds: the equations of motion from the past to the future, and the equations of motion from the future to the past.

Let us now come back to the physical condition $\mathrm{w} = -v_i u^i$ (1.195), which indicates the transition from the dynamical equations of motion in the extended space-time to those in the regular space-time. We have also found that $d\tau = dt$ (1.191) under this condition. On the other hand, we know that the equality $d\tau = dt$ is not imperative in the regular space-time. On the contrary, the physically observable time interval $d\tau$ is almost always a bit different from the coordinate time interval dt in the observed Universe.

Therefore, the ultimate transition from the extended space-time to the regular space-time, which occurs under the condition $\mathrm{w} = -v_i u^i$ is not a case of the conditions usual to the regular space-time.

Does that contain a contradiction between the equations of motion in the regular space-time and those in the extended space-time?

No it does not. All the laws applicable to the regular space-time ($g < 0$) are as well true in a non-degenerate region ($g < 0$) of the extended space-time ($g \leqslant 0$). At the same time, those two non-degenerate

regions *are not the same*. That is, the degenerate space-time added up to the regular space-time produces two absolutely segregated manifolds. The extended space-time is a different manifold and is absolutely independent of either strictly non-degenerate space-time or degenerate one. So, there is no surprise in the found fact that the transition from one to another occurs under very limited particular conditions.

The only question is what configuration of those manifolds exists in the observable Universe. Two options are possible here:

a) The non-degenerate space-time ($g < 0$) and the degenerate space-time ($g = 0$) exist as *two segregated manifolds*, i.e., as the regular space-time of the General Theory of Relativity with an "add-on" of the zero-space;
b) The non-degenerate space-time and the degenerate space-time exist as *two internal regions of the same manifold* — the extended space-time ($g \leqslant 0$) which we have considered.

In any case, the ultimate transition from the non-degenerate space-time into the degenerate space-time occurs under the physical conditions of degeneration (1.106). Future experiments and astronomical observations will show which of these two options actually exists.

1.12 Analysis of the equations: asymmetry of our Universe and the mirror universe

Compare the corpuscular equations of motion for the particles with $m > 0$ (1.164) and $\omega > 0$ (1.171) with those for the particles with $m < 0$ (1.165) and $\omega < 0$ (1.172).

Even a first look shows the obvious fact that the corpuscular equations of motion (ball-particles) from the past to the future differ from those from the future to the past. The same asymmetry exists for the wave form of the equations (wave-particles). Why?

From a purely geometric point of view, the asymmetry found in the equations of motion indicates the following:

> In the four-dimensional, inhomogeneous and curved space-time (pseudo-Riemannian space), there is a *primordial asymmetry* of the directions to the future and to the past.

To understand the origin of this primordial asymmetry of motion in time, consider the following example.

Assume that there is a *mirror* in the four-dimensional space-time, which coincides with the three-dimensional spatial section and, hence, separates the past from the future. Assume also that the mirror reflects all the particles and waves coming on it from the past and from the future. Thus, the particles and waves that travel from the past to the future ($m > 0$, $im > 0$, $\omega > 0$) always hit the mirror, then bounce back to the past so that their properties reverse ($m < 0$, $im < 0$, $\omega < 0$). At the same time, the particles and waves travelling from the future to the past ($m < 0$, $im < 0$, $\omega < 0$), when hitting the mirror change the sign of their properties ($m > 0$, $im > 0$, $\omega > 0$) to bounce back to the future.

With the aforementioned *mirror concept* everything becomes easy to understand. Look at the wave form of the equations of motion (1.167). After reflection from the mirror, the quantity $\frac{{}^{*}\partial\psi}{\partial t}$ changes its sign. As a result, the equations of wave propagation to the future ("plus" in the equations) become those of wave propagation to the past ("minus" in the equations), and vice versa, the equations of wave propagation to the past after reflection become those of wave propagation to the future.

Noteworthy, the equations of wave propagation to the future and those to the past transform *completely* into each other, i.e., no terms are vanished and no new terms are added up. Hence, the wave form of matter is *completely reflected* from the mirror.

However, this is not the case of the corpuscular equations of motion. After reflection from the mirror, the quantity $\varphi = \pm m$ for mass-bearing particles and also $\varphi = \pm k = \pm \frac{\omega}{c}$ for massless particles change their signs. At the same time, the corpuscular equations of motion to the future transform *incompletely* into those to the past. In the spatial chr.inv.-equations of motion to the future, there is an additional term. This term is absent in the spatial chr.inv.-equations of motion to the past. This term for mass-bearing and massless particles, respectively, has the form

$$2m\left(D^{i}_{k} + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^{k}, \qquad 2k\left(D^{i}_{k} + A^{\cdot i}_{k\cdot}\right)c^{k}. \tag{1.200}$$

Hence, a particle that travels from the past to the future hits the mirror and bounces back to lose the term in its spatial chr.inv.-equations of motion, and vice versa, a particle travelling from the future to the past bounces from the mirror to acquire the additional term in the spatial chr.inv.-equations of motion. So, we have obtained that the mirror itself affects the trajectories of particles!

As a result, a particle with a negative mass or frequency *is not* a simple mirror reflection of a particle, the mass or frequency of which is positive. In both the case of ball-particles and in the case of wave-particles we do not deal with simple reflection or bouncing from the mirror, but with *passing through the mirror* into the *mirror world*. There, in the mirror world, all particles have negative masses or frequencies and travel from the future to the past (from the viewpoint of an observer, whose home is our world).

The wave-particles of our world do not act on the mirror world, just as the wave-particles of the mirror world do not act on us. On the contrary, the ball-particles of our world can influence the mirror world, and the ball-particles of the mirror world can have influence on our world.

The complete isolation of our world from the mirror world, i.e., the absence of mutual influence between the particles of both worlds, takes place under the condition

$$D^i_k \mathrm{v}^k = -A^{\cdot i}_{k\cdot} \mathrm{v}^k, \tag{1.201}$$

that the asymmetric term (1.200) of the corpuscular equations of motion is equal to zero. This happens only if $A^{\cdot i}_{k\cdot} = 0$ and $D^i_k = 0$, i.e., in a region, where the space does not rotate or deform.

It is noteworthy that if particles of positive mass (frequency) coexisted in our world with particles of negative mass (frequency), then they would inevitably face with destroying each other, so there would be no particles left in our world. But we see nothing of the kind.

Therefore, in the second part of our analysis of the obtained equations of motion we arrive at the following conclusions:

1) The primordial (fundamental) asymmetry of the space-time directions to the future and to the past is due to a certain space-time mirror, which coincides geometrically with the spatial section of the observer and reflects all particles and waves that bounce off it from the past or the future. At the same time, the space-time mirror maintains such physical conditions that are very different from those in the regular space-time, and correspond to the particular physical conditions in a completely degenerate region of the space-time (zero-space), wherein the physically observable time stops. Therefore, we arrive at the obvious conclusion that the rôle of such a space-time mirror is played by the entire zero-space or a particular region in it;

2) The space-time is divided into our world and the mirror world. In our world (positive relativistic masses and frequencies) all particles and waves travel from the past to the future. In the mirror world (negative relativistic masses and frequencies), all particles and waves travel from the future to the past;
3) If you enter the mirror world through the mirror, then the particles and waves of our world will appear to have negative masses and frequencies and travel from the future to the past;
4) We do not observe either particles with negative masses or frequencies, or waves with negative phases, because they exist in the mirror world, i.e., beyond the mirror. The particles or waves that we can observe belong to our world, or those that are at the exit from the mirror (or when rebounding from the mirror, as it seems to us), since they came from the mirror world. Therefore, all particles and waves that we can observe travel from the past to the future.

1.13 The physical conditions characterizing the direct and reverse flow of time

Here we will consider the physical conditions under which: a) time flows from the past to the future, b) time flows in the opposite direction, i.e., from the future to the past, and c) time stops.

In the General Theory of Relativity, time is determined as the fourth coordinate $x^0 = ct$ of the four-dimensional space-time, where c is the velocity of light, and t is the time coordinate. This formula itself indicates that the coordinate time t changes with the velocity of light and does not depend on the physical conditions of observation. Therefore, the coordinate time t is also called the *ideal time*. In addition to the ideal time, there is the physically observable time τ (*real time*), which strongly depends on the conditions of observation. The theory of chronometric invariants determines the physically observable time interval as the chr.inv.-projection of the four-dimensional coordinate increment dx^α on the time line of the observer

$$d\tau = \frac{1}{c}\, b_\alpha\, dx^\alpha. \tag{1.202}$$

According to the chronometrically invariant formalism, in the reference frame accompanying an ordinary subluminal (substantial) ob-

server, $d\tau$ is determined on the basis of (1.22), i.e.

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i\, dx^i = dt - \frac{1}{c^2}\, \mathrm{w}\, dt - \frac{1}{c^2}\, v_i\, dx^i. \quad (1.203)$$

From here we see that $d\tau$ consists of three parts: a) the coordinate time interval dt, b) the "gravitational" time interval $dt_{\mathrm{g}} = \frac{1}{c^2}\,\mathrm{w}\, dt$, and c) the "rotational" time interval $dt_{\mathrm{r}} = \frac{1}{c^2}\, v_i\, dx^i$. The stronger the gravitational field of the reference body and the faster the observer's reference space rotates, the slower the observable time flow $d\tau$ of the observer. Theoretically, a strong enough gravitational field and a fast enough rotation of the space can stop the physically observable time flow.

We define the *mirror world* as the space-time, where time flows backward with respect to the time flow in our own reference frame, located in our space-time.

The direction of the coordinate time flow t, which describes the displacement along the time coordinate axis $x^0 = c\,t$, is indicated by the sign of the derivative $\frac{dt}{d\tau}$. Respectively, the sign of the derivative $\frac{d\tau}{dt}$ indicates the direction of the physically observable time flow τ.

In §1.3, we have obtained the coordinate time function $\frac{dt}{d\tau}$ (1.81), which comes from the conservation condition of the four-dimensional velocity of a subluminal, light-like and superluminal particle along its four-dimensional trajectory (1.77–1.79). On the other hand, the coordinate time function can also be obtained in another way by representing the space-time interval $ds^2 = c^2 d\tau^2 - d\sigma^2$ as

$$ds^2 = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 - 2\left(1 - \frac{\mathrm{w}}{c^2}\right) v_i\, dx^i dt + g_{ik}\, dx^i dx^k. \quad (1.204)$$

From here we see that the elementary space-time distance between two infinitely adjacent world-points consists of the three-dimensional coordinate distance $g_{ik}\, dx^i dx^k$, as well as two terms that depend on the physical properties of the space (space-time).

The term $\left(1 - \frac{\mathrm{w}}{c^2}\right) c\, dt$ is due to the fourth dimension (time) and the gravitational potential w that characterizes the field of the reference body. In the absence of gravitational fields, the time coordinate $x^0 = c\,t$ changes evenly with the velocity of light. As soon as $\mathrm{w} \neq 0$, the coordinate x^0 changes "slower" by the amount of $\frac{\mathrm{w}}{c^2}$. The stronger gravitational potential w, the slower time flows. At $\mathrm{w} = c^2$ the coordinate time t stops

completely. As is known, such a condition is realized in the state of gravitational collapse.

The term $\left(1-\frac{\mathrm{w}}{c^2}\right) v_i\, dx^i dt$ is due to the joint action of the gravitational inertial force and the space rotation. This term is non-zero only if $\mathrm{w} \neq c^2$ (i.e., out the state of gravitational collapse) and $v_i \neq 0$ (the space is non-holonomic, i.e., the three-dimensional space rotates).

Having both parts of (1.204) divided by $ds^2 = c^2 d\tau^2 \left(1-\frac{\mathrm{v}^2}{c^2}\right)$, we obtain a quadratic equation that is the same as (1.80). The equation has two solutions (1.81). Proceeding from the solutions (1.81), we see that the coordinate time increases $\frac{dt}{d\tau} > 0$, stops $\frac{dt}{d\tau} = 0$ and decreases $\frac{dt}{d\tau} < 0$ under the following conditions, respectively,

$$\frac{dt}{d\tau} > 0 \qquad \text{if} \quad v_i \mathrm{v}^i > \pm c^2, \tag{1.205}$$

$$\frac{dt}{d\tau} = 0 \qquad \text{if} \quad v_i \mathrm{v}^i = \pm c^2, \tag{1.206}$$

$$\frac{dt}{d\tau} < 0 \qquad \text{if} \quad v_i \mathrm{v}^i < \pm c^2. \tag{1.207}$$

As is known, the regular particles consisting of substance, which we observe, travel with the velocities that are slow to the velocity of light. So, the physical condition by which the coordinate time stops $v_i \mathrm{v}^i = \pm c^2$ (1.206) cannot be found in the world of substance, but is permitted for the other states of matter such as light-like matter, for instance.

The coordinate time increases, i.e., $\frac{dt}{d\tau} > 0$, at $v_i \mathrm{v}^i > \pm c^2$. In an ordinary laboratory, the linear velocity with which the space rotates, e.g., the linear velocity of the daily rotation of the Earth, is also slow to the velocity of light. Hence, in an ordinary laboratory we have $v_i \mathrm{v}^i > -c^2$, where the angle α between the space rotation velocity and the observable velocity of the particle that we observe is within the limits $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. In such a regular case, the coordinate time flows from the past to the future, and this is how the particle travels.

Respectively, the coordinate time decreases, i.e., $\frac{dt}{d\tau} < 0$ and the coordinate time flows from the future to the past (and this is how the particle travels), at $v_i \mathrm{v}^i < \pm c^2$.

Until now we have only considered the flow of the coordinate time t. Let us now analyse the possible directions of the physically observable time τ, which depends on the sign of the derivative $\frac{d\tau}{dt}$. To obtain a

formula for this derivative, we divide the formula that we have obtained for $d\tau$ (1.203) by dt. We obtain

$$\frac{d\tau}{dt} = 1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right). \tag{1.208}$$

By definition, the clock of any ordinary observer registers always positive intervals of time irrespective of the direction in which the clock's hands rotate. Therefore, in an ordinary laboratory bound on the Earth, the physically observable time may increase or stop, but it never decreases. Nevertheless, the decrease of the observable time, i.e., $\frac{d\tau}{dt} < 0$, is possible under certain circumstances.

From (1.208) we see that the observable time increases, stops, or decreases under the following conditions, respectively,

$$\frac{d\tau}{dt} > 0 \qquad \text{if} \quad \mathrm{w} + v_i u^i < c^2, \tag{1.209}$$

$$\frac{d\tau}{dt} = 0 \qquad \text{if} \quad \mathrm{w} + v_i u^i = c^2, \tag{1.210}$$

$$\frac{d\tau}{dt} < 0 \qquad \text{if} \quad \mathrm{w} + v_i u^i > c^2. \tag{1.211}$$

It is obvious that the condition by which the observable time stops $\mathrm{w} + v_i u^i = c^2$ is the space-time degeneration condition (1.106). In a particular case, where the space does not rotate, the physically observable time stops with gravitational collapse $\mathrm{w} = c^2$.

Generally speaking, the state of zero-space can be given by any of the whole scale of the physical conditions represented as $\mathrm{w} + v_i u^i = c^2$. The state of gravitational collapse ($\mathrm{w} = c^2$) is only a particular case in the scale of the conditions, which occurs in the absence of the space rotation ($v_i = 0$). In other words, the *mirror membrane* between the world with the direct flow of time and the mirror world with the reverse flow of time is not a specific zero-space region, wherein gravitational collapse occurs, but the zero-space as a whole.

So, what is the flow of the coordinate time t and what is the flow of the physically observable time τ?

In the coordinate time function $\frac{dt}{d\tau}$, we assume that the real time τ registered by an observer is the reference to which the coordinate time t is compared. In any calculation or observation, we are connected with the observer himself. So, the coordinate time function $\frac{dt}{d\tau}$ indicates the

motion of the observer along the time axis $x^0 = ct$, registered from his own viewpoint.

In the observable time function $\frac{d\tau}{dt}$, the reference to which the observer compares his measurements is his time coordinate t. That is, the physically observable time τ registered by the observer is determined with respect to the motion of the whole spatial section associated with him along the time axis (this motion occurs evenly at the velocity of light). Therefore, the observable time function $\frac{d\tau}{dt}$ indicates the observer's *true motion* along the time axis.

In other words, the coordinate time function $\frac{dt}{d\tau}$ shows the membrane between our world and the mirror world from the point of view of the observer himself (his logic recognizes the observable time to be always flowing from the past to the future).

On the contrary, the observable time function $\frac{d\tau}{dt}$ gives an abstract glimpse of the membrane from "outside". This means that the observable time function indicates the *true structure* of the space-time membrane that separates our world and the mirror world, wherein time flows in the opposite direction.

1.14 Introducing the mirror Universe

To obtain a more detailed view of the space-time membranes, we are going to use a *locally geodesic reference frame*. The fundamental metric tensor within the infinitesimal vicinity of any point in such a frame is

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{\mu\nu}}{\partial \tilde{x}^\rho \partial \tilde{x}^\sigma}\right)(\tilde{x}^\rho - x^\rho)(\tilde{x}^\sigma - x^\sigma) + \dots , \qquad (1.212)$$

i.e., the $g_{\mu\nu}$ components in the infinitesimal vicinity of any point have numerical values that differ from those at the point itself only in the 2nd order terms and higher-order terms, which can be neglected. Therefore, the fundamental metric tensor is constant (within the higher-order terms withheld) at any point of a locally geodesic reference frame, while the first derivatives of the metric tensor, i.e., the Christoffel symbols, are zero [3–5].

It is obvious that within the infinitesimal vicinity of any point in a Riemannian space a locally geodesic reference frame can be set up. As a result, at any point belonging to the locally geodesic reference frame, a flat space can be set up tangential to the Riemannian space so that the

locally geodesic reference frame in the Riemannian space is a globally geodesic reference frame in the tangential flat space. Since the fundamental metric tensor is constant in a flat space, in the vicinity of a point in the Riemannian space, the $\tilde{g}_{\mu\nu}$ components converge to those of the tensor $g_{\mu\nu}$ in the tangential flat space. This means that, in the tangential flat space, we can set up a system of basis vectors $\vec{e}_{(\alpha)}$ tangential to the curved coordinate lines of the Riemannian space. Since the coordinate lines of a Riemannian space are curved (in a general case), and, if the space is non-holonomic, are not even orthogonal to each other, the lengths of the basis vectors are sometimes substantially different from unit length.

Consider the world-vector $d\vec{r}$ of an infinitesimal displacement, i.e., $d\vec{r} = \left\{dx^0, dx^1, dx^2, dx^3\right\}$. Then $d\vec{r} = \vec{e}_{(\alpha)} dx^\alpha$, where the basis vectors $\vec{e}_{(\alpha)}$ have the following components

$$\left.\begin{aligned} \vec{e}_{(0)} &= \left\{e^0_{(0)}, 0, 0, 0\right\}, \qquad & \vec{e}_{(1)} &= \left\{0, e^1_{(1)}, 0, 0\right\} \\ \vec{e}_{(2)} &= \left\{0, 0, e^2_{(2)}, 0\right\}, \qquad & \vec{e}_{(3)} &= \left\{0, 0, 0, e^3_{(3)}\right\} \end{aligned}\right\}. \tag{1.213}$$

The scalar product of the vector $d\vec{r}$ with itself gives $d\vec{r}d\vec{r} = ds^2$. On the other hand, it is $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. So, we obtain a formula

$$g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)} e_{(\beta)} \cos\left(x^\alpha; x^\beta\right), \tag{1.214}$$

which facilitates our better understanding of the geometric structure of various regions within the Riemannian space and even beyond it. According to (1.214),

$$g_{00} = e^2_{(0)}\,, \tag{1.215}$$

while, on the other hand, $\sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$. Therefore, the length of the time basis vector $\vec{e}_{(0)}$ tangential to the real time line $x^0 = ct$ is

$$e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,. \tag{1.216}$$

The smaller $e_{(0)}$ is than 1, the stronger the gravitational potential w. In the case of gravitational collapse ($\mathrm{w} = c^2$), the length of the time basis vector $\vec{e}_{(0)}$ becomes zero.

According to (1.214) the g_{0i} components are

$$g_{0i} = e_{(0)} e_{(i)} \cos\left(x^0; x^i\right), \tag{1.217}$$

on the other hand, $g_{0i} = -\frac{1}{c}\, v_i \left(1 - \frac{\mathrm{w}}{c^2}\right) = -\frac{1}{c}\, v_i e_{(0)}$. Hence

$$v_i = -c\, e_{(i)} \cos\left(x^0; x^i\right). \tag{1.218}$$

Then according to the general formula (1.214)

$$g_{ik} = e_{(i)} e_{(k)} \cos\left(x^i; x^k\right), \tag{1.219}$$

we obtain the chr.inv.-metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2}\, v_i v_k$ in the form

$$h_{ik} = e_{(i)} e_{(k)} \left[\cos\left(x^0; x^i\right) \cos\left(x^0; x^k\right) - \cos\left(x^i; x^k\right)\right]. \tag{1.220}$$

Based on (1.218), we realize that, from a geometric point of view, v_i is the projection (scalar product) of the spatial basis vector $\vec{e}_{(i)}$ onto the time basis vector $\vec{e}_{(0)}$, multiplied by the velocity of light. If the spatial sections are everywhere orthogonal to the time lines (a holonomic space), then $\cos\left(x^0; x^i\right) = 0$ and $v_i = 0$. In a non-holonomic space, the spatial sections are non-orthogonal to the time lines, so $\cos\left(x^0; x^i\right) \neq 0$. Generally $\left|\cos\left(x^0; x^i\right)\right| \leqslant 1$. Hence, the linear velocity v_i (1.218) with which the space rotates cannot exceed the velocity of light.

If $\cos\left(x^0; x^i\right) = \pm 1$, then the space rotation velocity is

$$v_i = \mp c\, e_{(i)}\,, \tag{1.221}$$

and the time basis vector $\vec{e}_{(0)}$ coincides with the spatial basis vectors $\vec{e}_{(i)}$ (the time axis "falls" into the three-dimensional space). In the case of $\cos\left(x^0; x^i\right) = +1$, the time basis vector is co-directed with the spatial ones $\vec{e}_{(0)} \uparrow\uparrow \vec{e}_{(i)}$. At $\cos\left(x^0; x^i\right) = -1$ the time and spatial basis vectors are oppositely directed $\vec{e}_{(0)} \uparrow\downarrow \vec{e}_{(i)}$.

Let us have a closer look at the condition $\cos\left(x^0; x^i\right) = \pm 1$. If any spatial basis vector is co-directed (or oppositely directed) with the time basis vector, then the space is degenerate. A maximum degeneration occurs, when all three vectors $\vec{e}_{(i)}$ coincide with each other and with the time basis vector $\vec{e}_{(0)}$.

The terminal condition of the coordinate time $v_i \mathrm{v}^i = \pm c^2$ expressed through the basis vectors has the form

$$e_{(i)} \mathrm{v}^i \cos\left(x^0; x^i\right) = \mp c\,, \tag{1.222}$$

which is performed in practice, when we have $e_{(i)} = 1$, $\mathrm{v} = c$ and, hence, $\cos\left(x^0; x^i\right) = \pm 1$. In such a case, as soon as the linear velocity of the

space rotation reaches the velocity of light, the angle between the time line and the spatial coordinate lines becomes either zero or π (depending on the direction in which the space rotates).

Let us illustrate the above with a few examples.

The space does not rotate, i.e., is holonomic

In this case $v_i = 0$, therefore the spatial sections are everywhere orthogonal to the time lines, and the angle between them is $\alpha = \frac{\pi}{2}$. Hence, in the absence of the space rotation, the time basis vector $\vec{e}_{(0)}$ is orthogonal to all spatial basis vectors $\vec{e}_{(i)}$.

This means that all clocks can be synchronized: they display the same time (clock synchronization at different points in the space does not depend on the synchronization path). The linear velocity with which the space rotates is $v_i = -c\, e_{(i)} \cos\alpha = 0$. At $v_i = 0$ we have

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) c\, dt\,, \qquad h_{ik} = -g_{ik}\,, \tag{1.223}$$

and the space-time metric $ds^2 = c^2 d\tau^2 - d\sigma^2$ becomes

$$ds^2 = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k, \tag{1.224}$$

i.e., the physically observable time (1.223) depends only on the gravitational potential w. Two options are possible here:

a) The gravitational inertial force is $F_i = 0$, and $v_i = 0$ according to our initial assumption that the space does not rotate. Thus, according to the definitions of F_i and v_i (see §1.2), we obtain $\sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} = 1$ and $g_{0i} = -\frac{1}{c}\sqrt{g_{00}}\, v_i = 0$. The fact that the gravitational potential w vanishes means, in particular, that it does not depend on the three-dimensional coordinates (a homogeneously distributed gravitational field). In this case, the motion of an observer across the space leaves his clocks the same (the global synchronization of clocks remains unchanged with time);

b) If $F_i \neq 0$ and $v_i = 0$, then we have the derivative $\frac{\partial \mathrm{w}}{\partial x^i} \neq 0$ in the formula for F_i (1.34). This means that the gravitational potential w depends on the three-dimensional coordinates, i.e., the clock time differs at different points of the space. Hence, at $F_i \neq 0$ the clock synchronization at different points of a holonomic (non-rotating) space does not preserve with time.

In a holonomic (non-rotating) space, gravitational collapse may occur ($\mathrm{w} = c^2$) only if $F_i \neq 0$. If $F_i = 0$ in a holonomic space, according to the definition of F_i (1.34) we have $\mathrm{w} = 0$, therefore gravitational collapse is not possible.

The space rotates at subluminal speed

In this case, the spatial sections are not orthogonal to the time lines $v_i = -c\, e_{(i)} \cos\alpha \neq 0$. Because $-1 \leqslant \cos\alpha \leqslant +1$, we have $-c \leqslant v_i \leqslant +c$. Hence, $v_i > 0$ at $\cos\alpha > 0$, and also $v_i < 0$ at $\cos\alpha < 0$.

The space rotates at the speed of light (1st case)

The lesser α, the greater v_i. In the limiting case, where $\alpha = 0$, the linear velocity with which the space rotates is $v_i = -c$. In this case, the spatial basis vectors $\vec{e}_{(i)}$ coincide with the time basis vector $\vec{e}_{(0)}$ (the space coincides with time).

The space rotates at the speed of light (2nd case)

If $\alpha = \pi$, then $v_i = +c$ and the time basis vector $\vec{e}_{(0)}$ also coincides with the spatial basis vectors $\vec{e}_{(i)}$, but is oppositely directed to them. This case can be understood as a space coinciding with time flowing from the future to the past.

1.15 Who is a superluminal observer?

We can outline a few types of the reference frames which are possible in the space-time of General Relativity.

The particles and any observer travelling with a subluminal velocity (i.e., "inside" the light cone) have real relativistic masses. In other words, such particles, the observer and his reference body are in the state of matter that is commonly referred to as "substance". Therefore, we call any observer, whose reference frame is subluminal, a *subluminal observer*.

The particles and any observer travelling with the velocity of light (i.e., along the surface of the light cone) have $m_0 = 0$, but their relativistic masses (masses of motion) are $m \neq 0$. They are in the light-like state of matter. Therefore, we call any observer, whose reference frame is characterized by the light-like state, a *light-like observer*.

Respectively, we call the particles and any observer travelling with a superluminal velocity *superluminal particles* and a *superluminal observer*. They are in the state of matter, where $m_0 \neq 0$, and their relativistic masses are imaginary.

It is intuitively clear who a subluminal observer is; this term does not require further explanation. The same more or less applies to a light-like observer. From the viewpoint of a light-like observer, the world around looks like a colourful system of light waves. But who is a superluminal observer? To understand this, let us consider an example.

Imagine a new supersonic jet airplane to be commissioned into operation. All members of the commission are inborn blind. And so is the pilot. Thus, we may assume that all information about the surrounding world the pilot and the commission members gain from sound, i.e., from sound waves in the air. It is sound waves that create a picture that those people will perceive as their "real world".

Now the airplane has taken off and begun to accelerate. As long as its speed is less than the speed of sound in the air, the blind members of the commission match up its "heard" position in the sky with the one we see. But once the sound barrier is overcome, everything changes. The blind members of the commission still perceive the airplane's velocity as equal to the sound speed, regardless of its actual speed. For them, the speed of propagation of sound waves in the air is the *maximum speed of information propagation*, and the real supersonic jet airplane is outside their "real world", it is in the world of "imaginary objects" and all its properties are imaginary from their point of view. A blind pilot cannot hear anything either. Not a single sound reaches him from the past reality, and only local sounds from the cockpit (which also flies at the supersonic speed) break the silence. When overcoming the speed of sound, the blind pilot leaves the subsonic world for a new supersonic one. From his new point of view (supersonic reference frame), the old subsonic fixed world containing the airport and the commission members will simply disappear, turning into a region of "imaginary values".

What is light? These are transverse waves travelling in a certain medium at a constant speed. We perceive the world around us through vision, receiving light waves from other objects. It is the waves of light that create our picture of the "true real world".

Now imagine a spaceship accelerating faster and faster to eventually overcome the light barrier, while still increasing its speed. From a mathematical point of view, this is quite possible in the space-time of General Relativity. For us, the spaceship's speed will still be equal to the speed of light, whatever its actual speed. In the case, where we use light signals to "synchronize" the world around us, for us the velocity

of light is the maximum speed of information propagation, while the real spaceship for us stays in another "unreal" world of superluminal velocities, wherein all properties are imaginary. The same is true for the spaceship's pilot. From his viewpoint, overcoming the light barrier brings him into a new superluminal world, which becomes his "true reality", and the old world of subluminal speeds is gone, left behind in the region of "imaginary reality".

1.16 Gravitational collapse in different regions of space

We will call a *gravitational collapsar* a space-time region, wherein the gravitational collapse condition $g_{00} = 0$ is true. According to the theory of chronometric invariants, $\sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$. Hence, the collapse condition $g_{00} = 0$ also means $\mathrm{w} = c^2$. We will consider such a collapsing region "from outside", from the viewpoint of an ordinary observer.

Consider the formula for the four-dimensional interval so that it contains an explicit ratio of w and c^2, i.e.

$$ds^2 = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 - 2\left(1 - \frac{\mathrm{w}}{c^2}\right) v_i\, dx^i dt + g_{ik}\, dx^i dx^k. \quad (1.225)$$

Having substituted $\mathrm{w} = c^2$ into this formula, we obtain the space-time metric on the surface of a gravitational collapsar

$$ds^2 = g_{ik}\, dx^i dx^k. \quad (1.226)$$

From here we see that gravitational collapse in the four-dimensional space-time can be correctly determined only if the space-time is holonomic, i.e., the three-dimensional space of the observer does not rotate (his spatial section is everywhere orthogonal to the time lines).

Since $d\tau = \sqrt{g_{00}}\, dt = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt$ in a non-rotating space, the observable time stops ($d\tau = 0$) on the surface of a gravitational collapsar.

The linear velocity with which the space rotates

$$v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}} = -c\,\frac{g_{0i}}{1 - \frac{\mathrm{w}}{c^2}} \quad (1.227)$$

becomes infinite by the collapse condition $\mathrm{w} = c^2$. To avoid this problem, we assume $g_{0i} = 0$. Then the metric (1.225) takes the form

$$ds^2 = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k, \quad (1.228)$$

so the problem of a singular state of the space-time becomes automatically removed. Proceeding from the above, we obtain the metric on the surface of a gravitational collapsar (1.226) in the form

$$ds^2 = -d\sigma^2 = -h_{ik}\, dx^i dx^k, \qquad h_{ik} = -g_{ik}\,. \tag{1.229}$$

From here we see that on the surface of a gravitational collapsar the four-dimensional interval is space-like: the elementary distance between two points on its surface is imaginary

$$ds = i\,d\sigma = i\sqrt{h_{ik}\, dx^i dx^k}\,. \tag{1.230}$$

If the four-dimensional interval is $ds = 0$, then the observable three-dimensional distance $d\sigma$ between two points on the surface of a gravitational collapsar is zero.

Now we are going to consider gravitational collapse in different regions of the four-dimensional space-time.

Collapse in a subluminal region

In this region, $ds^2 > 0$. This is the home of ordinary (real) particles travelling with subluminal velocities. Hence, a gravitational collapsar located in this region is filled with a collapsed substance. Therefore, we call it a *substantial collapsar*. On its surface, the space-time metric is space-like: since $ds^2 < 0$ on the surface of a substantial collapsar, all particles on its surface have imaginary relativistic masses. The metric on the surface of a substantial collapsar is non-degenerate.

Collapse in a light-like region

In this region, $ds^2 = 0$. This is the home of light-like (massless) particles. A gravitational collapsar in this region is filled with light-like matter. Therefore, we call it a *light-like collapsar*. The metric (1.229) on its surface is $d\sigma^2 = -g_{ik}\, dx^i dx^k = 0$ that is possible in two cases:

a) The surface of the light-like collapsar is shrunk to a point (in other words, all $dx^i = 0$);
b) The three-dimensional spatial metric is degenerate ($\det \|g_{ik}\| = 0$). Since the four-dimensional metric is also degenerate, such a light-like collapsar in this case is a case of the zero-space.

Collapse in a degenerate region (zero-space)

It is obvious that the distributed matter that fills a completely degenerate space-time region (zero-space) can be in the state of gravitational

collapse. We call such gravitational collapsars *degenerate collapsars*. Strictly speaking, based on the degeneration conditions

$$\mathrm{w} + v_i u^i = c^2, \qquad g_{ik}\, dx^i dx^k = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2, \tag{1.231}$$

we see that in the case of collapse ($\mathrm{w} = c^2$) there is

$$v_i u^i = 0\,, \qquad g_{ik}\, dx^i dx^k = 0\,. \tag{1.232}$$

Hence, gravitational collapse in a zero-space region occurs in the absence of rotation ($v_i = 0$) and, due to the conditions (1.232), the entire surface of such a degenerate collapsar is shrunk to a point.

Chapter 2 Motion of Particles as a Result of Motion of the Space Itself

2.1 Preliminary words

Having substituted the gravitational potential w and the linear velocity v_i with which the space rotates into the definition of the physically observable time interval $d\tau$ (1.22), we obtain

$$\left(1 + \frac{1}{c^2}\, v_i \mathrm{v}^i\right) d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt\,, \tag{2.1}$$

so a significant difference between $d\tau$ and dt may result from either a strong gravitational field or high speeds comparable to the speed of light. Hence, in everyday life the difference between $d\tau$ and dt is small.

The physically observable time coincides with the coordinate time $dt = d\tau$ under the condition

$$\mathrm{w} = -v_i \mathrm{v}^i. \tag{2.2}$$

This condition actually means that the gravitational attraction of a particle by the reference body of an observer is completely compensated by the rotation of the reference body's space (reference space) together with the motion of the particle itself. That is, (2.2) is the mathematical formulation of the *weightlessness condition*. Substituting the gravitational potential according to Newton's formula, we obtain

$$\frac{GM}{r} = v_i \mathrm{v}^i. \tag{2.3}$$

If the orbital velocity of a particle is equal to the linear velocity with which the space of the gravitating body rotates in this orbit, then the weightlessness condition for the particle takes the form

$$\frac{GM}{r} = v^2, \tag{2.4}$$

Planet	Orbital velocity, km/sec	
	Measured	Calculated
Mercury	47.9	47.9
Venus	35.0	35.0
Earth	29.8	29.8
Mars	24.1	24.1
Jupiter	13.1	13.1
Saturn	9.6	9.6
Uranus	6.8	6.8
Neptune	5.4	5.4
Pluto	4.7	4.7
Moon	1.0	1.0

i.e., the farther the orbit is from the attracting body, the lower the speed of a satellite in this orbit.

Does this statement agree with experimental data? The Table shows the orbital velocities of the Moon and the planets, measured in astronomical observations and calculated from the state of weightlessness.

It can be seen from the Table that the weightlessness condition that we have obtained is valid for any satellite orbiting around a gravitating body. Note that the condition is satisfied if the planet's orbital velocity is equal to (or very close to) the linear rotation velocity of the gravitating body's space in this orbit (2.4). This means that the space of any gravitating body, if the body, and, hence, its space, rotates, *drags* all the bodies around it, causing their orbital rotation.

If the space of a gravitating body rotated like a solid, without any deformation, then its angular velocity would be constant ($\omega = const$), and the orbital velocities $\mathrm{v} = \omega r$ of accompanying satellites grew together with the radii of their orbits. However, as we have just seen with the example of the planets of the Solar System, the linear velocity of orbital rotation decreases with distance from the Sun.

This means that in reality the space of a gravitating body (reference space) rotates not as a solid, but as a viscous and deformable medium, and the layers far from the body do not rotate as rapid as those closer to the body's surface. As a result, the space of a rotating body is *twisted*, and the profile of orbital velocities simply repeats the structure of the twisted space. From here we see that the orbital motion of particles in

the gravitational field of a massive body is the result of the *rotation of the space itself of the gravitating body*.*

2.2 Problem statement

What are the possible consequences of our mathematical theory of particle motion after the conclusions we have just arrived at?

Assume that there is a metric space. Obviously, the motion of the space itself allows us to associate any point of this space with the motion vector Q^α of the point. It is also obvious that all points of the space have the same motion as the space itself. Therefore, Q^α can be considered as the motion vector of the space itself at the given point. Thus, we get a vector field that describes the motion of the entire space.

Of course, if the length of the vector Q^α remains constant while moving, then such a space moves in such a way that its metric also remains unchanged. Therefore, if in such a space the motion vector Q^α is given at a given point, then the space metric can be found based on the motion of the point (together with the motion of the space itself).

The way to solving this problem was paved at the end of the 19th century by Sophus Lie [18]. He had obtained the external derivative equations of the fundamental metric tensor $g_{\alpha\beta}$ in a space with respect to the trajectory of a motion vector Q^α, where the Q^α components were present as fixed coefficients. The number of the equations is equal to the number of the metric tensor components. Therefore, having a fixed vector Q^α, i.e., having a given motion of the space, we can solve the equations for finding the metric tensor components $g_{\alpha\beta}$ based on the components of the Q^α. Later, David van Danzig had proposed to call such a derivative of the metric the *Lie derivative*.

Now we will consider a particular case of the motion of a space, in which the space metric remains constant. This case was studied by Wil-

*Our conclusions are very close to the conclusions obtained due to the elastodynamics of the space-time continuum — an extension of General Relativity, which was introduced a decade ago by Pierre A. Millette based on the analysis of the deformation of the space-time in terms of continuum mechanics. In particular, he showed that the massive body itself is part of the spacetime fabric that is rotating. See his extensive paper and subsequent monograph on the subject: Millette P. A. Elastodynamics of the spacetime continuum. *The Abraham Zelmanov Journal*, 2012, vol. 5, 221–277. Millette P. A. Elastodynamics of the Spacetime Continuum. The 2nd expanded edition, American Research Press, Rehoboth (New Mexico), 2019, 415 pages.

helm Killing [19]. Obviously, such a motion is equivalent to equalizing the Lie derivative to zero (*Killing equations*). Therefore, if the motion of a space leaves its metric unchanged and we know the vector Q^α for any of the points of the space (i.e., the motion of the space is given at any of its points), the motion of the point (or points) can be used to obtain the space metric based on the Killing equations.

On the other hand, the motion of a particle is described by the equations of motion. On the contrary, these equations leave the space metric fixed, and the task here is to find the particle's dynamical vector Q^α. The fixed metric in the equations of motion leads to the fact that the Christoffel symbols, which are functions of the $g_{\alpha\beta}$ components of the space metric, appear in the equations as fixed coefficients. Therefore, once a particular space metric is given, we can use the equations of motion to obtain the vector Q^α for a particle travelling in that space.

So now we come to the next one. Since $g_{\alpha\beta}$ is a symmetric tensor ($g_{\alpha\beta} = g_{\beta\alpha}$), only 10 of the 16 components have different numerical values. In the Killing equations (10 equations), the motion vector of a point of the space is fixed, and the metric tensor components are unknown (10 unknowns). The equations of motion of a free particle (4 equations), on the contrary, leave the metric fixed, but the components of the particle motion vector (4 components) are unknown. Then, as soon as we consider the free motion of a particle as the motion of any of the points of the space, due to the motion of the space itself, we can compose a system of 10 Killing equations (equations of motion of the space itself) and 4 equations of the particle's motion. There will be 14 unknowns in the system of 14 equations, 10 of them are unknown components of the space metric and 4 unknown components of the dynamic vector of the particle. Therefore, by solving this system, we will obtain the motion of a particle in the space and the space metric at the same time.

In particular, when solving the mentioned equation system, we can find the motion of particles resulting from the motion of the space itself. For this type of motion, knowing the motion of a particular particle can explicitly provide the metric for the space itself.

For example, having solved the Killing equations and the dynamic equations of motion of a satellite (or a planet), we can use its motion to find the space metric of body, around which it is orbiting.

Next we will deduce the chr.inv.-form of the Killing equations according to the chronometrically invariant formalism.

2.3 The equations of motion and Killing's equations

Assume that there is a moving space (not necessarily a metric one). It is obvious that the motion vector Q^α of any point of the space is the vector of motion of the space itself at this point. The motion of a metric space is described by Lie's derivative

$$\underset{\mathrm{L}}{\delta}\, g_{\alpha\beta} = Q^\sigma \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} + g_{\alpha\sigma}\frac{\partial Q^\sigma}{\partial x^\beta} + g_{\beta\sigma}\frac{\partial Q^\sigma}{\partial x^\alpha}\,, \tag{2.5}$$

which is the derivative of the fundamental metric tensor of the space with respect to the parallel transport direction of the vector Q^α (direction of motion of the space itself).

Consider a point in the space. Since the space moves, the point is a subject to the action of the accompanying vector Q^α that is the motion vector of the space itself. For the point itself, the space rests and only the "wind" produced by the motion vector Q^α of the space discloses the motion of the entire space.

In a general case the Lie derivative is not zero, i.e., the motion of a space alters the space metric. But in a Riemannian space, the metric is fixed by definition, so the length of a vector parallel-transported to itself remains constant. This means that the parallel transport of a vector across the "bumpy road" in a Riemannian space will alter the vector along with the structure of the space. As a result, the Lie derivative of the metric in a Riemannian space should be zero

$$\underset{\mathrm{L}}{\delta}\, g_{\alpha\beta} = 0\,. \tag{2.6}$$

The Lie equations in a Riemannian space were first studied by Wilhelm Killing and are known as the *Killing equations*.

Then, 60 years later, A. Z. Petrov showed [20] that the Killing equations for any point of a Riemannian space are the necessary and sufficient condition for the motion of the point to be the motion of the space itself. In other words, if a point of a Riemannian space is dragged by the motion of the space and moves along with it, then the Killing equations must be true for that point.

It is obvious that to obtain the metric tensor components from the Killing equations we need to determine the particular motion vector Q^α of a point. Then we will have 10 Killing equations versus 10 unknown metric components, so we can solve the system.

There can be different types of motion in a Riemannian space. We will determine the motion vector Q^α to suit our problem.

Consider free (geodesic) motion. In this case, a point moves along a geodesic trajectory (the shortest of those between two points). Let us assume that any point of the Riemannian space is dragged by the motion of the space itself, i.e., it moves along a geodesic trajectory. Therefore, the motion of the entire Riemannian space in this case is geodesic. In this case, we can compare the motion of a point dragged by the motion of the space with the motion of a free particle.

Further, we call the motion of a space the *geodesic motion* of the space, if the free motion of particles is the result of their transport by the moving space.

Consider a system consisting of the dynamical equations of motion of free particles and the Killing equations

$$\left.\begin{aligned} \frac{\mathrm{D}Q^\alpha}{d\rho} &= 0 \\ \underset{\mathrm{L}}{\delta}\, g_{\alpha\beta} &= 0 \end{aligned}\right\}, \tag{2.7}$$

where Q^α is the dynamical vector of motion of a particle, ρ is the derivation parameter along the motion trajectory, and the Lie derivative can be expressed through the *Lie differential* as

$$\underset{\mathrm{L}}{\delta}\, g_{\alpha\beta} = \frac{\underset{\mathrm{L}}{\mathrm{D}}\, g_{\alpha\beta}}{d\rho}\,. \tag{2.8}$$

In fact, the system of equations (2.7) means that the motion of a free particle is geodesic and, at the same time, is the result of the fact that the particle is dragged by the moving space itself. The system is solved as a set of the dynamic vector components Q^α, as well as the metric tensor components $g_{\alpha\beta}$, for which the geodesic motion of the particle is the result of the geodesic motion of the space itself.

To solve the problem in a correct way, we need to present the Killing equations in the chr.inv.-form, i.e., to express them through the physical properties of the space. It is especially interesting to know what physical properties follow from the motion of the space itself.

The physical observables obtained from the Killing equations are the chr.inv.-projections of the equations onto the time line (1 component), the mixed projection (3 components), and the spatial projection

(6 components)

$$\left.\begin{aligned}
&\frac{\underset{\mathrm{L}}{\delta}\, g_{00}}{g_{00}} = 0 \\
&\frac{\underset{\mathrm{L}}{\delta}\, g_0^i}{\sqrt{g_{00}}} = \frac{g^{i\alpha}\,\underset{\mathrm{L}}{\delta}\, g_{0\alpha}}{\sqrt{g_{00}}} = 0 \\
&\underset{\mathrm{L}}{\delta}\, g^{ik} = g^{i\alpha} g^{k\beta}\,\underset{\mathrm{L}}{\delta}\, g_{\alpha\beta} = 0
\end{aligned}\right\}. \tag{2.9}$$

Here we are considering the motion of the space and particles from the viewpoint of an ordinary subluminal observer.

In the chr.inv.-projections of the Killing equations (2.9), we express the ordinary derivation operators in the Lie derivative through the chr. inv.-derivation operators, then we use a short notation for the chr.inv.-projections $\varphi = \frac{Q_0}{\sqrt{g_{00}}}$ and $q^i = Q^i$ of the dynamical vector Q^α of the particle. As a result, we obtain the *chr.inv.-Killing equations*

$$\left.\begin{aligned}
&\frac{{}^*\partial\varphi}{\partial t} - \frac{1}{c} F_i q^i = 0 \\
&\frac{1}{c}\frac{{}^*\partial q^i}{\partial t} - h^{im}\frac{{}^*\partial\varphi}{\partial x^m} - \frac{\varphi}{c^2} F^i + \frac{2}{c} A_{k\cdot}^{\cdot i}\, q^k = 0 \\
&\frac{2\varphi}{c} D^{ik} + h^{im} h^{kn} q^l \frac{{}^*\partial h_{mn}}{\partial x^l} + h^{im}\frac{{}^*\partial q^k}{\partial x^m} + h^{km}\frac{{}^*\partial q^i}{\partial x^m} = 0
\end{aligned}\right\}. \tag{2.10}$$

If the vector Q^α satisfies both the chr.inv.-Killing equations and the dynamical chr.inv.-equations of the motion of the particle, then this particle travels due to the geodesic motion of the space.

The joint solution of the equations in a general form is problematic. Therefore, we will confine ourselves to one special case, which is of great importance. Let the dynamic vector of the space motion Q^α be the dynamic vector of motion of a mass-bearing particle

$$Q^\alpha = m_0 \frac{dx^\alpha}{ds} = \frac{m}{c}\frac{dx^\alpha}{d\tau}, \tag{2.11}$$

and the observer accompanies the particle ($\mathrm{v}^i = 0$). In this case,

$$\varphi = m_0 = const, \qquad q^i = \frac{m}{c}\,\mathrm{v}^i, \tag{2.12}$$

and the chr.inv.-Killing equations (2.10) are simplified to

$$\left.\begin{aligned} F^i &= 0 \\ D^{ik} &= 0 \end{aligned}\right\}. \tag{2.13}$$

According to (1.43), $D^{ik} = 0$ means a stationary state of the observable metric: $h^{ik} = const$. The condition $F^i = 0$ means the fulfillment of the following equalities when transforming the time coordinate

$$g_{00} = 1\,, \qquad \frac{\partial g_{0i}}{\partial t} = 0\,. \tag{2.14}$$

Besides, the quantities F^i and A_{ik} are related by Zelmanov's identities, the first of which is (see formula 1.38 in §1.2)

$$\frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) + \frac{{}^*\partial A_{ik}}{\partial t} = 0\,, \tag{2.15}$$

from which we see that $F^i = 0$ means also

$$\frac{{}^*\partial A_{ik}}{\partial t} = 0\,, \tag{2.16}$$

so the space motion in this case is a *stationary rotation*.

As is seen from the Killing equations (2.13), the space deformation tensor is zero in this case. Hence, the stationary rotation of the space does not alter its structure. The vanishing of the gravitational inertial force in the Killing equations means that from the viewpoint of an observer associated with a particle dragged into motion by the moving space ($\mathrm{v}^i = 0$), this particle weighs nothing and is not attracted to anything (is in the state of weightlessness). This does not contradict the weightlessness condition $\mathrm{w} = -v_i \mathrm{v}^i$ obtained earlier, since from the viewpoint of the observer the gravitational potential of his reference body's field is $\mathrm{w} = 0$ and, therefore, $F^i = 0$.

Therefore, if Q^α is the motion vector of a mass-bearing particle travelling in a Riemannian space, then the geodesic motion of the space along this vector is a stationary rotation.

As you can see, the geodesic motion of mass-bearing particles is a stationary rotation. Such a stationary rotation arises as a result of dragging by rotation of the reference space surrounding the gravitating

body (reference body of the observer). At the same time, we know that the main type of motion in the Universe is orbital. Consequently, the main type of motion in the Universe is the geodesic motion resulting from the dragging of bodies by the stationary (geodesic) rotation of the space of the bodies that attract them.

2.4 Conclusions

So, what kind of space has a gravitational potential, deforms and, being in rotation, behaves like a viscous medium? It is worth noting that if we place a particle in such a space, the moving space will drag it in the same way that an ocean current carries a tiny boat and a giant iceberg.

The answer is this: according to the results that we have obtained above, the reference space of a body and its gravitational field are one and the same. Physically, reference space points can be considered as particles in the gravitational field of the reference body.

If the reference space does not rotate, the satellite will fall down on the reference body due to the influence of gravitational force. But if the reference space rotates, then the satellite will be under the action of a reference frame dragging force. This force acts like a wind or an ocean current, pushing the satellite forward, preventing it from falling down and causing it to rotate around the gravitating body along with the rotating space (of course, the extra speed given to the satellite will make it move faster than the rotating space).

Chapter 3 World-lines Deviation. Detecting Gravitational Waves

3.1 Gravitational wave detectors

In this Chapter, we explain a theory of detecting gravitational waves that was published, in brief, in our 2006 paper [21] and presented at the 2008 APS April Meeting [22]. The basics and main points of this theory were developed in 1968–1973 by one of us, L. Borissova, then the exact solutions to the equations were found in the 2000s by us together. For the general theory of gravitational waves and their criteria, the reader can be referred to the detailed paper [23] written in 1968 by L. Borissova.

Consider two particles having a rest-mass m_0, each one connected by a non-gravitational force Φ^α. Such particles travel along neighbouring non-geodesic world-lines with the same four-dimensional velocity U^α according to the non-geodesic equations of motion

$$\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = \frac{\Phi^\alpha}{m_0 c^2}\,, \tag{3.1}$$

while relative deviations of the world-lines (and the particles) are given by the Synge-Weber equation [24]

$$\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^\alpha_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \eta^\gamma = \frac{1}{m_0 c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}\, dv\,, \tag{3.2}$$

where $\mathrm{D}\eta^\alpha = d\eta^\alpha + \Gamma^\alpha_{\mu\nu}\eta^\mu dx^\nu$ is the absolute differential, $\eta^\alpha = \frac{\partial x^\alpha}{\partial v}\, dv$ is the relative deviation vector of the particles, and v is a derivation parameter having the same numerical value along a world-line and different as dv in the neighbouring world-lines.

If the particles are free ($\Phi^\alpha = 0$), they travel along neighbouring geodesics according to the geodesic equations of motion

$$\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = 0\,, \tag{3.3}$$

while the relative deviations of the geodesics (and the particles) are given by the Synge equations [25]

$$\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^\alpha_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \eta^\gamma = 0\,. \tag{3.4}$$

A gravitational wave as a wave of the space metric deforming the space should produce some effect in a two-particle system. The effect could be found as a solution to the deviation equations in the gravitational wave metric. Therefore, two kinds of gravitational wave detectors were presumed in 1960s by Joseph Weber, who pioneered experimental investigation on gravitational waves:

a) Solid-body detector — a freely suspended cylindrical bar, approximated by two masses connected by a spring. Such a detector should be deformed under the action of a gravitational wave. This deformation should lead to a piezoelectric effect therein;
b) Free-mass detector — a system consisting of two freely suspended mirrors, distantly separated within the visibility, and fitted with a laser range-finder. The expected deviation of the mirrors, derived from a gravitational wave, should be registered by the laser beam.

3.2 A brief history of the measurements

Initial interest in gravitational waves arose in 1968–1970 when Joseph Weber, Professor at the University of Maryland (USA), carried out his first experiments with solid-body gravitational wave detectors. He had registered several weak signals with his solid-body detectors that were located at a distance of up to 1000 km from each other. [26–28]. He supposed that some processes in the centre of the Galaxy were the source of the registered signals.

The experiments were continued in the next decades by many groups of researchers working at laboratories and research institutes throughout the world. The registering systems used in these attempts were much more sensitive than those of Weber. In his pioneering observations of 1968–1970, Weber used very simple and small size solid-body detectors in room-temperature conditions. To amplify the gravitational wave effect in measurements, the level of noise in all solid-body detectors of the second generation was lowered by cooling the cylinder bars down to a temperature close to 0 K. Besides gravitational antennae of the solid-body kind, many antennae based on free masses were constructed…

Theoretically, gravitational waves should be emitted by many processes in the Galaxy, and their magnitude [2] is such that it can be detected even by very simple gravitational wave detectors like those used by Weber. That is, theoretically, gravitational waves should be the object of daily observations. This is what Weber insisted on when he built his detectors in the late 1960s.

But even the second generation of gravitational wave detectors has not lead scientists to the expected results. In a few rare cases, when gravitational waves were registered, their magnitude was incredibly small. So far no one has registered gravitational waves emitted by many processes in the Galaxy.

Nonetheless it is accepted by most physicists that the discovery of gravitational waves should be expected as one of the main effects of General Relativity. The arguments in support of this thesis are [2]:

a) The energy of any gravitational field is determined by the gravitational field energy-momentum pseudotensor;
b) A linearized form of Einstein's equations permits a solution describing weak plane gravitational waves, which are transverse;
c) An energy flux, radiated by gravitational waves, can be calculated through the gravitational field energy-momentum pseudotensor.

Therefore, there is no doubt that gravitational radiation emitted by many objects in the Galaxy will be detected in the future.

The corner-stone of the problem was the fact that Weber's conclusions on the construction of the gravitational wave detectors were not based on an exact solution to the deviation equations, but on an *approximate analysis* of what could be expected: Weber expected that a plane weak wave of the space metric (gravitational wave) can displace two particles at rest with respect to each other.

Here we deduce *exact solutions* to both the Synge equation and the Synge-Weber equation (i.e., the *exact theory of free-mass and solid-body detectors*). The exact solutions show instead Weber's supposition that gravitational waves cannot displace resting particles; some effect can only be produced if the particles oscillate relative to each other.

According to the exact solutions we can alter the construction of both solid-body and free-mass detectors so that they can register oscillations produced by gravitational waves. Weber most probably detected them as claimed by him in 1968–1970, as his room-temperature

solid-body detectors may have had their own relative oscillations of the bar extremities, whereas the oscillations are inadvertently suppressed as noise in the detectors developed by his many followers, who have had no positive result in over 45 years.

3.3 Weber's approach and criticism thereof

Weber proposed the relative displacement of the particles η^α consisting of a constant distance r^α and an infinitely small displacement ζ^α caused by a gravitational wave

$$\eta^\alpha = r^\alpha + \zeta^\alpha, \qquad \zeta^\alpha \ll r^\alpha, \qquad \frac{\mathrm{D}r^\alpha}{ds} = 0\,. \tag{3.5}$$

Thus, the non-geodesic deviation equation, i.e., the Synge-Weber equation (3.2), takes the following particular form

$$\frac{\mathrm{D}^2\zeta^\alpha}{ds^2} + R^\alpha_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \left(r^\gamma + \zeta^\gamma\right) = \frac{\Phi^\alpha}{m_0 c^2}\,. \tag{3.6}$$

Then Weber considered the Φ^α as the sum of the returning elastic force $k^\alpha_\sigma \zeta^\sigma$ and the damping factor $d^\alpha_\sigma \frac{\mathrm{D}\zeta^\sigma}{ds}$, while k^α_σ and d^α_σ describe the properties of the spring. As a result, the equation (3.6) becomes

$$\frac{\mathrm{D}^2\zeta^\alpha}{ds^2} + \frac{d^\alpha_\sigma}{m_0 c^2}\frac{\mathrm{D}\zeta^\sigma}{ds} + \frac{k^\alpha_\sigma}{m_0 c^2}\zeta^\sigma = -R^\alpha_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \left(r^\gamma + \zeta^\gamma\right), \tag{3.7}$$

which is the equation of forced oscillations, where the curvature tensor $R^\alpha_{\cdot\beta\gamma\delta}$ is a forcing factor. After some simplifications, he transformed the non-geodesic deviation equation (3.7) to

$$\frac{d^2\zeta^\alpha}{dt^2} + \frac{d^\alpha_\sigma}{m_0}\frac{d\zeta^\sigma}{dt} + \frac{k^\alpha_\sigma}{m_0}\zeta^\sigma = -c^2 R^\alpha_{\cdot 0\sigma 0}\, r^\sigma. \tag{3.8}$$

Weber did not solve his equation (3.8). He limited himself by using the curvature tensor as a forcing factor in his calculations of expected resonant oscillations in solid-body detectors [24].

A solution to Weber's equation in the form (3.8) having all his simplifications was obtained in 1978 by L. Borissova [29]. She solved it in the field of a weak plane gravitational wave. Assuming, as Weber did, the r^α and its length $r = \sqrt{g_{\mu\nu} r^\mu r^\nu}$ to be covariantly constant $\frac{\mathrm{D}r^\alpha}{ds} = 0$, Borissova had obtained that for a gravitational wave linearly polarized

in the x^2 direction and propagating along x^1, the equation $\frac{\mathrm{D}r^\alpha}{ds} = 0$ gives $r^2 = r^2_{(0)}\left[1 - A\sin\frac{\omega}{c}(ct + x^1)\right]$ when the detector is oriented along the x^2 axis. Thus, she had obtained Weber's equation (3.8) in the form

$$\frac{d^2\zeta^2}{dt^2} + 2\lambda\frac{d\zeta^2}{dt} + \Omega_0^2\zeta^2 = -A\omega^2 r^2_{(0)}\sin\frac{\omega}{c}(ct + x^1), \tag{3.9}$$

which is the equation of forced oscillations, where the forcing factor is the relative motion of the particles caused by the gravitational wave. Here $2\lambda = \frac{b}{m_0}$ and $\Omega^2_{(0)} = \frac{k}{m_0}$ are determined by the non-gravitational force $\Phi^2 = -k\zeta^2 - b\dot{\zeta}^2$, acting along the x^2 axis, k is the elastic coefficient of the "spring", and b is the friction coefficient. Then she had obtained the exact solution to the equation — the relative displacement $\eta^2 = \eta_y$ of the detector's extremities transverse to the falling gravitational wave

$$\eta^2 = r^2_{(0)}\left[1 - A\sin\frac{\omega}{c}(ct + x^1)\right] + Me^{-\lambda t}\sin(\Omega t + \alpha) - \\ - \frac{A\omega^2 r^2_{(0)}}{(\Omega_0^2 - \omega^2)^2}\cos\left(\omega t + \delta + \frac{\omega}{c}x^1\right), \tag{3.10}$$

where $\Omega = \sqrt{\Omega_0^2 - \omega^2}$, $\delta = \arctan\frac{2\lambda\omega}{\omega^2 - \Omega_0^2}$, while M and α are constants.

In this solution, the relative oscillations consist of the "basic" harmonic oscillations and relaxing oscillations (first two terms), and also the resonant oscillations (third term).

As was shown by Borissova [29], Weber's final equation (3.8) can only be obtained under the following simplifications:

a) It can be considered that in fact there were two detectors in one: a long bar with a constant length r and a short bar with a length ζ, both of which change under the influence of the same gravitational wave. However, in real experiments, a solid rod responds to external influences as a whole;
b) The Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ are all zero. But, since the curvature tensor is non-zero, the $\Gamma^\alpha_{\mu\nu}$ cannot be reduced to zero in a finite region [20]. Therefore, in the neighbouring particle $\Gamma^\alpha_{\mu\nu} \neq 0$;
c) The extremities of the bar are at rest with respect to the observer ($U^i = 0$) all the time before a gravitational wave passes. Therefore, *only resonant oscillations* can be registered by such a detector. Parametric oscillations cannot appear there.

Since the same assumptions were applied to the geodesic deviation equation, all that has been said is applicable to a free-mass detector.

Thus, by his simplified equation (3.8), Weber actually postulated that gravitational waves force rest-particles to undergo relative resonant oscillations. His assumptions led to a specific interior of the solid-body and free-mass detectors, where parametric oscillations are obviated.

3.4 The main equations

Here we solve the deviation equations together with the equations of motion in the general case, where both particles in the pair move initially with respect to the observer ($U^i \neq 0$), and without Weber's simplifications. We solve the equations in terms of physically observable quantities (chronometric invariants). According to §1.2 of Chapter 1, any vector Q^α has two chr.inv.-projections: $\frac{Q_0}{\sqrt{g_{00}}}$ and Q^i. Thus, for the connecting force Φ^α, we denote

$$\sigma = \frac{\Phi_0}{\sqrt{g_{00}}}, \qquad f^i = \Phi^i, \tag{3.11}$$

and also, for the deviation vector η^α,

$$\varphi = \frac{\eta_0}{\sqrt{g_{00}}}, \qquad \eta^i \equiv \eta^i. \tag{3.12}$$

We consider the non-geodesic deviation equation in a general case, where the right hand side of the equation is non-zero.

The general covariant non-geodesic equations of motion (3.1) have two chr.inv.-projections, which have the form

$$\left.\begin{aligned} &\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = \frac{\sigma}{c} \\ &\frac{d}{d\tau}(m\mathrm{v}^i) - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right) + m\Delta^i_{kn}\mathrm{v}^k\mathrm{v}^n = f^i \end{aligned}\right\}, \tag{3.13}$$

where we still follow the conventional denotations: m is the relativistic mass of the particle, v^i is its physically observable chr.inv.-velocity, $d\tau$ is the interval of the physically observable time, F_i is the chr.inv.-vector of the gravitational inertial force, A_{ik} is the chr.inv.-tensor of the angular velocity with which the space rotates, D_{ik} is the tensor of the deformation rate of the space, and Δ^i_{kn} are the chr.inv.-Christoffel symbols (see §1.2 of Chapter 1).

We re-write the Synge-Weber equation of deviating non-geodesics (3.2) in the expanded form

$$\frac{d^2\eta^\alpha}{ds^2} + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma}U^\beta U^\delta\eta^\gamma = \frac{1}{m_0c^2}\frac{\partial\Phi^\alpha}{\partial x^\gamma}\eta^\gamma, \qquad (3.14)$$

where ds^2 can be expressed through the observable time interval $d\tau$ according to (1.30) as $ds^2 = c^2d\tau^2 - d\sigma^2 = c^2d\tau^2(1 - \mathrm{v}^2/c^2)$.

Consider the metric of weak plane gravitational waves

$$\begin{aligned} ds^2 = c^2dt^2 - (dx^1)^2 - (1+a)(dx^2)^2 + \\ + 2b\,dx^2dx^3 - (1-a)(dx^3)^2, \end{aligned} \qquad (3.15)$$

where a and b are functions of $ct + x^1$, i.e., we assume in the case under consideration that a weak plane gravitational wave propagates along the axis x^1. The functions a and b are small values, therefore their squares and the products of their derivatives vanish. The speed of both particles (extremities of a gravitational wave detector) is obviously slow. In this case, in the space of the gravitational wave metric (3.15),

$$\left.\begin{aligned} &d\tau = dt\,, && \eta^0 = \eta_0 = \varphi\,, && \Phi^0 = \Phi_0 = \sigma \\ &\Gamma^0_{kn} = \frac{1}{c}D_{kn}\,, && \Gamma^i_{0k} = \frac{1}{c}D^i_k\,, && \Gamma^i_{kn} = \Delta^i_{kn} \end{aligned}\right\}. \qquad (3.16)$$

With these, after some algebra, we obtain the chr.inv.-projections of the Synge-Weber equation (3.14)

$$\left.\begin{aligned} &\frac{d^2\varphi}{dt^2} + \frac{2}{c}D_{kn}\frac{d\eta^k}{dt}\mathrm{v}^n + \left(\varphi\frac{\partial D_{kn}}{\partial t} + c\frac{\partial D_{kn}}{\partial x^m}\eta^m\right)\frac{\mathrm{v}^k\mathrm{v}^n}{c^2} = \\ &\qquad = \frac{1}{m_0}\left(\frac{\varphi}{c}\frac{\partial\sigma}{\partial t} + \frac{\partial\sigma}{\partial x^m}\eta^m\right) \\ &\frac{d^2\eta^i}{dt^2} + \frac{2}{c}D^i_k\left(\frac{d\varphi}{dt}\mathrm{v}^k + c\frac{d\eta^k}{dt}\right) + 2\Delta^i_{kn}\frac{d\eta^k}{dt}\mathrm{v}^n + \\ &+ 2\left(\frac{\varphi}{c}\frac{\partial D^i_k}{\partial t} + \frac{\partial D^i_k}{\partial x^m}\eta^m\right)\mathrm{v}^k + \left(\frac{\varphi}{c}\frac{\partial\Delta^i_{kn}}{\partial t} + \frac{\partial\Delta^i_{kn}}{\partial x^m}\eta^m\right)\mathrm{v}^k\mathrm{v}^n = \\ &\qquad = \frac{1}{m_0}\left(\frac{\varphi}{c}\frac{\partial f^i}{\partial t} + \frac{\partial f^i}{\partial x^m}\eta^m\right) \end{aligned}\right\}. \qquad (3.17)$$

The obtained chr.inv.-deviation equations (3.17) in component notation form a system of four 2nd order differential equations with respect to φ, η^1, η^2, η^3, where the variable coefficients of the mentioned functions are the quantities $\dot{a}$, $\ddot{a}$, v^1, v^2, v^3. To solve this system we will get a from the gravitational wave metric (3.15), while the v^i components come as the solutions to the non-geodesic equations of motion (3.13).

3.5 The exact solution for a free-mass detector

First, we solve the chr.inv.-deviation equations (3.17) for a free-mass detector. In such a case, two particles associated with the extremities of the detector do not interact with each other ($\Phi^\alpha = 0$), i.e., the right hand side is zero in the equations.

We are looking for a solution in the field of a gravitational wave falling along the axis x^1 and linearly polarized in the x^2 direction ($b = 0$). With these, the gravitational wave metric (3.15) gives

$$\left.\begin{aligned} D_{22} = -D_{33} = \frac{1}{2}\,\dot{a}\,, \qquad & \frac{d}{dx^1} = \frac{1}{c}\frac{d}{dt} \\ \Delta^1_{22} = -\Delta^1_{33} = -\frac{1}{2c}\,\dot{a}\,, \qquad & \Delta^2_{12} = -\Delta^3_{13} = \frac{1}{2c}\,\dot{a} \end{aligned}\right\}. \tag{3.18}$$

In such a case and since $\Phi^\alpha = 0$, the chr.inv.-equations of motion (3.13) take the following form

$$\left.\begin{aligned} & (\mathrm{v}^2)^2 - (\mathrm{v}^3)^2 = 0 \\ & \frac{d\mathrm{v}^1}{dt} = 0\,, \quad \frac{d\mathrm{v}^2}{dt} + \dot{a}\,\mathrm{v}^2 = 0\,, \quad \frac{d\mathrm{v}^3}{dt} + \dot{a}\,\mathrm{v}^3 = 0 \end{aligned}\right\}. \tag{3.19}$$

Here $\mathrm{v}^1 = \mathrm{v}^1_{(0)} = const$. Hence, a transverse gravitational wave does not move a single particle in the longitudinal direction. Therefore,

$$\mathrm{v}^1 = \mathrm{v}^1_{(0)} = 0\,. \tag{3.20}$$

The last two spatial equations of (3.19) are also simple to integrate. After integration, we obtain

$$\mathrm{v}^2 = \mathrm{v}^2_{(0)}\, e^{-a}, \qquad \mathrm{v}^3 = \mathrm{v}^3_{(0)}\, e^{+a}. \tag{3.21}$$

Assuming the wave simple harmonic ($\omega = const$) with a constant amplitude $A = const$, i.e., $a = A \sin \frac{\omega}{c}(ct + x^1)$, then expanding the ex-

ponent into series (with higher-order terms withheld), we obtain

$$\mathrm{v}^2 = \mathrm{v}^2_{(0)}\left[1 - A\sin\frac{\omega}{c}\,(ct + x^1)\right], \tag{3.22}$$

$$\mathrm{v}^3 = \mathrm{v}^3_{(0)}\left[1 + A\sin\frac{\omega}{c}\,(ct + x^1)\right]. \tag{3.23}$$

Substituting these solutions into the chr.inv.-equations of deviating non-geodesics (3.17) and setting the right hand side to zero, as for geodesics, we obtain

$$\frac{d^2\varphi}{dt^2} + \frac{\dot{a}}{c}\left(\frac{d\eta^2}{dt}\,\mathrm{v}^2_{(0)} - \frac{d\eta^3}{dt}\,\mathrm{v}^3_{(0)}\right) = 0\,, \tag{3.24}$$

$$\frac{d^2\eta^1}{dt^2} - \frac{\dot{a}}{c}\left(\frac{d\eta^2}{dt}\,\mathrm{v}^2_{(0)} - \frac{d\eta^3}{dt}\,\mathrm{v}^3_{(0)}\right) = 0\,, \tag{3.25}$$

$$\frac{d^2\eta^2}{dt^2} + \dot{a}\,\frac{d\eta^2}{dt} + \frac{\dot{a}}{c}\left(\frac{d\varphi}{dt} + \frac{d\eta^1}{dt}\right)\mathrm{v}^2_{(0)} + \frac{\ddot{a}}{c}(\varphi + \eta^1)\mathrm{v}^2_{(0)} = 0\,, \tag{3.26}$$

$$\frac{d^2\eta^3}{dt^2} - \dot{a}\,\frac{d\eta^3}{dt} - \frac{\dot{a}}{c}\left(\frac{d\varphi}{dt} + \frac{d\eta^1}{dt}\right)\mathrm{v}^2_{(0)} - \frac{\ddot{a}}{c}(\varphi + \eta^1)\mathrm{v}^2_{(0)} = 0\,. \tag{3.27}$$

Summing up the first two equations of the above, then integrating the obtained sum, we obtain

$$\varphi + \eta^1 = B_1 t + B_2\,, \tag{3.28}$$

where B_1 and B_2 are integration constants. Substituting the result into the other two equations, we obtain two equations that differ only in the sign of the a and therefore can be solved in the same way

$$\frac{d^2\eta^2}{dt^2} + \dot{a}\,\frac{d\eta^2}{dt} + \frac{\dot{a}}{c}\,B_1\mathrm{v}^2_{(0)} + \frac{\ddot{a}}{c}(B_1 t + B_2)\mathrm{v}^2_{(0)} = 0\,, \tag{3.29}$$

$$\frac{d^2\eta^3}{dt^2} - \dot{a}\,\frac{d\eta^2}{dt} - \frac{\dot{a}}{c}\,B_1\mathrm{v}^3_{(0)} - \frac{\ddot{a}}{c}(B_1 t + B_2)\mathrm{v}^3_{(0)} = 0\,. \tag{3.30}$$

Introduce a new variable $y = \frac{d\eta^2}{dt}$. Then we have a linear uniform equation of the 1st order with respect to y

$$\dot{y} + \dot{a}y = -\frac{\dot{a}}{c}\,B_1\mathrm{v}^2_{(0)} - \frac{\ddot{a}}{c}(B_1 t + B_2)\mathrm{v}^2_{(0)}\,, \tag{3.31}$$

which has the solution

$$y = e^{-F}\left(y_0 + \int_0^t g(t)\, e^F dt\right), \quad F(t) = \int_0^t f(t)\, dt\,, \tag{3.32}$$

where $F(t) = \dot{a}$ and $g(t) = -\frac{\dot{a}}{c} B_1 \mathrm{v}_{(0)}^2 - (B_1 t + B_2)\mathrm{v}_{(0)}^2$. Expanding the exponent in y (3.32) into series, then integrating, we obtain

$$y = \dot{\eta}^2 = \dot{\eta}_{(0)}^2 \left[1 - A \sin \frac{\omega}{c}(ct + x^1)\right] - \\ - \frac{A\omega}{c}\mathrm{v}_{(0)}^2 (B_1 t + B_2) \cos \frac{\omega}{c}(ct + x^1) + \frac{A\omega}{c} B_2 \mathrm{v}_{(0)}^2\,. \tag{3.33}$$

We integrate this equation, then apply the same method for η^3. As a result, we obtain the physically observable relative displacements η^2 and η^3 in a free-mass detector

$$\eta^2 = \eta_{(0)}^2 + \left(\dot{\eta}_{(0)}^2 + \frac{A\omega B_2 \mathrm{v}_{(0)}^2}{c}\right) t + \frac{A}{\omega}\left(\dot{\eta}_{(0)}^2 - \frac{\mathrm{v}_{(0)}^2}{c} B_1\right) \times \\ \times \left[\cos \frac{\omega}{c}(ct + x^1) - 1\right] - \frac{A\mathrm{v}_{(0)}^2}{c}(B_1 t + B_2) \sin \frac{\omega}{c}(ct + x^1), \tag{3.34}$$

$$\eta^3 = \eta_{(0)}^3 + \left(\dot{\eta}_{(0)}^3 - \frac{A\omega B_2 \mathrm{v}_{(0)}^3}{c}\right) t - \frac{A}{\omega}\left(\dot{\eta}_{(0)}^3 - \frac{\mathrm{v}_{(0)}^3}{c} B_1\right) \times \\ \times \left[\cos \frac{\omega}{c}(ct + x^1) - 1\right] + \frac{A\mathrm{v}_{(0)}^3}{c}(B_1 t + B_2) \sin \frac{\omega}{c}(ct + x^1). \tag{3.35}$$

Getting $\dot{\eta}^2$ and $\dot{\eta}^3$, we obtain the physically observable relative displacement η^1 (3.25) in a free-mass detector and the physically observable time shift φ (3.24) at its ends

$$\eta^1 = \dot{\eta}_{(0)}^1 t - \frac{A}{\omega c}\left(\mathrm{v}_{(0)}^2 \dot{\eta}_{(0)}^2 - \mathrm{v}_{(0)}^3 \dot{\eta}_{(0)}^3\right)\left[1 - \cos \frac{\omega}{c}(ct + x^1)\right] + \eta_{(0)}^1, \tag{3.36}$$

$$\varphi = \dot{\varphi}_{(0)} t + \frac{A}{\omega c}\left(\mathrm{v}_{(0)}^2 \dot{\eta}_{(0)}^2 - \mathrm{v}_{(0)}^3 \dot{\eta}_{(0)}^3\right)\left[1 - \cos \frac{\omega}{c}(ct + x^1)\right] + \eta_{(0)}^1. \tag{3.37}$$

Finally, we substitute φ and η^1 into $\varphi + \eta^1 = B_1 t + B_2$ (3.28) to fix the integration constants. We obtain

$$B_1 = \dot{\varphi}_{(0)} + \dot{\eta}_{(0)}^1\,, \qquad B_2 = \varphi_{(0)} + \eta_{(0)}^1\,. \tag{3.38}$$

So, we have obtained the exact solutions φ, η^1, η^2, η^3 to the chr.inv.-equations of two deviating geodesics in the field of a weak plane gravitational wave. Proceeding from the exact solutions, we arrive at the following conclusions on free-mass detectors:

1) As is seen from the solutions for η^2 (3.34) and η^3 (3.35), gravitational waves can force the ends of a free-mass detector to undergo relative oscillations in the directions x^2 and x^3, transverse to that of the wave propagation. At the same time, this effect is possible only if the detector initially moves with respect to the observer ($\mathrm{v}^2_{(0)} \neq 0$ or $\mathrm{v}^3_{(0)} \neq 0$) or, alternatively, its ends initially move with respect to each other ($\dot{\eta}^2_{(0)} \neq 0$ or $\dot{\eta}^3_{(0)} \neq 0$). For instance, if the ends of a free-mass detector are at rest with respect to x^2, an x^1-propagating gravitational wave cannot displace them in the x^2 direction;
2) The solution for η^1 (3.36) means that gravitational waves can oscillatory bounce the ends of a free-mass detector even in the same direction as the wave propagation, if they initially move both with respect to the observer and each other in at least one of the transverse directions x^2 and x^3;
3) The solution for φ (3.37) is the time shift on the clocks located at the ends of a free mass detector caused by a gravitational wave. From (3.37) we see that this effect is possible if the ends initially move both with respect to the observer and each other in at least one of the transverse directions x^2 and x^3.

Based on the above results that we have obtained, we propose a new experimental statement for free-mass detectors:

New experiment (free-mass detector)

Use such a free mass detector, in which two mirrors, distant from each other, are suspended and vibrating so that they have free oscillations with respect to each other ($\dot{\eta}^i_{(0)} \neq 0$) or joint oscillations along parallel lines ($\mathrm{v}^i_{(0)} \neq 0$). According to the exact solution for a free-mass detector given above, a falling gravitational wave produces a parametric effect in the basic oscillations of the mirrors, which can be registered using a laser range-finder. Besides, as the solution predicts, a falling gravitational wave produces a time shift in the vibrating mirrors, that can be registered using synchronized clocks located with each of the mirrors: their de-synchronization means a gravitational wave detection.

3.6 The exact solution for a solid-body detector

We assume the elastic force $\Phi^\alpha = -k^\alpha_\sigma x^\sigma$ connecting two particles belonging to the opposite extremities of a solid-body detector to be independent of time ($k^0_\sigma = 0$). In such a case, when the chr.inv.-equations of motion (3.13) are applied to the particles, they take the form

$$(\mathrm{v}^2)^2 - (\mathrm{v}^3)^2 = 0\,, \tag{3.39}$$

$$\frac{d\mathrm{v}^1}{dt} = -\frac{k^1_\sigma}{m_0}\,x^\sigma, \tag{3.40}$$

$$\frac{d\mathrm{v}^2}{dt} + \dot{a}\,\mathrm{v}^2 = -\frac{k^2_\sigma}{m_0}\,x^\sigma, \tag{3.41}$$

$$\frac{d\mathrm{v}^3}{dt} - \dot{a}\,\mathrm{v}^3 = -\frac{k^3_\sigma}{m_0}\,x^\sigma, \tag{3.42}$$

where (3.40) means $\mathrm{v}^1 = \mathrm{v}^1_{(0)} = const$. Hence, in the detector,

$$\mathrm{v}^1 = \mathrm{v}^1_{(0)} = 0\,, \qquad k^1_\sigma = 0\,. \tag{3.43}$$

Only two equations, (3.41) and (3.42), are essential. They differ only in the sign of the $\dot{a}$, therefore we solve only (3.41).

Let the solid-body detector be elastic in only two directions transverse to the direction x^1, in which the gravitational wave propagates. In such a case the elastic coefficient components are $k^2_\sigma = k^3_\sigma = k = const$. With that, since $a = A\sin\frac{\omega}{c}(ct + x^1)$ as previously and denoting $x^2 \equiv x$, $\frac{k}{m_0} = \Omega^2$, $A\omega = -\mu$, we reduce (3.41) to

$$\ddot{x} + \Omega^2 x = \mu\cos\frac{\omega}{c}\,(ct + x^1)\,\dot{x}\,, \tag{3.44}$$

where μ is a "small parameter". We solve this equation using Poincaré's method, known also as the small parameter method or the perturbation method: we consider the right hand side as a forcing perturbation of a harmonic oscillation described by the left hand side. This is an exact-solution method, because a solution obtained with it is a power series expansion over the small parameter (see Chapter XII, §2 in Lefschetz [30]). Introducing a new variable $t' = \Omega t$ in order to make it dimensionless according to Lefschetz and $\mu' = \frac{\mu}{\Omega}$, we obtain

$$\ddot{x} + x = \mu'\cos\frac{\omega}{\Omega c}\,(ct' + \Omega x^1)\,\dot{x}\,. \tag{3.45}$$

A general solution of this equation, representable as

$$\dot{x} = y\,, \quad \dot{y} = -x + \mu' \cos \frac{\omega}{\Omega c}\,(ct' + \Omega x^1)\,y \tag{3.46}$$

with the initial data $x_{(0)}$ and $y_{(0)}$ at the moment of time $t' = 0$, is determined by the series pair (Lefschetz)

$$\left.\begin{aligned} x &= P_0(x_{(0)}, y_{(0)}, t') + \mu' P_1(x_{(0)}, y_{(0)}, t') + \dots \\ y &= \dot{P}_0(x_{(0)}, y_{(0)}, t') + \mu' \dot{P}_1(x_{(0)}, y_{(0)}, t') + \dots \end{aligned}\right\}. \tag{3.47}$$

Substituting these into (3.46) and equating coefficients in the same orders of μ', we obtain

$$\left.\begin{aligned} &\ddot{P}_0 + P_0 = 0 \\ &\ddot{P}_1 + P_1 = \dot{P}_0 \cos \frac{\omega}{\Omega c}\,(ct' + \Omega x^1) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{3.48}$$

with the initial data $P_0(0) = \xi, \dot{P}_0(0) = \vartheta$, $P_1(0) = \dot{P}_1(0) = 0$ (where $n > 0$) at the moment of time $t' = 0$. Because the amplitude A (it is a part of the variable $\mu' = -\frac{\omega}{\Omega} A$) is a small value, we consider only the first two equations into account. The first of them is the equation of harmonic oscillations, which has the solution

$$P_0 = \xi \cos t' + \vartheta \sin t', \tag{3.49}$$

and the second equation, with this solution, takes the form

$$\ddot{P}_1 + P_1 = (-\xi \sin t' + \vartheta \cos t') \cos \frac{\omega}{\Omega c}\,(ct' + \Omega x^1). \tag{3.50}$$

This is a linear uniform equation. The solution to this equation, according to Kamke (see Part III, Chapter II, §2.5 in [31]), is

$$\begin{aligned} P_1 = {} & \frac{\vartheta \Omega^2}{2} \left\{ \frac{\cos\left[(\Omega - \omega)t - \frac{\omega}{c} x^1\right]}{\Omega^2 - (\Omega - \omega)^2} + \frac{\cos\left[(\Omega + \omega)t + \frac{\omega}{c} x^1\right]}{\Omega^2 - (\Omega + \omega)^2} \right\} - \\ & - \frac{i \xi \Omega^2}{2} \left\{ \frac{\sin\left[(\Omega - \omega)t - \frac{\omega}{c} x^1\right]}{\Omega^2 - (\Omega - \omega)^2} + \frac{\sin\left[(\Omega + \omega)t + \frac{\omega}{c} x^1\right]}{\Omega^2 - (\Omega + \omega)^2} \right\}, \end{aligned} \tag{3.51}$$

where the brackets contain, respectively, the real and imaginary parts of the sum $e^{i(\Omega-\omega)t-\frac{\omega}{c}x^1}+e^{i(\Omega+\omega)t+\frac{\omega}{c}x^1}$. Substituting these into (3.47) and going back to $x=x^2$, we obtain the final solution in real numbers

$$x^2 = \xi\cos\Omega t + \vartheta\sin\Omega t - \\ -\frac{A\omega\Omega\vartheta}{2}\left\{\frac{\cos\left[(\Omega-\omega)t-\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega-\omega)^2}+\frac{\cos\left[(\Omega+\omega)t+\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega+\omega)^2}\right\}, \quad (3.52)$$

while the solution for x^3 differs only in the sign of A.

With this result we solve the chr.inv.-equations of deviating non-geodesics (3.17).

For the cylindrical bar (solid-body detector) under consideration, we assume $\mathrm{v}^1=0$, $\mathrm{v}^2=\mathrm{v}^3$, $\Phi^1=0$, $\Phi^2=-\frac{k}{m_0}\eta^2$, $\Phi^3=-\frac{k}{m_0}\eta^3$, where $\mathrm{v}^2=\mathrm{v}^3$ means that the initial conditions ξ and ϑ are the same in both the x^2 and x^3 directions. Thus, the chr.inv.-deviation equations along the $x^0=ct$ and x^1 directions, respectively, are

$$\frac{d^2\varphi}{dt^2}=0, \qquad \frac{d^2\eta^1}{dt^2}=0, \quad (3.53)$$

so we can put their solutions as $\varphi=0$ and $\eta^1=0$.

With all of the above, the chr.inv.-deviation equation along the x^2 direction (it differs from that along the x^3 direction by only the sign of the A) takes the following form

$$\frac{d^2\eta^2}{dt^2}+\frac{k}{m_0}\eta^2=-A\omega\cos\frac{\omega}{c}(ct+x^1)\frac{d\eta^2}{dt}, \quad (3.54)$$

which is like (3.44). So, the solutions η^2 and η^3 should be like (3.52). Thus, we obtain solutions identical to (3.52), which are

$$\eta^2 = \xi\cos\Omega t + \vartheta\sin\Omega t - \\ -\frac{A\omega\Omega\vartheta}{2}\left\{\frac{\cos\left[(\Omega-\omega)t-\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega-\omega)^2}+\frac{\cos\left[(\Omega+\omega)t+\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega+\omega)^2}\right\}, \quad (3.55)$$

$$\eta^3 = \xi\cos\Omega t + \vartheta\sin\Omega t - \\ -\frac{A\omega\Omega\vartheta}{2}\left\{\frac{\cos\left[(\Omega-\omega)t-\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega-\omega)^2}+\frac{\cos\left[(\Omega+\omega)t+\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega+\omega)^2}\right\}. \quad (3.56)$$

These are the exact solutions to the chr.inv.-equations of two deviating non-geodesics in the field of a weak plane gravitational wave. The solutions lead us to the following conclusions:

1) The solutions $\varphi = const$ and $\eta^1 = const$ mean that a gravitational wave falling on a horizontally suspended solid-body bar does not change the vertical size η^1 of the bar and does not produce a time shift φ on the clocks installed at its ends;
2) As is seen from the solutions for η^2 (3.55) and η^3 (3.56), gravitational waves can force the extremities of a solid-body bar to undergo relative oscillations, transverse to the wave propagation: a) forced relative oscillations at the frequency ω of the gravitational waves; b) resonant oscillations that occur as soon as the frequency of the gravitational wave becomes double the frequency of the basic oscillation of the bar extremities ($\omega = 2\Omega$). Both of these effects are of parametric origin: they are possible only if the extremities of the bar have an initial relative oscillation ($\Omega \neq 0$). In the absence of initial relative oscillation, such a solid-body detector does not respond to gravitational waves.

Owing to the theoretical results that we have obtained, we propose a new experimental statement for solid-body detectors:

New experiment (solid-body detector)

Use such a solid-body detector (cylindrical bar), which is horizontally suspended and having a laboratory induced oscillation of its body so that there are relative oscillations of the bar extremities ($\Omega \neq 0$). Such a system, according to the exact solution for a solid-body detector, can have a parametric effect in the basic oscillations of the bar extremities due to a falling gravitational wave, which can be registered as a piezo-effect in the bar.

3.7 Conclusions

The experimental statement on gravitational waves proceeds from the equation of deviating geodesic lines and the equation of deviating non-geodesics. Weber's result was not based on an exact solution to the equations, but on his approximate analysis of what could be expected: he expected that a plane weak wave of the space metric may displace two resting particles with respect to each other. Unlike Weber, here we have obtained exact solutions of the deviation equation for both free

and spring-bound particles. According to the obtained exact solutions, a gravitational wave can displace particles in a two-particle system only if they are in motion with respect to each other or the local space (there is no effect if they are at rest). In other words, gravitational waves produce a parametric effect on a two-particle system. According to the solutions, an altered detector construction can be proposed such that it might interact with gravitational waves. These are: a) a free-mass detector, in which the suspended mirrors have laboratory induced basic oscillations relative to each other; b) a horizontally suspended cylindrical bar, the extremities of which have basic relative oscillations induced by a laboratory source.

Chapter 4 Non-quantum Teleportation. Frozen Light

4.1 Trajectories of instant displacement. The zero-space and non-quantum teleportation

So, the basic space-time of General Relativity is a four-dimensional pseudo-Riemannian space, which, in general, is inhomogeneous, anisotropic, curved, non-holonomic (rotating) and deforming. The space-time interval in terms of physically observable quantities has the form

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \tag{4.1}$$

where

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i \tag{4.2}$$

is the physically observable time interval, $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ is the gravitational potential, v_i is the linear velocity with which the space rotates, $d\sigma^2 = h_{ik}\, dx^i dx^k$ is the square of the physically observable spatial interval, and h_{ik} is the physically observable chr.inv.-metric tensor.

Consider a particle moving at a space-time distance ds. Re-write ds^2 based on the formula (4.2). We obtain

$$ds^2 = c^2 d\tau^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right), \tag{4.3}$$

where $\mathrm{v}^2 = h_{ik}\, \mathrm{v}^i \mathrm{v}^k$, while $\mathrm{v}^i = \frac{dx^i}{d\tau}$ is the observable three-dimensional velocity of the particle. So, the numerical value of the space-time interval ds is a substantial number under $\mathrm{v} < c$, zero under $\mathrm{v} = c$, and an imaginary number under $\mathrm{v} > c$.

Particles with non-zero rest-masses ($m_0 \neq 0$) travel along real world-trajectories ($c\,d\tau > d\sigma$), if they have real relativistic masses, and along imaginary world-trajectories ($c\,d\tau < d\sigma$), if their relativistic masses are

imaginary (tachyons). The world-lines of both of these kinds are non-isotropic. In both cases relativistic masses are non-zero ($m \neq 0$). These are particles of substance.

Massless particles are particles with zero rest-masses ($m_0 = 0$), but having non-zero relativistic masses ($m \neq 0$). They travel with the velocity of light along world-trajectories of zero four-dimensional length ($ds = 0$, $c\,d\tau = d\sigma \neq 0$). These are isotropic trajectories. A particular case of massless particles are light-like particles — the quanta of an electromagnetic field (photons).

A condition under which a particle may realize an instant displacement (*teleportation*) is the vanishing of the observable time interval $d\tau$ (4.2). So, the *teleportation condition* is $d\tau = 0$ or, according to (4.2),

$$\mathrm{w} + v_i u^i = c^2, \tag{4.4}$$

where $u^i = \frac{dx^i}{dt}$ is its three-dimensional coordinate velocity of the particle. Hence, the space-time interval by which the particle is instantly displaced has the form

$$ds^2 = -d\sigma^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k \neq 0\,, \tag{4.5}$$

where $1 - \frac{\mathrm{w}}{c^2} = \frac{v_i u^i}{c^2}$ since $d\tau = 0$.

As is seen from this formula, in such a case, the signature $(+---)$ usual to the space-time region of an ordinary observer transforms into the signature $(-+++)$ of the space-time region where particles are teleported. In other words, the terms "time" and "three-dimensional space" are interchanged in the region of teleportation: "time" of a teleporting particle is "space" of an ordinary observer, and vice versa, "space" of the teleporting particle is "time" of the ordinary observer.

Further, we will refer to instant interaction or instant information transfer as the *long-range action*. A process in which a particle (mediator of the interaction) can realize the long-range action will be referred to as the *non-quantum teleportation*.

Long-range action mediators are particles inherent in a completely degenerate space-time (we have called it *zero-space*). We have called such particles *zero-particles*. See §1.4 of Chapter 1 for detail.

Once a particle has entered into a local zero-space region at one location of our regular space, it can be instantly connected to another

particle that has simultaneously entered into another zero-space "gate" at another distant location. From the viewpoint of an ordinary "external" observer, such a connexion is realized instantly. Meanwhile, inside the zero-space itself, completely degenerate particles transmit the interaction between these two locations with the coordinate velocities u^i that do not exceed the velocity of light.

Thus, we conclude that instant information transfer is naturally permitted in the framework of General Relativity, despite the fact that the real speeds of particles does not exceed the velocity of light. This is a "space-time trick" due to the space-time geometry and topology: we only see that information is transferred instantly, while it is transferred by not-faster-than-light particles travelling in another space that seems to us, the "external" observers, such as that there all intervals of time and all three-dimensional spatial intervals are zero*.

Let us first consider substantial particles. As is easy to see, instant displacement (teleportation) of such particles is possible along world-trajectories, on which $ds^2 = -d\sigma^2 \neq 0$ is true. So, these trajectories represented in terms of physically observable quantities are purely spatial lines of imaginary three-dimensional lengths $d\sigma$, although when considered in the ideal world-coordinates t and x^i the trajectories are four-dimensional. In a particular case, where the space does not rotate ($v_i = 0$) or the linear velocity of its rotation v_i is orthogonal to the coordinate velocity u^i of the teleporting particle and, hence, their scalar product is $v_i u^i = |v_i||u^i| \cos(v_i; u^i) = 0$, substantial particles can be teleported only if gravitational collapse occurs ($\mathrm{w} = c^2$). In this case, the world-trajectories of teleportation considered in the ideal world-coordinates also become purely spatial $ds^2 = g_{ik}\, dx^i dx^k$.

The second case is massless (light-like) particles, for example — photons. Since $ds^2 = 0$ for massless particles by definition, such particles can be teleported along world-trajectories located in a space having the metric

$$ds^2 = -d\sigma^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k = 0\,. \qquad (4.6)$$

*The most complete theoretical investigation of the teleportation condition in spaces of various metrics, including the real possibility of the non-quantum teleportation in an Earth-bound laboratory using a strong electromagnetic field, is given in our recent article: Rabounski D. and Borissova L. Non-quantum teleportation in a rotating space with a strong electromagnetic field. *Progress in Physics*, 2022, vol. 18, no. 1, 31–49.

As a result, we see that the photon teleportation space is characterized by the conditions $ds^2 = 0$ and $d\sigma^2 = c^2 d\tau^2 = 0$.

The obtained photon teleportation condition (4.6) is similar to the light cone equation $c^2 d\tau^2 - d\sigma^2 = 0$, where $d\sigma \neq 0$ and $d\tau \neq 0$. The light cone equation describes the light cone, elements of which are the world-trajectories of massless (light-like) particles*. Hence, teleporting photons actually travel along trajectories that are elements of a cone, similar to the light cone.

Considering the photon teleportation condition (4.6) from the viewpoint of an ordinary observer, we can see the obvious fact that, in such a case, the observable spatial metric $d\sigma^2 = h_{ik}\, dx^i dx^k$ becomes degenerate: $h = \det \|h_{ik}\| = 0$. This case means actually the degenerate light cone. Taking the relation $g = -h g_{00}$ [3–5] into account, we conclude that, in this case, the four-dimensional metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ becomes as well degenerate: $g = \det \|g_{\alpha\beta}\| = 0$. The latter means that the signature conditions that determine a pseudo-Riemannian space are broken, so photon teleportation is carried out outside the basic space-time of General Relativity. We considered such a completely degenerate space-time in §1.4 and §1.5 of Chapter 1 in this book, where we called it *zero-space* since, from the viewpoint of an ordinary observer, all spatial and time intervals in it are zero.

Under the conditions $d\tau = 0$ and $d\sigma = 0$, the observable relativistic mass m and frequency ω become zero. Hence, any particle with zero rest-mass $m_0 = 0$ when travelling in the zero-space (say, a teleporting photon) looks like it has zero relativistic mass $m = 0$ and frequency $\omega = 0$ to an ordinary observer. Therefore, particles of this kind can be considered the limiting case of massless (light-like) particles.

In §1.4 we have introduced a new term, *zero-particles*, for all particles that are inherent in the zero-space.

According to the wave-particle duality, every particle can be represented as a wave. In the framework of this concept, each mass-bearing particle is determined by its own four-dimensional wave vector $K_\alpha = \frac{\partial \psi}{\partial x^\alpha}$, where ψ is the wave phase known also as eikonal. The eikonal equation $K_\alpha K^\alpha = 0$ [2] manifests the fact that the length of a four-dimensional

*In contrast to the light cone equation, the photon teleportation equation (4.6) is expressed in terms of the ideal world-coordinates t and x^i, and not in terms of physically observable quantities.

vector remains unchanged in the four-dimensional pseudo-Riemannian space. As was shown in §1.3, the eikonal equation for regular massless light-like particles (regular photons) has the form

$$\frac{1}{c^2}\left(\frac{{}^*\partial\psi}{\partial t}\right)^2 - h^{ik}\frac{{}^*\partial\psi}{\partial x^i}\frac{{}^*\partial\psi}{\partial x^k} = 0\,, \tag{4.7}$$

which is a travelling wave equation. The eikonal equation in a zero-space region has the form (see §1.5 for detail)

$$h^{ik}\frac{{}^*\partial\psi}{\partial x^i}\frac{{}^*\partial\psi}{\partial x^k} = 0\,, \tag{4.8}$$

because there we have $\omega = \frac{{}^*\partial\psi}{\partial t} = 0$ and thus, the time term of the equation (4.7) becomes zero. This is a standing wave equation. Therefore, all particles located in a zero-space region appear to an ordinary observer as *standing light waves*, and the entire zero-space appears to him as a system of standing light waves (a *light-like hologram*). This means that an experiment for discovering the non-quantum teleportation of photons should be linked to stationary (stopped) light.

At the end, we conclude that instant displacements of particles are naturally permitted in the space-time of General Relativity. As was shown above, the teleportation trajectories of real particles and photons lie in different regions of the space-time. But it would be a mistake to think that for teleportation it is necessary to accelerate a substantial particle to a superluminal speed (making it a tachyon), and to accelerate a photon to infinite speed. No — as it is easy to see from the teleportation condition $\mathrm{w} + v_i u^i = c^2$, if the gravitational potential is strong enough and the space rotates at a velocity comparable with the velocity of light, then substantial particles can be teleported at regular subluminal speeds. Photons can reach the teleportation condition easier, because they initially travel with the velocity of light. From the viewpoint of an ordinary observer, as soon as the teleportation condition is realized in the neighbourhood of a travelling particle, the particle "disappears" from his observed world, although it continues its motion with a subluminal (or light) coordinate velocity u^i in another space-time region invisible to us. Then, as soon as the particle's velocity decreases, or if something else violates the teleportation condition (for example, lowering the gravitational potential or the linear velocity with which the space rotates),

the particle "appears" at the same observable moment of time at a different point in our observable space at that distance and in the direction in which it travelled.

There is no problem with photon teleportation, since it is realized along completely degenerate world-trajectories ($g = 0$) outside the basic pseudo-Riemannian space ($g < 0$). On the other hand, the teleportation trajectories of substantial particles are strictly non-degenerate ($g < 0$), hence such trajectories lie in the pseudo-Riemannian space*. It presents no problem because at any point in the pseudo-Riemannian space we can place a tangential space of $g \leqslant 0$ consisting of the regular pseudo-Riemannian space ($g < 0$) and the zero-space ($g = 0$) as two different regions of the same manifold. A space of $g \leqslant 0$ is a natural generalization of the basic space-time of General Relativity, permitting the non-quantum teleportation of both photons and substantial particles.

Until this day, teleportation has had an explanation given only by Quantum Mechanics [32]. It was previously achieved only in the strict quantum fashion — quantum teleportation of photons in 1998 [33] and of atoms in 2004 [34, 35]. Now the situation changes: with our theory we can find physical conditions for teleportation of photons in a non-quantum way, which is not due to the probabilistic laws of Quantum Mechanics but according to the exact (non-quantum) laws of the General Theory of Relativity following the space-time geometry. We therefore suggest referring to this phenomenon as the *non-quantum teleportation*.

The only difference is that from the viewpoint of an ordinary observer the length of any parallel transported vector remains unchanged. It is also an "observable truth" for vectors in a zero-space region, because the observer reasons only the standards (properties) of his pseudo-Riemannian space anyway. The eikonal equation in a zero-space region, expressed in his observable world-coordinates, is $K_\alpha K^\alpha = 0$. However, the internal zero-space metric $ds^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k = 0$, expressed in terms of the ideal world-coordinates t and x^i, degenerates into a three-dimensional metric $d\mu^2$ that, depending on the gravitational

*Any space of Riemannian geometry has a strictly non-degenerate metric ($g < 0$) by definition. Pseudo-Riemannian spaces are a particular case of Riemannian spaces, where the metric is sign-alternating. Einstein had chosen a four-dimensional pseudo-Riemannian space with the signature $(+---)$ or $(-+++)$ as a basis to his theory. Therefore, the basic space-time of the General Theory of Relativity has a strictly non-degenerate metric ($g < 0$).

potential w uncompensated by something else, is not invariant

$$d\mu^2 = g_{ik}\, dx^i dx^k = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 \neq inv\,. \tag{4.9}$$

As a result of the zero-space metric, a four-dimensional vector, say, the coordinate velocity vector U^α, degenerates in the zero-space into a three-dimensional spatial vector U^i, and its length when transporting the vector parallel to itself does not remain unchanged

$$U_i U^i = g_{ik}\, U^i U^k = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 \neq const\,. \tag{4.10}$$

This means that although the observed geometry inside the zero-space is Riemannian for an ordinary observer, the real geometry inside the zero-space is non-Riemannian.

In connexion with the above results, it is important to remember the "Infinite Relativity Principle", introduced by Abraham Zelmanov. Proceeding from his studies of relativistic cosmology [36–38], he had arrived at the following conclusion:

Zelmanov's Infinite Relativity Principle

> In homogeneous isotropic cosmological models, the spatial infinity of the Universe depends on our choice of the reference frame from which we observe the Universe (i.e., the observer's reference frame). If the three-dimensional space of the Universe, observed in one reference frame, is infinite, it may be finite in another reference frame. The same is as well true for the time during which the Universe evolves.

We have come to the "Finite Relativity Principle" here. As we have showed, since there is a difference between the physically observable world-coordinates and the ideal world-coordinates, the same space-time region may look very different, when considered in different reference frames. So, in the observable world-coordinates, the entire zero-space is a point ($d\tau = 0$, $d\sigma = 0$). On the other hand, $d\tau = 0$ and $d\sigma = 0$ considered in the ideal world-coordinates is $-\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k = 0$, which is a four-dimensional cone equation similar to the light cone equation $c^2 d\tau^2 - d\sigma^2 = 0$. Actually, the Finite Relativity Principle for observed objects is manifested here — an observed point is the entire zero-space when considered in the ideal coordinates.

4.2 The geometric structure of the zero-space

So, we have obtained that an ordinary real observer perceives the entire zero-space as a space-time region determined by the observable conditions of degeneration, which are $d\tau = 0$ and $d\sigma^2 = h_{ik}\, dx^i dx^k = 0$. See §1.4 for detail.

The physical sense of the first condition $d\tau = 0$ is that an ordinary observer perceives any two events in the zero-space as simultaneous, at whatever distance from each other they are. We called such a way of instant information transfer the long-range action.

The second condition $d\sigma^2 = 0$ means the absence of any observable distance between the event and the observer. Such "superposition" of an observer and an object observed by him is only possible, if we assume that our regular four-dimensional pseudo-Riemannian space meets the zero-space at each point (as is "stuffed" with the zero-space).

Let us now turn to the mathematical interpretation of the degeneration conditions.

The quantity $c\,d\tau$ is the chr.inv.-projection of the four-dimensional coordinate interval dx^α onto the time line: $c\,d\tau = b_\alpha dx^\alpha$. The proper world-vector of the observer b^α by definition is not zero and dx^α is not zero as well. Then $d\tau = 0$ is true at $d\sigma^2 = 0$ only if the space-time metric $ds^2 = c^2 d\tau^2 - d\sigma^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ is degenerate, i.e., the determinant of the fundamental metric tensor is zero

$$g = \det \|g_{\alpha\beta}\| = 0\,. \tag{4.11}$$

Similarly, the condition $d\sigma^2 = h_{ik}\, dx^i dx^k = 0$ means that the observable three-dimensional metric is degenerate

$$h = \det \|h_{ik}\| = 0\,. \tag{4.12}$$

Having both of the space-time degeneration conditions $\mathrm{w} + v_i u^i = c^2$ and $g_{ik}\, dx^i dx^k = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2$ substituted into $d\sigma^2 = h_{ik}\, dx^i dx^k = 0$, we obtain the zero-space metric

$$ds^2 = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 - g_{ik}\, dx^i dx^k = 0\,. \tag{4.13}$$

Hence, inside the zero-space (from the viewpoint of an "internal" observer) the three-dimensional space is holonomic, and the rotation of

the zero-space is present in the time component of its metric

$$\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 = \left(\frac{v_i u^i}{c^2}\right)^2 c^2 dt^2. \tag{4.14}$$

If $\mathrm{w} = c^2$ (the gravitational collapse condition), then the zero-space metric (4.13) takes the form

$$ds^2 = -g_{ik}\, dx^i dx^k = 0\,, \tag{4.15}$$

i.e., the space-time metric becomes purely three-dimensional, and the three-dimensional space becomes as well degenerate

$$g_{(3\mathrm{D})} = \det \|g_{ik}\| = 0\,. \tag{4.16}$$

Here the condition $g_{(3\mathrm{D})} = 0$ originates in the fact that $g_{ik}\, dx^i dx^k$ is sign-definite, so it can become zero only if the determinant of the three-dimensional metric tensor g_{ik} is zero.

Because $\mathrm{w} + v_i u^i = c^2$ in the zero-space, in the case of gravitational collapse the condition $v_i u^i = 0$ is true.

We call the quantity $v_i u^i = v u \cos(v_i; u^i)$, which is the scalar product of the linear velocity with which the space rotates and the coordinate velocity of a zero-particle, the *zero-particle chirality*. Three cases of the zero-particle chirality are possible:

1) If the zero-particle chirality is $v_i u^i > 0$, then the angle α between the v_i and u^i is within $\frac{3\pi}{2} < \alpha < \frac{\pi}{2}$. Since the second degeneration condition $g_{ik}\, u^i u^k = c^2 \left(1 - \frac{\mathrm{w}}{c^2}\right)^2$ means $u = c\left(1 - \frac{\mathrm{w}}{c^2}\right)$, hence the gravitational potential is $\mathrm{w} < c^2$ in this case. This is the case of a regular gravitational field;
2) If the zero-particle chirality is $v_i u^i < 0$, then the angle α is within the range $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$, so $\mathrm{w} > c^2$ that means a super-strong gravitational field;
3) The zero-particle chirality is $v_i u^i = 0$ only if $\alpha = \left\{\frac{\pi}{2}; \frac{3\pi}{2}\right\}$ or $\mathrm{w} = c^2$ (gravitational collapse). Hence, the zero-particle chirality is zero if either the particle's velocity is orthogonal to the linear velocity with which the space rotates, or the state of gravitational collapse takes place (since under the gravitational collapse condition, the modulus of the particle's coordinate velocity is zero, $u = 0$).

To obtain an illustrated view of the zero-space geometry, we use a locally geodesic reference frame (see §1.14 for detail).

First, consider the geometric structure of the isotropic (light-like) space. It is characterized by the condition $c^2 d\tau^2 = d\sigma^2 \neq 0$. According to this condition, time and the regular three-dimensional space meet each other. Geometrically, this means that the time basis vector $\vec{e}_{(0)}$ coincides with all three spatial basis vectors $\vec{e}_{(i)}$, i.e., time "falls" into space (this fact does not mean that the spatial basis vectors coincide, because the time basis vector is the same for the entire spatial frame). In other words, $\cos(x^0; x^k) = \pm 1$ everywhere in the isotropic space. At $\cos(x^0; x^i) = +1$ the time basis vector is co-directed with the spatial ones: $\vec{e}_{(0)} \uparrow\uparrow \vec{e}_{(i)}$. If $\cos(x^0; x^i) = -1$, then the time and spatial basis vectors are oppositely directed: $\vec{e}_{(0)} \uparrow\downarrow \vec{e}_{(i)}$. The condition $\cos(x^0; x^k) = \pm 1$ can be expressed through the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$, because, in a general case, $e_{(0)} = \sqrt{g_{00}}$ (1.216). Finally, in a locally geodesic reference frame (according to §1.14), we obtain the geometric conditions characteristic of the isotropic space

$$\cos(x^0; x^k) = \pm 1\,, \qquad e_{(i)} = e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{4.17}$$

and, hence,

$$v_i = \mp c\, e_{(i)} = \mp \sqrt{g_{00}}\, c_i = \mp \left(1 - \frac{\mathrm{w}}{c^2}\right) c_i\,, \tag{4.18}$$

$$h_{ik} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 \left[\, 1 - \cos(x^i; x^k) \right], \tag{4.19}$$

where c^i is the three-dimensional chr.inv.-vector of the physically observable velocity of light, for which $c_i c^i = h_{ik} c^i c^k = c^2$.

The isotropic space exists at any point in the four-dimensional regular space as the light cone — a hypersurface with the metric

$$g_{\alpha\beta}\, dx^\alpha dx^\beta = 0\,, \tag{4.20}$$

or, in the extended form,

$$\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 - 2\left(1 - \frac{\mathrm{w}}{c^2}\right) v_i\, dx^i dt + g_{ik}\, dx^i dx^k = 0\,, \tag{4.21}$$

according to the formulae of the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ and the linear velocity $v_i = -\frac{c\, g_{0i}}{\sqrt{g_{00}}}$ with which the space rotates.

This is a subspace of the four-dimensional space, which hosts massless (light-like) particles travelling with the velocity of light. Because the space-time interval in such a region is zero, all four-dimensional directions inside it are equal (in other words, they are isotropic). Therefore, this subspace is commonly referred to as the *isotropic cone*.

Let us now turn to the geometric structure of the zero-space. Since w and v_i in the basis form are $\mathrm{w} = c^2(1 - e_{(0)})$ and $v_i = -c\,e_{(i)} \cos(x^0; x^i)$, the degeneration condition $\mathrm{w} + v_i u^i = c^2$ in the basis form is

$$c\,e_{(0)} = -e_{(i)}\,u^i \cos(x^0; x^i). \tag{4.22}$$

The number of dimensions of a space is determined by the number of the linearly independent basis vectors in it. In our formula (4.22), which is the basis notation of the degeneration condition $\mathrm{w} + v_i u^i = c^2$, the time basis vector $\vec{e}_{(0)}$ is linearly dependent on all of the spatial basis vectors $\vec{e}_{(i)}$. This means actual degeneration of the space-time. Hence, our formula (4.22) is the *geometric condition of degeneration*.

Since the four-dimensional metric is zero in the zero-space, such a space exists at any point of the isotropic (light) cone as a completely degenerate subspace of it. Such a *completely degenerate isotropic cone* is described by a somewhat different equation

$$\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 - g_{ik}\,dx^i dx^k = 0\,, \tag{4.23}$$

or, due to the degeneration condition $\mathrm{w} + v_i u^i = c^2$, by the equation

$$\frac{v_i v_k u^i u^k}{c^2}\,dt^2 - g_{ik}\,dx^i dx^k = 0\,. \tag{4.24}$$

The difference between the completely degenerate isotropic cone and the regular isotropic (light) cone is that the first satisfies the degeneration condition $\mathrm{w} + v_i u^i = c^2$. Therefore, the physical conditions inside a zero-space region are the ultimately degenerate case of the conditions in the regular isotropic (light-like) space, which is the home of photons. In other words, the long-range action is transmitted by special photons — *completely degenerate photons* that exist under the physical conditions of complete degeneration.

Since v_i has the same formulation (4.18) both in the case of the completely degenerate isotropic cone and in the case of the regular isotropic (light) cone, we arrive at the following important conclusion:

The completely degenerate isotropic cone is a cone of rotation at the speed of light, just like the regular isotropic cone. In other words, the zero-space rotates at each of its points with a linear velocity equal to the velocity of light. Its rotation becomes slower than light in the presence of a gravitational potential.

This conclusion is exactly the same as that we have arrived at in our previous study [39].

Finally, we conclude that the regular isotropic (light) cone contains the degenerate isotropic cone, which is the entire zero-space, as a subspace embedded into it at its each point. This is a clear illustration of the fractal structure of the world presented here as a system of the isotropic cones embedded into each other.

4.3 Gravitational collapse in the zero-space. Completely degenerate black holes

As is known, a *gravitational collapsar* or *black hole* is a local region of space (space-time), wherein the condition $g_{00} = 0$ is true. Because the gravitational potential is defined as $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$, the gravitational collapse condition $g_{00} = 0$ means that the gravitational potential is $\mathrm{w} = c^2$ in the region. We are going to consider how this condition can be realized in the zero-space.

As mentioned above, the first degeneration condition is $\mathrm{w} + v_i u^i = c^2$. According to the condition, if $v_i u^i = 0$ in a local zero-space region, then the gravitational potential is $\mathrm{w} = c^2$ therein. This means that, in the case of $v_i u^i = 0$, the gravitational potential is strong enough to bring the local region of the zero-space to gravitational collapse. We suggest referring to such a region as a *completely degenerate gravitational collapsar* or, equivalently, as a *completely degenerate black hole*.

Under the gravitational collapse condition $\mathrm{w} = c^2$, the second degeneration condition becomes $g_{ik}\, dx^i dx^k = 0$. Together with the above, this means that three physical and geometric conditions are realized in completely degenerate black holes

$$\mathrm{w} = c^2, \qquad v_i u^i = 0\,, \qquad g_{ik}\, dx^i dx^k = 0\,, \tag{4.25}$$

the physical sense of which is as follows:

1) The gravitational potential w inside every completely degenerate black hole is strong enough to stop the regular light-speed rotation

of the local region of the zero-space, i.e.

$$v_i = \mp c\, e_{(i)} = \mp \sqrt{g_{00}}\, c_i = \mp \left(1 - \frac{\mathrm{w}}{c^2}\right) c_i = 0\,; \tag{4.26}$$

2) In this case, the time basis vector $\vec{e}_{(0)}$ has zero length (time intervals are zero inside completely degenerate black holes)

$$e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} = 0\,; \tag{4.27}$$

3) In the zero-space, the condition $\cos\,(x^0; x^k) = \pm 1$ is true: the time basis vector $\vec{e}_{(0)}$ matches all three spatial basis vectors $\vec{e}_{(i)}$ (time "falls" into space). Hence, the previous condition $e_{(0)} = 0$ means that all three three-dimensional (spatial) basis vectors $\vec{e}_{(i)}$ have zero length $e_{(i)} = 0$ inside completely degenerate black holes

$$e_{(i)} = e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} = 0\,; \tag{4.28}$$

4) The condition $e_{(i)} = 0$ means that all three-dimensional coordinate intervals are $dx^i = 0$, i.e., the entire three-dimensional space inside completely degenerate black holes is shrunk to a point. Hence, the third condition $g_{ik}\, dx^i dx^k = 0$ of the conditions inside completely degenerate black holes (4.25) is due to $dx^i = 0$.

Therefore, completely degenerate black holes are point-like objects that keep light inside themselves due to their own extremely strong gravity. In other words, they are "absolute black holes" of all gravitational collapsars that are conceivable due to General Relativity.

4.4 Zero-particles as virtual photons. The geometric interpretation of Feynman diagrams

As is known, Feynman diagrams are a graphical description of the interactions between elementary particles. The diagrams show that the actual carriers of the interactions are virtual particles. In other words, almost all physical processes are based on the emission and absorption of virtual particles (say, virtual photons) by real particles of our world.

Another notable property of Feynman diagrams is that they are capable of describing particles and antiparticles (e.g., the electron and the positron) at the same time. In this example, a positron is represented as an electron which moves back in time.

According to Quantum Electrodynamics, the interaction of particles at every branching point of Feynman diagrams conserves their four-dimensional momentum. This suggests a possible geometric interpretation of Feynman diagrams in General Relativity.

In the four-dimensional pseudo-Riemannian space, which is the basic space-time of General Relativity, the following objects can get correct, formal definitions:

1) Mass-bearing particles — particles, the rest-masses of which are non-zero ($m_0 \neq 0$), and the trajectories are non-isotropic ($ds \neq 0$). There are subluminal mass-bearing particles (real particles) and superluminal mass-bearing particles (tachyons). Mass-bearing particles include both particles and antiparticles, realizing their motion from the past to the future and from the future to the past, respectively;
2) Massless particles — particles with zero rest-masses ($m_0 = 0$), but non-zero relativistic masses ($m \neq 0$). Such particles travel along isotropic trajectories ($ds = 0$) with the velocity of light. These are light-like particles, e.g., photons. Massless particles include both particle and antiparticle options as well;
3) Zero-particles — particles with zero rest-mass and zero relativistic mass, which travel along trajectories in the completely degenerate space-time (zero-space). From the viewpoint of an ordinary observer, whose home is our world, the physically observable time stops on zero-particles. Therefore, both particle and anti-particle options become nonsense for zero-particles.

Hence, to give a geometric interpretation of Feynman diagrams in the space-time of General Relativity, we only need to give a formal definition of virtual particles. Here is how to do it.

As is known according to Quantum Electrodynamics, virtual particles are those for which, contrary to regular ones, the regular relation between energy and momentum

$$E^2 - c^2 p^2 = E_0^2 \tag{4.29}$$

is not true. In other words, for virtual particles,

$$E^2 - c^2 p^2 \neq E_0^2 , \tag{4.30}$$

where $E = mc^2$, $p^2 = m^2 \mathrm{v}^2$ and $E_0 = m_0 c^2$.

In a pseudo-Riemannian space, the regular relation (4.29) is true. It follows from the condition $P_\alpha P^\alpha = m_0^2 = const \neq 0$ for mass-bearing particles (non-isotropic trajectories), and from the condition $P_\alpha P^\alpha = 0$ for massless particles (isotropic trajectories). Substituting the respective components of the momentum vector P^α, we obtain the regular relation in the chr.inv.-form for mass-bearing particles

$$E^2 - c^2 m^2 \mathrm{v}_i \mathrm{v}^i = E_0^2 , \tag{4.31}$$

and that for massless ones, $E^2 - c^2 m^2 \mathrm{v}_i \mathrm{v}^i = 0$, that is the same as

$$h_{ik} \mathrm{v}^i \mathrm{v}^k = c^2. \tag{4.32}$$

But this is not true in the completely degenerate space (zero-space), because the zero-space metric $d\mu^2$ (4.9) is not invariant: $d\mu^2 \neq inv$. As a result, from the viewpoint of a hypothetical observer, whose home is the zero-space, a degenerate four-velocity vector transported parallel to itself does not conserve its length: $U_\alpha U^\alpha \neq const$ (4.10). Therefore, the regular relation between energy and momentum $E^2 - c^2 p^2 = const$ (4.29) is not applicable to zero-particles, but another relation, which is a sort of $E^2 - c^2 p^2 \neq const$ (4.30), is true. Because the latter is the main property of virtual particles, we arrive at the conclusion:

> Zero-particles can play the rôle of virtual particles, which, according to Quantum Electrodynamics, are the material carriers transmitting the interactions between regular particles of our world. If this is so, then the entire zero-space is an "exchange buffer", through the capacity of which zero-particles transmit the interactions between regular mass and massless particles of our world.

We have concluded in §4.2 that zero-particles are completely degenerate photons. They can also exist in collapsed regions of the zero-space, wherein the gravitational collapse condition is true (see §4.3 for detail). Hence, virtual particles of two kinds can be presupposed according to General Relativity:

1) Virtual photons — regular completely degenerate photons;
2) Virtual collapsars — completely degenerate photons located in collapsed regions of the zero-space.

As a result, we arrive at a conclusion that all interactions between regular mass-bearing and massless particles in the basic space-time of

General Relativity (four-dimensional pseudo-Riemannian space), are affected through an *exchange buffer*, inside which the zero-space acts. The material carriers of the interactions inside such a buffer are virtual particles of the two aforementioned kinds.

In §1.5 of Chapter 1, when considering particles in the framework of the wave-particle duality, we have obtained that the eikonal equation for zero-particles is a standing wave equation of stopped light (1.130). Hence, virtual particles are actually *standing light waves*, and the interaction between regular particles of our regular space-time is transmitted through a system of standing light-like waves (a *standing-light hologram*) that fills the exchange buffer (zero-space).

Everything that we have proposed here is so far the only explanation of virtual particles and virtual interactions given by the geometric method of General Relativity and in accordance with the geometric structure of the four-dimensional space (space-time). It is possible that this method will create a link between Quantum Electrodynamics and the General Theory of Relativity.

4.5 Frozen light

Here we summarize our recent results, detailed in our 2011 publication [40], in which we presented a theory of frozen light in the framework of General Relativity. This result has also been presented at the 2011 APS March Meeting [41].

In the summer of 2000, Lene V. Hau, who pioneered light-slowing experiments over many years in the 1990s at Harvard University, first obtained light slowed down to a rest state. In her experiment, light was stored, for milliseconds, in ultracold atoms of sodium (with a gaseous cloud of the atoms cooled down to within a millionth of a degree of absolute zero). This state was then referred to as *frozen light* or *stopped light*. An anthology of the primary experiments is given in her publications [42–46]. After the first success of 2000, Lene Hau still continues the study: in 2009, light was stopped for 1.5 second at her laboratory [47].

Then the frozen light experiment was repeated, during one year, by two other groups of experimentalists. The group headed by Ronald L. Walsworth and Mikhail D. Lukin of the Harvard-Smithsonian Center for Astrophysics stopped light in a room-temperature gas [48]. In the

experiments conducted by Philip R. Hemmer at the Air Force Research Laboratory in Hanscom (Massachusetts), light was stopped in a cooled-down solid [49].

The best-of-all survey of all experiments on this subject was given in Lene Hau's *Frozen Light*, which was first published in 2001, in *Scientific American* [45]. Then an extended version of this paper was reprinted in 2003, in a special issue of the journal [46].

On the other hand, the frozen light problem meets our theoretical research of the 1990s, which was produced independently of the experimentalists (we knew nothing about the experiments until January 2001, when the first success in stopping light was widely advertised in the scientific press). Our task was to reveal what kinds of particles could theoretically inhabit the space (space-time) of General Relativity. See Chapter 1 of this book for detail. We have obtained that, aside for mass-bearing and massless (light-like) particles, particles of the third kind can also exist. Such particles (we called them *zero-particles*) inhabit a space with a completely degenerate metric, which is the limiting case of the light-like (particularly degenerate) space. This means that zero-particles are the limiting case of photons: they are completely degenerate photons, in other words. Zero-particles can be hosted by both regular regions and collapsed regions of the completely degenerate space. In the latter case, they exist under the gravitational collapse condition. From the viewpoint of an ordinary observer, the completely degenerate space (*zero-space*) looks like a local volume, wherein all observable time intervals and all three-dimensional observable intervals are zero. Once a photon has entered into such a zero-space "gate" at one location of our regular space, it can be instantly connected to another photon which has entered into a similar "gate" at another location. This is the way for the *non-quantum teleportation*. Also, the regular relation between energy and momentum is not true for zero-particles. This means that zero-particles may play the rôle of virtual particles, which are the material carriers of the interaction between regular particles of our world. All this has been explained in detail earlier in this Chapter.

In addition, from the point of view of an ordinary observer, zero-particles should appear as standing light waves — waves of stopped light. The latter corresponds to the result registered in the frozen light experiment: in this experiment, a stopped light beam is "stored" in the atomic vapour and remains invisible to the observer until the point in

time when it is released again in its regularly "traveling state". (See the original reports about the frozen light experiments referred above.)

This means that the frozen light experiment pioneered at Harvard by Lene Hau is an experimental "foreword" to the discovery of zero-particles and, hence, a way for the non-quantum teleportation.

With these we can mean frozen light as a new state of matter, which differs from the others (solid, gas, liquid, plasma).

4.6 Conclusions

The geometric structure of the four-dimensional space (space-time) of General Relativity therefore allows the possibility of such particles that are the limiting case of photons and endowed with zero rest-mass (like photons), but their relativistic masses are also zero. Therefore, we call them *zero-particles*. Such particles are inherent in a space with the completely degenerate metric (*zero-space*), which is the limiting case of the (particularly degenerate) light-like space. In other words, these particles are completely degenerate photons.

Zero-particles can belong to two types of regions of the zero-space: ordinary regions of the zero-space and those in the state of gravitational collapse. In the latter case, they exist only under the gravitational collapse condition.

The completely degenerate space (zero-space) looks like a local space volume, in which all observable time intervals and observable spatial three-dimensional intervals are identically zero. As soon as a photon enters the zero-space through a "gateway" at a point in our regular (non-degenerate) space, it can be instantly connected to another photon that has entered an analogous "gateway" at another point. This is a form of the *non-quantum teleportation* of photons.

The classical relation between energy and momentum is not true for zero-particles. It follows that the zero-particles play a rôle of virtual particles which are the material carriers of the interaction between regular particles of our Universe.

The frozen light experiment, first performed in 2000 by Lene Hau, holds the key to the discovery of zero particles and therefore to the non-quantum teleportation.

Epilogue

In the novel *Far Rainbow*, written by Arcady and Boris Strugatsky 60 years ago, one of the characters remembers the following...

> "...Being a schoolboy he was surprised by the problem: move things across vast spaces in no time. The goal was set to contradict any existing views of absolute space, space-time, kappa-space... At that time they called it "punch of Riemannian fold". Later it would be dubbed "hyper-infiltration", "sigma-infiltration", or "zero-contraction". At length it was named zero-transportation or "zero-T" for short. This produced "zero-T-equipment", "zero-T-problems", "zero-T-tester", "zero-T-physicist".
> — What do you do?
> — I'm a zero-physicist.
> A look full of surprise and admiration.
> — Excuse me, could you explain what zero-physics is? I don't understand a bit of it.
> — Well... Neither do I."

This passage might be a good afterword to our research study. In the early 1960s, words like "zero-space" or "zero-transportation" sounded science-fiction or at least something to be brought to real life generations from now.

But science is progressing faster then we think. The results obtained in this book suggest that the variety of existing particles, along with the types of their interactions, is not limited to those known to contemporary physics. We should expect that further advancements in experimental technique will discover zero-particles, which inhabit the degenerate space-time (zero-space) and can be observed as waves of "stopped light" (standing light waves). From the viewpoint of an ordinary observer, zero-particles travel instantly, despite the fact that they actually travel with the velocity of light in zero-space. For this reason, they can realize zero-transportation.

Therefore, we are sure that it would be a mistake to believe or take for granted that most Laws of Nature have already been discovered by contemporary science. What seems more likely is that we are just at the very beginning of a long, long road to the Unknown World.

Bibliography

1. Levi-Civita T. Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura Riemanniana. *Rendiconti del Circolo Matematico di Palermo*, 1917, tome 42, 173–205.
2. Landau L. D. and Lifshitz E. M. The Classical Theory of Fields. Pergamon Press, Oxford, 1951.
   Translated from the 1st Russian edition published in 1939. Section references are given from the final 4th English edition, expanded twicely from the 1st edition, Butterworth-Heinemann, 1979.
3. Zelmanov A. L. Chronometric Invariants. Translated from the 1944 PhD thesis, American Research Press, Rehoboth, New Mexico, 2006.
4. Zelmanov A. L. Chronometric invariants and accompanying frames of reference in the General Theory of Relativity. *Soviet Physics Doklady*, 1956, vol. 1, 227–230.
   Translated from: *Doklady Academii Nauk SSSR*, 1956, vol. 107, no. 6, 815–818.
5. Zelmanov A. L. On the relativistic theory of an anisotropic inhomogeneous universe. *The Abraham Zelmanov Journal*, 2008, vol. 1, 33–63.
   Translated from: *Proceedings of the 6th Soviet Conference on the Problems of Cosmogony*, held in 1957 in Moscow, USSR Acad. Science Publishers, Moscow, 1959, 144–174 (in Russian).
6. Cattaneo C. General Relativity: relative standard mass, momentum, energy, and gravitational field in a general system of reference. *Nuovo Cimento*, 1958, vol. 10, 318–337.
7. Cattaneo C. On the energy equation for a gravitating test particle. *Nuovo Cimento*, 1959, vol. 11, 733–735.
8. Cattaneo C. Conservation laws in General Relativity. *Nuovo Cimento*, 1959, vol. 13, 237–240. vol. 11, 733–735.
9. Cattaneo C. Problèmes d'interprétation en relativité générale. *Colloques internationaux du CNRS*, no. 170 "Fluides et champ gravitationel en relativité générale", Éditions du CNRS, Paris, 1969, 227–235.
10. Hafele J. C. Relativistic behaviour of moving terrestrial clocks. *Nature*, July 18, 1970, vol. 227, 270–271.

11. Hafele J. Performance and results of portable clocks in aircraft. PTTI 3rd Annual Meeting, November 16–18, 1971, 261–288.
12. Hafele J. C. Relativistic time for terrestrial circumnavigations. *American Journal of Physics*, 1972, vol. 40, 81–85.
13. Hafele J. and Keating R. Around the world atomic clocks: predicted relativistic time gains. *Science*, July 14, 1972, vol. 177, 166–168.
14. Hafele J. and Keating R. Around the world atomic clocks: observed relativistic time gains. *Science*, July 14, 1972, vol. 177, 168–170.
15. Demonstrating relativity by flying atomic clocks. *Metromnia*, the UK's National Measurement Laboratory Newsletter, issue 18, Spring 2005.
16. Zelmanov A. L. Relationship of the orthometric monad formalism to chronometric and kinemetric invariants. *Soviet Physics Doklady*, 1976, vol. 21, 147–150.
Translated from: *Doklady Academii Nauk SSSR*, 1976, vol. 227, no. 1, 78–81.
17. Schouten J. A. und Struik D. J. Einführung in die neuren Methoden der Differentialgeometrie. Noordhoff, Groningen, 1938.
First published in *Zentralblatt für Mathematik*, 1935, Bd. 11 und Bd. 19.
18. Lie S. Theorie der Transformationsgruppen. First published by Teubner, Leipzig, in 3 volumes: vol. 1, 1888; vol. 2, 1890; vol. 3, 1893 (reprinted by Chelsea Publishing, American Mathematical Society, New York, 1970).
19. Killing W. Über die Grundlagen der Geometrie. *Crelle's Journal für die reine und angewandte Mathematik*, 1892, Bd. 109, 121–186.
20. Petrov A. Z. Einstein Spaces. Pergamon Press, Oxford, 1969.
21. Rabounski D. and Borissova L. Exact theory to a gravitational wave detector. New experiments proposed. *Progress in Physics*, 2006, vol. 2, no. 2, 31–38.
22. Rabounski D. and Borissova L. Exact solution for a gravitational wave detector. 2008 APS April Meeting, April 12–15, 2008, St. Louis, Missouri.
23. Borissova L. Gravitational waves and gravitational inertial waves according to the General Theory of Relativity. *The Abraham Zelmanov Journal*, 2010, vol. 3, 25–70.
24. Weber J. General Relativity and Gravitational Waves. Interscience Publishers, New York, 1961.
25. Synge J. L. Relativity: the General Theory. North Holland, Amsterdam, 1960.
26. Weber J. Gravitational-wave-detector events. *Physical Review Letters*, 1968, vol. 20, 1307–1308.
27. Weber J. Evidence for discovery of gravitational radiation. *Physical Review Letters*, 1969, vol. 22, 1320–1324.

28. Weber J. Gravitational radiation experiments. *Physical Review Letters*, 1970, vol. 24, 276–279.
29. Borissova L. B. Quadrupole mass-detector in a field of weak plane gravitational waves. *Russian Physics Journal*, 1978, vol. 21, no. 10, 1341–1344. Translated from: *Izvestiia Vysshikh Uchebnykh Zavedenii*, ser. *Fizika*, 1978, no. 10, 109–114.
30. Lefschetz S. Differential equations: geometric theory. Interscience Publishers, New York, 1957.
31. Kamke E. Differentialgleichungen: Lösungsmethoden und Lösungen. Chelsea Publishing Co., New York, 1959.
First published as: Kamke E. Differentialgleichungen. Lösungsmethoden und Lösungen. Gewöhnliche Differentialgleichungen, S. Hirzel Verlag, Leipzig 1942.
32. Bennett C. H., Brassard G., Crepeau C., Jozsa R., Peres A., and Wootters W. K. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Physical Review Letters*, 1993, vol. 70, 1895–1899.
33. Boschi D., Branca S., De Martini F., Hardy L., and Popescu S. Experimental realization of teleporting an unknown pure quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Physical Review Letters*, 1998, vol. 80, 1121–1125.
34. Riebe M., Häffner H., Roos C. F., Hänsel W., Benhelm J., Lancaster G. P. T., Korber T. W., Becher C., Schmidt-Kaler F., James D. F. V., and Blatt R. Deterministic quantum teleportation with atoms. *Nature*, June 17, 2004, vol. 429, 734–736.
35. Barrett M. D., Chiaverini J., Schaetz T., Britton J., Itano W. M., Jost J. D., Knill E., Langer C., Leibfried D., Ozeri R., and Wineland D. J. Deterministic quantum teleportation of atomic qubits. *Nature*, June 17, 2004, vol. 429, 737–739.
36. Zelmanov A. L. On the formulation of the problem of the infinity of space in the General Theory of Relativity. *Soviet Physics Doklady*, 1959, vol. 4, 161–164. Translated from: *Doklady Academii Nauk SSSR*, 1959, vol. 124, no. 5, 1030–1033.
37. Rabounski D. Zelmanov's Anthropic Principle and the Infinite Relativity Principle. *Progress in Physics*, 2006, vol. 2, no. 1, 35–37.
38. Rabounski D. Biography of Abraham Zelmanov (1913–1987). *The Abraham Zelmanov Journal*, 2008, vol. 1, xx–xxvi.
39. Rabounski D. On the speed of rotation of isotropic space: insight into the redshift problem. *The Abraham Zelmanov Journal*, 2009, vol. 2, 208–223.
40. Rabounski D. and Borissova L. A theory of frozen light according to General Relativity. *The Abraham Zelmanov Journal*, 2011, vol. 4, 3–27.

41. Rabounski D. and Borissova L. A theory of frozen light according to General Relativity. 2011 APS March Meeting, March 21–25, 2011, Dallas, Texas.

42. Hau L. V., Harris S. E., Dutton Z., Behroozi C. H. Light speed reduction to 17 metres per second in an ultracold atomic gas. *Nature*, February 18, 1999, vol. 397, 594–598.

43. Liu C., Dutton Z., Behroozi C. H., Hau L. V. Observation of coherent optical information storage in an atomic medium using halted light pulses. *Nature*, January 25, 2001, vol. 409, 490–493.

44. Dutton Z., Budde M., Slowe C., Hau L. V. Observation of quantum shock waves created with ultra-compressed slow light pulses in a Bose-Einstein condensate. *Science*, July 27, 2001, vol. 293, 663–668.

45. Hau L. V. Frozen light. *Scientific American*, July 17, 2001, vol. 285, no. 1, 52–59.

46. Hau L. V. Frozen light. *Scientific American Special Edition* "*The Edge of Physics*", May 31, 2003, 44–51.

47. Zhang R., Garner S. R., Hau L. V. Creation of long-term coherent optical memory via controlled nonlinear interactions in Bose-Einstein condensates. *Physical Review Letters*, December 4, 2009, vol. 103, 233602.

48. Phillips D. F., Fleischhauer A., Mair A., Walsworth R. L., Lukin M. D. Storage of light in atomic vapor. *Physical Review Letters*, January 29, 2001, vol. 86, no. 5, 783–786.

49. Turukhin A. V., Sudarshanam V. S., Shahriar M. S., Musser J. A., Ham B. S., Hemmer P. R. Observation of ultraslow and stored light pulses in a solid. *Physical Review Letters*, January 14, 2002, vol. 88, no. 2, 023602.

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