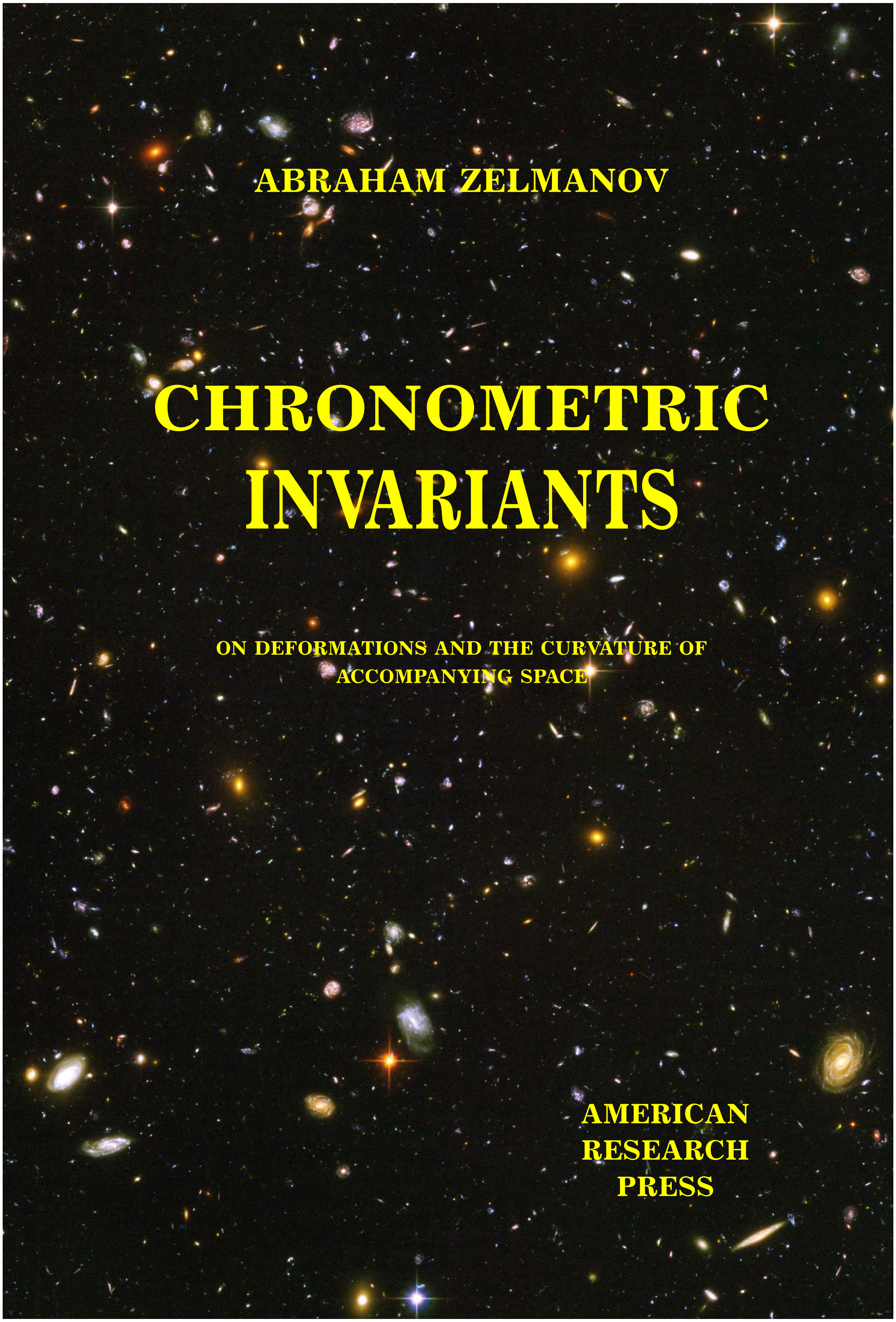

ABRAHAM ZELMANOV

CHRONOMETRIC INVARIANTS

ON DEFORMATIONS AND THE CURVATURE OF ACCOMPANYING SPACE

AMERICAN RESEARCH PRESS

Abraham Zelmanov

CHRONOMETRIC INVARIANTS

On Deformations and the Curvature of
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Contents

Chapter 2
The Mathematical Methods

Chapter 3
Relativistic Physical Equations

Foreword of the Editor

Abraham Zelmanov in the 1940's

Abraham Leonidovich Zelmanov was born on May 15, 1913 in Poltava Gubernya of the Russian Empire. His father was a Judaic religious scientist, a specialist in comments on Torah and Kabbalah. In 1937 Zelmanov completed his education at the Mechanical Mathematical Department of Moscow University. After 1937 he was a research-student at the Sternberg Astronomical Institute in Moscow, where he presented his dissertation in 1944. In 1953 he was arrested for "cosmopolitism" in Stalin's campaign against Jews. However, as soon as Stalin died, Zelmanov was set free, after some months of imprisonment. For several decades Zelmanov and his paralyzed parents lived in a room in a flat shared with neighbours. He took everyday care of his parents, so they lived into old age. Only in the 1970's did he obtain a personal municipal flat. He was married three times. Zelmanov worked on the academic staff of the Sternberg Astronomical Institute all his life, until his death on the winter's day, 2nd of February, 1987.

He was very thin in physique, like an Indian yogi, rather shorter than average, and a very delicate man. From his appearance it was possible to think that his life and thoughts were rather ordinary or uninteresting. However, in acquaintance with him and his scientific discussions in friendly company one formed another opinion about him. Those were discussions with a great scientist and humanist who reasoned in a very unorthodox way. Sometimes we thought that we were not speaking with a contemporary scientist of the 20th century, but some famous philosopher from Classical Greece or the Middle Ages. So the themes of those discussions are eternal — the interior of the Universe, the place of a human being in the Universe, the nature of space and time.

Zelmanov liked to remark that he preferred to make mathematical "instruments" than to use them in practice. Perhaps thereby his main goal in science was the mathematical apparatus of physical observable quantities in the General Theory of Relativity known as the *theory of chronometric invariants* [1]. In developing the apparatus he also created other mathematical methods, namely — *kinemetric invariants* [5] and *monad formalism* [6]. Being very demanding of himself, Zelmanov published less than a dozen scientific publications during his life (see References), so every publication is a concentrate of his fundamental scientific ideas.

Most of his time was spent in scientific work, but he sometimes gave lectures on the General Theory of Relativity and relativistic cosmology as a science for the geometrical structure of the Universe. Stephen Hawking, a young scientist in the 1960's, attended Zelmanov's seminars on cosmology at the Sternberg Astronomical Institute in Moscow. Zelmanov presented him as a "promising young cosmologist". Hawking read a brief report at one of those seminars.

Because Zelmanov made scientific creation the main goal of his life, writing articles was a waste of time to him. However he never regretted time spent on long discussions in friendly company, where he set forth his philosophical concepts on the geometrical structure of the Universe and the process of human evolution. In those discussions he formulated his famous *Anthropic Principle* and the *Infinite Relativity Principle*.

His Anthropic Principle is stated here in his own words, in two versions. The first version sets forth the idea that the law of human evolution is dependent upon fundamental physical constants:

> Humanity exists at the present time and we observe world constants completely because the constants bear their specific numerical values at this time. When the world constants bore other values humanity did not exist. When the constants change to other values humanity will disappear. That is, humanity can exist only with the specific scale of the numerical values of the cosmological constants. Humanity is only an episode in the life of the Universe. At the present time cosmological conditions are such that humanity develops.

In the second form he argues that any observer depends on the Universe he observes in the same way that the Universe depends on him:

> The Universe has the interior we observe, because we observe the Universe in this way. It is impossible to divorce the Universe from the observer. The observable Universe depends on

> the observer and the observer depends on the Universe. If the contemporary physical conditions in the Universe change then the observer is changed. And vice versa, if the observer is changed then he will observe the world in another way, so the Universe he observes will also change. If no observers exist then the observable Universe as well does not exist.

It is probable that by proceeding from his Anthropic Principle, in the years 1941–1944, Zelmanov solved the well-known problem of physical observable quantities in the General Theory of Relativity.

It should be noted that many researchers were working on the theory of observable quantities in the 1940's. For example, Landau and Lifshitz, in their famous *The Classical Theory of Fields*, introduced observable time and the observable three-dimensional interval, similar to those introduced by Zelmanov. But they limited themselves only to this particular case and did not arrive at general mathematical methods to define physical observable quantities in pseudo-Riemannian spaces. It was only Cattaneo, an Italian mathematician, who developed his own approach to the problem, not far removed from Zelmanov's solution. Cattaneo published his results on the theme in 1958 and later [9–12]. Zelmanov knew those articles, and he highly appreciated Cattaneo's works. Cattaneo also knew of Zelmanov's works, and even cited the theory of chronometric invariants in his last publication [12].

In 1944 Zelmanov completed his mathematical apparatus for calculating physical observable quantities in four-dimensional pseudo-Riemannian space, in strict solution of that problem. He called the apparatus the *theory of chronometric invariants*.

Solving Einstein's equations with this mathematical apparatus, Zelmanov obtained the total system of all cosmological models (scenarios of the Universe's evolution) which could be possible as derived from the equations. In particular, he had arrived at the possibility that infinitude may be relative. Later, in the 1950's, he enunciated the *Infinite Relativity Principle*:

> In homogeneous isotropic cosmological models spatial infinity of the Universe depends on our choice of that reference frame from which we observe the Universe (the observer's reference frame). If the three-dimensional space of the Universe, being observed in one reference frame, is infinite, it may be finite in another reference frame. The same is just as well true for the time during which the Universe evolves.

In other words, using purely mathematical methods of the General Theory of Relativity, Zelmanov showed that any observer forms his world-picture from a comparison between his observational results and some standards he has in his laboratory — the standards of different objects and their physical properties. So the "world" we see as a result of our observations depends directly on that set of physical standards we have, so the "visible world" depends directly on our considerations about some objects and phenomena.

The mathematical apparatus of physical observable quantities and those results it gave in relativistic cosmology were the first results of Zelmanov's application of his Anthropic Principle to the General Theory of Relativity. To obtain the results with general covariant methods (standard in the General Theory of Relativity), where observation results do not depend on the observer's reference properties, would be impossible.

Unfortunately, Zelmanov's scientific methods aren't very popular with today's physicists. Most theoreticians working in General Relativity don't use his very difficult methods of chronometric invariants, although the methods afford more opportunities than regular general covariant methods. The reason is that Zelmanov put his scientific ideas into the "code" of this difficult mathematical terminology. It is of course possible to understand Zelmanov's ideas using his mathematical apparatus in detail; he patiently taught several of his pupils. For all other scientists it has proved very difficult to understand Zelmanov's mathematical methods from his very compressed scientific articles with formulae, without his personal comments.

Herein I present Zelmanov's dissertation of 1944, where his mathematical apparatus of chronometric invariants has been described in all the necessary details. The dissertation also contains numerous results in cosmology which Zelmanov had obtained using the mathematical methods. It is impossible to find a more detailed and systematic description of the theory of chronometric invariants, than the dissertation. Even the book *Elements of the General Theory of Relativity* [8], which Vladimir Agakov had composed from Zelmanov's lectures and articles, gives a very fragmented account of the mathematical methods that prevents a reader from learning it on his own. The same can be said about Zelmanov's original papers, each no more than a few pages in length. Anyway the dissertation is the best for depth of detail. Sometimes Zelmanov himself said that to use the mathematical methods of chronometric invariants in its full power would be possible only after studying his dissertation.

The sole surviving manuscript of Zelmanov's dissertation is kept in the library of the Sternberg Astronomical Institute in Moscow, and the manuscript is in very poor condition. This is the fourth or the fifth typescript with handwritten formulae. From the handwriting we can conclude that the formulae were inscribed by Zelmanov personally. Some fragments of the manuscript are so faded that it is almost impossible to read. I asked Larissa Borissova, who knew Zelmanov closely, beginning from 1963, to make a copy of the manuscript for me. She did so, and I therefore extend my thanks to her.

In preparation for publication I reconstructed the damaged text fragments in accordance with context. Besides this, I introduced numerous necessary changes to the manuscript, because the terminology Zelmanov used in 1944 has became obsolete. For instance, Zelmanov initially called quantities invariant with respect to transformation of time "in-invariants", however in the 1950's he introduced the more useful term "chronometric invariants". The latter term has become fixed in the annals of science. Symbols for numerous tensor quantities have also became obsolete. Therefore I put the old terms in order in accordance with the contemporary terminology of chronometric invariants, which Zelmanov finished in the 1960's.

This book mainly targets an experienced reader, who knows the basics of the theory of chronometric invariants and wants to study the theory in detail. For such a reader the book will be a true mathematical delicatessen. I invite the reader to this delicate dinner table. Surely Zelmanov's mathematical delicatessen will satisfy the requirements of all true gourmets.

D. R.

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1. **Zelmanov A. L.** Chronometric invariants and co-moving coordinates in the general relativity theory. *Doklady Acad. Nauk USSR*, <u>107</u> (6), 815–818, 1956.
2. **Zelmanov A. L.** On the relativistic theory of anisotropic inhomogeneous Universe. *Proceedings of the 6th Soviet Conference on Cosmogony*, Nauka, Moscow, 144–174, 1959 (*in Russian*).
3. **Zelmanov A. L.** On the statement of the problem of the infinity of space in the general relativity theory. *Doklady Acad. Nauk USSR*, <u>124</u> (5), 1030–1034, 1959.

4. **Zelmanov A. L.** On the problem of the deformation of the co-moving space in Einstein's theory of gravitation. *Doklady Acad. Nauk USSR*, 135 (6), 1367–1370, 1960.
5. **Zelmanov A. L.** Kinemetric invariants and their relation to chronometric invariants in Einstein's theory of gravitation. *Doklady Acad. Nauk USSR*, 209 (4), 822–825, 1973.
6. **Zelmanov A. L.** Orthometric form of monad formalism and its relations to chronometric invariants and kinemetric invariants. *Doklady Acad. Nauk USSR*, 227 (1), 78–81, 1976.
7. **Zelmanov A. L. and Khabibov Z. R.** Chronometrically invariant variations in Einstein's gravitation theory. *Doklady Acad. Nauk USSR*, 268 (6), 1378–1381, 1982.
8. **Zelmanov A. L. and Agakov V. G.** Elements of the General Theory of Relativity. Nauka, Moscow, 1989 (*in Russian*).
9. **Cattaneo C.** General Relativity: Relative standard mass, momentum, energy, and gravitational field in a general system of reference. *Il Nuovo Cimento*, 10 (2), 318–337, 1958.
10. **Cattaneo C.** On the energy equation for a gravitating test particle. *Il Nuovo Cimento*, 11 (5), 733–735, 1959.
11. **Cattaneo C.** Conservation laws in General Relativity. *Il Nuovo Cimento*, 13 (1), 237–240, 1959.
12. **Cattaneo C.** Problèmes d'interprétation en Relativité Générale. *Colloques Internationaux du Centre National de la Recherche Scientifique*, No. 170 "Fluides et champ gravitationel en Relativité Générale", Éditions du Centre National de la Recherche Scientifique, Paris, 227–235, 1969.

Introducing Chronometric Invariants
(by the Editor)

The essence of Zelmanov's mathematical method — the theory of chronometric invariants — is as follows.

A regular observer perceives four-dimensional space as the three-dimensional spatial section $x^0 = \text{const}$, pierced at each point by time lines $x^i = \text{const}$.* Therefore, physical quantities perceived by an observer are actually *projections* of four-dimensional quantities onto his own time line and spatial section. The spatial section is determined by a three-dimensional coordinate net spanning a real reference body. Time lines are determined by clocks at those points where the clocks are located. If time lines are everywhere orthogonal to the spatial section, the space is known as *holonomic*. If not, there is a field of the space non-holonomity — the non-orthogonality of time lines to the spatial section, manifest as a three-dimensional rotation of the reference body's space. Such a space is said to be *non-holonomic*. In the general case, the space is curved, inhomogeneous, and deforming.

By mathematical means, four-dimensional quantities can be projected onto an observer's time line by the projecting operator

$$b^{\alpha} = \frac{dx^{\alpha}}{ds}\,,$$

the observer's four-dimensional velocity vector tangential to his world-line, while the projection onto his spatial section is made by the operator

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_{\alpha} b_{\beta}\,,$$

which are satisfying to the properties $b_{\alpha} b^{\alpha} = 1$ and $h^{i}_{\alpha} b^{\alpha} = 0$ required to such projecting operators. (Other components of the tensor $h_{\alpha\beta}$ are: $h^{\alpha\beta} = -g^{\alpha\beta} + b^{\alpha} b^{\beta}$, $h^{\alpha}_{\beta} = -g^{\alpha}_{\beta} + b^{\alpha} b_{\beta}$.)

A real observer rests with respect to his reference body ($b^i = 0$). In other word, he accompanies to his reference body in all its

*Greek suffixes are the space-time indices $0, 1, 2, 3$, Latin ones are the spatial indices $1, 2, 3$. So the space-time interval is $ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$.

motions. Such person is known as *accompanying observer*. Projections of four-dimensional quantities onto the time line and spatial section of such an accompanying observer (i. e. the observable projections) are invariant in respect to transformations of time along the spatial section. Zelmanov therefore called such projections *chronometrically invariant quantities*, i. e. "bearing the property of chronometric invariance". Therefore all quantities observed by a real observer (who rests in respect to his references) are chronometric invariants.

So, meaning a real observer ($b^i = 0$), we have

$$b^0 = \frac{1}{\sqrt{g_{00}}}\,, \quad b_0 = g_{0\alpha}b^\alpha = \sqrt{g_{00}}\,, \quad b_i = g_{i\alpha}b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}}\,,$$

and also

$$\begin{aligned} &h_{00} = 0\,, && h^{00} = -g^{00} + \frac{1}{g_{00}}\,, && h^0_0 = 0\,, \\ &h_{0i} = 0\,, && h^{0i} = -g^{0i}, && h^i_0 = \delta^i_0 = 0\,, \\ &h_{i0} = 0\,, && h^{i0} = -g^{i0}, && h^0_i = \frac{g_{i0}}{g_{00}}\,, \\ &h_{ik} = -g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}}\,, && h^{ik} = -g^{ik}, && h^i_k = -g^i_k = \delta^i_k\,. \end{aligned}$$

Thus, the chr.inv.-projections of a world-vector Q^α are

$$b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha Q^\alpha = Q^i,$$

while chr.inv.-projections of a symmetric world-tensor of the 2nd rank, for instance $Q^{\alpha\beta}$, are

$$b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}\,, \qquad h^{i\alpha}b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}.$$

Physically observable properties of the space are derived from the fact that the chr.inv.-differential operators

$$\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\,v_i\,\frac{^*\partial}{\partial t}$$

are non-commutative

$$\frac{^*\partial^2}{\partial x^i\,\partial t} - \frac{^*\partial^2}{\partial t\,\partial x^i} = \frac{1}{c^2}\,F_i\,\frac{^*\partial}{\partial t}\,, \qquad \frac{^*\partial^2}{\partial x^i\partial x^k} - \frac{^*\partial^2}{\partial x^k\partial x^i} = \frac{2}{c^2}\,A_{ik}\,\frac{^*\partial}{\partial t}\,,$$

and also from the fact that the chr.inv.-metric tensor h_{ik} may not be stationary. The observable characteristics are the chr.inv.-vector of gravitational inertial force F_i, the chr.inv.-tensor of angular velocities of the space rotation A_{ik}, and the chr.inv.-tensor of rates of the space deformations D_{ik}, namely

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \qquad v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,,$$

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \qquad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \qquad D^k_k = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,,$$

where w is gravitational potential, v_i is the linear velocity of the space rotation,

$$h_{ik} = -g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}} = -g_{ik} + \frac{1}{c^2}\,v_i\,v_k$$

is the chr.inv.-metric tensor, which possesses all the properties of the fundamental metric tensor $g_{\alpha\beta}$ in the spatial section. (Here $h = \det\|h_{ik}\|$, $hg_{00} = -g$, while $g = \det\|g_{\alpha\beta}\|$). Observable inhomogeneity of the space is set up by the chr.inv.-Christoffel symbols

$$\Delta^i_{jk} = h^{im}\Delta_{jk,m} = \frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^j} - \frac{{}^*\partial h_{jk}}{\partial x^m}\right)$$

which are built just like Christoffel's usual symbols $\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$ using h_{ik} instead of $g_{\alpha\beta}$. A four-dimensional generalization of the main chr.inv.-quantities F_i, A_{ik}, and D_{ik} (by Zelmanov) is:

$$F_\alpha = -2c^2 b^\beta a_{\beta\alpha}\,, \qquad A_{\alpha\beta} = ch^\mu_\alpha h^\nu_\beta a_{\mu\nu}\,, \qquad D_{\alpha\beta} = ch^\mu_\alpha h^\nu_\beta d_{\mu\nu}\,,$$

where $a_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha\, b_\beta - \nabla_\beta\, b_\alpha\right)$, $d_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha\, b_\beta + \nabla_\beta\, b_\alpha\right)$.

In this way, for any equations obtained using general covariant methods, we can calculate their physically observable projections on the time line and the spatial section of any particular reference body and formulate the projections in terms of their real physically observable properties, from which we obtain equations containing only quantities measurable in practice.

Zelmanov deduced chr.inv.-formulae for the space curvature. He followed that procedure by which the Riemann-Christoffel tensor was built: proceeding from the non-commutativity of the second derivatives of an arbitrary vector

$${}^*\nabla_i{}^*\nabla_k\, Q_l - {}^*\nabla_k{}^*\nabla_i\, Q_l = \frac{2A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdots j}Q_j\,,$$

he obtained the chr.inv.-tensor

$$H_{lki\cdot}^{\cdot\cdot\cdot j} = \frac{{}^*\partial\Delta_{il}^{j}}{\partial x^k} - \frac{{}^*\partial\Delta_{kl}^{j}}{\partial x^i} + \Delta_{il}^{m}\Delta_{km}^{j} - \Delta_{kl}^{m}\Delta_{im}^{j}\,,$$

which is similar to Schouten's tensor from the theory of non-holonomic manifolds. The tensor $H_{lki}^{\cdot\cdot\cdot j}$ differs algebraically from the Riemann-Christoffel tensor because of the presence of the space rotation A_{ik} in the formula for non-commutativity. Nevertheless its generalization gives the chr.inv.-tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right),$$

which possesses all the algebraic properties of the Riemann-Christoffel tensor in this three-dimensional space and, at the same time, the property of chronometric invariance. Therefore Zelmanov called C_{iklj} the *chr.inv.-curvature tensor* as the tensor of the observable curvature of the observer's spatial section. Its contraction term-by-term

$$C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im}C_{kimj}\,, \qquad C = C_j^j = h^{lj}C_{lj}$$

gives the chr.inv.-scalar C, which is the *observable curvature* of this three-dimensional space. Chr.inv.-projections of the Riemann-Christoffel tensor are:

$$X^{ik} = -c^2\,\frac{R_{0\cdot 0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \qquad Y^{ijk} = -c\,\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \qquad Z^{ijkl} = c^2 R^{ijkl}.$$

In this way, for any equations obtained using general covariant methods, we can calculate their physically observable projections on the time line and the spatial section of any particular reference body and formulate the projection in terms of their real physically observable properties, from which we obtain equations containing only quantities measurable in practice.

This completes the brief introduction to Zelmanov's mathematical apparatus of chronometric invariants, that is required to better understanding of the Zelmanov book.

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Chapter 1

PRELIMINARY NOTICES

§ 1.1 The initial suppositions of today's relativistic cosmology

At the present time two main cosmological theories, referred to as "relativistic", exist. Both are theories of a homogeneous universe. They are also known as *theories of an expanding universe*. One of the theories is derived from Einstein's General Theory of Relativity (see, for instance, [1]), the other — from Milne's Kinematic Theory of Relativity [2]. Usually one calls the second of the cosmological theories "kinematic", while "relativistic" is reserved for only the first of them. We adhere to this terminology herein.

The General Theory of Relativity and the Kinematic Theory of Relativity, being continuations of Einstein's Special Theory of Relativity, extend the Einstein theory in two different directions. From the logical viewpoint the theories exclude one another, and from the physical viewpoint they are absolutely inequivalent. Inequivalent also are the contemporary theories of a homogeneous universe — the relativistic and the kinematic ones. The first is one of possible cosmological constructions, based on a confirmed physical theory. Future cosmological constructions, built on its base, could be theories of an inhomogeneous universe. The second is a section of the Kinematic Theory of Relativity, pretending to be a physical theory, one of the main points of which is the *cosmological principle* (this principle leads to the necessity of homogeneity). It is possible to maintain from the relativistic theory that cosmology is deduced from physics there. On the contrary, the kinematic theory deduces physics from cosmology. Experimental disproof of the theory of a homogeneous universe must: (1) create the theory of an inhomogeneous universe on the basis of the General Theory of Relativity, in the first case; (2) overthrow the Kinematic Theory of Relativity itself, in the second case.

We will not consider the Kinematic Theory of Relativity here. We will also have no use for the Special Theory of Relativity. So the term "relativistic" will always connote a relationship to Einstein's General Theory of Relativity.

The relativistic theory of a homogeneous universe is derived from the following suppositions. The first of them is:

> Einstein's equations of gravitation are applicable to the Universe as a whole.

This supposition defines the cosmological theory as relativistic, because the supposition considers the Einstein equations

$$G_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T\right) + \Lambda\, g_{\mu\nu}\,, \quad \mu,\nu = 0,1,2,3 \qquad (1.1)$$

and their applicability at different scales. Suppositions, defining the theory as a theory of a homogeneous universe, can be formulated in different ways. Tolman (see [1], p. 362) merges them into the united supposition:

> If we take any point, in relation to which matter located near the point is at rest (on the average) at any moment of time, then observations we make at the point show that spatial directions are independent of one another in the "large scale" — in other words, the space is isotropic.

A mathematically more useful statement of this supposition, but less obviously physical, has been introduced by Robertson [3].

This statement of the first initial supposition contains Einstein's equations with the cosmological constant, the numerical value of which, or even its sign, is an open problem in contemporary cosmology. In general, the constant can be negative, zero, or positive. The cosmological constant of a positive numerical value had been introduced by Einstein [4] in his theory of a static universe*. Later, Friedmann [5] developed the theory of a non-static universe, adducing arguments by which the positive sign of the cosmological constant, or even its non-zero numerical value in general, could be eliminated. Therefore Einstein himself [6] subsequently excluded the constant from the equations. However, because the relativistic gravitational equations of the second order, in their general form (for instance, see [7], p. 269), are equations (1.1), relativistic cosmology regularly uses the equations with the cosmological constant.

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*Cosmology uses the term "static" to mean a metric independent of time, while the General Theory of Relativity usually means it to be the non-orthogonality of time to space.

At the same time those cases where the constant is non-zero have mathematical interest rather than physical meaning.

Contemporary cosmology peculiarly identifies the Metagalaxy, supposed infinite, with the Universe as a whole. For this reason we use the terms "neighbourhood" and "large scale" in the second initial supposition*, according to which we assume volumes elementary, if the volumes contain so many galaxies that matter inside the volumes can be assumed to be continuously distributed (see also §1.11).

§ 1.2 The world-metric

The second initial supposition implies the following:

> Any point, in relation to which matter located in the neighbourhood is at rest, can be considered as the centre of a spatially spherical symmetry (see [1], p. 368).

This supposition, taking the relativistic equations of motion and Shur's theorem into consideration (for instance, see [8], p. 136), gives the possibility of taking a coordinate frame which, being at rest (on the average) with respect to the matter, measures cosmic universal time — such time satisfies the conditions

$$g_{00} = 1\,, \qquad g_{0i} = 0\,, \qquad i = 1, 2, 3\,. \tag{2.1}$$

The space of this coordinate frame is a constant curvature space of curvature $\frac{1}{3}C$. This space, undergoing homologous expansions and contractions with

$$C = 3\,\frac{k}{R^2}\,, \qquad k = 0, \pm 1\,, \qquad R = R\,(t)\,, \tag{2.2}$$

in conformally Euclidean spatial coordinates has the metric

$$ds^2 = c^2 dt^2 - R^2\,\frac{dx^2 + dy^2 + dz^2}{\left[1 + \frac{k}{4}\,(x^2 + y^2 + z^2)\right]^2}\,. \tag{2.3}$$

The case of $k = +1$ had first been considered by Einstein [4] for his static model, and by Friedman [5] for the non-static models. The case of $k = 0$ had first been considered by de Sitter [10] for the

*We use the terms "neighbourhood" and "large scale" in this sense throughout, unless otherwise stated.

empty static model, by Lemaître [11] for the non-empty non-static model, and by Robertson [3] for the non-empty models. The case of $k = -1$ had first been considered by Friedmann [12].

It is known from geometry that a space of $k = +1$ is locally spherical, a space of $k = 0$ is locally Euclidean, and a space of $k = -1$ is locally hyperbolic. Taking spherical symmetry with respect to any point, and applying to it the properties of coherence, we conclude that the space of $k = +1$ is elliptic (actually, doubly-connected) or spherical (actually, simply connected), and the spaces of $k = 0$ and of $k = -1$ are Euclidean and hyperbolic respectively (both spaces are infinite and simply connected).

§ 1.3 Properties of matter

Taking $g_{\alpha\beta}$ from (2.3) and substituting it into the Einstein equations of gravitation

$$G^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} G = -\kappa\, T^{\mu\nu} - \Lambda\, g^{\mu\nu}, \tag{3.1}$$

we deduce that only two functions of time, namely — the functions

$$\rho = \rho\,(t)\,, \qquad p = p\,(t)\,, \tag{3.2}$$

exist, where $T^{\mu\nu}$ can be expressed by the formulae

$$T^{00} = \frac{1}{g_{00}}\left(\rho + \frac{p}{c^2}\right) - \frac{p}{c^2}\, g^{00}, \tag{3.3}$$

$$T^{0j} = -\frac{p}{c^2}\, g^{0j}, \tag{3.4}$$

$$T^{ik} = -\frac{p}{c^2}\, g^{ik}, \tag{3.5}$$

or, in other words,

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{p}{c^2}\, g^{\mu\nu}, \tag{3.6}$$

where

$$\frac{dx^\sigma}{ds} = \frac{1}{\sqrt{g_{00}}}\,,\ 0,\ 0,\ 0 \tag{3.7}$$

is a four-dimensional velocity, which characterizes the mean motion of matter in the neighbourhood of every point (irregular deviations from the mean motion are taken into account with the scale parameter p). Formula (3.6) coincides with the formula for the energy-momentum tensor of an ideal fluid*, the density of which is ρ and the pressure is p. Hence matter, in the idealized universe we are considering, can be considered as an ideal fluid, which, having a homogeneous density and pressure in accordance with (3.2), is at rest with respect to the non-static space. In other words, the ideal fluid undergoes expansions/contractions, accompanied by the non-static space. The pioneering work in relativistic cosmology undertaken by Einstein and Friedmann†, did not take the pressure into account. Friedmann was the first to consider the static models, so the case of $\rho > 0$, $p = 0$ is known as *Friedmann's case* of an inhomogeneous universe. The first to introduce $p > 0$, was Lemaître [15, 16].

§ 1.4 The law of energy

One of the consequences, which could be deduced from the Einstein equations of gravitation, is the relativistic law of energy

$$\frac{\partial T_\mu^\nu}{\partial x^\nu} - \Gamma_{\mu\nu}^\sigma T_\sigma^\nu + \frac{\partial \ln\sqrt{-g}}{\partial x^\sigma} T_\mu^\sigma = 0 \,. \tag{4.1}$$

As a result of (2.3) and (3.3–3.5), the equations (4.1) with $\mu = 0$ take the form

$$\dot{\rho} + 3\,\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0\,, \tag{4.2}$$

where the dot denotes differentiation with respect to time. The equations (4.1) with $\mu = 1, 2, 3$ become the identities

$$0 = 0\,. \tag{4.3}$$

Equation (4.2) is one of the main equations of the relativistic theory of a homogeneous universe (the *cosmological equations*). We will refer to this equation as the *cosmological equation of energy*.

Let us take any fixed volume of space, i. e. a space volume limited by surfaces, equations of which are independent of time. We can write the value V of such a volume as follows

$$V = R^3 \Pi_0\,, \qquad \Pi_0 \not\parallel t\,, \tag{4.4}$$

*Ideal in the sense of the absence of viscosity, not in the sense of incompressibility.

†We do not take the works of de Sitter [9, 10], Lanczos [13], Weyl [14], and Lemaître [11] into account, because they considered the empty models.

then the energy E inside this volume is

$$E = \rho R^3 \Pi_0 c^2. \tag{4.5}$$

As a result of (4.4) and (4.5), we can transform the cosmological equation of energy (4.2) into the form

$$dE + p dV = 0\,, \tag{4.6}$$

which is the condition for adiabatic expansion and contraction of the space.

In the Friedmann case, where

$$p = 0\,, \tag{4.7}$$

it is evident that

$$\frac{dE}{dt} = 0\,, \tag{4.8}$$

and because

$$E = Mc^2, \tag{4.9}$$

where M is the mass of a matter inside the volume V, we have

$$\frac{dM}{dt} = 0\,. \tag{4.10}$$

So in the absence of pressure the mass and the energy of any fixed volume remain unchanged.

§ 1.5 The law of gravitation

As a result of (2.3) and (3.3–3.5), the equations of the relativistic law of gravitation with $\mu, \nu = 0$, with $\mu = 0$, $\nu = 1, 2, 3$, and with $\mu, \nu = 1, 2, 3$ give the following formulae, respectively: the equation

$$3\frac{\ddot{R}}{c^2 R} = -\frac{\kappa}{2}\left(\rho + 3\frac{p}{c^2}\right) + \Lambda\,, \tag{5.1}$$

the identity

$$0 = 0\,, \tag{5.2}$$

and the equation

$$\frac{\ddot{R}}{c^2 R} + 2\frac{\dot{R}^2}{c^2 R^2} + 2\frac{k}{R^2} = \frac{\kappa}{2}\left(\rho - \frac{p}{c^2}\right) + \Lambda\,. \tag{5.3}$$

Eliminating $\ddot{R}$ from (5.1) and (5.3), we obtain

$$3\frac{\dot{R}^2}{c^2 R^2} + 3\frac{k}{R^2} = \kappa\rho + \Lambda\,. \tag{5.4}$$

Equations (5.1) and (5.3), as well as (5.4) and (4.2), are the main equations of the relativistic theory of a homogeneous universe (the cosmological equations). So, we will refer to the equations (5.1) and (5.3), and also to their consequence (5.4), as the *cosmological equations of gravitation*.

It is easy to see that the cosmological equation of energy is a consequence of the cosmological equations of gravitation.

§ 1.6 The system of the cosmological equations

There are only two independent equations of the cosmological equations, namely (4.2) and (5.4)

$$\left.\begin{aligned} &\dot{\rho} + 3\,\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0 \\ &3\,\frac{\dot{R}^2}{c^2 R^2} + 3\,\frac{k}{R^2} = \kappa\rho + \Lambda \end{aligned}\right\}. \tag{6.1}$$

The remaining equations can be deduced as their consequences. So we have a system of two equations, where three functions are independent, namely, the functions

$$\rho = \rho\,(t)\,, \qquad p = p\,(t)\,, \qquad R = R\,(t)\,. \tag{6.2}$$

To find the functions we need to add a third independent equation to the system. It could be, for instance, the equation of state, which links density and pressure. A form of the curves themselves (6.2) could be found without the third equation, if we were to make valid limitations on density and pressure. Usually one studies the curve of the third function of (6.2) under the supposition that

$$\rho \geqslant 0\,, \qquad p \geqslant 0\,, \qquad \frac{dp}{dR} \leqslant 0\,. \tag{6.3}$$

On the one hand, for any substance, we have

$$T > 0\,, \tag{6.4}$$

and also for radiations (including electromagnetic fields) we have

$$T = 0\,. \tag{6.5}$$

On the other hand, for ideal fluids (3.6) we have

$$T = \rho - 3\,\frac{p}{c^2}\,. \tag{6.6}$$

So, in general, we have

$$\rho \geqslant 3\,\frac{p}{c^2}\,. \tag{6.7}$$

Therefore, instead of (6.3), we can write

$$\rho \geqslant 3\,\frac{p}{c^2}\,, \qquad p \geqslant 0\,, \qquad \frac{dp}{dR} \leqslant 0\,. \tag{6.8}$$

We will limit our tasks here, because we will consider only non-empty models — those models, where

$$\rho > 0\,. \tag{6.9}$$

§ 1.7 Its solutions. The main peculiarities.

The main properties of solutions like $R = R\,(t)$ can be routinely found for different Λ and k, by studying nulls of the functions

$$f\,(R,\Lambda) = \dot{R}^2 = \frac{\kappa}{2}\,\rho c^2 R^2 + \frac{\Lambda}{3}\,R^2 - k\,, \tag{7.1}$$

see [5, 12, 17, 18] and [1], p. 359–405, for instance.

In this case one makes a tacit supposition (which is physically reasonable) that the essential positive function $R\,(t)$ is continuous everywhere, where the function exists. Moreover, one supposes its derivative continuous under $R \neq 0$ (besides, see §1.13). The main results the case $\rho > 0$ (6.9) give will be discussed in the next section. We will show here only several of the main properties of the function $R\,(t)$, obtained in another way, similar to the way we will consider in §4.19–§4.23. First of all, it is evident that numerical values of R, between which the function $R\,(t)$ (under the aforementioned supposition) is monotone, are:

(1) zero;

(2) finite minimal values;

(3) finite values, which are asymptotic approximations of R from above or from below (under the conditions $t \to +\infty$ or $t \to -\infty$);

(4) finite maximal values;

(5) infinity.

For all the finite values of R we write respectively, instead of (3.1) and (3.4),

$$3\ddot{R} = -\frac{\kappa}{2}\left(\rho c^2 + 3p\right)R + \Lambda c^2 R\,, \tag{7.2}$$

$$3\dot{R}^2 + 3kc^2 = \kappa\rho R^2 + \Lambda c^2 R^2. \tag{7.3}$$

It is seen from (4.4–4.8) that the density changes as fast as R^{-3} under zero pressure. It changes faster than R^{-3} when the pressure becomes positive. Note also, because of the first and the second conditions (6.8), we have

$$\rho R \leqslant \left(\rho + 3\frac{p}{c^2}\right) R \leqslant 2\rho R\,, \tag{7.4}$$

so, if ρR approaches zero or infinity, then $\left(\rho + 3\frac{p}{c^2}\right) R$ also approaches zero or infinity, respectively.

Similarly, in the case of the density changes (7.3), it follows that when R approaches zero, ρ and $\dot{R}^2$ approach infinity — the model evolves into the special state of infinite density ($\dot{R}^2 = \infty$). As seen from (7.2) and (7.3), this case is possible under any numerical values of Λ and k.

From the aforementioned density changes we can also conclude, when R approaches infinity, then the density and the pressure respectively approach zero — the model evolves into the ultimate state of the infinite rarefaction. Looking at (7.2) and (7.3) we note that the abovementioned evolution scenario is possible with $\Lambda > 0$ under any numerical values of k, with $\Lambda = 0$ under only $k = 0$ or $k = -1$, and the scenario is impossible with $\Lambda < 0$.

When R transits its minimum value, i. e. when the model transits the state of minimum volume, then $\dot{R}$ becomes zero and $\ddot{R}$ is nonnegative. When R approaches a non-minimum numerical value, i. e. when the model evolves asymptotically to a static state, then $\dot{R}$ and $\ddot{R}$ approach zero. It is evident that if R is static, the model will also be static. The formulae (7.2) and (7.3) for all the three cases give

$$3k > \Lambda R^2, \tag{7.5}$$

$$0 < \Lambda R\,. \tag{7.6}$$

From this we can conclude that the transit of the model through the state of minimum volume, the asymptotic approach of the model to a static state, and the static models in general, can occur only when $\Lambda > 0$, $k = +1$.

When R transits its maximum numerical value, i. e. when the model evolves through the state of its maximum volume, then $\dot{R}$ becomes zero and $\ddot{R}$ is nonpositive. Then, by formulae (7.2) and (7.3), we obtain

$$3k > \Lambda R^2, \tag{7.7}$$

$$0 \gtreqless \Lambda R\,. \tag{7.8}$$

So the transit of the model through the state of its maximum volume is possible with $\Lambda \geqslant 0$ only for $k = +1$, and it is possible for $\Lambda < 0$ under any numerical values of k.

Let us suppose that the model undergoes monotone transformations of its volume. The transformations are limited by the state of the minimum volume or a static state from below, and they are limited by the state of the maximal volume or another static state from above. We will mark the lower and the upper ultimate states by indices 1 and 2, respectively. Then

$$\ddot{R}_1 \geqslant 0 \geqslant \ddot{R}_2\,, \qquad \dot{R}_1 < \dot{R}_2\,, \qquad \rho_1 > \rho_2\,, \tag{7.9}$$

and formula (5.1) gives

$$p_1 < p_2\,, \tag{7.10}$$

which contradicts the third of the conditions (6.8). So the supposed kinds of the evolution of the model are impossible (see also §1.13).

§ 1.8 Types of non-empty universes

Let us make a list of the kinds of non-empty universes according to the contemporary classification of cosmological models* (for instance, see [18]).

Static models or, in other words, Einstein's models (Type E)

In this case R remains unchanged. Equations (5.1) and (5.4) give, respectively

$$\frac{\kappa}{2}\left(\rho + 3\,\frac{p}{c^2}\right) = \Lambda\,, \tag{8.1}$$

$$3\,\frac{k}{R^2} = \kappa\rho + \Lambda\,, \tag{8.2}$$

consequences of which are that ρ and p remain unchanged as well, and that $\Lambda > 0$, $k = +1$.

The Einstein models are unstable. Consequently, their fluctuations, their homogeneity unchanged, alter one or two of the quantities R, ρ, and p, so the models compress themselves up to the special state of infinite density, or, alternatively, up to the ultimate state of infinite rarefaction. Eddington [19] was the first to discover the instability of the Einstein models (as a matter of fact, the instability is evident if you consider Friedmann's equations [5]).

*This classification, in its main properties, is in accordance with Friedmann [5].

Models of the first kind — asymptotic, monotone, and oscillating models

All the models evolve through the special state of infinite density.

Asymptotic models of the first kind (type A_1) change their volumes monotonically during their expansions or contractions between the special state of infinite density, when time has a finite value $t = t_0$, and a static state, corresponding to the Einstein model when $t = +\infty$ or $t = -\infty$. The aforementioned models are possible only when $\Lambda > 0$, $k = +1$.

Monotone models of the first kind (type M_1) change their volumes monotonically during their expansions or contractions between the special state of infinite density, when time has a finite value $t = t_0$, and the ultimate state of infinite rarefaction, when $t = +\infty$ or $t = -\infty$. The models are possible with $\Lambda > 0$ for any numerical value of k, and with $\Lambda = 0$ for $k = 0$, $k = -1$.

Oscillating models of the first kind (type O_1) expand their volumes from the special state of infinite density, when $t = t_1$, and they continue expanding up to their maximum volumes, when $t = t_0$. Then they contract themselves into the special state of infinite density, reaching that state when $t = t_2$ (t_1, t_0, t_2 have finite values). The models are possible with $\Lambda \geqslant 0$ for $k = +1$, or with $\Lambda < 0$ under any numerical value of k.

When a model of any of the aforementioned kinds evolves through the special state, three cases are possible:

(a) the type of model remains unchanged (it is always known for $p = 0$);

(b) the type of the model changes from one to another; at the same time the new kind will also be of the first kind;

(c) the numerical value of R, which is usually real, becomes imaginary (R^2 changes its sign).

Models of the second kind — asymptotic and monotone ones*

The models do not transit through the special state of infinite density.

Asymptotic models of the second kind (type A_2) change their volumes during their monotone expansions or contractions between a static state, corresponding to the Einstein model, when $t = \mp\infty$,

*Read about oscillating models of the second kind in §1.13.

and the ultimate state of infinite rarefaction, when $t = \pm\infty$. The models are possible only for $\Lambda > 0$ and $k = +1$.

Monotone models of the second kind (type M_2) contract their volumes from the ultimate state of pure vacuum, when $t = -\infty$, up to the state of their maximum volumes, when time has a finite value $t = t_0$. Then the models go into expansion again up to the ultimate state of infinite rarefaction, when $t = +\infty$. The models are possible only for $\Lambda > 0$ and $k = +1$.

Table 1.1 shows which kinds of non-static non-empty universes are possible under different numerical values of Λ and k.

	$k = +1$	$k = 0$	$k = -1$
$\Lambda > 0$	A_1 M_1 O_1 A_2 M_2	M_1	M_1
$\Lambda = 0$	O_1	M_1	M_1
$\Lambda < 0$	O_1	O_1	O_1

Table 1.1 Types of the models of non-static non-empty universes.

In the cases of $p > 0$ or $p = 0$, the possible kinds of the models and their locations in the table's cells are the same.

§ 1.9 Sections of relativistic cosmology

Considering contemporary cosmology, we select five main problems, which define its main sections. Three of them define sections of the relativistic theory itself — the dynamics, the three-dimensional geometry, and the thermodynamics of models of the Universe. The fourth links the relativistic theory to Classic Mechanics. The fifth compares the theory with observational data.

The main part of relativistic cosmology is the dynamics of the Universe. It consists of: (1) studies of the evolution of the models, the main results of which are given in §1.7–§1.8; (2) studies of the stability of the models (this problem is not actually in the literature yet, besides the aforementioned stability of the Einstein models, see §1.8). The dynamics borders on the space geometry from one side, and on the thermodynamics from the other side.

Related to the space geometry are the questions: which are the space curvature and the topological structure of the space? The main results of the studies are given in that part of §1.2, which is related to the properties of three-dimensional space.

The thermodynamics of the relativistic homogeneous universe is historically associated with Tolman, who introduced the relativistic generalization of Classical Thermodynamics [1]. The first law of Tolman's relativistic thermodynamics is, naturally, the law of the conservation of energy (see [20] or [1], p. 292), one of the forms of which are equations (4.1). The second law of thermodynamics (see [21] or [1], p. 293) can be represented in the form

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\nu}}\left(\varphi\,\frac{dx^{\nu}}{ds}\sqrt{-g}\right)d\Sigma \geqslant \frac{dQ}{\Theta}\,, \tag{9.1}$$

where φ and $\frac{dx^{\nu}}{ds}$ are the entropy density and the for-dimensional velocity of an elementary four-dimensional volume $d\Sigma$, respectively. The quantity dQ here is the gain of heat between the time borders of the volume through its spatial borders (Θ is their own absolute temperature). The inequality sign here is related, as it is evident, to reversible processes. The equality sign is related to reversible processes. In the case of the cosmological models we are considering, the first law leads to the condition of an adiabatic state (4.6). Under this condition, we can clearly see that there is no gain of heat in the model (it is also a result of the initial supposition that the model is isotropic, see §1.1). Taking it into consideration in a space of the metric (2.3), the second law of thermodynamics leads to the condition that the entropy of any fixed (in the sense of §1.4) volume does not decrease (compare with [1], p. 424)

$$\frac{\partial}{\partial t}\left(\varphi V\right) \geqslant 0\,. \tag{9.2}$$

Matter in the model is being considered in general as a mixture of two interacting components — a pressured substance and isotropic radiations. In particular cases, interactions between the two components or the pressure of the substance can be absent.

Let us mention another of Tolman's main results related to the cosmological models we are considering. Arising from the fact that the models do not contain heat fluxes, friction, or pressure gradients, Tolman discovered that expansions and contractions of the models can be reversible (see [22] and [23]; and also [1], p. 322, p. 426 etc.). This does not contradict models containing radiation, since every observer who is at rest can detect a flux of the radiation through his environment's surface, of an arbitrary constant radius — from within under the expansions, and from outside under the contractions of the model (see [1], p. 432 etc.). Sources of irreversibility can be contained in only physical chemical processes within every element of the matter. The reversible expansions must accompany transformations of a portion of the substance into radiation (a portion of the substance is annihilated), the reversible contractions must accompany the inverse process — the reconstitution of the substance (see [23] or [1], p. 434). Finally, because every fixed (in the sense of §1.4) volume has no constant energy, the cosmological model has no fixed maximum entropy, so the irreversible processes can be infinite — the model does not evolve into a state of the maximum entropy (see [24] or [1], p. 326 etc., p. 439 etc.).

Relativistic cosmology is linked to Classical Mechanics by the analogy between the relativistic cosmological equations of a homogeneous universe and the classical equations of a homologous expanding/contracting gravitating homogeneous sphere of an arbitrary radius. This analogy can apply only for the case of an ideal fluid — it has no tensions or pressure ($p=0$). The analogy was first created by Milne in the case of $\Lambda=0$, $k=0$ (see [25] or [2], p. 304). Then McCrea and Milne studied the analogy in the cases of $\Lambda=0$, $k\neq 0$ (see [26] or [2], p. 311). Finally, it was generalized for the case of $\Lambda\neq 0$ (see [2], p. 319). In the last case, Newton's law of gravitation must be generalized by introducing an additional force, producing an additional relative acceleration of two interacting particles (the acceleration equals the product of $\frac{1}{3}\Lambda c^2$ and the distance between the particles). If r, θ, and φ are the polar coordinates of an arbitrary point on the spherical mass we are considering, where the origin of the coordinates can be fixed at any other point of the sphere (ρ is its density, γ is Gauss' constant, and ε is the integrating constant), then in that case we obtain the condition of homology

$$\frac{\dot{r}}{r} = f(t) \not\parallel r, \theta, \varphi\,, \tag{9.3}$$

the condition of continuity

$$\dot{\rho} + 3\frac{\dot{r}}{r}\,\rho = 0\,, \tag{9.4}$$

the equation of motion (in the form, transformed for the small)

$$3\frac{\ddot{r}}{r} = -4\pi\gamma\rho + \Lambda c^2, \tag{9.5}$$

and the integral of energy

$$3\frac{\dot{r}^2}{r^2} - 6\frac{\varepsilon}{r^2} = 8\pi\gamma\rho + \Lambda c^2. \tag{9.6}$$

Introducing the real function of time $R(t)$, which satisfies the conditions

$$\frac{\dot{R}}{R} = \frac{\dot{r}}{r}\,, \qquad \frac{k}{R^2} = -2\,\frac{\varepsilon}{c^2 r^2}\,, \qquad k = 0, \pm 1\,, \tag{9.7}$$

and taking into account

$$\kappa = \frac{8\pi\gamma}{c^2}\,, \tag{9.8}$$

we transform (9.4), (9.5), and (9.6) to the form, which is identical to equations (4.2), (5.1), and (5.4) in the case of $p=0$. It is evident that the analogy we have considered facilitates determination of a link between the dynamics and the geometry of the relativistic homogeneous universe — the link between the evolution scenario of the models and the space curvature.

The question "Does the cosmological theory correlate to observational data?" remains. We reserve this question for §1.14.

§1.10 Accompanying space

As we have seen, the three-dimensional space in which the relativistic theory of a homogeneous universe operates is an accompanying space — a space which moves in company with matter at any of its points. Therefore the matter is at rest (on the average) with respect of the space. So expansions or contractions of the space imply the analogous expansions or contractions of the matter itself.

It is evident that the accompanying space is primary from the physical viewpoint. For this reason determining the geometric properties of such a space is of the greatest importance. In particular, the question "Is the space infinite, or not?" is the same as the question "Is the real Universe spatially infinite, or not?", in a reasonable physical sense of the words. Geometric characteristics of the accompanying space are directly linked to observational data, because the data characterize cosmic objects, which are at rest (on the average) with respect of the space.

We can say that relativistic cosmology studies relative motions of the elements of matter as the analogous motions of the elements of the accompanying space, and deformations of matter are considered as the analogous deformations of the accompanying space.

The idea of studying motions in this way is very simple. In fact, let us consider, for instance, two points

$$x = \alpha\,, \qquad x = \beta\,, \tag{10.1}$$

located on the x axis. The distance between the points along the axis can be expressed by the integral

$$J = \int_{\alpha}^{\beta} \sqrt{a}\, dx\,, \tag{10.2}$$

where a is the necessary coefficient of the spatial quadratic form. In the coordinate frame, which accompanies the points (10.1), α and β are constant, so the motion of the points with respect to one another along the x-axis will be described by the function

$$a = a\,(t)\,. \tag{10.3}$$

This method of studying motions is conceivable, of course, in Classical Mechanics too. At the same time in Classical Mechanics we have: (1) this method studying motions, as well as the standard studies of motions with respect to the static space of Classical Mechanics, as the final result must be based on equations of motion; (2) the geometrical properties of the accompanying space are the same as the properties of the static space. On the contrary, in the relativistic theory we have: (1) this method of studying deformations of the accompanying space realizes itself by the equations of gravitation, the equations of motion remain unused; (2) in general, the geometric properties of the accompanying space are different from the properties of any other space, moreover the static space can be introduced in only an infinitesimal location.

§ 1.11 The cosmological equations: the regular interpretation

From the usual viewpoint the cosmological equations are considered as the equations of the whole Universe, defining its "radius" R. From this viewpoint* the quantities

$$C = 3\,\frac{k}{R^2}\,, \tag{11.1}$$

$$D = 3\,\frac{\dot{R}}{R} \tag{11.2}$$

are the tripled curvature and the relative rate of the volume expansion of the accompanying space as a whole. The quantities ρ and p

*Note, one sometimes means by "the radius of the Universe", the radius of the space curvature which, as is evident, equals $\frac{R}{\sqrt{k}}$.

are the average density and the average pressure of matter in the Universe. From such a viewpoint, of course, the next supposition is essential:

Supposition A The Einstein equations are applicable to the whole Universe.

This supposition, of course, is a far-reaching extrapolation (for instance, see [1], p. 331). At the same time, building the theory of the world as a whole, the supposition cannot be replaced by another now, because the others have proved less fruitful than it at this time.

When we, from this standpoint, apply the homogeneous cosmological equations to the real Universe, we suppose, first, that:

Supposition B If we consider the Universe in the frames of the large scale, we can assume the Universe homogeneous.

In this case, of course, in general, it is not supposed that this scale coincides with that scale we have mentioned in §1.1. This supposition cannot be justified by any theoretical suppositions (physical or astronomical, or otherwise), at least at this time. Moreover, it is unknown as to whether the homogeneous state of the whole Universe is stable or not (for instance, see [1], p. 482). Applying the homogeneous models to the real Universe in any special case, we need to set forth a scale, starting from which the Universe will be supposed homogeneous. Contemporary applications of the aforementioned models to the real Universe identify the Universe with the Metagalaxy. In this case it is supposed:

Supposition C The Metagalaxy contains the same quantity of matter inside volumes less than the sphere of radius 10^8 parsecs and more than the sphere of radius 10^5 parsecs*.

Next, the contemporary applications of the relativistic models to the real Universe suppose that:

Supposition D The red shift in the spectra of extragalactic nebulae, proportional (at least, in the first approximation) to the distances between them and our Galaxy, is a result of mutual galactic recession (of the expansion of the accompanying space).

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*The radius 10^8 parsecs is in order of the distances up to the weakest in their light intensity of the extragalactic nebulae, observed with the 100in reflector. The radius 10^5 parsecs is in order of distances between the galaxies nearest to one another. It is evident that "elementary" must be supposed volumes of the order of 10^{20} cubic parsecs.

Finally, the contemporary applications of the relativistic models to the real Universe suppose:

Supposition E Almost all the mass of the Universe is concentrated in galaxies. Alternatively, at least, to find the average density of matter in the Universe it is enough to take into account only the masses of the galaxies.

Note that suppositions A and B are sufficient for deducing the cosmological equations (compare with §1.1). Suppositions C, D, and E play a part in comparing (in the qualitative comparison especially) the theory and observations. Supposition B is covered by Supposition C. The last supposition is a powered expression of the *sample principle* (for instance, see [1], p. 363 and [2], p. 123), which is very fuzzy. This principle, specific for contemporary cosmology, sets up the known part of the Universe as a sufficient sample for studying the main properties of the whole Universe.

§ 1.12 Numerous drawbacks of homogeneous models

The contemporary relativistic cosmology has numerous advantages in relation to pre-relativistic cosmological theories. In particular, the contemporary theory is free of both of the classical paradoxes, namely — the photometric* and the gravitational), which are irremovable in pre-relativistic cosmology if the average density of matter in the Universe is non-zero†. So the contemporary theory eliminates the heat death of the Universe, because it does not consider the contemporary state of the known fragment of the Universe as a fluctuation. Finally, the theory naturally explains the red shift, because expansions or contractsions of the accompanying space are inevitable (static states of the Universe are unstable). In particular, it provides a possibility of linking the expansions of the space to the fact that the substance we observe transforms itself partially into radiations‡.

Besides its advantages, the contemporary relativistic cosmology

*The contemporary theory is free of this paradox, because the finite pressure of radiations in the Universe results in the finite brightness of the sky we observe.

†Extrapolating the red shift up to infinity, we can also remove the photometric paradox under finite densities in non-relativistic cosmology. At the same time it creates the problem of the nature of the red shift itself.

‡Eigenson's non-relativistic theory [27] sets forth the same link, however his theory requires an excessive loss of stellar masses; the stars transform into radiations.

has numerous drawbacks, which indicate that the theory is inapplicable to the real Universe. Some of the drawbacks were mentioned in the previous §1.11, where we discussed Suppositions A and B. Other the drawbacks (they will be discussed in §1.14 and §1.15) are derived from comparison between the theory and observational data. Finally, the remaining drawbacks can be formulated as notations on those properties of the models which are unreasonable from their physical sense or from more general considerations. As examples of such properties of the cosmological models, we mentioning the follows:

1. The special state of infinite density. From the physical viewpoint, it is senseless to consider the real Universe under transit through this state. So it is senseless to apply the cosmological models to the real Universe under those conditions;
2. To consider the contemporary state of the Universe we observe as exceptional. This point, as we will see in the next section, is a peculiarity of all the asymptotic and monotone models (see [1], p. 399 etc.);
3. To suppose the cosmological constant non-zero, having physically unproved suppositions as a basis. Eddington (see [28], Chapter XIV) attempted to join relativistic cosmology and Quantum Mechanics. He attempted to show that the value of the cosmological constant is positive $\Lambda > 0$, $k = +1$ in this case, so the real Universe must be of the type A_2. However his speculations are very artificial and questionable;
4. The finiteness (the closure) of the space is a drawback from the general viewpoint. The supposition that space is closed, as Einstein and Friedmann made in their first models, had been recognized by them as artificial after the theory of a non-static infinite universe had been created [12, 29].

As we can see from §1.7 and §1.8 (for instance, see Table 1.1 in p. 28), there are no non-static models* which would be free of all the aforementioned properties, or even of three of them. Naturally:

- All the models, which are free of the 1st property (the models of the second kind), possess all the remaining three properties

$$\left.\begin{array}{l} A_2 \\ M_2 \end{array}\right\} \Lambda > 0\,, \;\; k = +1\,; \tag{12.1}$$

*We exclude the Einstein models from consideration, because of their instability.

- All the models, which are free of the 2nd property (the oscillating models), have the 1st property and also one of the others — the 3rd or the 4th property

$$\mathrm{O}_1 \left\{ \begin{array}{ll} \Lambda \geqslant 0\,, & k=+1 \\ \Lambda < 0\,, & k=0,\pm 1 \end{array} \right\}; \tag{12.2}$$

- All the models, which are free of the 3rd property (the models with $\Lambda = 0$), have the 1st property and also one of the others — the 2nd or the 4th property

$$\Lambda = 0 \left\{ \begin{array}{ll} k=+1\,, & \mathrm{O}_1 \\ k=0,-1\,, & \mathrm{M}_1 \end{array} \right\}; \tag{12.3}$$

- All the models, which are free of the 4th property (the spatially infinite models), have the 1st property and also one of the others — the 2nd or the 3rd property

$$k = 0,-1 \left\{ \begin{array}{ll} \Lambda \geqslant 0\,, & \mathrm{M}_1 \\ \Lambda < 0\,, & \mathrm{O}_1 \end{array} \right\}. \tag{12.4}$$

§ 1.13 Some peculiarities of the evolution of the models

Let us consider some peculiarities of the evolution of the homogeneous models we have enunciated in §1.12.

Essential is that R, and consequently any fixed volume, becomes zero under the special state of infinite density. Let us consider the lower border of the changes of R, where $\dot{R}$ has a break, however R itself is not zero (see §1.7), reaching a numerical value $R_s > 0$. Then we can assume that ρ has a singularity, namely, it becomes infinite when $R = R_s > 0$ (a positive pressure is necessary, see [3] p. 72 and p. 82, or [1] p. 399 etc.).

Examining (7.3) we can see that infinite density under finite R requires infinite $\dot{R}^2$ and, hence, infinite $\frac{\dot{R}}{R}$. The latter implies that two infinitesimally close points of the accompanying space move one with respect to one another with the velocity of light. At the same time, we know that the masses of all particles become infinite at the velocity of light. So any finite volume acquires infinite density at the lower border of its changes ($R_s > 0$).

Considering the lower condition $R_s > 0$, we obtain the special state of infinite super-density

$$\rho = \infty\,, \qquad \left|\frac{\dot{R}}{R}\right| = \infty\,, \tag{13.1}$$

where, to remove $R=0$ with $R_s>0$ in the lower limit of the changes of R, does not give any advantage.

According to Einstein's opinion [6], which has been accepted by other scientists, the special states of infinite density can be a criterion for which some idealizations, such as the homogeneity of the Universe under the conditions of its maximum contraction, are absolutely inapplicable.

Looking at the observable state of the Universe from the viewpoint of the asymptotic and monotone models, we can say that the exclusivity of the state, where ρ and $\frac{\dot{R}}{R}$ have finite numerical values (see §1.14), consists of the following. We assume ε_1 and ε_2 any infinitesimal values of the quantities ρ and $\left|\frac{\dot{R}}{R}\right|$, respectively. Then that time interval, during which the conditions

$$\rho \geqslant \varepsilon_1\,, \qquad \left|\frac{\dot{R}}{R}\right| \geqslant \varepsilon_2 \tag{13.2}$$

are true, is infinite in any of the aforementioned models. Therefore we can say of the asymptotic and homogeneous models:

> Supposing that observable properties of the Universe we know from its known fragment are law for all space, we conclude that the asymptotic and homogeneous models are rare and unique situations in time.

As we saw it in §1.12, no homogeneous model, which would be free of the 1st and the 2nd properties we have mentioned above, exist. The models with the properties could be the oscillating models of the 2nd kind (type O_2), which change R between the finite minimum and the finite maximum. Such evolution of R, according to Tolman (see [30] or [1], p. 401 etc.), is possible for $\Lambda>0$, $k=+1$ if the third of the conditions (4.8) is violated in the contractions and the expansions. However, he found (see [1], p. 402, 430 etc.) that the said violations of the condition (6.8), and hence also the possibility that the homogeneous models of kind O_2 can exist, cannot be justified from the physical viewpoint. Moreover, the models have also the 3rd and the 4th properties anyhow.

Assuming $\Lambda \neq 0$, we obtain more possible homogeneous models than it would possible for $\Lambda=0$. For instance, we can obtain the models which, being free of the 1st property or of the 2nd and 4th properties in their sum, are impossible under $\Lambda = 0$.

Similar considerations stimulate one to consider the cases of $\Lambda \neq 0$ (for instance, see [16]).

If we wish to obtain models which are free of the 1st property or of the 2nd and the 3th properties, then we arrive at the models with $k = +1$. Let us recall the final remarks in §1.2 in relation to it. In accordance with §1.2, the positive space curvature leads to the closure of space, necessary in only those models which are symmetric with respect of any of their points (the homogeneous models). In general, the relation between the curvature and the coherence properties, consisting of many more factors, has much greater possibilities.

§ 1.14 The homogeneous models and observational data

The red shift discovered by Slipher (see [10] and [7], p. 301 etc.) increases with distance and was initially considered in relation to the empty models [10, 31, 14, 11]. Lemaître [15] began to consider the non-empty non-static models in the time between two events, which suggested qualitative grounds for them, namely, Hubble's discovery that galaxies are approximate distributed homogeneous in space [32], and Hubble's discovery that red shift is approximately proportional to distance [33, 34].

The observational data he used gave numerical bounds of the following quantities:

- Using Supposition E (see §1.11), the average density of matter in the Universe, $\rho \sim 10^{-31}\,\text{gram}\times\text{cm}^{-3}$. In accordance with the late bounding, the average density is $\sim 10^{-30}\,\text{gram}\times\text{cm}^{-3}$ [35, 36];
- Using Supposition D, the data gave the relative speed of linear expansions of space, $\frac{\dot{R}}{R} = 1.8\times 10^{-17}\,\text{sec}^{-1}$, as the factor of proportionality between the expansion speed and the distance (in this bounding, he used measurements of the absolute brightness of all galaxies and their number in a unit of volume).

Those bounds, in connection with some of other bounds, lead to the conclusion that in the contemporary epoch (even if we reject Supposition E),

$$p \ll \rho c^2\,, \qquad \kappa p \ll \frac{\dot{R}^2}{R^2}\,, \tag{14.1}$$

hence, using the cosmological equations for numerical calculations, we can set $p=0$.

Moreover, as it is easy to see,

$$3\,\frac{\dot{R}^2}{c^2R^2} > \kappa\rho\,. \tag{14.2}$$

Therefore, the following cases are possible in a homogeneous universe*

$$\left.\begin{array}{lll} \Lambda > 0 \; \begin{cases} k=+1 & \mathrm{A}_1\,,\ \mathrm{M}_1\,,\ \mathrm{M}_2 \\ k=0,-1\,, & \mathrm{M}_1 \end{cases} & & \\ \Lambda = 0\,, & k=-1\,, & \mathrm{M}_1 \\ \Lambda < 0\,, & k=-1\,, & \mathrm{O}_1 \end{array}\right\}. \tag{14.3}$$

To draw conclusions which would be more likely than the above, and to make quantitative tests of the relativistic cosmological equations, more detailed observational data are needed on one hand and, on the other hand, some additional theoretical correlations between the observable values we use in the cosmological models.

More detailed statistical data had been collected by Hubble [37]. Those data concern:

(a) the average spectral type of galaxies;

(b) the numerical values of the red shift $\delta=\frac{\Delta\lambda}{\lambda}$ in the spectra of the galaxies; their photographic stellar magnitudes go up to $m=17$;

(c) the number of the galaxies $N(m)$, absolute stellar magnitudes of which go up to $m=21$.

From the necessary theoretical correlations, we select the following

$$\Phi_1\left\{l_m\,,\ m\,,\ \delta\,;\ T\right\}=0\,, \tag{14.4}$$

$$\Phi_2\left\{\delta\,,\ l_m\,;\ \frac{\dot{R}}{R}\,,\ \frac{\ddot{R}}{R}\,,\ \dots\right\}=0\,, \tag{14.5}$$

$$\Phi_3\left\{N(l_m)\,,\ l_m\,;\ n\,,\ \frac{k}{R^2}\right\}=0\,, \tag{14.6}$$

*These are true, because the cosmological equations under conditions (14.1), (14.2) with $\Lambda>0$, $k=+1$ become $\ddot{R}>0$ (which is impossible in the models A_1 and O_1).

and also

$$\Phi_4\left\{N(\delta)\,,\ \delta\,;\ n\,,\ \frac{\dot{R}}{R}\,,\ \frac{\ddot{R}}{R}\,,\ldots\right\}=0\,, \tag{14.7}$$

using which, we obtain

$$\Phi_5\left\{\frac{dN}{d\delta}\,,\ \delta\,;\ n\,,\ \frac{\dot{R}}{R}\,,\ \frac{\ddot{R}}{R}\,,\ldots\right\}=0\,. \tag{14.8}$$

Photometric distances l_m here are the distances we calculate, using the inverse square law in photometry, from the stellar magnitudes m corrected with the red shift in accordance with (14.4)*. The explicit temperature T can characterize a galaxy, if its spectrum is approximated to Planck's spectrum. The number of galaxies in an unit of volume is n. The number of the galaxies, which range up to stellar magnitude m, is $N(l_m)$. The number of galaxies which range up to the numerical value δ of their red shift, is $N(\delta)$. Numerical values of all the quantities are taken in the epoch of the observations.

It is possible to compare the theory with the observations in different ways. Let us consider two of them:

1. Assuming the numerical value $T\approx 6000^\circ$ the data (a) gave, employing (14.4), we can compare the data (b) with formula (14.5). As a result we can obtain, besides the known numerical value of $\frac{\dot{R}}{R}$, the numerical value of $\frac{\ddot{R}}{R}$. Next, using (14.4) again and extrapolating the data (b) up to $m=21$, i. e. for all the area of the data (c), we can compare the data (c) with (14.6). A result will be the numerical value of $\frac{k}{R^2}$. Substituting the obtained values into the cosmological equations (5.1) and (5.4), we can obtain the numerical values of Λ and ρ. The value of ρ, as it is easy to see, will be obtained without Supposition E. So we can compare ρ with its known numerical value, obtained in the framework of Supposition E;
2. Extrapolating the data (b) for all data (c) and eliminating m, we can compare the results with (14.7) and (14.8). A result will be, besides the numerical values of $\frac{\dot{R}}{R}$ and ρ we knew before, the numerical value of $\frac{\ddot{R}}{R}$. Substituting the values into the cosmological equations (5.1) and (5.4), we can obtain the numerical values of Λ and $\frac{k}{R^2}$. Using the last value, we can

*In non-Euclidean spaces and non-static spaces, the photometric distances are the same as regular distances in only the infinitesimal scale.

compare (14.4) and (14.6) with the data (b). We can also obtain the numerical value of T, which is in (14.4). This numerical value can be compared with the data (a).

Hubble [37] followed the first approach. He used a method developed by himself and Tolman [38]. Using the assumed numerical value $T \approx 6000^\circ$, he found the values of Λ and $\frac{k}{R^2}$ to be positive for $R \approx 1.45 \times 10^8$ parsecs (this value of R is in the order of the radius of the contemporary volume of the space). He had also obtained the large value $\rho \approx 6 \times 10^{-27}$ gram×cm^{-3} and the type M_1.

The second approach had been realised in McVittie's works*, who used McCrea's formulae [43]. Assuming $\rho \sim 10^{-30}$ gram×cm^{-3}, he obtained negative values of Λ and $\frac{k}{R^2}$ negative (he assumed $R \sim 10^8$–10^9 parsecs), the Universe's kind to be O_1, and $T \approx 7000^\circ$–7500°.

The results we have mentioned above show that the theory of a homogeneous universe, with suppositions C, D, and E, deviates from the contemporary observational data. Moreover, as it follows from Hubble's work, this difference applies to the classical mechanical theory of the Metagalaxy as well as to the relativistic cosmological theory. McVittie explained the difference, supposing that the reason is that data (a), which, affecting the quantitative bounds significantly, are inexact. Eddington [44] explained it by arguing that other observational data are inexact. At the same time, to eliminate the difference, we actually need to take the extreme numerical values of the possible empirical data. Therefore we will examine Suppositions C, D, and E.

To explain the difference via the falsity of Supposition E, we need very large masses of dark intergalactic matter (which does not undergo any interactions that we could observe). To eliminate the difference we can reject Supposition D (as had been shown in the Hubble study), however it requires the "degeneration of photons" which is hypothetical and physically unexplained. On the contrary, by the falsity of Supposition C, Shapley proposed [45, 46] his explanation of the difference, having a basis in observational data.

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*The numerical values of ρ, $\frac{\ddot{R}}{R}$, $\frac{k}{R^2}$, and Λ used by McVittie had been obtained in [39, 40], the bounds of T had been found by him in [41, 42], where he used an intermediate method to calculate other values. Therefore the method McVittie followed is logically contradictory.

§1.15 Non-uniformity of the visible part of the Universe

The Harvard studies (Shapley and others) differ from the studies made at the Mount Wilson observatory (Hubble and others). Although the former studies penetrated less into the depths of space than did the latter studies, they do however take more facts into consideration (higher percentage of galaxies in the region studied). For these reasons the Harvard studies complement the Mount Wilson data. In points of dispute, the Harvard studies take, possibly, greater weight. Collecting the results, we make the following points:

1. Many galactic clusters of up to hundreds of galaxies exist. The tendency for galaxies to concentrate themselves into clusters is under discussion. Hubble thinks the tendency less clear than does Shapley. At the same time, the fact that the tendency itself exists is indisputable [47, 48];
2. From the galaxies which are not more distant than about 3×10^6 parsecs, about two thirds are located in the northern galactic hemisphere [35]. This results from the fact that the massive galactic cluster in the Virgo constellation is in the northern hemisphere;
3. From the galaxies which are not more distant than about 3×10^7 parsecs, the larger part is located in the southern hemisphere [49, 47, 50, 46]. As shown by Shapley (see ibid.), this result does not contradict the Hubble data that both hemispheres have the same quantity of galaxies, which are not more distant than 10^8 parsecs [47, 36];
4. The number of galaxies (not more distant than about 3×10^7 parsecs) in the strip $30°\times120°$, which covers the South Pole, being calculated for 1 square degree of the strip, has a gradient from the beginning of the strip to its end [50, 46];
5. The coefficient of proportionality between the red shift and the galactic distances is different in the two hemispheres [50, 46].

The 1st and 2nd results do not contradict Supposition C. They suggest, exclusively, that the Universe is certainly inhomogeneous and anisotropic on a scale less than the aforementioned. The 3rd result indicates that a density gradient in the Metagalaxy along one of its radial directions, and also an anisotropy produced by the gradient, exist. So Supposition C is very inaccurate. The 4th result indicates that an essential density gradient in the Metagalaxy, and also an essential anisotropy in the observed direction, exist. So

Supposition C is severely violated. Finally, the 5th result shows that the anisotropy of matter in the Metagalaxy exists in company with the anisotropy of its deformations.

Taking Shapley's results into account, the 4th result mainly, McCrea [51] set up the next problem: find theoretical correlations between observable values, first, between the numerical values of the red shift and photometric distances, calculated with astronomical methods. He set up this problem within the framework of the suppositions: (1) the General Theory of Relativity is true in the region where the observable space objects are located; (2) some additional conditions, which do not need homogeneity and anisotropy, take a place in the region. In relation to the first supposition, McCrea took $\Lambda = 0$ in the Einstein equations. An additional condition he took into account was that matter can be assumed to be an ideal fluid without inner pressure. As a result of the latter, it is possible to utilise accompanying coordinates, where

$$g_{00} = 1\,, \qquad \frac{\partial g_{0i}}{\partial t} = 0\,, \tag{15.1}$$

so, besides (15.1), it would be possible to suppose time orthogonal to space everywhere, namely

$$g_{0i} = 0\,, \qquad i = 1, 2, 3\,. \tag{15.2}$$

As a reason of the first of the additional conditions, McCrea pointed out that the peculiar velocities of galaxies are low. The second additional condition of McCrea has no simple physical interpretation. The condition had been introduced by him for only mathematical simplification*.

Using his suppositions, McCrea found an approximate correlation between the numerical values of the red shift and the "projected distances" (which characterize distances to the observable space objects), and the density of matter at the point of the observations. He pointed out that he was not successful in linking the "projected distances" to the distances astronomical data give, and so he was unable to solve the problem he had set up†.

McCrea's failure is nonrandom and has also no explanation in mathematical difficulties. It is doubtful, in general, that McCrea's

*An analogous declaration had been made by Friedmann [5].

†"Projected distance" coincides with regular photometric distance in only the first approximation. At the same time it is insufficient, if we want to apply the correlation McCrea had found.

suppositions would be sufficient for determining theoretical correlations between the observable values. His suppositions could be sufficient for some qualitative conclusions on the evolution of the observable fragment of the Universe. So these would be enough for some steps toward building the theory of an inhomogeneous universe. As a result, it is possible that the aforementioned correlations in which McCrea was interested can be obtained only in the framework of the completed theory of an inhomogeneous universe.

§ 1.16 The theory of an inhomogeneous universe

The initial suppositions of the theory of a homogeneous universe, namely — the supposition we have mentioned in §1.1 or Suppositions A and B (see §1.11), cannot be justified. Those models, which can be obtained under the suppositions, have the properties we have considered in §1.12 and §1.13 to be inapplicable to the real Universe. The supposition that the known fragment of the Universe is homogeneous and, hence, is isotropic (see Supposition C, §1.11) is in contradiction to observational data. The arbitrariness of the existing theory and the necessity of building the relativistic theory of an inhomogeneous universe had been understood even before the contradiction had been found. However, the development was delayed because of mathematical difficulties in the problem*. In this consideration, the thesis that the Universe as a whole is identified with the Metagalaxy remained valid. At the same time various astronomers, namely — Mason, Fesenkov, Eigenson, and Krat [52, 53, 54, 27, 55], conceive of the Metagalaxy spatially as limited (that the Metagalaxy is inhabited by numerous metagalaxies). One can link this viewpoint, actually unnecessarily, to Lambert's concept, according to which the Universe has a hierarchical interior of infinitely numerous levels [53, 54, 56].

The following suppositions for a method of building the theory of an inhomogeneous universe are reasonable.

Suppositions All previous suppositions relate to the Universe as a whole. To replace them with the following suppositions on a

*For instance, see [1], p. 330, 332, 363–364, 482, 486–488. Note that besides the "technical" difficulties, there is the principal difficulty that no universal boundary conditions for infinity exist in the relativistic theory of gravitation. If the properties of symmetry exist in, for instance, a homogeneous universe, then the properties compensate the absence of the aforementioned boundary conditions. At the same time we cannot attribute the properties to an inhomogeneous universe.

fragment of the Universe we are considering at each stage of this study*.

1. Einstein's General Theory of Relativity is valid everywhere in the fragment we are considering. In addition, although only the case $\Lambda = 0$ has an explicit physical sense, we can consider the cases $\Lambda \neq 0$, aiming to compare them with the former. So we take the cosmological constant valid in the initial equations.
2. The volume we are considering is filled with matter, consisting of a monotonic distribution of substance† and radiations, in general. A difficulty with this however, is that the four-dimensional energy-momentum tensor, corrected for heat fluxes, has not yet been found (see [1], p. 330). This difficulty can be passed over, if we consider the transparency of the intergalactic space within the framework of the supposition that radioactive transfer dominates other transfers of energy. Supposing the interaction energy of two elements of matter negligible in comparison with the energy of each element, we can write
$$T_\mu^\nu = \left(T_\mu^\nu\right)_m + \left(T_\mu^\nu\right)_r \,, \tag{16.1}$$
where the first term here describes the substance free of heat fluxes, the second term describes the radiations. So we have
$$T = (T)_m \,. \tag{16.2}$$
3. The fragment we are considering can be spanned, step-by-step, by coordinate nets which accompany the substance. In other words, numerous conditions, making such coordinates possible, take a place in the given fragment of the Universe. Because the substance (galaxies) is the "skeleton" of the Universe, and because we use the accompanying coordinates in observational tests of the theory, the aforementioned coordinates have more physical meaning than coordinates accompanying the whole of the matter on the average‡. We note that under the suppositions we have made above, we have
$$\left(T_0^i\right)_m = 0 \,, \tag{16.3}$$

*By studying an element of the Universe, we mean "a fragment we are considering" as any infinitesimal (finite) volume, which contains this element.

†This is a "liquid", or better, a "gas", where galaxies are "molecules".

‡Both kinds of the accompanying coordinates are the same in a homogeneous universe.

thus we obtain, finally

$$T_0^i = \left(T_0^i\right)_r \,. \tag{16.4}$$

The way to build the theory of an inhomogeneous universe To retain considerations on the Universe as a whole from the beginning of the study. To split the study into the following stages.

1. Consider the evolution of an element of an inhomogeneous anisotropic universe. The main point here is the following problem: is the evolution of universes of kind O_2 really possible, and what are the conditions of their evolution (if this kind of evolution can exist)? Actually, this is the question: can the oscillating models, which do not transit through the special state of infinite density, exist, and what are the conditions? If the answer is in the negative, then the assumed idealization is not applicable to the real Universe.

The minimum of those scales where the second of the assumed suppositions is reasonable, is of the same order as that mentioned in §1.11. To study the Universe on this scale would be most desirable. At the same time, we can also use a scale, larger than the minimum, only if the "elements" of the Universe we have in the scale are infinitesimal in comparison to the Metagalaxy (if the Metagalaxy is infinite, then its elements can be assumed as large as desirable). Taking the latter into account, results of the 1st stage of the study can be applicable to very large fragments of the Universe.

From the 1st stage of this study, we can go to the 2nd stage or, by passing this stage, directly to the 3rd stage, where the problem of space curvature has a special meaning. For this reason, we must take space curvature into account in the 1st stage of the study.

2. Consider the case where the Metagalaxy is spatially finite. Analogously to the previous stage, the question about applicability of the type O_2 to all the elements of the Metagalaxy (and the conditions of its evolution, if it is applicable) has a special meaning here. In other words, we ask: are the special states of the infinite density absent throughout the four-dimensional volume, where the Metagalaxy is? If the answer is in the negative, then the assumed idealization is not applicable or, probably, the Metagalaxy's spatial boundedness cannot exist eternally.

3. Consider the case where the Metagalaxy is spatially unbounded and infinite (that is the Universe). The next question has a special meaning here: are any peculiarities in this four-dimensionally infinite volume, the Universe, absent, and what are the conditions of the peculiarities (if they exist)? To answer this question in the negative implies inapplicability of the accepted idealization, or that the Lambert concept (see p. 44) is true. In connection with the aforementioned question about the spatial infinitude of the Universe, the question about the space curvature has a special meaning. Unfortunately, the correlation between the curvature and the topological structure of space has not been studied sufficiently*: the data we have under consideration are apparently, exhausted with some of the sufficient conditions of the spatial infinitude (see [57], p. 234 and 239). It is necessary to note that to apply any result we obtain in three-dimensional geometry to the case where time is everywhere orthogonal to space, requires further study.

As a matter of fact, before we study an element of an inhomogeneous universe, we need to build a mathematical apparatus, according to the physical contents of the problem†. We need to first find those equations by which we will study the Universe in each of its elements.

§ 1.17 The cosmological equations: the local interpretation

Let us consider the cosmological equations of a homogeneous universe we know from the foregoing. The quantities ρ, p, $D=3\frac{\dot{R}}{R}$ (and their derivatives), and also $C=3\frac{k}{R^2}$, constitute the equations that characterize a homogeneous universe as a whole, as well as any of its elements. Under such "local interpretation" of the cosmological equations, the quantity R does not keep its physical meaning as the "radius of the Universe". For this reason, using (11.1) and (11.2) we exclude the radius R from the cosmological equations of gravitation (5.1) and (5.4), and also from the cosmological equation of energy

*Except for constant curvature spaces, which play a part only in the theory of a homogeneous universe.

†Of course, we need to develop this apparatus step-by-step at each stage of the study.

(4.2). As a result, the equations take the form

$$\frac{1}{c^2}\left(\frac{\partial D}{\partial t}+\frac{1}{3}\,D^2\right)=-\frac{\kappa}{2}\left(\rho+3\,\frac{p}{c^2}\right)+\Lambda\,, \tag{17.1}$$

$$\frac{1}{3c^2}\,D^2+C=\kappa\rho+\Lambda\,, \tag{17.2}$$

$$\frac{\partial\rho}{\partial t}+D\left(\rho+\frac{p}{c^2}\right)=0\,. \tag{17.3}$$

The correlation between space curvature and expansions or contractions of space, which could eliminate formula (2.2), can be obtained from (17.1–17.3). Eliminating Λ, ρ, and p from them, we obtain

$$\frac{\partial C}{\partial t}+\frac{2}{3}\,DC=0\,, \tag{17.4}$$

then, because of (4.4) and (14.2), we have

$$D=\frac{1}{V}\frac{\partial V}{\partial t}\,, \tag{17.5}$$

whence

$$\frac{\partial}{\partial t}\big(\sqrt[3]{V}\,C\big)=0\,. \tag{17.6}$$

The density ρ and the pressure p describe:

(a) the state of matter throughout the space (D is the relative rate of the volume expansion of the matter);

(b) the evolution (namely — the deformations) of the substance and its accompanying space;

(c) the curvature $\frac{1}{3}C$ characterizes the geometrical properties of the accompanying space.

So having the local interpretation of the cosmological equations, we can say:

- The cosmological equations of gravitation are the correlations the equations of the law of gravitation set up* between the quantities ρ, p, D, and $\frac{1}{3}C$ (the quantities characterize the state of matter, its deformations, and the geometrical properties of the accompanying space);

*That is, the correlations are set up by the equations of the law of gravitation, not by the equations of the law of energy, which is a consequence of the law of gravitation.

- The cosmological equation of energy is the correlation the equations of the law of energy set up between the aforementioned quantities ρ, p, and D. In other words, this is the correlation between the state of the matter and deformations of the accompanying space.

We are now going to consider the general case of the equations of the law of gravitation and of the law of energy in the accompanying coordinates.

The quantities ρ and p, characterizing the state of matter, are contained in components of $T_{\mu\nu}$, which consist of the equation of gravitation and the equation of energy. The quantity D, describing the evolution (i. e. deformations) of the substance and of its accompanying space, must evidently be linked to the quantities $\frac{\partial g_{ik}}{\partial x^0}$, which are contained in the equation of gravitation and the equation of energy as well. The curvature $\frac{1}{3}C$, characterizing the geometrical properties of the accompanying space, is linked to the quantities $\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l}$, which are contained in the equation of gravitation, however it is absent in the equation of energy. Besides those mentioned above, some other quantities, which are linked to $\frac{\partial g_{0\alpha}}{\partial x^\beta}$, can appear in both equations (the quantities characterize the force field* of the accompanying space).

Thus, in the general case, we can obtain:

- The correlations the equations of gravitation set up between the quantities ρ, p, D, and $\frac{1}{3}C$, characterizing the state of matter, deformations and the geometrical properties of the accompanying space, and, possibly, the force field;
- The correlations the equations of the law of energy set up between the quantities ρ, p, and D, describing the state of matter, deformations of the space, and, possibly, the force field.

By analogy with the equations of a homogeneous universe, we will refer to the first as the *cosmological equations of gravitation*, and to the second as the *cosmological equations of energy*. In general, we call the correlations between the aforementioned quantities (which characterize the state of matter, deformations and the geometrical properties of the accompanying space, and also the force field) the *cosmological equations*. It is evident that the cosmological

*In the sense of Classical Mechanics.

equations of an inhomogeneous universe, being a generalization of the cosmological equations of a homogeneous universe, permit only their local interpretation.

We will use the cosmological equations for studying elements of inhomogeneous universes.

§ 1.18 Friedmann's case in an inhomogeneous universe

This study realises only the beginning of the program we have mentioned above. We will limit ourselves to the first stage (see §1.16), where the main task is to consider the cosmological equations and some of their consequences. From the consequences we will consider, mainly, the evolution of the volume of an element of an inhomogeneous universe, where we assume the matter a substance* of positive density without any tensions or pressure. We also suppose the substance to be free of heat fluxes. Then, within the framework of arbitrary coordinates, its energy-momentum tensor is the tensor of an ideal fluid of infinitesimal pressure, that is

$$T^{\mu\nu} = \rho \frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}\,, \qquad \rho > 0\,. \tag{18.1}$$

This case correlates the condition (4.7) in a homogeneous universe† and, like the same case in homogeneous universe, can be called *Friedmann's case*.

To clarify that the Friedmann case in an inhomogeneous universe relates to the same case in a homogeneous universe and to the general case of an inhomogeneous universe, we are going to make a list of those factors which are expected in the transfer from a homogeneous universe to an inhomogeneous one.

Factors which do not link to the presence of pressure

1. Anisotropy of deformations of elements of the accompanying space.
2. Inhomogeneity of their density‡.

*That is, here Zelmanov supposed the matter a substance without radiations. — Editor's comment. D. R.

†Compare (18.1) with formula (3.6) under the condition (4.7).

‡Note that the density gradient in explicit form cannot appear in the cosmological equations, because no derivatives of the time component of the energy-momentum tensor with respect to spatial coordinates exist in the initial relativistic equations. This can be explained from the physical viewpoint, because the term "density gradient" has no meaning with respect to a single element of the Universe.

3. A field of forces, which act on test-bodies only in their motions with respect to the coordinates we are considering (the coordinates which accompany to substance).
4. The correlation between the Riemannian curvature and two-dimensional directions (i. e. anisotropy of the curvature).
5. Inhomogeneity of the Riemannian curvature along all two-dimensional directions (i. e. inhomogeneity of the curvature)*.

Factors linked to the presence of pressure

6. Viscosity, which manifests under anisotropy of the deformations†.
7. Divergence of the three-dimensional stress tensor; in other words, inhomogeneity of pressure, linked to inhomogeneity of density.
8. A field of forces, which put the inhomogeneity of pressure into equilibrium.
9. A heat flux, linked to the inhomogeneity of pressure.
10. A flux of momentum, linked to the heat flux.

It is evident that the facts of the first group are different to the case we are considering from the Friedmann case in a homogeneous universe, the facts of the second group are different to the general case of an inhomogeneous universe.

The Friedmann case in the theory of a homogeneous universe is inapplicable to thermodynamics. At the same time, this case is sufficient for dynamics and observational tests of the theory, and is that sole case which is applicable for comparing the theory with Classical Mechanics in strong form. Taking these as a basis for building the theory of an inhomogeneous universe, we can say that the Friedmann case cannot be applicable to thermodynamics; it is

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*The mean curvature gradient in explicit form cannot appear in the cosmological equations, because it must include the third derivatives of components of the metric tensor.

†As it has been obtained in the theory of a homogeneous universe (see §1.3), matter can be assumed to be an ideal fluid. Then, from the physical viewpoint, it can be a consequence of not only the absence of the viscosity, but also the presence of the viscosity under isotropy of the deformations. In other words, this is the fact we have also in Classical Hydrodynamics (for instance, see [58], p. 544, under the conditions $a = b = c \neq 0$ and $f = g = h = 0$). Note that the viscosity of the "fluid", consisting of "molecules-galaxies", is four orders greater than the viscosity of water. It can easy be calculated, taking masses, volumes, and peculiar velocities of galaxies into the calculation.

the sole case whereby we have any hope of finding a strong analogy to the equations of Classical Mechanics. At the same time, because numerous factors are linked to light pressure in an inhomogeneous universe, we have no knowledge of how much the Friedmann case (and its consequences) differ from the general case in the spheres of dynamics, geometry, and its observational tests. We are limiting our attention by the Friedmann case here, because mathematical simplicity and physical obviousness recommend to first consider influences of only one of the groups of the factors, namely the first group of the factors, because those factors can be in action even in the absence of any factors of the second group (the last factors of the aforementioned are in action under only the presence of some of the factors of the first group). In our opinion, any reasons that pressure and its linked factors can be infinitesimal in the fragment of the Universe we observe in the present time, have secondary meanings.

The next case, after the Friedmann case, could be considered as an approximation to the case of an inhomogeneous universe*. This is the case of

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{p}{c^2}\,g^{\mu\nu}, \quad \rho > 0\,, \quad p > 0\,, \qquad (18.2)$$

i. e. an ideal fluid — the mix of a substance and radiations under the 7th and the 8th factors. This case corresponds to the general case of a homogeneous universe, and it is the general case, where the condition (4.6) remains unchanged.

To obtain the cosmological equations and their analysis we need to prepare the necessary mathematical apparatus and also methods to interpret numerous mathematical results we will obtain. These tasks will also be included in this work.

We have given a brief survey of the theory and the tasks of our work. Looking forwards, we propose the following plan. To prepare the necessary mathematical apparatus (Chapter 2); To apply the apparatus to equations of physics (Chapter 3); To use the equations to obtain the cosmological equations and their consequences, considering the Friedmann case in an inhomogeneous universe (Chap. 4).

Keeping in mind that programme we have set forth in §1.16, we will complete each of the Chapters with results, more than it would be necessary.

*It has been partially developed by the author, but the results are not included in this work.

§ 1.19 The mathematical methods

Considering the proposed contents of Chapter 2, we must first consider the following problem: is the use of some special coordinate frames of the systems accompanying the Universe's substance, possible and appropriate?

Most simple are those coordinates where

$$g_{0\alpha} = \delta_{0\alpha} \,. \tag{19.1}$$

Very simple and physically rational are the coordinates where

$$\frac{\partial}{\partial x^{\beta}}\left(g^{\alpha\beta}\sqrt{-g}\right) = 0 \,, \tag{19.2}$$

proposed first by Lanczos [59]. Such coordinates have been used in physical studies by Fock (the "harmonic coordinates" [60]).

However, generally speaking, both coordinate frames of the above are not of the accompanying coordinates kind*. So, considering the conditions (19.1) and (19.2), we can associate the accompaniment condition with only the case $\alpha = 0$, where the choice of time coordinate is fixed. So, using the special coordinates (of the accompanying ones) in general equations cannot give much simplification. Moreover, this is not good mathematically, because a circle of those simplifications, which could be possible in the partial cases, is much narrower under the special coordinates. Finally, this is inappropriate, for the following physical reasons. The special coordinate frames obstruct selection of physical quantities which:

(1) are invariant with respect to those transformations, where the accompaniment of the coordinates to the substance is not violated;

(2) contain not only the transformations of the spatial coordinates

$$x^{i'} = x^{i'}(x^1, x^2, x^3) \,, \qquad i = 1, 2, 3 \,, \tag{19.3}$$

but also any transformation of the time coordinate, i. e.

$$x^{\alpha'} = x^{\alpha'}(x^0, x^1, x^2, x^3) \,, \qquad \alpha = 0, 1, 2, 3 \,. \tag{19.4}$$

Because of the physical equivalence of all time coordinates, among the things measured in the body of reference that we use,

*We will use the harmonic coordinates, moving with respect to the accompanying ones, in §4.26.

the invariance with respect to transformations (19.4), in other words, the property of "chronometric invariance", must be a property of all the main quantities and equations which characterize this body of reference. For this reason we retain attempts to apply any special coordinates, and so we set forth the main task of Chapter 2 as the introduction of a three-dimensional tensor calculus, the main quantities and operators of which have this additional property of chronometric invariance with respect to transformations (19.4). We will refer to this new mathematical apparatus as the *chronometrically invariant tensor calculus*. Accordingly, we will refer to its objects and operators as *chronometrically invariant tensors**, which characterize deformations of the accompanying space, its curvature, and its metric.

Chapter 2 will be arranged into numerous thematic sections.

A. §2.1–§2.4 introduce three-dimensional quantities, both invariant with respect to (19.4) and non-invariant. The non-invariant quantities will be very helpful because of their property to take, as a result of our choice of a specific coordinate of time, the necessary numerical values†.

B. §2.5–§2.7 introduce the generalized operators of differentiation with respect to time and spatial coordinates, invariant with respect to (19.4). The sections also introduce quantities ("force values" — the vector F_i and the antisymmetric tensor A_{jk}) which characterize non-commutativity of the operators‡.

C. §2.8–§2.12 introduce the chr.inv.-metric tensor, the chr.inv.-tensor of the space deformations, the correlation between the tensors, and also consequences of this correlation. Equation (11.22), giving the correlation, is the most important result of the entire study§.

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*Chr.inv.-tensors, in brief. Initially, Zelmanov called chronometric invariant tensors *in-tensors*, and the whole mathematical apparatus the *in-tensor calculus*. However the terms were unsuccessful, so Zelmanov replaced them with the more reasonable terms *chronometrically invariant tensors* and the *chronometrically invariant calculus*. — Editor's comment. D. R.

†See the method to vary the potentials — §3.7, §3.12, §3.17, and §3.20.

‡§2.7 gives the necessary and sufficient conditions of $g_{00}=1$ and $\frac{\partial g_{0i}}{\partial t}=0$ (15.1), and also $g_{0i}=0$ (15.2) can be written as $F_i\equiv 0$ and $A_{jk}\equiv 0$, respectively. The operators give a possibility of bringing two cases, where the world-lines of an ideal fluid's current are Eisenhart [61] geodesics, together into a single case, which is defined by the condition $\frac{{}^*\partial p}{\partial x^i}\equiv 0$ in the accompanying coordinates (p is pressure).

§Under accompanying coordinates (not only in relativistic mechanics but also in

D. §2.13–§2.15 introduce generalized covariant differentiation in self-deforming spaces, invariant with respect to (19.4). The sections also give the necessary generalizations of the Riemann-Christoffel tensor and Einstein's tensor.

E. §2.16–§2.19 introduce chr.inv.-tensor quantities, which describe rotational motions. The correlation between relative rotations of the elements of the self-deforming accompanying space and the space deformations is given there. This correlation, as seen from formula (19.19) in §2.19, implies a result similar to the well-known equation

$$\mathrm{rot\,rot}\;\vec{v} = \mathrm{grad\,div}\;\vec{v} - \nabla^2\,\vec{v} \tag{19.5}$$

if: (1) we re-write the equation in the form

$$\frac{1}{2}\,\mathrm{rot\,rot}\;\vec{v} = \left(\mathrm{grad\,div}\;\vec{v} - \frac{1}{2}\,\mathrm{rot\,rot}\;\vec{v}\right) - \nabla^2\,\vec{v}\,, \tag{19.6}$$

and (2) we take $\vec{v}$ to be the vector of velocity.

F. §2.20–§2.22 introduce chr.inv.-tensor quantities which characterize the curvature of the self-deforming accompanying space. The quantities are the chr.inv.-tensor generalizations of the curvature tensors, the curvature scalar, and Ricci's tensor.

Using only the relativistic equation of the four-dimensional interval ds^2 as a basis for this study, Chapter 2 does not use any additional equations or additional physical requirements.

§ 1.20 Relativistic physical equations

Because we will deduce the cosmological equations in Chapter 4, we must consider relativistic physical equations before that Chapter. This will be done in Chapter 3. We must first clarify the physical (mechanical) sense of the "power quantities" F_i and A_{jk} on the one hand, and, on the other hand, obtain chr.inv.-tensor quantities which describe force fields. Both tasks need to establish the equations of geodesic world-lines in chr.inv.-tensor form and to compare them with the equations of Classical Mechanics. We must introduce chr.inv.-tensor quantities which characterize states of matter. This task requires deduction of components of the energy-momentum

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Classical Mechanics), this correlation implies that a third interpretation of motions of continuous media exists, besides those of Euler and Lagrange.

world-tensor in chr.inv.-tensor form. Finally, introducing chr.inv.-tensor quantities which characterize states of matter, deformations of space, space curvature, and forces acting in the space, we must obtain equations of the law of gravitation and the law of energy in chr.inv.-tensor form. So the main task of Chapter 3 is to translate the main relativistic tensor quantities and the equations into the "specific language" of the mathematical apparatus of chronometric invariants (see the apparatus itself in Chapter 2), clarifying the mechanical sense of the "power quantities" simultaneously.

Let us make a list of the main themes contained in Chapter 3.

A. §3.1–§3.3 give three-dimensional tensor equations for components of the world-metric tensor and also for Christoffel's world-symbols of the 1st and the 2nd kind. Very significant in the consequences (see §3.14–§3.17) is the fact that the aforementioned tensor world-quantities Q, all indices of which are not zero, are linked to the appropriate three-dimensional tensor quantities T of the same indices by the linear equations

$$Q = \pm T + a\,, \tag{20.1}$$

where the additional term a can be a three-dimensional quantity or zero.

B. §3.4–§3.6 introduce (after the chr.inv.-vector of the velocity of light introduced in §2.9) chr.inv.-tensor characteristic of a point-mass — its mass, energy, and momentum, invariant with respect to (19.4). The generalized total derivatives of the quantities with respect to time are also given there. As we expect, after we apply the chr.inv.-tensor apparatus, these relations between the physical quantities considered are similar to the relations we know from the Special Theory of Relativity.

C. §3.7–§3.9 deduce the equations of world-geodesics in chr.inv.-tensor form. The chr.inv.-equations obtained give the dynamic equations and the theorem of energy of a point-mass, which can be compared with the appropriate equations of Classical Mechanics. This comparison justifies the terminology we have used for "power quantities", because they are characteristics of force fields; namely, the chr.inv.-vector F_i plays the part of gravitational inertial force, calculated with a unit mass, the chr.inv.-tensor A_{jk} (or the chr.inv.-vector Ω^i, appropriating the A_{jk}) plays the part of the momentary angular velocity of the absolute rotation of the reference frame in the formula for the Coriolis force. As it is easy to see that the "power quantities" F_i and Ω^i relate those factors (see §1.18), which can be expected in the transfer from a homogeneous universe to an inhomogeneous one: F_i correlates the 8th factor (the field of forces, which put the effect of the spatial inhomogeneity into equilibrium), Ω^i correlates the 3rd factor (the field of forces, which act only upon bodies in motion with respect to the coordinates accompanying the substance). In our opinion, this result is very important*.

D. §3.10–§3.13 obtain components of the energy-momentum world-tensor in chr.inv.-form, express them with the mass density, the momentum density, and the kinematic stress tensor, invariant with respect to (19.4). Using the results, we obtain world-equations of the law of energy in chr.inv.-tensor form. The equations obtained for the law of energy (the scalar equation for the energy density, the vector equations for the momentum density) contain only chr.inv.-tensor quantities, which characterize states of matter, its evolution, the space deformations, and forces acting in the space.

E. §3.14–§3.17 give components of Einstein's world-tensor in chr.inv.-tensor form. This operation requires tedious formal algebra, but results in numerous very compact formulae. As a result of §3.1–§3.3, spatial components of the Einstein world-tensor are linked to components of the appropriate three-dimensional tensor by equations like (20.1).

F. §3.18–§3.24 develop the equations of Einstein's law of gravitation into chr.inv.-tensor form. As a result, we obtain the scalar equation of gravitation, the vector equation, and the tensor equation. The equations consist of only chr.inv.-tensor quantities, which characterize states of matter, its evolution, the space deformations, and forces acting in the space. It is interesting that the vector equation can be considered as a differential equation with respect to the angular velocity vector of the "geodesic precession" (see §4.10).

*In particular, this result has the following consequences. The physical sense of the supposition (see §1.15) that time is everywhere orthogonal to space $g_{0i}=0$ (15.2), which was unknown to Friedmann and McCrea, is the absence of the Coriolis effect (the absence of the "dynamic absolute rotation", in the terminology of the Chapter 4). The physical sense of the conditions $g_{00}=1$, $\frac{\partial g_{0i}}{\partial t}=0$ (15.1) is merely that those forces which could put the pressure gradient into equilibrium, are absent.

All the results of Chapter 3 are not linked to any limiting suppositions, related to kinds of matter or relative motions of the coordinate systems we use. The entire problem statement and the results are independent of both the mathematical viewpoint and the physical viewpoint. This fact, in our opinion, suggests the following. The mathematical apparatus of chronometric invariants we give in Chapter 2 could be a natural form of physical equations in those cases, where the complex of coordinate systems, which are at rest with respect to each other, plays a special part. One such case is relativistic cosmology, because it uses accompanying coordinate frames.

§ 1.21 Numerous cosmological consequences

Beginning with Chapter 4, which is actually the cosmological part of this study, we will give numerous notes on the cosmological equations in the Friedmann case, namely — on those 10 factors (see §1.18), which differentiate the inhomogeneous universe from the homogeneous one.

The absence of pressure and also of the 6th, 7th, 9th, and the 10th factors, linked to it, must be taken into account. It is necessary to do this when we introduce quantities which characterize states of matter and its evolution (if the quantities are taken in coordinates accompanying the substance), into the cosmological equations. The absence of the 8th factor, i. e. when the vector F_i is zero, must be obtained as a consequence of the cosmological equations. Looking at the factors of inhomogeneity, we can say that the 2nd and the 5th factors cannot appear in explicit form in the cosmological equations. For this reason the factors manifest their links to the other factors of inhomogeneity (the 1st, 3rd, and the 4th factors) only under consideration of a finite or infinite volume of the space. The remaining factors of inhomogeneity (the 3rd of them appears as the vector Ω^i) are linked to the nonequivalence of two-dimensional directions*, so they can be called the *factors of anisotropy*. These factors can affect the evolution of "isotropic" characteristics of any element of the Universe — its volume, density, mass, and the mean curvature of the space, linked to the factors and also one to another by the cosmological equations. From this we can appreciate that, concerning the evolution of one of the "isotropic" characteristics, we need to consider not only the evolution of other "isotropic" char-

*The nonequivalence of two-dimensional directions in three-dimensional space is the same as the nonequivalence of one-dimensional directions.

acteristics (just as we did in the theory of homogeneous universe) but also the factors of anisotropy. The factors of inhomogeneity can be expressed as relations between themselves on the one hand, and the relations between them and the evolution of the "isotropic" characteristics on the other hand. To do this we must take the cosmological equations and replace the tensor equation we are writing in the abstract form

$$B_i^k = 0 \tag{21.1}$$

with the equivalent system of equations: (1) the scalar equation

$$B = 0 \tag{21.2}$$

which we obtain by contraction of equation (21.1); and (2) the tensor equation

$$B_i^k - \frac{1}{3}\, h_i^k B = 0\,, \tag{21.3}$$

which becomes an identity after contraction (see the "main form" of the cosmological equations).

After the foregoing has been established, we formulate the main task of our cosmological problem (see the beginning of §1.18) more precisely. So the main task of Chapter 4 is to obtain the cosmological equations for the Friedmann case in an inhomogeneous universe, and also numerous their consequences thereof, which characterize: (1) factors, which are unnatural in homogeneous models; (2) effects of the unnatural factors on the evolution of the main "isotropic" characteristics of an arbitrary element of an inhomogeneous universe (the evolution of its volume being the main task).

Chapter 4 will also be arranged into numerous thematic sections.

A. §4.1–§4.4 introduce suppositions on those kinds of matter which pertain to the Friedmann case, and also coordinates accompanying the substance. To describe deformations and relative rotations of elements of the substance, we use quantities characterizing the analogous motions of the elements of the space. We also introduce the possibility of using the accompanying coordinates, on a basis of general suppositions on motions of the substance.

B. §4.5–§4.8 give the cosmological equations in their "main form", and also set forth the possibility of introducing the primary coordinate of time at any given point of the space (this possibility is derived from the fact that the vector F_i

can be reduced to zero at any point). The "main form" of the cosmological equations shows that if the scalar cosmological equations characterize an element of the Universe, independent of spatial directions, and the tensor equations characterize anisotropy of the element, then the vector cosmological equation of gravitation links the evolution of the element to the evolution of its neighbouring elements.

C. §4.9–§4.13 consider the factors of anisotropy and also relations between anisotropy and inhomogeneity. Let us mention of those results that: (1) the vector cosmological equation, in its mechanical sense, is a differential equation with respect to the angular velocity vector of the "geodesic precession"; (2) dynamic absolute rotations cannot disappear or appear, which is a peculiarity of this factor of anisotropy; (3) the law of transformations of the vector Ω^i under deformations of the element; (4) the deformations can be anisotropic in the absence of other factors of anisotropy, which is a peculiarity of only the anisotropy of deformations; (5) the presence of inhomogeneities in a finite volume is necessary, if the volume is isotropic; (6) a static model, which differs from the Einstein static model, can manifest when $\Omega^i \equiv 0$.

D. §4.14–§4.18 consider the scalar cosmological equations, generalizing the equations of the theory of a homogeneous universe. Using the scalar equations, we obtain the laws of transformations of the mean curvature of the space and of the density of the substance under transformations of the volume of the element. We also make some preparations in consideration of possible kinds of evolution of the volume under transformations of an arbitrary time coordinate. Let us mention of those results that: (1) anisotropy of the space curvature does not act directly on the evolution of the "isotropic" characteristic of the element; (2) the mass and energy of the element remain unchanged (as for a homogeneous universe when $\rho = 0$); (3) the mean curvature of the space can change sign, if the anisotropy factors are present in the space; (4) the mean curvature (taken in an element) undergoes changes due to the space deformations, just as in homogeneous universe, if the element of an inhomogeneous universe is isotropic.

E. §4.19–§4.23, as well as §1.7, consider states between which monotonic transformations of the volume of an arbitrary element are conceivable. For an isotropic element, we obtain the same results as in a homogeneous universe. The anisotropy factors lead to: (1) additional different states, which are possible for given numerical values of the cosmological constant and the mean curvature of the space; (2) states which are unknown in the theory of a homogeneous universe; (3) removal of the prohibition of the transformations, limited from above and below, which is most important.

F. §4.24–§4.28 consider possible kinds of evolution of the volume of an arbitrary element. First, we consider the evolution in the interval of monotonic deformations. Then, in that interval, where the volume and its density remain unchanged. The anisotropy factors make the types of evolution which are possible for given numerical values of the cosmological constant, more numerous. The aforementioned factors result new kinds of evolution, which are impossible in the case of a homogeneous universe. In particular, it is very essential to note that the anisotropy factors result in a new oscillating model of the O_2 kind, which does not transit through the special state of infinite density. Let us mention the most important results: (1) for an element of an inhomogeneous isotropic universe, the possible kinds of transformations of its volume which are possible for given numerical values of the cosmological constant and the given sign of the mean curvature (or, when the mean curvature is zero), are the same as those in a homogeneous universe; (2) for anisotropic deformations in the absence of dynamic absolute rotations, new kinds of evolution (in comparison with a homogeneous universe) can exist only for $\Lambda > 0$,* and the kind O_2 can exist only when the mean curvature is positive; (3) in the general case of anisotropy, including also dynamic absolute motions, new kinds of evolution (between which is the kind O_2) are possible for any numerical values of the cosmological constant, so that there are no limitations on the sign of the mean curvature.

The results show that dynamic absolute rotations (the Coriolis effect) can play a very important part in the Universe.

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*For this reason, only the kinds M_1 and O_1 are possible under McCrea's suppositions (see §1.15).

Chapter 2

THE MATHEMATICAL METHODS

§ 2.1 Reference frames

The general transformations of coordinates of a system S into coordinates of any other system S'

$$x^{\alpha\prime} = x^{\alpha\prime}(x^0, x^1, x^2, x^3)\,, \quad \alpha = 0, 1, 2, 3 \tag{1.1}$$

can be written in the form

$$\left.\begin{aligned} x^{0\prime} &= x^{0\prime}(x^0, x^1, x^2, x^3) \\ x^{i\prime} &= x^{i\prime}(x^0, x^1, x^2, x^3)\,, \quad i = 1, 2, 3 \end{aligned}\right\}. \tag{1.2}$$

Greek indices can be 0, 1, 2, 3, Latin indices only 1, 2, 3.* Let us consider a particular case of the transformations (1.2), where the condition

$$\frac{\partial x^{i\prime}}{\partial x^0} \equiv 0 \tag{1.3}$$

holds, i. e. transformations like

$$\left.\begin{aligned} x^{0\prime} &= x^{0\prime}(x^0, x^1, x^2, x^3) \\ x^{i\prime} &= x^{i\prime}(x^1, x^2, x^3) \end{aligned}\right\}. \tag{1.4}$$

Because of (1.3), we have

$$dx^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^j}\,dx^j \tag{1.5}$$

and also

$$dx^j = 0\,, \tag{1.6}$$

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*In other words, Zelmanov denotes space-time indices in Greek, and spatial indices Latin. This implies that he considers a four-dimensional space-time of signature (+−−−), where time is real and spatial coordinates are imaginary. Because of the latter fact, the three-dimensional *observable impulse* (the projection of the four-dimensional impulse vector on the observer's spatial section) is positive, for benefit in practical calculations. — Editor's comment. D. R.

so we obtain

$$dx^{i'} = 0\,. \tag{1.7}$$

On the other hand

$$dx^{j} = \frac{\partial x^{j}}{\partial x^{0'}}dx^{0'} + \frac{\partial x^{j}}{\partial x^{i'}}dx^{i'}. \tag{1.8}$$

Because equation (1.6) gives (1.7), we have

$$\frac{\partial x^{j}}{\partial x^{0'}} \equiv 0\,. \tag{1.9}$$

We distinguish the term "reference frame" from the term "coordinate frame". We will say that coordinate frames S and S' are of the same reference frame, if the conditions (1.3) and (1.9) are true for them. Besides this, the systems S and S' are of the same reference frame, because

$$\frac{\partial x^{k''}}{\partial x^{0'}} \equiv 0\,. \tag{1.10}$$

Since in the general case we have

$$\frac{\partial x^{k''}}{\partial x^{0}} = \frac{\partial x^{k''}}{\partial x^{0'}}\frac{\partial x^{0'}}{\partial x^{0}} + \frac{\partial x^{k''}}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial x^{0}}\,, \tag{1.11}$$

(1.3) and (1.10) give

$$\frac{\partial x^{k''}}{\partial x^{0}} \equiv 0\,, \tag{1.12}$$

i. e. the coordinate frames S and S'' are of the same reference frame. In other words, if the coordinate frames S and S' are linked by the transformations (1.4) and the coordinate frames S and S'' are linked by the same transformations (1.4), then the systems S and S'' are also linked by the transformations (1.4).

In this way we can define a reference frame as the sum of those coordinate systems which are linked to one another by the transformations (1.4). From the viewpoint of mechanics, we can specify a physical sense of a reference frame in general, via equation (1.5):

> A reference frame is a complex of those coordinate frames which are at rest with respect to one another.

Let us take any reference frame and operate only with transformations (1.4). We can modify them with the system of transformations:

(1) where the spatial coordinates change but the time coordinate remains unchanged

$$\left.\begin{aligned} &x^{0'} = x^{0'} \\ &x^{i'} = x^{i'}(x^1, x^2, x^3) \end{aligned}\right\}, \tag{1.13}$$

and (2) where the time coordinate changes but the spatial coordinates remain unchanged

$$\left.\begin{aligned} &x^{0'} = x^{0'}(x^0, x^1, x^2, x^3) \\ &x^{i'} = x^{i'} \end{aligned}\right\}. \tag{1.14}$$

Let us suppose that we have a relation which is true in this reference frame. For its form to remain unchanged under any point-transformations of the four coordinates inside the reference frame it is a necessary and sufficient condition that the relation must retain its form, first, under the transformations (1.13) and second, under the transformations (1.14). So the relation must retain its form under the transformations

$$x^{i'} = x^{i'}(x^1, x^2, x^3)\,, \tag{1.15}$$

$$x^{0'} = x^{0'}(x^0, x^1, x^2, x^3)\,. \tag{1.16}$$

§2.2 An observer's reference space. Sub-tensors

We will use the three-dimensional space of a reference frame, the *reference space* in brief, defining any point of this space by a world-line

$$x^i = a^i\,, \qquad i = 1, 2, 3\,, \tag{2.1}$$

in this reference frame (a^i are some numbers). Then the transformations (1.15) can be considered as coordinate transformations in this space, so there is the possibility of using three-dimensional tensor calculus here. We call tensors of this three-dimensional calculus *sub-tensors* to distinguish them from the usual four-dimensional tensors. It is evident that the world-invariants and time components of world-tensors are sub-invariants (the three-dimensional invariants), while space-time components and spatial components of any world-tensor consist of sub-tensors, the ranks of which are the numbers of their respective significant (non-zero) indices. As a matter of fact, the subscripts imply covariance and the superscripts contravariance. In general, we can write the symbolic equality

$$(1+t)^r = t^0 + rt^1 + \frac{r\,(r-1)}{1\times 2}\,t^2 + \frac{r\,(r-1)\,(r-2)}{1\times 2\times 3}\,t^3 + \,\dots\,, \qquad (2.2)$$

which implies the following: any $(1+t)$-dimensional tensor of rank r splits into t-dimensional tensors, namely — one t-dimensional tensor of zero rank (the invariant), r tenors of the 1st rank (the vectors), $\frac{r(r-1)}{1\times 2}$ tensors of the 2nd rank, $\frac{r(r-1)(r-2)}{1\times 2\times 3}$ tensors of the 3rd rank, etc. ($t=3$ in this case). It is to be understood that if the initial $(1+t)$-dimensional tensor is symmetric, then some of the t-dimensional tensors of the initial tensor can be the same. It is easy to see, for instance, that the covariant metric world-tensor $g_{\mu\nu}$ splits into the sub-invariant g_{00}, the covariant sub-vector g_{0i}, and the covariant symmetric sub-tensor g_{ik} of the 2nd rank. The contravariant metric world-tensor $g^{\mu\nu}$ splits into the sub-invariant g^{00}, the contravariant sub-vector g^{0i}, and the contravariant symmetric sub-tensor of the 2nd rank g^{ik}. Naturally, the transformations (1.13) give

$$g'_{00} = g_{00}\,, \qquad g'_{0i} = g_{0j}\,\frac{\partial x^j}{\partial x^{i'}}\,, \qquad g'_{ik} = g_{jl}\,\frac{\partial x^j}{\partial x^{i'}}\,\frac{\partial x^l}{\partial x^{k'}}\,, \qquad (2.3)$$

$$g^{00\prime} = g^{00}\,, \qquad g^{0i\prime} = g^{0j}\,\frac{\partial x^{i'}}{\partial x^j}\,, \qquad g^{ik\prime} = g^{jl}\,\frac{\partial x^{i'}}{\partial x^j}\,\frac{\partial x^{k'}}{\partial x^l}\,. \qquad (2.4)$$

Numerous Christoffel world-symbols of the 1st and the 2nd rank are also sub-tensor quantities. Really, considering the general transformations of the Christoffel symbols*

$$\Gamma'_{\mu\nu,\sigma} = \Gamma_{\alpha\beta,\,\xi}\,\frac{\partial x^\alpha}{\partial x^{\mu'}}\,\frac{\partial x^\beta}{\partial x^{\nu'}}\,\frac{\partial x^\xi}{\partial x^{\sigma'}} + g_{\varepsilon\xi}\,\frac{\partial^2 x^\varepsilon}{\partial x^{\mu'} x^{\nu'}}\,\frac{\partial x^\xi}{\partial x^{\sigma'}}\,, \qquad (2.5)$$

$$\Gamma^{\sigma\,\prime}_{\mu\nu} = \Gamma^{\xi}_{\alpha\beta}\,\frac{\partial x^\alpha}{\partial x^{\mu'}}\,\frac{\partial x^\beta}{\partial x^{\nu'}}\,\frac{\partial x^{\sigma'}}{\partial x^\xi} + \frac{\partial^2 x^\xi}{\partial x^{\mu'} x^{\nu'}}\,\frac{\partial x^{\sigma'}}{\partial x^\xi} \qquad (2.6)$$

and taking (1.13) into account, we obtain

$$\left.\begin{aligned} &\Gamma'_{00,0} = \Gamma_{00,0}\,, && \Gamma'_{00,i} = \Gamma_{00,j}\,\frac{\partial x^j}{\partial x^{i'}}\,, \\ &\Gamma'_{0i,0} = \Gamma_{0j,0}\,\frac{\partial x^j}{\partial x^{i'}}\,, && \Gamma'_{0i,k} = \Gamma_{0j,l}\,\frac{\partial x^j}{\partial x^{i'}}\,\frac{\partial x^l}{\partial x^{k'}} \end{aligned}\right\}, \qquad (2.7)$$

*Compare formula (2.6) with (33) of [62], p. 412. Formula (2.5) can be obtained from (2.6) without problems.

$$\left.\begin{array}{l} \Gamma_{00}^{0}{}' = \Gamma_{00}^{0}\,, \qquad \Gamma_{00}^{i}{}' = \Gamma_{00}^{j}\dfrac{\partial x^{i'}}{\partial x^{j}}\,, \qquad \Gamma_{0i}^{0}{}' = \Gamma_{0j}^{0}\dfrac{\partial x^{j}}{\partial x^{i'}}\,, \\ \Gamma_{0i}^{k}{}' = \Gamma_{0j}^{l}\dfrac{\partial x^{j}}{\partial x^{i'}}\dfrac{\partial x^{k'}}{\partial x^{l}}\,, \qquad \Gamma_{ik}^{0}{}' = \Gamma_{jl}^{0}\dfrac{\partial x^{j}}{\partial x^{i'}}\dfrac{\partial x^{l}}{\partial x^{k'}} \end{array}\right\}. \tag{2.8}$$

In other words, we have: $\Gamma_{00,0}$ and Γ_{00}^{0} are sub-invariants; $\Gamma_{00,i}$, $\Gamma_{0i,0}$, and Γ_{0i}^{0} are covariant sub-vectors; Γ_{00}^{i} is a contravariant sub-vector; $\Gamma_{0i,k}$ and Γ_{0i}^{k} are mixed symmetric covariant sub-tensors of the 2nd rank, Γ_{ik}^{0} is a symmetric covariant sub-tensor of the 2nd rank. At the same time, $\Gamma_{ik,0}$, $\Gamma_{ik,j}$, and Γ_{ik}^{j} are not sub-tensors.

We will introduce new sub-invariants, sub-vectors, and sub-tensors. For instance, the sub-invariant w and the sub-vector v_i, which are given by the equations

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2, \tag{2.9}$$

$$g_{0i} = -\left(1 - \frac{\mathrm{w}}{c^2}\right)\frac{v_i}{c}\,, \tag{2.10}$$

$$1 - \frac{\mathrm{w}}{c^2} > 0\,. \tag{2.11}$$

§ 2.3 Time, co-quantities, and chr.inv.-quantities

We are going to consider the transformations of time (1.16). We will refer to sub-invariants, sub-vectors, and sub-tensors as well as any other quantities, which change their form under the transformations (1.16), as *co-quantities*, that is, *co-invariants*, *co-vectors*, and *co-tensors**. Sub-invariants, sub-vectors, and sub-tensors, invariant with respect to the transformations of time (1.16), will be referred to as *chronometrically invariant quantities* (in brief — chronometric invariants) thus, *chr.inv.-invariants*, *chr.inv.-vectors*, and *chr.inv.-tensors*. It is easy to see that g_{00}, g_{0i}, g_{ik}, g^{00}, and g^{0i} are co-quantities, while g^{ik} is chr.inv.-tensor.

THEOREM† *We assume that $A^{ij\dots k}_{00\dots 0}$ is the component of a world-tensor, all superscripts of which are significant, while all m subscripts are zero. Next, we assume that $B_{00\dots 0}$ is the time component*

*As you can see, the prefix "co" here and below has a different sense to that in Weyl's geometry (for instance, see [7], p. 380).

†We call this theorem *Zelmanov's theorem*. — Editor's comment. D. R.

of a covariant world-tensor of n-th rank. Then, because of (1.16) or (1.14), we have

$$A^{ij\ldots k\,\prime}_{00\ldots0} = A^{ij\ldots k}_{00\ldots0}\left(\frac{\partial x^0}{\partial x^{0'}}\right)^m, \qquad (3.1)$$

$$B'_{00\ldots0} = B_{00\ldots0}\left(\frac{\partial x^0}{\partial x^{0'}}\right)^n, \qquad (3.2)$$

and so the quantity

$$Q^{ij\ldots k} = \frac{A^{ij\ldots k}_{00\ldots0}}{\left(B_{00\ldots0}\right)^{\frac{m}{n}}} \qquad (3.3)$$

is the component of a contravariant chr.inv.-tensor. We will use g_{00} for $B_{00\ldots0}$, i. e.

$$Q^{ij\ldots k} = \frac{A^{ij\ldots k}_{00\ldots0}}{\left(g_{00}\right)^{\frac{m}{2}}}\,. \qquad (3.4)$$

It should be noted, $\frac{\Gamma^i_{00}}{g_{00}}$ is chr.inv.-tensor quantity (namely — the chr.inv.-vector), as it easy to see.

§ 2.4 The potentials

We will refer to the co-invariant w and the co-vector v_i as the *scalar potential* and the *vector potential*, respectively.

Let us show, taking any reference frame, that we can transform the time coordinate in that way, where the quantities $\widetilde{\mathrm{w}}$, $\tilde{v}_1$, $\tilde{v}_2$, $\tilde{v}_3$, for any pre-assigned world-point*

$$x^\sigma = a^\sigma\,, \qquad a^\sigma = const^\sigma\,, \qquad \sigma = 0,1,2,3, \qquad (4.1)$$

can take any preassigned system of their numerical values $(\widetilde{\mathrm{w}})_a$, $(\tilde{v}_1)_a$, $(\tilde{v}_2)_a$, $(\tilde{v}_3)_a$ the condition (2.11) permits. Then, assuming that

$$\tilde{x}^0 = A_\sigma x^\sigma\,, \qquad A_\sigma = const_\sigma\,, \qquad (4.2)$$

we obtain

$$\left(g_{00}\right)_a = \left(\tilde{g}_{00}\right)_a\left(A_0\right)^2, \qquad \left(g_{0i}\right)_a = \Big[\left(\tilde{g}_{00}\right)_a A_i + \left(\tilde{g}_{0i}\right)_a\Big]A_0\,, \qquad (4.3)$$

$$A_0 = \frac{\sqrt{\left(g_{00}\right)_a}}{\sqrt{\left(\tilde{g}_{00}\right)_a}}\,, \quad A_i = \frac{1}{\sqrt{\left(\tilde{g}_{00}\right)_a}}\left[\frac{\sqrt{\left(g_{0i}\right)_a}}{\sqrt{\left(g_{00}\right)_a}} - \frac{\sqrt{\left(\tilde{g}_{0i}\right)_a}}{\sqrt{\left(\tilde{g}_{00}\right)_a}}\right], \qquad (4.4)$$

---

*We denote the transformed quantities by a *tilde*, while the symbol a marks the same quantities at the world-point (4.1).

or, in the alternative form,

$$A_0 = \frac{1}{1-\dfrac{(\tilde{\mathrm{w}})_a}{c^2}}\left[1-\frac{(\mathrm{w})_a}{c^2}\right], \qquad A_i = \frac{(\tilde{v}_i)_a-(v_i)_a}{c\left[1-\dfrac{(\tilde{\mathrm{w}})_a}{c^2}\right]}\,. \tag{4.5}$$

So the univalent numbers A_σ can be found independently of the numerical values of the potentials at the world-point under the previous time coordinate and their possible numerical values under any new time coordinate.

We will refer to this method as the *method of variation of potentials*. As a matter of fact, this method provides a means by which the potentials any necessary numerical values at any world-point we are considering. Later in this study we will use this method of variation of potentials many times. Of particular importance will be the case where we make the potentials zero at a pre-assigned world-point.

§ 2.5 Chr.inv.-differentiation

The standard operators for differentiation with respect to the time coordinate and spatial coordinates, namely

$$\frac{\partial}{\partial x^0}\,, \qquad \frac{\partial}{\partial x^i}\,, \tag{5.1}$$

are, generally speaking, non-invariant with respect of the transformations (1.14). In fact, we have

$$\frac{\partial}{\partial x^{0'}} = \frac{\partial}{\partial x^0}\frac{\partial x^0}{\partial x^{0'}} \neq \frac{\partial}{\partial x^0}\,, \tag{5.2}$$

$$\frac{\partial}{\partial x^{i'}} = \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^0}\frac{\partial x^0}{\partial x^{i'}} \neq \frac{\partial}{\partial x^i}\,. \tag{5.3}$$

So we will refer to the usual differentiation as *co-differentiation*. Besides this, we can introduce a generalization of the usual differentiation, namely — *chronometrically invariant differentiation* with respect to time and spatial coordinates, operators of which are invariant with respect of the transformations (1.14).

We assume that an arbitrary frame of coordinates x^0, x^1, x^2, x^3 exists. Let us introduce a new system of coordinates $\tilde{x}^0$, $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$, which is of the same reference frame. Their differences are only that this new (tilde) coordinate frame has: (1) another time coordinate

$$\left.\begin{aligned} x^0 &= x^0(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \\ x^i &= \tilde{x}^i \end{aligned}\right\} \tag{5.4}$$

and (2) zero potentials at the world-point (4.1) so that

$$(\tilde{g}_{00})_a = 1\,, \qquad (\tilde{g}_{0i})_a = 0\,. \tag{5.5}$$

Hence we have

$$(g_{00})_a \left(\frac{\partial x^0}{\partial \tilde{x}^0}\right)^2_a = 1\,, \qquad (g_{00})_a \left(\frac{\partial x^0}{\partial \tilde{x}^i}\right)_a + (g_{0i})_a = 0\,. \tag{5.6}$$

Considering the world-point (4.1), let us take the derivatives with respect to time and spatial coordinates of the new system at the given point. Next, let us transform the new coordinates into the old coordinate frame. As a result, we obtain

$$\left(\frac{\partial}{\partial \tilde{x}^0}\right)_a = \left(\frac{\partial}{\partial x^0}\right)_a \left(\frac{\partial x^0}{\partial \tilde{x}^0}\right)_a, \tag{5.7}$$

$$\left(\frac{\partial}{\partial \tilde{x}^i}\right)_a = \left(\frac{\partial}{\partial x^i}\right)_a + \left(\frac{\partial}{\partial x^0}\right)_a \left(\frac{\partial x^0}{\partial \tilde{x}^i}\right)_a. \tag{5.8}$$

At the same time, formula (5.6) gives

$$\left(\frac{\partial x^0}{\partial \tilde{x}^0}\right)_a = \frac{1}{\sqrt{(g_{00})_a}}\,, \qquad \left(\frac{\partial x^0}{\partial \tilde{x}^i}\right)_a = -\frac{(g_{0i})_a}{(g_{00})_a}\,, \tag{5.9}$$

therefore

$$\left(\frac{\partial}{\partial \tilde{x}^0}\right)_a = \frac{1}{\sqrt{(g_{00})_a}} \left(\frac{\partial}{\partial x^0}\right)_a, \tag{5.10}$$

$$\left(\frac{\partial}{\partial \tilde{x}^i}\right)_a = \left(\frac{\partial}{\partial x^i}\right)_a - \frac{(g_{0i})_a}{(g_{00})_a} \left(\frac{\partial}{\partial x^0}\right)_a. \tag{5.11}$$

Because (5.10) and (5.11) are true at any world-point in any system of coordinates x^0, x^1, x^2, x^3, the differential operators

$$\frac{^*\partial}{\partial x^0} \equiv \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial x^0}\,, \tag{5.12}$$

$$\frac{^*\partial}{\partial x^i} \equiv \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial x^0} \tag{5.13}$$

must be invariant with respect to the transformations (1.14) as well. We can see this fact directly. Naturally, we have

$$\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial x^0}=\frac{1}{\sqrt{g'_{00}\left(\dfrac{\partial x^{0'}}{\partial x^0}\right)^2}}\frac{\partial}{\partial x^{0'}}\frac{\partial x^{0'}}{\partial x^0}=\frac{1}{\sqrt{g'_{00}}}\frac{\partial}{\partial x^{0'}}\,, \tag{5.14}$$

$$\begin{gathered}\frac{\partial}{\partial x^i}-\frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0}=\frac{\partial}{\partial x^i}-\frac{g_{0i}}{\sqrt{g_{00}}}\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial x^0}=\frac{\partial}{\partial x^{i'}}+\frac{\partial}{\partial x^{0'}}\frac{\partial x^{0'}}{\partial x^i}-\\ -\frac{\partial x^{0'}}{\partial x^0}\frac{\left(g'_{00}\dfrac{\partial x^{0'}}{\partial x^0}+g'_{0i}\right)}{\sqrt{g'_{00}\left(\dfrac{\partial x^{0'}}{\partial x^0}\right)^2}}\frac{1}{\sqrt{g'_{00}}}\frac{\partial}{\partial x^{0'}}=\frac{\partial}{\partial x^{i'}}-\frac{g'_{0i}}{g'_{00}}\frac{\partial}{\partial x^{0'}}\,.\end{gathered} \tag{5.15}$$

For this reason we accept the asterisk-marked operators (5.12) and (5.13) the *chr.inv.-differential operators* with respect to time and spatial coordinates, respectively. The method we employed for obtaining the operators, namely — formulae (5.10) and (5.11), shows the geometrical sense which chr.inv.-differentiation has in the four-dimensional world. Chr.inv.-differentiation with respect to the time coordinate is the same as differentiation with respect to the local time of an observer at the point of his observations. Chr.inv.-differentiation with respect to spatial coordinates is spatial differentiation along a curve, which is orthogonal to the time line of the observer's reference frame. In fact, we have

$$\left(\frac{d}{ds}\right)_{x^i=const}=\frac{\partial}{\partial x^\alpha}\left(\frac{dx^\alpha}{ds}\right)_{x^i=const}=\frac{\partial}{\partial x^0}\left(\frac{dx^0}{ds}\right)_{x^i=const}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial x^0}\,, \tag{5.16}$$

$$\left(\frac{^*\partial}{\partial x^i}\right)_{g_{0i}=0}=\frac{\partial}{\partial x^i}\,. \tag{5.17}$$

Let us give other formulae expressing the operators of chr.inv.-differentiation. It is evident that

$$\frac{^*\partial}{\partial x^0}=\frac{c^2}{c^2-\mathrm{w}}\frac{\partial}{\partial x^0}\,, \tag{5.18}$$

$$\frac{^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}+\frac{c\,v_i}{c^2-\mathrm{w}}\frac{\partial}{\partial x^0}\,. \tag{5.19}$$

If we introduce

$$v_0 = \frac{\mathrm{w}}{c}\,, \tag{5.20}$$

then we have

$$\frac{^*\partial}{\partial x^\sigma} = \frac{\partial}{\partial x^\sigma} + \frac{v_\sigma}{c - v_0}\frac{\partial}{\partial x^0}\,. \tag{5.21}$$

If we introduce

$$t = \frac{x^0}{c}\,, \tag{5.22}$$

then we have

$$\frac{^*\partial}{\partial t} = \frac{c^2}{c^2 - \mathrm{w}}\frac{\partial}{\partial t}\,, \tag{5.23}$$

$$\frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{v_i}{c^2 - \mathrm{w}}\frac{\partial}{\partial t}\,. \tag{5.24}$$

In addition to the above, it should be noted that if we assume that Q is a chr.inv.-quantity, i. e.

$$Q' = Q\,, \tag{5.25}$$

the quantity satisfies the transformations (1.14), i .e.

$$\frac{^*\partial Q'}{\partial x^{\sigma\prime}} = \frac{^*\partial Q}{\partial x^\sigma}\,, \tag{5.26}$$

so the chr.inv.-derivative of any chr.inv.-quantity is also a chr.inv.-quantity.

It easy to see that the chr.inv.-derivative of a chr.inv.-tensor with respect to a time coordinate is a chr.inv.-tensor of the same rank. The chr.inv.-derivative of a chr.inv.-invariant with respect to spatial coordinates is a covariant chr.inv.-vector (chr.inv.-differentiation of non-zero rank tensors with respect to spatial coordinates will be considered below).

§ 2.6 Changing the order of chr.inv.-differentiation

Let us suppose that quantities we are differentiating satisfy the conditions

$$\frac{\partial^2}{\partial x^i \partial t} = \frac{\partial^2}{\partial t\, \partial x^i}\,, \tag{6.1}$$

$$\frac{\partial^2}{\partial x^i \partial x^k} = \frac{\partial^2}{\partial x^k \partial x^i}\,. \tag{6.2}$$

Then we have, respectively, for the first of the conditions

$$\frac{{}^*\partial^2}{\partial x^i \partial t} - \frac{{}^*\partial^2}{\partial t\,\partial x^i} = \frac{{}^*\partial}{\partial x^i}\left(\frac{{}^*\partial}{\partial t}\right) - \frac{{}^*\partial}{\partial t}\left(\frac{{}^*\partial}{\partial x^i}\right) =$$
$$= \frac{\partial}{\partial x^i}\left(\frac{c^2}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\right) + \frac{v_i}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\left(\frac{c^2}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\right) -$$
$$- \frac{c^2}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x^i}\right) - \frac{c^2}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\left(\frac{v_i}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\right) =$$
$$= \frac{c^2}{(c^2-\mathrm{w})^2}\frac{\partial \mathrm{w}}{\partial x^i}\frac{\partial}{\partial t} + \frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2}{\partial x^i \partial t} + \frac{c^2 v_i}{(c^2-\mathrm{w})^3}\frac{\partial \mathrm{w}}{\partial t}\frac{\partial}{\partial t} + \tag{6.3}$$
$$+ \frac{c^2 v_i}{(c^2-\mathrm{w})^2}\frac{\partial^2}{\partial t^2} - \frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2}{\partial t\,\partial x^i} - \frac{c^2}{(c^2-\mathrm{w})^2}\frac{\partial v_i}{\partial t}\frac{\partial}{\partial t} -$$
$$- \frac{c^2 v_i}{(c^2-\mathrm{w})^3}\frac{\partial \mathrm{w}}{\partial t}\frac{\partial}{\partial t} - \frac{c^2 v_i}{(c^2-\mathrm{w})^2}\frac{\partial^2}{\partial t^2} =$$
$$= \frac{c^2}{(c^2-\mathrm{w})^2}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right)\frac{\partial}{\partial t} = \frac{1}{c^2-\mathrm{w}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right)\frac{{}^*\partial}{\partial t},$$

and for the second

$$\frac{{}^*\partial^2}{\partial x^i \partial x^k} - \frac{{}^*\partial^2}{\partial x^k \partial x^i} = \frac{{}^*\partial}{\partial x^i}\left(\frac{{}^*\partial}{\partial x^k}\right) - \frac{{}^*\partial}{\partial x^k}\left(\frac{{}^*\partial}{\partial x^i}\right) =$$
$$= \frac{\partial^2}{\partial x^i \partial x^k} + \frac{\partial}{\partial x^i}\left(\frac{v_k}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\right) + \frac{v_i}{c^2-\mathrm{w}}\frac{\partial^2}{\partial t\,\partial x^k} +$$
$$+ \frac{v_i}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\left(\frac{v_k}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\right) - \frac{\partial^2}{\partial x^k \partial x^i} - \frac{\partial}{\partial x^k}\left(\frac{v_i}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\right) -$$
$$- \frac{v_k}{c^2-\mathrm{w}}\frac{\partial^2}{\partial t\,\partial x^i} - \frac{v_k}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\left(\frac{v_i}{c^2-\mathrm{w}}\frac{\partial}{\partial t}\right) = \frac{1}{c^2-\mathrm{w}}\frac{\partial v_k}{\partial x^i}\frac{\partial}{\partial t} +$$
$$+ \frac{v_k}{(c^2-\mathrm{w})^2}\frac{\partial \mathrm{w}}{\partial x^i}\frac{\partial}{\partial t} + \frac{v_k}{c^2-\mathrm{w}}\frac{\partial^2}{\partial x^i \partial t} + \frac{v_i}{c^2-\mathrm{w}}\frac{\partial^2}{\partial t\,\partial x^k} +$$
$$+ \frac{v_i}{(c^2-\mathrm{w})^2}\frac{\partial v_k}{\partial t}\frac{\partial}{\partial t} + \frac{v_i v_k}{(c^2-\mathrm{w})^3}\frac{\partial \mathrm{w}}{\partial t}\frac{\partial}{\partial t} + \frac{v_i v_k}{(c^2-\mathrm{w})^2}\frac{\partial^2}{\partial t^2} -$$
$$- \frac{1}{c^2-\mathrm{w}}\frac{\partial v_i}{\partial x^k}\frac{\partial}{\partial t} - \frac{v_i}{(c^2-\mathrm{w})^2}\frac{\partial \mathrm{w}}{\partial x^k}\frac{\partial}{\partial t} - \frac{v_i}{c^2-\mathrm{w}}\frac{\partial^2}{\partial x^k \partial t} -$$
$$- \frac{v_k}{c^2-\mathrm{w}}\frac{\partial^2}{\partial t\,\partial x^i} - \frac{v_k}{(c^2-\mathrm{w})^2}\frac{\partial v_i}{\partial t}\frac{\partial}{\partial t} - \frac{v_k v_i}{(c^2-\mathrm{w})^3}\frac{\partial \mathrm{w}}{\partial t}\frac{\partial}{\partial t} -$$

$$-\frac{v_k v_i}{(c^2-\mathrm{w})^2}\frac{\partial^2}{\partial t^2}=\left[\frac{1}{c^2-\mathrm{w}}\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)+\frac{v_k}{(c^2-\mathrm{w})^2}\times\right.$$
$$\left.\times\left(\frac{\partial \mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}\right)-\frac{v_i}{(c^2-\mathrm{w})^2}\left(\frac{\partial \mathrm{w}}{\partial x^k}-\frac{\partial v_k}{\partial t}\right)\right]\frac{\partial}{\partial t}=$$
$$=\left[\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)+\frac{v_k}{c^2-\mathrm{w}}\left(\frac{\partial \mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}\right)-\frac{v_i}{c^2-\mathrm{w}}\times\right.$$
$$\left.\times\left(\frac{\partial \mathrm{w}}{\partial x^k}-\frac{\partial v_k}{\partial t}\right)\right]\frac{1}{c^2}\frac{{}^*\partial}{\partial t}\,. \tag{6.4}$$

Introducing the notations

$$F_i=\frac{c^2}{c^2-\mathrm{w}}\left(\frac{\partial \mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}\right), \tag{6.5}$$

$$A_{ik}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)+\frac{1}{2c^2}\left(F_i\,v_k-F_k\,v_i\right), \tag{6.6}$$

we can write

$$\frac{{}^*\partial^2}{\partial x^i\partial t}-\frac{{}^*\partial^2}{\partial t\,\partial x^i}=\frac{1}{c^2}F_i\frac{{}^*\partial}{\partial t}\,, \tag{6.7}$$

$$\frac{{}^*\partial^2}{\partial x^i\partial x^k}-\frac{{}^*\partial^2}{\partial x^k\partial x^i}=\frac{2}{c^2}A_{ik}\frac{{}^*\partial}{\partial t}\,. \tag{6.8}$$

Inspection of (6.5) and (6.6), reveals that the quantity F_i is a covariant vector and the quantity A_{ik} is an antisymmetric covariant tensor of the 2nd rank. It is not difficult to verify that the quantities are a chr.inv.-vector and a chr.inv.-tensor, respectively. Let us assume that Q is an arbitrary chr.inv.-invariant. So formulae (6.7) and (6.8) are true for this quantity, i. e.

$$\frac{{}^*\partial^2 Q}{\partial x^i\partial t}-\frac{{}^*\partial^2 Q}{\partial t\,\partial x^i}=\frac{1}{c^2}F_i\frac{{}^*\partial Q}{\partial t}\,, \tag{6.9}$$

$$\frac{{}^*\partial^2 Q}{\partial x^i\partial x^k}-\frac{{}^*\partial^2 Q}{\partial x^k\partial x^i}=\frac{2}{c^2}A_{ik}\frac{{}^*\partial Q}{\partial t}\,. \tag{6.10}$$

The left-hand sides of the equalities consist of chr.inv.-quantities, while the right-hand sides are the products of the chr.inv.-quantity $\frac{{}^*\partial Q}{\partial t}$ and the quantities F_i and A_{ik}, respectively. Hence, F_i and A_{ik} are chr.inv.-quantities as well. Because they contain the potentials w and v_i, we will refer to them as the "power quantities" — the *power vector* F_i and the *power tensor* A_{ik}.

§ 2.7 The power quantities

Let us suppose that

$$\mathrm{w}' = 0 \tag{7.1}$$

is true everywhere inside a four-dimensional volume in a coordinate system S'. Then, in another coordinate frame S of the same reference frame, we have

$$g_{00}\left(\frac{\partial x^{0'}}{\partial x^0}\right)^2 = 1\,. \tag{7.2}$$

The inverse is also true: formula (7.1) is a consequence of (7.2). Thus, we can always set the scalar potential to zero throughout the given four-dimensional volume, if we introduce a new time coordinate $x^{0'}$ in accordance with the condition (7.2).

Setting the power vector to zero in the given four-dimensional volume in the given reference frame is the necessary and sufficient condition for making the scalar potential and the derivative from the vector potential with respect to time, zero (throughout the volume).

Really, if we have

$$\mathrm{w} \equiv 0\,, \qquad \frac{\partial v_i}{\partial t} \equiv 0 \tag{7.3}$$

under a choice of time coordinate, then

$$F_i \equiv 0\,. \tag{7.4}$$

Conversely, let us assume that the condition (7.4) holds. Then, introducing time coordinate x^0 in accordance with the first of the equalities (7.3), we obtain the second of them as a consequence of the condition (7.4).

Setting the power vector to zero in the given four-dimensional volume in the reference frame is the necessary and sufficient condition for making the vector potential zero (throughout the volume). Really, we assume that a coordinate frame

$$\left.\begin{aligned} \tilde{x}^{i'} &= \tilde{x}^{0'}(x^0, x^1, x^2, x^3) \\ \tilde{x}^{i'} &= x^{i'} \end{aligned}\right\}, \tag{7.5}$$

exists, where we have

$$\tilde{v}_i \equiv 0\,. \tag{7.6}$$

Then, everywhere inside the volume, we have

$$g_{00}=\tilde{g}_{00}\left(\frac{\partial\tilde{x}^0}{\partial x^0}\right)^2, \qquad g_{0i}=\tilde{g}_{00}\frac{\partial\tilde{x}^0}{\partial x^0}\frac{\partial\tilde{x}^0}{\partial x^i}, \tag{7.7}$$

thus

$$\frac{\partial\tilde{x}^0}{\partial x^0}=\frac{c^2-\mathrm{w}}{c^2-\widetilde{\mathrm{w}}}, \qquad \frac{\partial\tilde{x}^0}{\partial x^i}=-\frac{c\,v_i}{c^2-\widetilde{\mathrm{w}}}. \tag{7.8}$$

On the other hand

$$d\tilde{x}^0=\frac{\partial\tilde{x}^0}{\partial x^0}dx^0+\frac{\partial\tilde{x}^0}{\partial x^i}dx^i. \tag{7.9}$$

Therefore we can write

$$d\tilde{x}^0=\frac{(c^2-\mathrm{w})dx^0-cv_i dx^i}{c^2-\widetilde{\mathrm{w}}} \tag{7.10}$$

or, introducing

$$t=\frac{x^0}{c}, \qquad \tilde{t}=\frac{\tilde{x}^0}{c}, \tag{7.11}$$

in the alternative form

$$d\tilde{t}=\frac{(c^2-\mathrm{w})dt-cv_i dx^i}{c^2-\widetilde{\mathrm{w}}}. \tag{7.12}$$

In order that $d\tilde{t}$ exists, it is necessary and sufficient to realize the exact conditions of the total integrability of the Pfaffian equation

$$-(c^2-\mathrm{w})dt+v_i dx^i=0. \tag{7.13}$$

In general, Pfaffian equations

$$N du+P dx+Q dy+R dz=0 \tag{7.14}$$

can be totally integrable under the necessary and sufficient conditions (any three of the four equations below can be accepted as the conditions*),

$$\left.\begin{aligned} N\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)+P\left(\frac{\partial N}{\partial y}-\frac{\partial Q}{\partial u}\right)+Q\left(\frac{\partial P}{\partial u}-\frac{\partial N}{\partial x}\right)\equiv 0\\ N\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial N}{\partial z}-\frac{\partial R}{\partial u}\right)+R\left(\frac{\partial Q}{\partial u}-\frac{\partial N}{\partial y}\right)\equiv 0\\ N\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial N}{\partial x}-\frac{\partial P}{\partial u}\right)+P\left(\frac{\partial R}{\partial u}-\frac{\partial N}{\partial z}\right)\equiv 0 \end{aligned}\right\}, \tag{7.15}$$

*The fourth equation is a consequence of the other three.

$$P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\equiv 0\,. \tag{7.16}$$

Assuming

$$N=-(c^2-\mathrm{w})\,,\quad P=v_1\,,\quad Q=v_2\,,\quad R=v_3\,, \tag{7.17}$$

$$u=t\,,\qquad x=x^1,\qquad y=x^2,\qquad z=x^3 \tag{7.18}$$

in the equations, after dividing them by $2(c^2-\mathrm{w})$, we obtain

$$A_{ik}\equiv 0\,. \tag{7.19}$$

Note that the condition (7.16) take the form

$$A_{12}v_3+A_{23}v_1+A_{31}v_2\equiv 0\,. \tag{7.20}$$

It is evident that the simultaneous conditions $F_i\equiv 0$ (7.4) and $A_{ik}\equiv 0$ (7.19) are the necessary and sufficient conditions for making the scalar potential and the vector potential both zero throughout the four-dimensional volume. The condition (7.19) provides a means by which the vector potential v_i becomes zero. In addition, because of the condition (7.4), the scalar potential w becomes a function of only the time coordinate t. Then we introduce a new time coordinate $\tilde{t}$, which is

$$d\tilde{t}=\left(1-\frac{\mathrm{w}}{c^2}\right)dt\,. \tag{7.21}$$

§2.8 The space metric

In any reference frame we have

$$ds^2=g_{00}(dx^0)^2+2g_{0i}\,dx^0dx^i+g_{ik}\,dx^idx^k \tag{8.1}$$

under an arbitrary coordinate of time. Going to a new time coordinate $\tilde{x}^0$, we set the potentials to zero at the world-point we are considering. As a result we obtain

$$ds^2=(d\tilde{x}^0)^2-d\sigma^2, \tag{8.2}$$

where

$$d\sigma^2=-\tilde{g}_{ik}\,dx^idx^k \tag{8.3}$$

is, evidently, the square of a spatial linear element. Let us transform (8.3) to an arbitrary time coordinate x^0. At the world-point we are considering, we have

$$\left.\begin{aligned} & g_{00}\left(\frac{\partial x^0}{\partial \tilde{x}^0}\right)^2 = 1 \\ & g_{00}\frac{\partial x^0}{\partial \tilde{x}^0}\frac{\partial x^0}{\partial \tilde{x}^i} + g_{0i}\frac{\partial x^0}{\partial \tilde{x}^0} = 0 \\ & g_{00}\frac{\partial x^0}{\partial \tilde{x}^i}\frac{\partial x^0}{\partial \tilde{x}^k} + g_{0i}\frac{\partial x^0}{\partial \tilde{x}^k} + g_{0k}\frac{\partial x^0}{\partial \tilde{x}^i} + g_{ik} = \tilde{g}_{ik} \end{aligned}\right\} . \tag{8.4}$$

Eliminating $\frac{\partial x^0}{\partial \tilde{x}^0}$, $\frac{\partial x^0}{\partial \tilde{x}^i}$, $\frac{\partial x^0}{\partial \tilde{x}^k}$ from (8.4), we obtain

$$\tilde{g}_{ik} = g_{ik} - \frac{g_{0i}\, g_{0k}}{g_{00}} . \tag{8.5}$$

Hence, generally speaking*,

$$d\sigma^2 = \left(-g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}}\right) dx^i dx^k . \tag{8.6}$$

Thus we can introduce the covariant metric sub-tensor h_{ik}

$$d\sigma^2 = h_{ik}\, dx^i dx^k , \tag{8.7}$$

$$h_{ik} = -g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}} , \tag{8.8}$$

where we have

$$g_{ik} = -h_{ik} + \frac{v_i v_k}{c^2} . \tag{8.9}$$

Formula (8.5) shows that h_{ik} must be invariant with respect to the transformations (1.14). We can also see this fact directly, thus

$$\begin{aligned} h_{ik} &= -g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}} = \\ &= -\left(g'_{00}\frac{\partial x^{0'}}{\partial x^0}\frac{\partial x^{0'}}{\partial x^0} + g'_{0i}\frac{\partial x^{0'}}{\partial x^k} + g'_{0k}\frac{\partial x^{0'}}{\partial x^i} + g'_{ik}\right) + \\ &+ \frac{\left(g'_{00}\frac{\partial x^{0'}}{\partial x^0}\frac{\partial x^{0'}}{\partial x^0} + g'_{0i}\frac{\partial x^{0'}}{\partial x^0}\right)\left(g'_{00}\frac{\partial x^{0'}}{\partial x^0}\frac{\partial x^{0'}}{\partial x^0} + g'_{0k}\frac{\partial x^{0'}}{\partial x^0}\right)}{g'_{00}\left(\frac{\partial x^{0'}}{\partial x^0}\right)^2} = \end{aligned}$$

*This equation can also be obtained from other considerations — for instance, see [64], p. 200–201.

$$
\begin{aligned}
&= -g'_{00}\frac{\partial x^{0'}}{\partial x^i}\frac{\partial x^{0'}}{\partial x^k} - g'_{0i}\frac{\partial x^{0'}}{\partial x^k} - g'_{0k}\frac{\partial x^{0'}}{\partial x^i} - g'_{ik} + \\
&+ \frac{1}{g'_{00}\left(\frac{\partial x^{0'}}{\partial x^0}\right)^2}\left[\left(g'_{00}\frac{\partial x^{0'}}{\partial x^0}\right)^2\frac{\partial x^{0'}}{\partial x^i}\frac{\partial x^{0'}}{\partial x^k} + \right. \\
&+ g'_{00}g'_{0i}\left(\frac{\partial x^{0'}}{\partial x^0}\right)^2\frac{\partial x^{0'}}{\partial x^k} + g'_{00}g'_{0k}\left(\frac{\partial x^{0'}}{\partial x^0}\right)^2\frac{\partial x^{0'}}{\partial x^i} + \\
&\left. + g'_{0i}g'_{0k}\left(\frac{\partial x^{0'}}{\partial x^0}\right)^2\frac{\partial x^{0'}}{\partial x^i}\frac{\partial x^{0'}}{\partial x^k}\right] = -g'_{ik} + \frac{g'_{0i}g'_{0k}}{g'_{00}} = h'_{ik}\,.
\end{aligned}
\tag{8.10}
$$

Hence, the covariant metric sub-tensor h_{ik} is the *chr.inv.-metric tensor*. It is evident that the contravariant metric sub-tensor h^{ik}, the components of which are given by adjuncts of the determinant

$$
h = \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix}, \tag{8.11}
$$

divided by h, is also a chr.inv.-tensor. Because the determinants (8.11) and

$$
\tilde{g} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} \\ 0 & \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} \\ 0 & \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} \end{vmatrix} \tag{8.12}
$$

are different only in their signs, then their adjuncts, derived from the matrix elements with the same indices $i, k = 1, 2, 3$, are equal. So we have

$$
h^{ik} = -\tilde{g}^{ik}. \tag{8.13}
$$

Formula (8.13) contains chr.inv.-tensors in both parts. Hence, in general, we have

$$
h^{ik} = -g^{ik}. \tag{8.14}
$$

We can also introduce the mixed metric sub-tensor h_i^k, which is also a chr.inv.-tensor,

$$
h_i^k = +g_i^k\,. \tag{8.15}
$$

Let us find relations of h to world-quantities. It is known that

$$\begin{vmatrix} g^{00} & g^{01} & g^{02} & g^{03} \\ g^{10} & g^{11} & g^{12} & g^{13} \\ g^{20} & g^{21} & g^{22} & g^{23} \\ g^{30} & g^{31} & g^{32} & g^{33} \end{vmatrix} = \frac{1}{g}\,, \tag{8.16}$$

$$\begin{vmatrix} h^{11} & h^{12} & h^{13} \\ h^{21} & h^{22} & h^{23} \\ h^{31} & h^{32} & h^{33} \end{vmatrix} = \frac{1}{h}\,, \tag{8.17}$$

and also that the quantities $g_{\mu\nu}$ and h_{ik} equal to the adjuncts of the determinants of $g^{\mu\nu}$ (3.16) and h^{ik} (3.17), multiplied by g or h, respectively. Then, because of (8.14), we have

$$g_{00} = -\frac{g}{h}\,, \tag{8.18}$$

or, in the alternative form,

$$\sqrt{-g} = \left(1 - \frac{\text{w}}{c^2}\right)\sqrt{h}\,, \tag{8.19}$$

where h, as is evident, is not sub-invariant, but a chr.inv.-quantity.

Using the chr.inv.-metric tensors, we can contract, substitute, raise and lower significant indices invariant with respect to the transformations (1.14). In this way we can transform chr.inv.-quantities into chr.inv.-quantities, and co-quantities into co-quantities*. Thus, we can build the contravariant vector-potential (the co-vector)

$$v^i = h^{ij} v_j \tag{8.20}$$

and the square of the length of the vector-potential itself (the co-invariant)

$$v_i v^i = h_{ik}\, v^i v^k = h^{ik} v_i v_k\,. \tag{8.21}$$

Because of

$$g_{00}\, g^{0i} + g_{0j}\, g^{ji} = g_0^i = 0\,, \tag{8.22}$$

$$g_{00}\, g^{00} + g_{0j}\, g^{j0} = g_0^0 = 1\,, \tag{8.23}$$

we have

$$g^{0i} = -\frac{1}{1 - \frac{\text{w}}{c^2}}\,\frac{v^i}{c}\,, \tag{8.24}$$

*An exception is the case where a co-quantity vanishes after its contraction.

$$g^{00} = \frac{1}{\left(1 - \frac{\mathrm{w}}{c^2}\right)^2} \left(1 - \frac{v_j v^j}{c^2}\right). \tag{8.25}$$

It is also possible to introduce operations of general covariant differentiation. The operations are non-invariant with respect to the transformations of the time coordinate. For this reason we will introduce chr.inv.-differentiation, which, being the generalization of general covariant differentiation, is invariant with respect to the aforementioned transformations of time. In this process we will need to differentiate h_{ik}, h^{ik}, and h with respect to the time coordinate. Therefore, we will first clarify the kinematic sense of those quantities, which are derived from this differentiation. The first such quantity is chr.inv.-velocity, which will be introduced in the next section.

§ 2.9 The chr.inv.-vector of velocity

We assume that a point-mass moves with respect to a given reference frame. The vector of its velocity with respect to this reference frame is

$$u^i = \frac{dx^i}{dt} = c\,\frac{dx^i}{dx^0} \tag{9.1}$$

which, as it is easy to see, is a co-vector. Let us take a world-point on the world-line of this particle. Applying the necessary choice of time coordinate $\tilde{x}^0$, we set the potentials to zero at the point. Next, we transform the velocity co-vector

$$\tilde{u}^i = c\,\frac{dx^i}{d\tilde{x}^0}\,, \tag{9.2}$$

calculated in the time coordinate $\tilde{x}^0$, into an arbitrary time coordinate x^0. Because

$$d\tilde{x}^0 = \frac{\partial \tilde{x}^0}{\partial x^0} dx^0 + \frac{\partial \tilde{x}^0}{\partial x^j} dx^j, \tag{9.3}$$

$$g_{00} = \left(\frac{\partial \tilde{x}^0}{\partial x^0}\right)^2, \qquad g_{0i} = \frac{\partial \tilde{x}^0}{\partial x^0}\frac{\partial \tilde{x}^0}{\partial x^i}\,, \tag{9.4}$$

we have

$$d\tilde{x}^0 = \sqrt{g_{00}}\,dx^0 + \frac{g_{0j}}{\sqrt{g_{00}}} dx^j, \tag{9.5}$$

$$\frac{dx^i}{d\tilde{x}^0} = \frac{1}{\sqrt{g_{00}} + \frac{g_{0j}}{\sqrt{g_{00}}}\frac{dx^j}{dx^0}}\,\frac{dx^i}{dx^0}\,, \tag{9.6}$$

$$\tilde{u}^i = \frac{c^2 u^i}{c^2 - \mathrm{w} - v_j u^j}\,. \tag{9.7}$$

Equation (9.6) has a chr.inv.-quantity on its left side, so this quantity, and also equation (9.7), must be invariant with respect to the transformations (1.14). Really, because

$$\left.\begin{aligned} \frac{\partial x^0}{\partial x^0} &= \frac{\partial x^0}{\partial x^{\alpha'}}\frac{\partial x^{\alpha'}}{\partial x^0} = \frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^{0'}}{\partial x^0} = 1 \\ \frac{\partial x^0}{\partial x^j} &= \frac{\partial x^0}{\partial x^{\alpha'}}\frac{\partial x^{\alpha'}}{\partial x^j} = \frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^{0'}}{\partial x^j} + \frac{\partial x^0}{\partial x^{j'}} = 0 \end{aligned}\right\}, \tag{9.8}$$

$$g'_{00} = g_{00}\left(\frac{\partial x^0}{\partial x^{0'}}\right)^2, \qquad g'_{0j} = g_{00}\frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^0}{\partial x^{j'}} + g_{0j}\frac{\partial x^0}{\partial x^{0'}}\,, \tag{9.9}$$

we obtain

$$\frac{1}{\sqrt{g'_{00}} + \frac{g'_{0j}}{\sqrt{g'_{00}}}\frac{dx^{j'}}{dx^0}}\frac{dx^{i'}}{dx^{0'}} = \frac{\sqrt{g'_{00}}\,dx^{i'}}{g'_{00}dx^{0'} + g'_{0j}dx^{j'}} =$$

$$= \frac{\sqrt{g_{00}\left(\frac{\partial x^0}{\partial x^{0'}}\right)^2}\,dx^i}{g_{00}\left(\frac{\partial x^0}{\partial x^{0'}}\right)^2\left(\frac{\partial x^{0'}}{\partial x^0}dx^0 + \frac{\partial x^{0'}}{\partial x^j}dx^j\right) + \left(g_{00}\frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^0}{\partial x^{j'}} + g_{0j}\frac{\partial x^0}{\partial x^{0'}}\right)dx^j} =$$

$$= \frac{\sqrt{g_{00}}\,dx^i}{g_{00}\frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^{0'}}{\partial x^0}dx^0 + g_{00}\left(\frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^{0'}}{\partial x^j} + \frac{\partial x^0}{\partial x^{j'}}\right)dx^j + g_{0j}dx^j} =$$

$$= \frac{\sqrt{g_{00}}\,dx^i}{g_{00}dx^0 + g_{0j}dx^j} = \frac{1}{\sqrt{g_{00}} + \frac{g_{0j}}{\sqrt{g_{00}}}\frac{dx^j}{dx^0}}\frac{dx^i}{dx^0}\,. \tag{9.10}$$

Hence, we can introduce the *chr.inv.-velocity vector*, in brief — the *chr.inv.-velocity*

$$^*u^i \equiv \frac{c^2 u^i}{c^2 - \mathrm{w} - v_j u^j}\,. \tag{9.11}$$

The chr.inv.-velocity can also be introduced from other considerations. Let us introduce the covariant differential of time*

$$dx_0 = g_{0\alpha}dx^\alpha = g_{00}dx^0 + g_{0j}dx^j\,. \tag{9.12}$$

*This quantity, generally speaking, is not a total differential.

Because $\dfrac{dx_0}{\sqrt{g_{00}}}$ is a chr.inv.-invariant (see §2.3), the quantity

$$\sqrt{g_{00}}\,\frac{dx^i}{dx_0} = \frac{\sqrt{g_{00}}\,dx^i}{g_{00}dx^0 + g_{0j}dx^j} \tag{9.13}$$

is a chr.inv.-vector: the chr.inv.-velocity we have introduced, divided by c.

§2.10 The chr.inv.-tensor of the rate of space deformations

Considering deformations of a continuous medium in any given reference frame, we introduce the covariant three-dimensional tensor of the rate of its deformations Δ_{ik} in the regular way, i. e.

$$2\Delta_{ik} = \nabla_i\, u_k + \nabla_k\, u_i \tag{10.1}$$

or, in the component form,

$$2\Delta_{ik} = h_{kl}\frac{\partial u^l}{\partial x^i} + h_{il}\frac{\partial u^l}{\partial x^k} + (\Delta_{il,k} + \Delta_{kl,i})\, u^l, \tag{10.2}$$

where $\Delta_{pl,q}$ are three-dimensional Christoffel symbols (the sub-symbols) of the 1st kind

$$\Delta_{pl,q} = \frac{1}{2}\left(\frac{\partial h_{pq}}{\partial x^l} + \frac{\partial h_{lq}}{\partial x^p} - \frac{\partial h_{pl}}{\partial x^q}\right). \tag{10.3}$$

This sub-tensor of the deformation rates, as is easy to see, is a co-tensor. However, we can introduce the *chr.inv.-tensor of the deformation rates* ${}^*\Delta_{ik}$, replacing the co-velocity with the chr.inv.-velocity, and also co-differentiation with chr.inv.-differentiation, respectively. So we have

$$2\,{}^*\Delta_{ik} = h_{kl}\frac{{}^*\partial\, {}^*u^l}{\partial x^i} + h_{il}\frac{{}^*\partial\, {}^*u^l}{\partial x^k} + ({}^*\Delta_{il,k} + {}^*\Delta_{kl,i})\,{}^*u^l, \tag{10.4}$$

where we denote*

$${}^*\Delta_{pl,q} = \frac{1}{2}\left(\frac{{}^*\partial h_{pq}}{\partial x^l} + \frac{{}^*\partial h_{lq}}{\partial x^p} - \frac{{}^*\partial h_{pl}}{\partial x^q}\right). \tag{10.5}$$

*Zelmanov subsequently mentioned that the Christoffel sub-symbols (10.3) area an unnecessary intermediate stage of the algebra. For this reason his scientific articles of the 1960's contain non-asterisk notations for the chr.inv.-Christoffel symbols. For instance, Zelmanov denotes the chr.inv.-Christoffel symbols of the 1st kind (10.5) simply by $\Delta_{pl,q}$. We retain the old notation here, for obvious reasons. — Editor's comment. D. R.

We will have particular interest in the case where

$$^*u^l \equiv 0 \tag{10.6}$$

at the point we are considering. Let us introduce the following special notation for the chr.inv.-tensor of the deformation rates

$$^*\Delta_{ik} = D_{ik}\,. \tag{10.7}$$

In this case* we have

$$u^l \equiv 0\,, \tag{10.8}$$

$$\frac{^*\partial\, ^*u^l}{\partial x^s} = \frac{c^2}{c^2 - \mathrm{w}}\frac{\partial u^l}{\partial x^s}\,, \tag{10.9}$$

$$2D_{ik} = \frac{c^2}{c^2 - \mathrm{w}}\left(h_{kl}\frac{\partial u^l}{\partial x^i} + h_{il}\frac{\partial u^l}{\partial x^k}\right). \tag{10.10}$$

Hence, it is clear that the equality (10.10) holds in all coordinate frames of this reference frame.

Introducing the coordinates

$$\left.\begin{aligned} &\tilde{x}^0 = \tilde{x}^0\left(x^0, x^1, x^2, x^3\right) \\ &\tilde{x}^i = x^i \end{aligned}\right\}, \tag{10.11}$$

where

$$\widetilde{\mathrm{w}} = 0\,, \tag{10.12}$$

we will have,

$$2\widetilde{D}_{ik} = \tilde{h}_{kl}\frac{\partial \tilde{u}^l}{\partial \tilde{x}^i} + \tilde{h}_{il}\frac{\partial \tilde{u}^l}{\partial \tilde{x}^k}\,. \tag{10.13}$$

On the other hand, we can write the co-tensor of the deformation rates under the condition $u^l \equiv 0$ (10.8), in general, as follows

$$2\Delta_{ik} = h_{kl}\frac{\partial u^l}{\partial x^i} + h_{il}\frac{\partial u^l}{\partial x^k}\,, \tag{10.14}$$

and so we have

$$\widetilde{D}_{ik} = \widetilde{\Delta}_{ik}\,. \tag{10.15}$$

§ 2.11 Deformations of a space

Let us consider a volume around an arbitrary point a in the space of coordinates

$$x^i = a^i\,, \qquad a^i = const^i. \tag{11.1}$$

---

*Zelmanov assumes that the derivative of u^i can be finite and essentially non-zero under $u^i \to 0$. He also assumes that this quantity is static, i. e. $\dot{u}^i = 0$. — Editor's comment. D. R.

Taking a small time interval, we can make this volume so small that any of its points at any moment of time (inside the time interval) has the univalent defined geodesic distance σ to the point a. The quantity σ^2 differs from $(h_{pq})_a\,(x^p-a^p)(x^q-a^q)$ with high order terms in x^i-a^i sufficiently small quantities. So, with the volume around the point a sufficiently small, we can write

$$\sigma^2=\left[(h_{pq})_a+\alpha_{pq,j}(x^j-a^j)\right](x^p-a^p)(x^q-a^q)\,, \tag{11.2}$$

where $\alpha_{pq,j}$ are finite* (as it easy to see, we can assume $\alpha_{pq,j}$ symmetric with respect to its indices p and q). Generally speaking, h_{ik} are functions of t (the space undergoes deformations), so the geodesic distances between the given points in the space and the point a change with time. Let us also introduce an auxiliary reference frame, defining it by the conditions: (1) this auxiliary system is fixed with respect to the initial reference frame of the point a in such a way that if ${}^*u^i$ is the chr.inv.-velocity of the auxiliary system with respect to the initial system (the ${}^*u^i$ is measured in the initial system), then at the point a we have

$${}^*u^i\equiv 0\,; \tag{11.3}$$

and (2) the geodesic distances (measured in the initial system) between the point a and all given points of the auxiliary system near a, remain unchanged, so

$$\frac{{}^*\partial\sigma^2}{\partial t}=0\,. \tag{11.4}$$

We will refer to this auxiliary system as the *locally-stationary reference frame at the point a.*

It is evident that the locally-stationary at the point a system is not uniquely defined, but in the order of its arbitrary rotation near the point. The following speculations are related to any reference frame of infinitely numerous system, which are locally-orthogonally at this point.

Formula (11.4), because of (11.2), gives

$$\begin{gathered}\left[\frac{{}^*\partial(h_{pq})_a}{\partial t}+\frac{{}^*\partial\alpha_{pq,j}}{\partial t}(x^j-a^j)+\frac{c^2\alpha_{pq,j}}{c^2-\mathrm{w}}u^j\right](x^p-a^p)(x^q-a^q)+\\+\,2c^2\,\frac{(h_{pq})_a+\alpha_{pq,j}\,(x^j-a^j)}{c^2-\mathrm{w}}\,u^p\,(x^q-a^q)=0\,,\end{gathered} \tag{11.5}$$

*We suppose that the derivatives we are considering exist and are finite.

where

$$u^i = \frac{dx^i}{dt} \tag{11.6}$$

is the velocity of the space (measured at the point x^i in the reference frame), which is locally-stationary at the point a with respect to the space. Introducing

$$\Theta_{pq} = \frac{{}^*\partial(h_{pq})_a}{\partial t} + \frac{{}^*\partial \alpha_{pq,j}}{\partial t}\,(x^j - a^j) + \frac{c^2 \alpha_{pq,j}}{c^2 - \mathrm{w}}\,u^j, \tag{11.7}$$

$$\Xi_{pq} = 2c^2\,\frac{(h_{pq})_a + \alpha_{pq,j}(x^j - a^j)}{c^2 - \mathrm{w}}\,, \tag{11.8}$$

we can re-write (11.5) in the form

$$\Theta_{pq}(x^p - a^p)(x^q - a^q) + \Xi_{pq}u^p(x^q - a^q) = 0\,. \tag{11.9}$$

Differentiating this equality term-by-term twice (with respect to x^k and with respect to x^i, respectively), we obtain

$$\begin{gathered}\frac{\partial^2\Theta_{pq}}{\partial x^i \partial x^k}\,(x^p - a^p)(x^q - a^q) + 2\left(\frac{\partial\Theta_{kq}}{\partial x^i} + \frac{\partial\Theta_{iq}}{\partial x^k}\right)(x^q - a^q) + \\ + 2\,\Theta_{ik} + \frac{\partial^2\Xi_{pq}}{\partial x^i \partial x^k}\,u^p(x^q - a^q) + \left(\frac{\partial\Xi_{pq}}{\partial x^k}\frac{\partial u^p}{\partial x^i} + \frac{\partial\Xi_{pq}}{\partial x^i}\frac{\partial u^p}{\partial x^k}\right)\times \\ \times (x^q - a^q) + \Xi_{pq}\frac{\partial^2 u^p}{\partial x^i \partial x^k}\,(x^q - a^q) + \left(\frac{\partial\Xi_{iq}}{\partial x^k} + \frac{\partial\Xi_{kq}}{\partial x^i}\right)u^q + \\ + \left(\Xi_{iq}\frac{\partial u^q}{\partial x^k} + \Xi_{kq}\frac{\partial u^q}{\partial x^i}\right) = 0\,.\end{gathered} \tag{11.10}$$

At the point a, taking the first of the equalities (11.3) into account, we have

$$u^i = 0\,, \tag{11.11}$$

then

$$2\,(\Theta_{ik})_a + (\Xi_{iq})_a\left(\frac{\partial u^q}{\partial x^k}\right)_a + (\Xi_{kq})_a\left(\frac{\partial u^q}{\partial x^i}\right)_a = 0\,. \tag{11.12}$$

In other words, because

$$(\Theta_{ik})_a = \frac{{}^*\partial\,(h_{ik})_a}{\partial t}\,, \tag{11.13}$$

$$(\Xi_{ik})_a = 2\,\frac{c^2\,(h_{ik})_a}{c^2 - (\mathrm{w})_a}\,, \tag{11.14}$$

we have

$$\frac{{}^*\partial\,(h_{ik})_a}{\partial t}+\frac{c^2}{c^2-(\mathrm{w})_a}\left[(h_{kq})_a\left(\frac{\partial u^q}{\partial x^i}\right)_a+(h_{iq})_a\left(\frac{\partial u^q}{\partial x^k}\right)_a\right]=0 \qquad (11.15)$$

or, in the final form

$$\frac{{}^*\partial\,(h_{ik})_a}{\partial t}=-\left[\frac{c^2}{c^2-\mathrm{w}}\left(h_{kq}\frac{\partial u^q}{\partial x^i}+h_{iq}\frac{\partial u^q}{\partial x^k}\right)\right]_a. \qquad (11.16)$$

We now introduce the chr.inv.-tensor of the deformation rate of the space of the reference frame (defined in this reference frame) with respect to the space, which is locally-stationary at the point a. We assume that

$$2\,{}^*\Delta_{ik}=h_{kq}\frac{{}^*\partial\,{}^*\bar{u}^q}{\partial x^i}+h_{iq}\frac{{}^*\partial\,{}^*\bar{u}^q}{\partial x^k}+\left({}^*\Delta_{iq,k}+{}^*\Delta_{kq,i}\right){}^*\bar{u}^q, \qquad (11.17)$$

where ${}^*\bar{u}^q$ is the chr.inv.-velocity of the space (measured in the reference frame) with respect to the locally stationary space at the point a. It is evident that

$${}^*\bar{u}^q=-{}^*u^q, \qquad (11.18)$$

where ${}^*u^q$ is the chr.inv.-velocity of the locally-stationary space at the point a, measured with respect to the space. Hence

$$2\,{}^*\Delta_{ik}=-\left(h_{kq}\frac{{}^*\partial\,{}^*u^q}{\partial x^i}+h_{iq}\frac{{}^*\partial\,{}^*u^q}{\partial x^k}\right)-\left({}^*\Delta_{iq,k}+{}^*\Delta_{kq,i}\right){}^*u^q \qquad (11.19)$$

characterizes deformations of the space at any point near a. Formula (11.3) holds at the point a, so as well as (10.10) we obtain

$$2\left({}^*\Delta_{ik}\right)_a=2D_{ik}=-\left[\frac{c^2}{c^2-\mathrm{w}}\left(h_{kq}\frac{\partial u^q}{\partial x^i}+h_{iq}\frac{\partial u^q}{\partial x^k}\right)\right]_a. \qquad (11.20)$$

We have thus obtained the chr.inv.-tensor of the deformation rates of the space at the point a with respect to the locally-stationary space at the point a. Comparing (11.20) and (11.16), we obtain

$$\frac{{}^*\partial\,(h_{ik})_a}{\partial t}=2\,(D_{ik})_a\,. \qquad (11.21)$$

Because the point a is arbitrary, we can, in general, write*

$$\frac{{}^*\partial h_{ik}}{\partial t}=2D_{ik}\,. \qquad (11.22)$$

*Zelmanov called equation (11.22) the *theorem on the space deformations*, see §2.13. — Editor's comment. D. R.

Thus, at any point of the space, the chr.inv.-derivative of the covariant chr.inv.-metric tensor with respect to time is twice the covariant chr.inv.-tensor of the deformation rates of this space (with respect to the locally-stationary space at this point).

Let us consider $\frac{{}^*\partial h^{ik}}{\partial t}$. Because

$$h_{ij}h^{jk} = h_i^k = \delta_i^k\,, \tag{11.23}$$

we have

$$h_{ij}\frac{{}^*\partial h^{jk}}{\partial t} = -\frac{{}^*\partial h_{ij}}{\partial t}\,h^{jk}, \tag{11.24}$$

and finally, taking (11.22) into account, we obtain

$$h_{ij}\frac{{}^*\partial h^{jk}}{\partial t} = -2D_i^k\,. \tag{11.25}$$

Raising the index i, we obtain

$$\frac{{}^*\partial h^{ik}}{\partial t} = -2D^{ik}. \tag{11.26}$$

Thus, in any point of the space, the chr.inv.-derivative of the contravariant chr.inv.-metric tensor with respect to time is twice the negative of the contravariant chr.inv.-tensor of the deformation rates of this space (with respect to the locally-stationary space at this point).

Let us consider $\frac{{}^*\partial h}{\partial t}$. Applying the rule for differentiating determinants and taking (11.22) into account, gives

$$\frac{{}^*\partial h}{\partial t} = \begin{vmatrix} 2D_{11} & h_{12} & h_{13} \\ 2D_{21} & h_{22} & h_{23} \\ 2D_{31} & h_{32} & h_{33} \end{vmatrix} + \begin{vmatrix} h_{11} & 2D_{12} & h_{13} \\ h_{21} & 2D_{22} & h_{23} \\ h_{31} & 2D_{32} & h_{33} \end{vmatrix} + \begin{vmatrix} h_{11} & h_{12} & 2D_{13} \\ h_{21} & h_{22} & 2D_{23} \\ h_{31} & h_{32} & 2D_{33} \end{vmatrix} =$$

$$= 2h\left(D_{i1}h^{i1} + D_{i2}h^{i2} + D_{i3}h^{i3}\right) = 2h\,h^{ik}D_{ik}\,. \tag{11.27}$$

Introducing the chr.inv.-invariant of the deformation rates at the point (with respect to the locally-stationary space at this point)

$$D = h_{ik}\,D^{ik} = h^{ik}D_{ik} = D_j^j\,, \tag{11.28}$$

we obtain

$$\frac{1}{h}\frac{{}^*\partial h}{\partial t} = 2D\,. \tag{11.29}$$

Thus, at any point of the space, the derivative of the logarithm of the fundamental determinant with respect to time is twice the chr.inv.-invariant of the deformation rates of this space (with respect to the locally-stationary space at this point).

Clearly, the fact that the locally-stationary reference system at the point is not unique does not affect our conclusions.

Furthermore, dropping the terms "locally-stationary space" and "its reference systems" for brevity, we will merely refer to the space deformations.

§ 2.12 Transformations of space elements

We will now deduce numerous equations describing deformations of space by analogy with equations for deformations of continuous media.

We assume that $\delta\mathcal{L}_a^1$ is the length of an elementary coordinate interval on the x^1 axis, i. e.

$$\delta\mathcal{L}_a^1 = \sqrt{h_{11}}\,\left|\delta_a x^1\right|, \qquad \delta_a x^1 = const_a^1\,, \tag{12.1}$$

then (for instance, see [62], p. 365)

$$\frac{^*\partial(\delta\mathcal{L}_a^1)}{\partial t} = \frac{1}{2\sqrt{h_{11}}}\frac{^*\partial h_{11}}{\partial t}\left|\delta_a x^1\right| = \frac{D_{11}}{\sqrt{h_{11}}}\left|\delta_a x^1\right|, \tag{12.2}$$

$$\frac{1}{\delta\mathcal{L}_a^1}\frac{^*\partial(\delta\mathcal{L}_a^1)}{\partial t} = \frac{D_{11}}{h_{11}}\,. \tag{12.3}$$

Thus, $\dfrac{D_{11}}{h_{11}}$ is the rate of the relative lengthening (because of the space deformations) of the linear element along the x^1 axis.

Next, we assume that δS_{ab}^{23} is the square of an element of the coordinate surface x^2, x^3

$$\left.\begin{aligned} \delta S_{ab}^{23} &= \sqrt{\begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix}}\,\left|\delta\Pi_{ab}^{23}\right| \\ \delta\Pi_{ab}^{23} &= \begin{vmatrix} \delta_a x^2 & \delta_a x^3 \\ \delta_b x^1 & \delta_b x^2 \end{vmatrix}, \quad \delta_a x^i = const_a^i\,, \;\; \delta_b x^i = const_b^i \end{aligned}\right\}, \tag{12.4}$$

then we have

$$\begin{aligned}\frac{{}^*\partial\left(\delta S^{23}_{ab}\right)}{\partial t} &= \frac{1}{2\sqrt{hh^{11}}}\left(\frac{{}^*\partial h}{\partial t}h^{11} + h\frac{{}^*\partial h^{11}}{\partial t}\right)\left|\delta\Pi^{23}_{ab}\right| = \\ &= \frac{Dhh^{11} - hD^{11}}{\sqrt{hh^{11}}}\left|\delta\Pi^{23}_{ab}\right| ,\end{aligned} \tag{12.5}$$

$$\frac{1}{\delta S^{23}_{ab}}\frac{{}^*\partial\left(\delta S^{23}_{ab}\right)}{\partial t} = D - \frac{D^{11}}{h^{11}}\,. \tag{12.6}$$

Thus, $D - \dfrac{D^{11}}{h^{11}}$ is the rate of the relative expansion (because of the space deformations) of the element of the surface x^2, x^3.

Finally, we assume that δV_{abc} is the value of a volume element

$$\left.\begin{aligned} &\delta V_{abc} = \sqrt{h}\left|\delta\Pi^{123}_{abc}\right| \\ &\delta\Pi^{123}_{abc} = \begin{vmatrix} \delta_a x^1 & \delta_a x^2 & \delta_a x^3 \\ \delta_b x^1 & \delta_b x^2 & \delta_b x^3 \\ \delta_c x^1 & \delta_c x^2 & \delta_c x^3 \end{vmatrix} \qquad \begin{aligned} \delta_a x^i &= const^i_a \\ \delta_b x^i &= const^i_b \\ \delta_c x^i &= const^i_c \end{aligned} \end{aligned}\right\}. \tag{12.7}$$

Because

$$\frac{{}^*\partial\left(\delta V_{abc}\right)}{\partial t} = \frac{1}{2\sqrt{h}}\frac{{}^*\partial h}{\partial t}\left|\delta\Pi^{123}_{abc}\right| = \frac{hD}{\sqrt{h}}\left|\delta\Pi^{123}_{abc}\right| , \tag{12.8}$$

we have (for instance, see [62], p. 366)

$$\frac{1}{\delta V_{abc}}\frac{{}^*\partial\left(\delta V_{abc}\right)}{\partial t} = D\,. \tag{12.9}$$

So the chr.inv.-invariant D is the rate of the relative expansions of the volume element (because of the space deformations).

§2.13 Christoffel's symbols in chr.inv.-form

We will use two species of Christoffel symbols of the 1st kind, see (10.3) and (10.5); the co-symbols

$$\Delta_{ij,k} = \frac{1}{2}\left(\frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k}\right) \tag{13.1}$$

and the chr.inv.-symbols

$$^{*}\Delta_{ij,k} = \frac{1}{2}\left(\frac{^{*}\partial h_{ik}}{\partial x^{j}} + \frac{^{*}\partial h_{jk}}{\partial x^{i}} - \frac{^{*}\partial h_{ij}}{\partial x^{k}}\right). \tag{13.2}$$

In accordance with the foregoing, we will use two species of Christoffel sub-symbols of the 2nd kind, namely, the co-symbols

$$\Delta^{k}_{ij} = h^{kl}\Delta_{ij,l} \tag{13.3}$$

and the chr.inv.-symbols

$$^{*}\Delta^{k}_{ij} = h^{kl}\,{}^{*}\Delta_{ij,l}\,. \tag{13.4}$$

Let us find formulae for the Christoffel chr.inv.-symbols, employing the theorem on the space deformations (11.22). Because

$$\frac{^{*}\partial}{\partial x^{j}} = \frac{\partial}{\partial x^{j}} + \frac{v_j}{c^2}\frac{^{*}\partial}{\partial t}\,, \tag{13.5}$$

we have

$$\frac{^{*}\partial h_{ik}}{\partial x^{j}} = \frac{\partial h_{ik}}{\partial x^{j}} + \frac{2}{c^2}\,D_{ik}\,v_j\,. \tag{13.6}$$

Therefore, we obtain

$$^{*}\Delta_{ij,k} = \Delta_{ij,k} + \frac{1}{c^2}\left(D_{ik}v_j + D_{jk}v_i - D_{ij}v_k\right), \tag{13.7}$$

$$^{*}\Delta^{k}_{ij} = \Delta^{k}_{ij} + \frac{1}{c^2}\left(D^{k}_{i}\,v_j + D^{k}_{j}\,v_i - D_{ij}v^{k}\right). \tag{13.8}$$

It is necessary to highlight various properties of the chr.inv.-Christoffel symbols corresponding to the analogous properties of the co-symbols. The first of these is their symmetry

$$^{*}\Delta_{ij,k} = {}^{*}\Delta_{ji,k}\,, \tag{13.9}$$

$$^{*}\Delta^{k}_{ij} = {}^{*}\Delta^{k}_{ji}\,. \tag{13.10}$$

Then, considering (13.2), we have

$$^{*}\Delta_{ij,k} + {}^{*}\Delta_{kj,i} = \frac{^{*}\partial h_{ik}}{\partial x^{j}}\,. \tag{13.11}$$

Finally, because

$$\Delta^{j}_{ij} + \frac{1}{c^2}\left(D^{j}_{i}\,v_j + D^{j}_{j}\,v_i - D_{ij}v^{j}\right) = \frac{\partial \ln\sqrt{h}}{\partial x^{i}} + \frac{v_i}{c^2}\,D\,, \tag{13.12}$$

and taking (13.8) and (13.5) into account, we obtain

$$^{*}\Delta^{j}_{ij} = \frac{^{*}\partial \ln\sqrt{h}}{\partial x^{i}}\,. \tag{13.13}$$

§ 2.14 General covariant differentiation in chr.inv.-form

We now introduce operations of *chr.inv.-differentiation*, defining the operations by the requirements*: (1) they must be invariant with respect to the transformations of the time coordinate; and (2) they must coincide with the operations of regular covariant differentiation when the potentials are set to zero. To realize the requirements in the operations of regular covariant differentiation, it is necessary and sufficient to replace all co-derivatives with chr.inv.-derivatives and, respectively, to replace the usual symbols Christoffel with the chr.inv.-Christoffel symbols.

Modifying the symbol ∇ denoting regular covariant differentiation, we denote chr.inv.-differentiation by $^*\nabla$. Accordingly, we have the chr.inv.-differential operations for sub-vectors

$$^*\nabla_i\, Q_k = \frac{^*\partial Q_k}{dx^i} - {^*\Delta}^l_{ik} Q_l\,, \tag{14.1}$$

$$^*\nabla_i\, Q^k = \frac{^*\partial Q^k}{dx^i} + {^*\Delta}^k_{il} Q^l, \tag{14.2}$$

for sub-tensors of the 2nd rank

$$^*\nabla_i\, Q_{jk} = \frac{^*\partial Q_{jk}}{dx^i} - {^*\Delta}^l_{ij} Q_{lk} - {^*\Delta}^l_{ik} Q_{jl}\,, \tag{14.3}$$

$$^*\nabla_i\, Q^k_j = \frac{^*\partial Q^k_j}{dx^i} - {^*\Delta}^l_{ij} Q^k_l + {^*\Delta}^k_{il} Q^l_j\,, \tag{14.4}$$

$$^*\nabla_i\, Q^{jk} = \frac{^*\partial Q^{jk}}{dx^i} + {^*\Delta}^j_{il} Q^{lk} + {^*\Delta}^k_{il} Q^{jl}, \tag{14.5}$$

and so forth. In general, chr.inv.-derivative notation is different to regular covariant derivative notation merely by an asterisk before their ∇ and Christoffel symbols.

It is clear that the chr.inv.-derivative of a sub-tensor quantity will be a chr.inv.-quantity if the differentiated sub-tensor is also a chr.inv.-quantity.

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*Zelmanov uses two different terms here, denoting chronometrically invariant differential operations, namely — *chr.inv.-derivation* for chr.inv.-derivatives with respect to time and spatial coordinates (see §2.5), and *chr.inv.-covariant differentiation*, generalizing covariant four-dimensional differentiation into its three-dimensional analogue, which has the property of chronometric invariance. Later, Zelmanov did not use the separate terms, because it was unnecessary, and so only used the term "chr.inv.-differentiation" for any differential operations which have the property of chronometric invariance. — Editor's comment. D. R.

Regular divergence can be defined as a covariant derivative contracted with one of the upper indices of the differentiated quantity. For this reason we will refer to the quantity, obtained from chr.inv.-differentiation in this way, as *chr.inv.-divergence*. For instance,

$$^*\nabla_i\, Q^i = \frac{^*\partial Q^i}{\partial x^i} + \frac{^*\partial \ln\sqrt{h}}{\partial x^i}\, Q^i, \tag{14.6}$$

$$^*\nabla_i\, Q^i = \frac{1}{\sqrt{h}}\frac{^*\partial\big(\sqrt{h}\, Q^i\big)}{\partial x^i}\,, \tag{14.7}$$

$$^*\nabla_i\, Q^i_j = \frac{^*\partial Q^i_j}{\partial x^i} - {}^*\!\Delta^l_{ij} Q^i_l + \frac{^*\partial \ln\sqrt{h}}{\partial x^i}\, Q^i_j\,, \tag{14.8}$$

$$^*\nabla_i\, Q^{ji} = \frac{^*\partial Q^{ji}}{\partial x^i} + {}^*\!\Delta^j_{il} Q^{il} + \frac{^*\partial \ln\sqrt{h}}{\partial x^i}\, Q^{ji}. \tag{14.9}$$

In relation to chr.inv.-differentiation, the metric sub-tensors are the same as for regular covariant differentiation. Really,

$$^*\nabla_j\, h_{ik} = \frac{^*\partial h_{ik}}{\partial x^j} - {}^*\!\Delta^l_{ji} h_{lk} - {}^*\!\Delta^l_{jk} h_{il} = \frac{^*\partial h_{ik}}{\partial x^j} - {}^*\!\Delta_{ji,k} - {}^*\!\Delta_{jk,i} \tag{14.10}$$

and, because of (13.11), we have

$$^*\nabla_j\, h_{ik} = 0\,. \tag{14.11}$$

We therefore have

$$^*\nabla_j\, h^k_i = \frac{^*\partial h^k_i}{\partial x^j} - {}^*\!\Delta^l_{ji} h^k_l + {}^*\!\Delta^k_{jl} h^l_i = -\,{}^*\!\Delta^k_{ji} + {}^*\!\Delta^k_{ji}\,, \tag{14.12}$$

and thus obtain

$$^*\nabla_j\, h^k_i = 0\,. \tag{14.13}$$

Because

$$h^k_i = h_{qi}\, h^{qk}, \tag{14.14}$$

and taking (14.11) and (14.13) into account, we obtain

$$h_{qi}\, {}^*\nabla_j\, h^{qk} = 0 \tag{14.15}$$

or, raising the index i,

$$h^i_q\, {}^*\nabla_j\, h^{qk} = 0\,. \tag{14.16}$$

Finally, because

$$h^i_q h^{qk} = h^{ik}, \tag{14.17}$$

formula (14.16), taking (14.13) into account, gives

$$^*\nabla_j\, h^{ik} = 0\,. \tag{14.18}$$

So the operation of chr.inv.-differentiation, as well as the operation of regular covariant differentiation, is commutative with respect to the raising, lowering, or substitution of indices.

We can accordingly re-write formula (10.4) as follows

$$2^*\!\Delta_{ik} = h_{kl}{}^*\nabla_i\, {}^*u^l + h_{il}{}^*\nabla_k\, {}^*u^l \tag{14.19}$$

or, in the alternative form

$$2^*\!\Delta_{ik} = {}^*\nabla_i\, {}^*u_k + {}^*\nabla_k\, {}^*u_i\,. \tag{14.20}$$

These are equations for general covariant differentiation in chr.inv.-form, obtained from the chr.inv.-tensor of the deformation rates as an example. It easy to see that the equations are different from the analogous equations for the co-tensor of the deformation rates only by the presence of an asterisk.

§ 2.15 The Riemann-Christoffel tensor in chr.inv.-form

Let us denote

$$^*\nabla_{pq} = {}^*\nabla_p\, ({}^*\nabla_q)\,. \tag{15.1}$$

We are going to take any sub-vectors Q_k and Q^k in order to change the sequence of their chr.inv.-covariant differentiation. As a result, for Q_k, changing its dummy indices, we obtain

$$\begin{aligned}
&{}^*\nabla_{ij}\, Q_k - {}^*\nabla_{ji}\, Q_k = {}^*\nabla_i\, ({}^*\nabla_j\, Q_k) - {}^*\nabla_j\, ({}^*\nabla_i\, Q_k) = \\
&= \frac{{}^*\partial}{\partial x^i}\, ({}^*\nabla_j\, Q_k) - {}^*\Delta^l_{ij}\, ({}^*\nabla_l\, Q_k) - {}^*\Delta^l_{ik}\, ({}^*\nabla_j\, Q_l) - \\
&- \frac{{}^*\partial}{\partial x^j}\, ({}^*\nabla_i\, Q_k) + {}^*\Delta^l_{ji}\, ({}^*\nabla_l\, Q_k) + {}^*\Delta^l_{jk}\, ({}^*\nabla_i\, Q_l) = \\
&= \frac{{}^*\partial}{\partial x^i}\left(\frac{{}^*\partial Q_k}{\partial x^j} - {}^*\Delta^l_{jk} Q_l\right) - \frac{{}^*\partial}{\partial x^j}\left(\frac{{}^*\partial Q_k}{\partial x^i} - {}^*\Delta^l_{ik} Q_l\right) - \\
&- {}^*\Delta^l_{ik}\left(\frac{{}^*\partial Q_l}{\partial x^j} - {}^*\Delta^m_{jl} Q_m\right) + {}^*\Delta^l_{jk}\left(\frac{{}^*\partial Q_l}{\partial x^i} - {}^*\Delta^m_{il} Q_m\right) = \\
&= \left(\frac{{}^*\partial^2 Q_k}{\partial x^i \partial x^j} - \frac{{}^*\partial^2 Q_k}{\partial x^j \partial x^i}\right) - \left(\frac{{}^*\partial\, {}^*\Delta^l_{jk}}{\partial x^i} - \frac{{}^*\partial\, {}^*\Delta^l_{ik}}{\partial x^j}\right) Q_l\, -
\end{aligned}$$

$$- \left({}^*\!\Delta^l_{jk} \frac{{}^*\partial Q_l}{\partial x^i} - {}^*\!\Delta^l_{ik} \frac{{}^*\partial Q_l}{\partial x^j} \right) - \left({}^*\!\Delta^l_{ik} \frac{{}^*\partial Q_l}{\partial x^j} - {}^*\!\Delta^l_{jk} \frac{{}^*\partial Q_l}{\partial x^i} \right) +$$

$$+ \left({}^*\!\Delta^l_{ik} {}^*\!\Delta^m_{jl} - {}^*\!\Delta^l_{jk} {}^*\!\Delta^m_{il} \right) Q_m = \frac{2}{c^2} A_{ij} \frac{{}^*\partial Q_k}{\partial t} + \tag{15.2}$$

$$+ \left(\frac{{}^*\partial\, {}^*\!\Delta^l_{ik}}{\partial x^j} - \frac{{}^*\partial\, {}^*\!\Delta^l_{jk}}{\partial x^i} + {}^*\!\Delta^m_{ik} {}^*\!\Delta^l_{jm} - {}^*\!\Delta^m_{jk} {}^*\!\Delta^l_{im} \right) Q_l \,.$$

Doing the same for the contravariant sub-vector Q^k, we have

$$\begin{aligned}
&{}^*\nabla_{ij}\, Q^k - {}^*\nabla_{ji}\, Q^k = {}^*\nabla_i \left({}^*\nabla_j\, Q^k \right) - {}^*\nabla_j \left({}^*\nabla_i\, Q^k \right) = \\
&= \frac{{}^*\partial}{\partial x^i} \left({}^*\nabla_j\, Q^k \right) - {}^*\!\Delta^l_{ij} \left({}^*\nabla_l\, Q^k \right) + {}^*\!\Delta^k_{il} \left({}^*\nabla_j\, Q^l \right) - \\
&- \frac{{}^*\partial}{\partial x^j} \left({}^*\nabla_i\, Q^l \right) + {}^*\!\Delta^l_{ji} \left({}^*\nabla_l\, Q^k \right) - {}^*\!\Delta^k_{jl} \left({}^*\nabla_i\, Q^l \right) = \\
&= \frac{{}^*\partial}{\partial x^i} \left(\frac{{}^*\partial Q^k}{\partial x^j} + {}^*\!\Delta^k_{jl} Q^l \right) - \frac{{}^*\partial}{\partial x^j} \left(\frac{{}^*\partial Q^k}{\partial x^i} + {}^*\!\Delta^k_{il} Q^l \right) + \\
&+ {}^*\!\Delta^k_{il} \left(\frac{{}^*\partial Q^l}{\partial x^j} + {}^*\!\Delta^l_{jm} Q^m \right) - {}^*\!\Delta^k_{jl} \left(\frac{{}^*\partial Q^l}{\partial x^i} + {}^*\!\Delta^l_{im} Q^m \right) = \\
&= \left(\frac{{}^*\partial^2 Q^k}{\partial x^i \partial x^j} - \frac{{}^*\partial^2 Q^k}{\partial x^j \partial x^i} \right) - \left(\frac{{}^*\partial\, {}^*\!\Delta^k_{jl}}{\partial x^i} - \frac{{}^*\partial\, {}^*\!\Delta^k_{il}}{\partial x^j} \right) Q^l + \\
&+ \left({}^*\!\Delta^k_{jl} \frac{{}^*\partial Q^l}{\partial x^i} - {}^*\!\Delta^k_{il} \frac{{}^*\partial Q^l}{\partial x^j} \right) + \left({}^*\!\Delta^k_{il} \frac{{}^*\partial Q^l}{\partial x^j} - {}^*\!\Delta^k_{jl} \frac{{}^*\partial Q^l}{\partial x^i} \right) + \\
&+ \left({}^*\!\Delta^k_{il} {}^*\!\Delta^l_{jm} - {}^*\!\Delta^k_{jl} {}^*\!\Delta^l_{im} \right) Q^m = \frac{2}{c^2} A_{ij} \frac{{}^*\partial Q^k}{\partial t} - \\
&- \left(\frac{{}^*\partial\, {}^*\!\Delta^k_{il}}{\partial x^j} - \frac{{}^*\partial\, {}^*\!\Delta^k_{jl}}{\partial x^i} + {}^*\!\Delta^m_{il} {}^*\!\Delta^k_{jm} - {}^*\!\Delta^m_{jl} {}^*\!\Delta^k_{im} \right) Q^l.
\end{aligned} \tag{15.3}$$

Introducing the notation

$$H_{kji}^{\cdot\cdot\cdot l} = \frac{{}^*\partial\, {}^*\!\Delta^l_{ik}}{\partial x^j} - \frac{{}^*\partial\, {}^*\!\Delta^l_{jk}}{\partial x^i} + {}^*\!\Delta^m_{ik} {}^*\!\Delta^l_{jm} - {}^*\!\Delta^m_{jk} {}^*\!\Delta^l_{im} \,, \tag{15.4}$$

we can write

$${}^*\nabla_{ij}\, Q_k - {}^*\nabla_{ji}\, Q_k = \frac{2}{c^2} A_{ij} \frac{{}^*\partial Q_k}{\partial t} + H_{kji}^{\cdot\cdot\cdot l} Q_l \,, \tag{15.5}$$

$${}^*\nabla_{ij}\, Q^k - {}^*\nabla_{ji}\, Q^k = \frac{2}{c^2} A_{ij} \frac{{}^*\partial Q^k}{\partial t} - H_{lji}^{\cdot\cdot\cdot k} Q^l. \tag{15.6}$$

Because Q_k (and Q^k, respectively) is an arbitrary sub-vector, then, according to the theorem of fractions, $H_{kji}^{\cdot\cdot\cdot l}$ is a sub-tensor of the 4th rank, which is thrice covariant and once contravariant. Since Q_k (and Q^k) is an arbitrary chr.inv.-vector, we are assured that $H_{kji}^{\cdot\cdot\cdot l}$ is a chr.inv.-tensor. By the relationship of its structure to the structure of the Riemann-Christoffel mixed sub-tensor (the co-tensor)

$$K_{kji}^{\cdot\cdot\cdot l} = \frac{\partial \Delta_{ik}^{l}}{\partial x^j} - \frac{\partial \Delta_{jk}^{l}}{\partial x^i} + \Delta_{ik}^{m}\,\Delta_{jm}^{l} - \Delta_{jk}^{m}\,\Delta_{im}^{l}\,, \tag{15.7}$$

as a base, we call $H_{kji}^{\cdot\cdot\cdot l}$ the *chr.inv.-Riemann-Christoffel mixed tensor*. Lowering the upper index and using (13.11)

$$\begin{aligned} h_{nl}H_{kji}^{\cdot\cdot\cdot l} &= h_{nl}\left(\frac{{}^*\partial\,{}^*\Delta_{ik}^{l}}{\partial x^j} - \frac{{}^*\partial\,{}^*\Delta_{jk}^{l}}{\partial x^i} + {}^*\Delta_{ik}^{m}\,{}^*\Delta_{jm}^{l} - {}^*\Delta_{jk}^{m}\,{}^*\Delta_{im}^{l}\right) = \\ &= \frac{{}^*\partial\,{}^*\Delta_{ik,n}}{\partial x^j} - \left({}^*\Delta_{nj,l} + {}^*\Delta_{lj,n}\right){}^*\Delta_{ik}^{l} - \frac{{}^*\partial\,{}^*\Delta_{jk,n}}{\partial x^i} + \\ &+ \left({}^*\Delta_{ni,l} + {}^*\Delta_{li,n}\right){}^*\Delta_{jk}^{l} + {}^*\Delta_{ik}^{m}\,{}^*\Delta_{jm,n} - {}^*\Delta_{jk}^{m}\,{}^*\Delta_{im,n} = \\ &= \frac{{}^*\partial\,{}^*\Delta_{ik,n}}{\partial x^j} - \frac{{}^*\partial\,{}^*\Delta_{jk,n}}{\partial x^i} - {}^*\Delta_{nj,l}\,{}^*\Delta_{ik}^{l} + {}^*\Delta_{ni,l}\,{}^*\Delta_{jk}^{l}\,, \end{aligned} \tag{15.8}$$

we obtain, after rearrangement of its dummy indices, the *chr.inv.-Riemann-Christoffel covariant tensor*

$$H_{kjin} = \frac{{}^*\partial\,{}^*\Delta_{ik,n}}{\partial x^j} - \frac{{}^*\partial\,{}^*\Delta_{jk,n}}{\partial x^i} - {}^*\Delta_{ik,l}\,{}^*\Delta_{jn}^{l} + {}^*\Delta_{jk,l}\,{}^*\Delta_{in}^{l}\,, \tag{15.9}$$

the structure of which has the relationship to the Riemann-Christoffel covariant co-tensor*

$$K_{kjin} = \frac{\partial \Delta_{ik,n}}{\partial x^j} - \frac{\partial \Delta_{jk,n}}{\partial x^i} - \Delta_{ik,l}\,\Delta_{jn}^{l} + \Delta_{jk,l}\,\Delta_{in}^{l}\,. \tag{15.10}$$

Some of properties of the chr.inv.-tensor H_{kjin} are given below as follows. Because

$$\left(\frac{{}^*\partial\,{}^*\Delta_{ik,n}}{\partial x^j} - \frac{{}^*\partial\,{}^*\Delta_{jk,n}}{\partial x^i}\right) + \left(\frac{{}^*\partial\,{}^*\Delta_{in,k}}{\partial x^j} - \frac{{}^*\partial\,{}^*\Delta_{jn,k}}{\partial x^i}\right) =$$

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*Sites of different indices in the Riemann-Christoffel tensor, and also their meanings, are different in various publications, because of different notation for the Riemann-Christoffel tensor (for instance, see [8], p. 91 and [7], p. 130). We are following Eddington here.

$$= \frac{^*\partial}{\partial x^j}\left({}^*\Delta_{ik,n} + {}^*\Delta_{in,k}\right) - \frac{^*\partial}{\partial x^i}\left({}^*\Delta_{jk,n} + {}^*\Delta_{jn,k}\right) =$$
$$= \frac{^*\partial^2 h_{kn}}{\partial x^j \partial x^i} - \frac{^*\partial^2 h_{kn}}{\partial x^i \partial x^j} = \frac{2}{c^2} A_{ji} \frac{^*\partial h_{kn}}{\partial t} = \frac{4}{c^2} A_{ji} D_{kn}\,, \tag{15.11}$$

we have

$$\frac{1}{2}\left(H_{kjin} + H_{njik}\right) = \frac{2}{c^2} A_{ji} D_{kn}\,, \tag{15.12}$$

and also

$$\frac{1}{2}\left(H_{kjin} + H_{kijn}\right) = 0\,. \tag{15.13}$$

Thus, the chr.inv.-tensor H_{kjin} as well as K_{kjin} is antisymmetric with respect to the inner pair of indices and, in contrast to K_{kjin}, generally speaking, is antisymmetric with respect to the outer pair of indices. It is possible to show that H_{kjin} has, generally speaking, no other symmetric properties that are typical for K_{kjin}. However, to set one of the tensors A_{ik} or D_{ik} to zero will be sufficient for obtaining the tensor H_{kjin} by the symmetric properties.

We will refer to the chr.inv.-tensor of the 2nd rank

$$H_{kj} = H_{kjl}^{\cdots l} = H_{kjin} h^{in}, \tag{15.14}$$

obtained as a result of the contraction of the chr.inv.-Riemann-Christoffel tensor by the second pair of indices, as *Einstein's chr.inv.-tensor*. Because of (15.4), we have

$$H_{kj} = \frac{^*\partial\, {}^*\Delta_{kl}^l}{\partial x^j} - \frac{^*\partial\, {}^*\Delta_{kj}^l}{\partial x^l} + {}^*\Delta_{kl}^m\, {}^*\Delta_{jm}^l - {}^*\Delta_{kj}^m\, {}^*\Delta_{ml}^l\,, \tag{15.15}$$

or, in the alternative form,

$$H_{kj} = -\frac{^*\partial\, {}^*\Delta_{kj}^l}{\partial x^l} + {}^*\Delta_{kl}^m\, {}^*\Delta_{jm}^l + \frac{^*\partial^2 \ln\sqrt{h}}{\partial x^j \partial x^k} - {}^*\Delta_{kj}^l \frac{^*\partial \ln\sqrt{h}}{\partial x^l}\,, \tag{15.16}$$

which is like the formulae for *Einstein's co-tensor*

$$K_{kj} = K_{kjl}^{\cdots l} = K_{kjin} h^{in}, \tag{15.17}$$

$$K_{kj} = \frac{\partial \Delta_{kl}^l}{\partial x^j} - \frac{\partial \Delta_{kj}^l}{\partial x^l} + \Delta_{kl}^m \Delta_{jm}^l - \Delta_{kj}^m \Delta_{ml}^l\,, \tag{15.18}$$

$$K_{kj} = -\frac{\partial \Delta_{kj}^l}{\partial x^l} + \Delta_{kl}^m \Delta_{jm}^l + \frac{\partial^2 \ln\sqrt{h}}{\partial x^j \partial x^k} - \Delta_{kj}^l \frac{\partial \ln\sqrt{h}}{\partial x^l}\,, \tag{15.19}$$

obtained as a result of the contraction of the Riemann-Christoffel co-tensor by the second pair of its indices.

In contrast to K_{kj}, the chr.inv.-tensor H_{kj}, generally speaking, is not symmetric. Indeed,

$$\frac{{}^*\partial^2 \ln\sqrt{h}}{\partial x^j \partial x^k} - \frac{{}^*\partial^2 \ln\sqrt{h}}{\partial x^k \partial x^j} = \frac{2}{c^2}\,A_{jk}\,D\,, \tag{15.20}$$

so we have

$$\frac{1}{2}\left(H_{kj} - H_{jk}\right) = \frac{1}{c^2}\,A_{jk}\,D\,. \tag{15.21}$$

We also introduce the chr.inv.-invariant

$$H = h^{kj} H_{kj}\,. \tag{15.22}$$

Using the chr.inv.-Riemann-Christoffel tensor and also those quantities result from its contractions, we will consider the problem of space curvature and the problem of space rotations. We begin these studies from the latter problem, where we will need to introduce chr.inv.-angular velocity.

§ 2.16 Chr.inv.-rotor

Let us introduce by analogy with regular three-dimensional calculus, the contravariant sub-tensor of the 3rd rank ε^{ijk}, which is completely defined by its component

$$\varepsilon^{123} = \frac{1}{\sqrt{h}} \tag{16.1}$$

and its antisymmetry with respect to any pair of its indices. We also introduce the covariant sub-tensor ε_{ijk}. The sub-tensor ε_{ijk} has the same symmetric properties (for instance, see [8], p. 78), and

$$\varepsilon_{123} = \sqrt{h}\,. \tag{16.2}$$

Because $\sqrt{h}$ is a chr.inv.-quantity, the sub-tensors ε^{ijk} and ε_{ijk} are chr.inv.-tensors as well. These are some of their properties: (1) the tensors are linked the index operators (the components of the chr.inv.-metric tensor)

$$\varepsilon^{pqk}\varepsilon_{ijk} = h^p_i h^q_j - h^p_j h^q_i\,, \tag{16.3}$$

and (2) the chr.inv.-derivative of them is zero

$${}^*\nabla_p\,\varepsilon_{ijk} = 0\,. \tag{16.4}$$

Formula (16.3) is known from regular tensor analysis (see [8], p. 111). Formula (16.4) is derived from the fact that the regular covariant derivative of the tensor ε_{ijk} is zero (see [8], p. 88)

$$\nabla_p \,\varepsilon_{ijk} = 0\,. \tag{16.5}$$

Actually, considering coordinates $\tilde{x}^0$, $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$, which set the potentials to zero at the world-point we are considering, we have

$${}^*\widetilde{\nabla}_p \,\tilde{\varepsilon}_{ijk} = \widetilde{\nabla}_p \,\tilde{\varepsilon}_{ijk} \tag{16.6}$$

so, because of (16.5), we obtain

$${}^*\widetilde{\nabla}_p \,\tilde{\varepsilon}_{ijk} = 0\,. \tag{16.7}$$

The left side of the equality (16.7) is chr.inv.-tensor of the 4th rank. Hence, in arbitrary coordinates, it becomes zero — the formula (16.4) is true.

Just as in regular tensor algebra, using the chr.inv.-tensors ε^{ijk} and ε_{ijk}, we can set sub-vectors ω^k or ψ_k dual to any antisymmetric sub-tensor of the 2nd rank a_{ij} or b^{ij}

$$\omega^k = \frac{1}{2}\,\varepsilon^{ijk} a_{ij}\,, \tag{16.8}$$

$$\psi_k = \frac{1}{2}\,\varepsilon_{ijk}\, b^{ij}. \tag{16.9}$$

The converse is also true. Using the chr.inv.-tensors ε^{ijk} and ε_{ijk}, we can uniquely set the antisymmetric tensors of the 2nd rank dual to any sub-vector.

If the sub-tensors a_{ij} and b^{ij} are mutually conjuncted, then the sub-vectors ω^k and ψ_k are mutually conjuncted as well. If a_{ij} and b^{ij} are chr.inv.-tensors, then ω^k and ψ_k are chr.inv.-vectors. Multiplying (16.8) term-by-term by ε_{pqk}, and multiplying (16.9) term-by-term by ε^{pqk}, after the results are contracted, we obtain

$$\varepsilon_{pqk}\,\omega^k = \frac{1}{2}\,\varepsilon^{ijk}\varepsilon_{pqk}\,a_{ij}\,, \tag{16.10}$$

$$\varepsilon^{pqk}\psi_k = \frac{1}{2}\,\varepsilon^{pqk}\varepsilon_{ijk}\,b^{ij}, \tag{16.11}$$

and, because of (16.3), we have

$$a_{pq} = \varepsilon_{pqk}\,\omega^k, \tag{16.12}$$

$$b^{pq} = \varepsilon^{pqk}\psi_k \,. \tag{16.13}$$

It is evident that if the sub-vectors ω^k and ψ_k are mutually conjuncted, then the sub-tensors a_{pq} and b^{pq} are mutually conjuncted as well. If ω^k and ψ_k are chr.inv.-vectors, then a_{pq} and b^{pq} are chr.inv.-tensors. It is clear that formulae (16.8–16.9) and (16.12–16.13) set the unique mutual link between sub-vectors and antisymmetric dual sub-tensors of the 2nd rank.

The regular definition of the rotor (vortex) r^p of a vector ω_k is*

$$r^p(\omega) = \varepsilon^{qkp}\,\nabla_q\,\omega_k \,. \tag{16.14}$$

We introduce the *chr.inv.-rotor*

$$^*r^p(\omega) = \varepsilon^{qkp}\,{}^*\nabla_q\,\omega_k \,. \tag{16.15}$$

It is evident that the operation of building chr.inv.-rotors is invariant with respect to transformations of the time coordinate. This operation coincides with the operation of building regular co-rotors when the potentials are set to zero at the world-point we are considering. It is also evident that the chr.inv.-rotor of a chr.inv.-vector is a chr.inv.-vector.

Let us introduce the antisymmetric tensor of the 2nd rank a^{ij} dual to the vector ω_k. So we have

$$\omega_k = \frac{1}{2}\,\varepsilon_{ijk}\,a^{ij}. \tag{16.16}$$

Substituting (16.16) into (16.15), we have

$$\begin{aligned} {}^*r^p(\omega) &= \frac{1}{2}\,\varepsilon^{qkp}\varepsilon_{ijk}\,{}^*\nabla_q\,a^{ij} + \frac{1}{2}\,\varepsilon^{qkp}a^{ij}\,{}^*\nabla_q\,\varepsilon_{ijk} = \\ &= \frac{1}{2}\,\varepsilon^{qkp}\varepsilon_{ijk}\,{}^*\nabla_q\,a^{ij} = -\frac{1}{2}\,\varepsilon^{qpk}\varepsilon_{ijk}\,{}^*\nabla_q\,a^{ij} = \\ &= -\frac{1}{2}\left(h^q_i h^p_j - h^q_j h^p_i\right){}^*\nabla_q\,a^{ij} = -\frac{1}{2}\left({}^*\nabla_i\,a^{ip} - {}^*\nabla_j\,a^{pj}\right), \end{aligned} \tag{16.17}$$

and so

$$^*r^p(\omega) = {}^*\nabla_j\,a^{pj}, \tag{16.18}$$

or, in the alternative form,

$$\varepsilon^{qkp}\,{}^*\nabla_q\,\omega_k = {}^*\nabla_j\,a^{pj}. \tag{16.19}$$

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*In general, the vortex of a vector is antisymmetric tensor of the 2nd rank. However, in the particular case of three-dimensional space, the vortex can be assumed the vector dual to the tensor. We also use this representation here (see [8], p. 79–80).

So the chr.inv.-rotor of any sub-vector is the chr.inv.-divergence of its antisymmetric dual sub-tensor of the 2nd rank.

§ 2.17 The chr.inv.-vector of the angular velocities of rotation. Its chr.inv-rotor

The regular relation between the contravariant sub-vector ω^k of the angular velocities of a volume element and the covariant sub-vector u_j of the linear velocities of its points is

$$\omega^k = \frac{1}{2}\, r^k (u) \tag{17.1}$$

or, in the alternative form

$$\omega^k = \frac{1}{2}\, \varepsilon^{ijk}\, \nabla_i\, u_j\,. \tag{17.2}$$

The sub-vector ω^k has the dual antisymmetric sub-tensor

$$a_{ij} = \frac{1}{2} \left(\nabla_i\, u_j - \nabla_j\, u_i\right), \tag{17.3}$$

and so we have

$$\omega^k = \frac{1}{2}\, \varepsilon^{ijk} a_{ij}\,. \tag{17.4}$$

It easy to see that the sub-vector (17.2) and the sub-tensor (17.3) are co-vector and co-tensor, respectively. At the same time, we can introduce the chr.inv.-vector of the angular velocities ${}^*\omega^k$ and its chr.inv.-dual tensor ${}^*a_{ij}$, by replacing this co-vector with the chr.inv.-vector and co-differentiation with chr.inv.-differentiation in the equations (17.2) and (17.3)

$${}^*\omega^k = \frac{1}{2}\, {}^*r^k \left({}^*u\right), \tag{17.5}$$

$${}^*\omega^k = \frac{1}{2}\, \varepsilon^{ijk}\, {}^*\nabla_i\, {}^*u_j\,, \tag{17.6}$$

$${}^*a_{ij} = \frac{1}{2} \left({}^*\nabla_i\, {}^*u_j - {}^*\nabla_j\, {}^*u_i\right), \tag{17.7}$$

so we arrive at

$${}^*\omega^k = \frac{1}{2}\, \varepsilon^{ijk}\, {}^*a_{ij}\,. \tag{17.8}$$

Taking the chr.inv.-rotor from the chr.inv.-vector of the angular velocities, because of (16.18) or (16.19), we obtain

$${}^*r^i \left({}^*\omega\right) = {}^*\nabla_j {}^*a^{ij}, \tag{17.9}$$

or, in the alternative form

$$\varepsilon^{qki}\,{}^*\nabla_i\,{}^*\omega_k = {}^*\nabla_j\,{}^*a^{qj}, \tag{17.10}$$

where

$${}^*a^{ij} = \frac{1}{2}\left(h^{il}\,{}^*\nabla_l\,{}^*u^j - h^{jl}\,{}^*\nabla_l\,{}^*u^i\right). \tag{17.11}$$

We will have interest in that case where ${}^*u^i \equiv 0$ (10.6) holds at the world-point we are considering. Hence, $u^i \equiv 0$ (10.8) and $\frac{{}^*\partial\,{}^*u^i}{\partial x^k} = \frac{c^2}{c^2 - \mathrm{w}}\frac{\partial u^i}{\partial x^k}$ (10.9) must also hold at the point. In this case we have

$${}^*\omega^k = \frac{1}{2}\frac{c^2}{c^2 - \mathrm{w}}\,\varepsilon^{ijk} h_{jl}\frac{\partial u^l}{\partial x^i}, \tag{17.12}$$

$$\omega^k = \frac{1}{2}\,\varepsilon^{ijk} h_{jl}\frac{\partial u^l}{\partial x^i}. \tag{17.13}$$

Introducing the coordinates (10.11), where the condition $\tilde{\mathrm{w}} = 0$ (10.12) is true, this case gives

$${}^*\tilde{\omega}^k = \tilde{\omega}^k. \tag{17.14}$$

In the case we are considering, we introduce the special bond for the chr.inv.-rotor of the vector of the angular velocities

$${}^*r^i\left({}^*\omega\right) = R^i\left({}^*\omega\right). \tag{17.15}$$

§ 2.18 Differential rotations. Differential deformations

Because of (17.11), we have

$${}^*\nabla_j\,{}^*a^{ij} = \frac{1}{2}\left(h^{il}\,{}^*\nabla_{jl}\,{}^*u^j - h^{jl}\,{}^*\nabla_{jl}\,{}^*u^i\right). \tag{18.1}$$

In accordance with (15.6), we can write

$${}^*\nabla_{jl}\,{}^*u^i = {}^*\nabla_{lj}\,{}^*u^i + \frac{2}{c^2}A_{jl}\frac{{}^*\partial\,{}^*u^i}{\partial t} - H_{nlj}^{\cdots i}\,{}^*u^n \tag{18.2}$$

$${}^*\nabla_{jl}\,{}^*u^j = {}^*\nabla_{lj}\,{}^*u^j + \frac{2}{c^2}A_{jl}\frac{{}^*\partial\,{}^*u^j}{\partial t} - H_{nl}\,{}^*u^n. \tag{18.3}$$

For this reason, we have

$${}^*\nabla_j\,{}^*a^{ij} = \frac{1}{2}\,h^{il}\,{}^*\nabla_{lj}\,{}^*u^j + \frac{1}{c^2}A_{j\cdot}^{\cdot i}\frac{{}^*\partial\,{}^*u^j}{\partial t} - \frac{1}{2}\,H_n^i\,{}^*u^n\; -$$

$$-\frac{1}{2}\,h^{jl}\,{}^*\nabla_{lj}\,{}^*u^i = \frac{1}{2}\,{}^*\nabla_l\left(h^{il}\,{}^*\nabla_j\,{}^*u^j - h^{jl}\,{}^*\nabla_j\,{}^*u^i\right) + + \frac{1}{c^2}\,A_{j\cdot}^{\cdot i}\,\frac{{}^*\partial\,{}^*u^j}{\partial t} - \frac{1}{2}\,H_n^{\cdot i}\,{}^*u^n. \tag{18.4}$$

At the same time,

$${}^*\nabla_j\,{}^*u^i = h^{il}\,{}^*\nabla_j\,{}^*u_l = h^{il}\left({}^*\Delta_{jl} + {}^*a_{jl}\right) = {}^*\Delta_j^i + {}^*a_j^{\cdot i}\,, \tag{18.5}$$

where ${}^*\Delta_{jl}$ is derived from (10.4). Hence

$${}^*\nabla_j\,{}^*u^j = {}^*\Delta\,, \tag{18.6}$$

where

$${}^*\Delta = h^{ik}\,{}^*\Delta_{ik} = h_{ik}\,{}^*\Delta^{ik} = {}^*\Delta_j^j\,. \tag{18.7}$$

Therefore we arrive at

$${}^*\nabla_j\,{}^*a^{ij} = \frac{1}{2}\,{}^*\nabla_l\left(h^{il}\,{}^*\Delta - {}^*\Delta^{il} + {}^*a^{il}\right) + \frac{1}{c^2}\,A_{j\cdot}^{\cdot i}\,\frac{{}^*\partial\,{}^*u^j}{\partial t} - \frac{1}{2}\,H_n^{\cdot i}\,{}^*u^n, \tag{18.8}$$

$${}^*\nabla_j\,{}^*a^{ij} = {}^*\nabla_j\left(h^{ij}\,{}^*\Delta - {}^*\Delta^{ij}\right) + \frac{2}{c^2}\,A_{j\cdot}^{\cdot i}\,\frac{{}^*\partial\,{}^*u^j}{\partial t} - H_j^{\cdot i}\,{}^*u^j, \tag{18.9}$$

or, in the alternative form

$${}^*r^i\left({}^*\omega\right) = {}^*\nabla_j\left(h^{ij}\,{}^*\Delta - {}^*\Delta^{ij}\right) + \frac{2}{c^2}\,A_{j\cdot}^{\cdot i}\,\frac{{}^*\partial\,{}^*u^j}{\partial t} - H_j^{\cdot i}\,{}^*u^j. \tag{18.10}$$

§2.19 Differential rotations of a space

We assume that ${}^*\Delta^{ij}$ is the contravariant chr.inv.-tensor of the deformation rates of the space near the point

$$x^i = a^i, \tag{19.1}$$

measured with respect to the reference frame that is locally-stationary at this point. We can then write

$${}^*\Delta^{ij} = {}^*\Delta^{ij}\left(t;\, x^1,\, x^2,\, x^3;\, \xi^1,\, \xi^2,\, \xi^3\right), \tag{19.2}$$

where

$$\xi^k = x^k - a^k. \tag{19.3}$$

Let us suppose that for any numerical value of the time coordinate in the time interval we are considering, it is possible to find such a coordinate system (of this reference frame), at the point (19.1) of which all the functions (19.2) and their first derivative with respect to t and also all the x^k are continuous with respect to all the ξ^l. In this coordinate frame we have*

$$\left(\frac{\partial\,{}^*\!\Delta^{ij}}{\partial x^k}\right)_0 = \frac{\partial\left({}^*\!\Delta^{ij}\right)_0}{\partial x^k}\,, \tag{19.4}$$

$$\left(\frac{\partial\,{}^*\!\Delta^{ij}}{\partial t}\right)_0 = \frac{\partial\left({}^*\!\Delta^{ij}\right)_0}{\partial t}\,, \tag{19.5}$$

and hence

$$\left(\frac{{}^*\partial\,{}^*\!\Delta^{ij}}{\partial x^k}\right)_0 = \frac{{}^*\partial\left({}^*\!\Delta^{ij}\right)_0}{\partial x^k}\,, \tag{19.6}$$

At the same time, the definition of the chr.inv.-tensor D_{pq} gives

$$\left({}^*\!\Delta_{pq}\right)_0 = D_{pq} \tag{19.7}$$

and hence

$$\left({}^*\!\Delta^{ij}\right)_0 = D^{ij}. \tag{19.8}$$

Therefore we have

$$\left(\frac{{}^*\partial\,{}^*\!\Delta^{ij}}{\partial x^k}\right)_0 = \frac{\partial D^{ij}}{\partial x^k}\,. \tag{19.9}$$

Because of (19.9) and (19.8), we have

$$\left({}^*\nabla_k\,{}^*\!\Delta^{ij}\right)_0 = {}^*\nabla_k\, D^{ij}. \tag{19.10}$$

Equality (19.10), owing to its chr.inv.-tensor nature, is valid in all coordinate frames of the reference frame, not only in that specific coordinate system we have been considering from the beginning of this section. Since the point (19.10)is arbitrary, equality (19.10) holds for all points (under the continuity conditions we have formulated above). The equality (19.10) gives

$$\left({}^*\nabla_k\,{}^*\!\Delta\right)_0 = {}^*\nabla_k\, D\,, \tag{19.11}$$

$$\left({}^*\nabla_j\,{}^*\!\Delta^{ij}\right)_0 = {}^*\nabla_j\, D^{ij}, \tag{19.12}$$

and therefore we have

$$\left[{}^*\nabla_j\left(h^{ij}\,{}^*\!\Delta - {}^*\!\Delta^{ij}\right)\right]_0 = {}^*\nabla_j\left(h^{ij}D - D^{ij}\right). \tag{19.13}$$

*The subscript zero indicates that all the ξ^k equal zero.

Let us again consider the neighbourhood of any point (19.1). We assume that in an arbitrary point in the neighbourhood, $^*u^j$ is the chr.inv.-velocity of the space with respect to the locally-stationary system at the given point (19.1). As found before, $^*\Delta^{ij}$ is the chr.inv.-tensor of the deformation rates of the space with respect to the aforementioned locally-stationary system, $^*\omega_k$ is the chr.inv.-vector of the angular velocities of the space rotations with respect to the same system and $^*a^{ij}$ is the antisymmetric chr.inv.-tensor, dual to the chr.inv.-vector $^*\omega_k$. The quantities are related by equations (18.9) and (18.10). At the point (19.1) itself, we have

$$\xi^i \equiv 0\,, \tag{19.14}$$

$$^*u^i \equiv 0 \tag{19.15}$$

and hence

$$\left(^*\nabla_j\, {}^*a^{ij}\right)_0 = \left[^*\nabla_j \left(h^{ij}\, {}^*\Delta - {}^*\Delta^{ij}\right)\right]_0\,, \tag{19.16}$$

$$R^i\left(^*\omega\right) = \left[^*\nabla_j \left(h^{ij}\, {}^*\Delta - {}^*\Delta^{ij}\right)\right]_0\,. \tag{19.17}$$

Because of (19.13), we can finally write

$$\left(^*\nabla_j\, {}^*a^{ij}\right)_0 = {}^*\nabla_j \left(h^{ij} D - D^{ij}\right), \tag{19.18}$$

$$R^i\left(^*\omega\right) = {}^*\nabla_j \left(h^{ij} D - D^{ij}\right). \tag{19.19}$$

It is apparent from this algebra that the fact that the locally-stationary reference frame at the point is not unique, does not affect our conclusions*. This result also holds in the case where the locally-stationary system does not rotate at this point, with respect to the space. In this case, at the point, we have

$$^*\omega_k = 0\,, \tag{19.20}$$

$$^*a^{ij} = 0\,. \tag{19.21}$$

§ 2.20 The system of locally independent quantities

Considering the problem of the space curvature, we will need to use a number of suppositions. We will give the suppositions here in a more general form than needed for solving the aforementioned

*As mentioned in §2.11, the locally-stationary system is defined in order of its arbitrary rotation near the point.

problem. In accordance with §2.4, at any given world-point of any reference frame, where the spatial coordinate frame is fixed, the potentials can be set to any preassigned numerical values by appropriate transformations of the time coordinate*

It easy to see that we cannot set the first derivatives of the potentials in this way, because the 16 derivatives are related by 6 conditions, such that the power quantities are invariant with respect to transformations of the time coordinate

$$\widetilde{F}_i = F_i\,, \tag{20.1}$$

$$\widetilde{A}_{ik} = A_{ik}\,. \tag{20.2}$$

It is evident that we can superimpose only 10 additional conditions on the 16 derivatives of the potentials. For this reason we introduce 10 chr.inv.-quantities, which, being dependent on the potentials and their first derivatives, can take any numerical values at any given world-point as a result of the transformations of the time coordinate. Since we have interest in the first derivatives of the potentials, we can write

$$\left.\begin{aligned} \tilde{x}^0 &= \frac{1}{B}\Big(B_\alpha \xi^\alpha + \frac{1}{2}B_{\beta\gamma}\,\xi^\beta\xi^\gamma\Big) \\ \tilde{x}^i &= x^i \end{aligned}\right\}, \tag{20.3}$$

where B, B_α, and $B_{\beta\gamma}$ are constants and $B_0 = 1$,

$$\xi^\alpha = x^\alpha - a^\alpha, \tag{20.4}$$

while coordinates of the world-point are

$$x^\alpha = a^\alpha. \tag{20.5}$$

It is evident that in general,

$$\frac{\partial\tilde{x}^0}{\partial x^\sigma} = \frac{B_\sigma}{B} + \frac{B_{\beta\sigma}}{B}\,\xi^\beta\,, \qquad \frac{\partial^2\tilde{x}^0}{\partial x^\mu\partial x^\nu} = \frac{B_{\mu\nu}}{B}\,, \tag{20.6}$$

so at the world-point we have

$$\left(\frac{\partial\tilde{x}^0}{\partial x^\sigma}\right)_a = \frac{B_\sigma}{B}\,, \qquad \left(\frac{\partial^2\tilde{x}^0}{\partial x^\mu\partial x^\nu}\right)_a = \frac{B_{\mu\nu}}{B}\,. \tag{20.7}$$

*In this consideration, we mean that the scalar potential w cannot take numerical values greater than or equal to c^2.

Because

$$g_{00}=\tilde{g}_{00}\left(\frac{\partial\tilde{x}^0}{\partial x^0}\right)^2, \quad g_{0i}=\left(\tilde{g}_{00}\frac{\partial\tilde{x}^0}{\partial x^i}+\tilde{g}_{0i}\right)\frac{\partial\tilde{x}^0}{\partial x^0}\,, \tag{20.8}$$

we have

$$c^2-\widetilde{\mathrm{w}}=B\frac{c^2-\mathrm{w}}{1+B_{0\beta}\,\xi^\beta}\,, \quad \tilde{v}_i=v_i+\frac{1}{c}\left(c^2-\mathrm{w}\right)\frac{B_i+B_{i\gamma}\,\xi^\gamma}{1+B_{0\beta}\,\xi^\beta}\,, \tag{20.9}$$

$$\left(c^2-\widetilde{\mathrm{w}}\right)_a=B\left(c^2-\mathrm{w}\right)_a\,, \quad \left(\tilde{v}_i\right)_a=\left(v_i\right)_a+\frac{B_i}{c}\left(c^2-\mathrm{w}\right)_a\,. \tag{20.10}$$

Next, let us consider the chr.inv.-derivatives and the chr.inv.-covariant derivatives of the potentials. First, this is

$$\frac{^*\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^0}=\frac{^*\partial\widetilde{\mathrm{w}}}{\partial x^0}=\frac{B}{1+B_{0\beta}\,\xi^\beta}\,\frac{c^2}{c^2-\mathrm{w}}\,\frac{\partial\mathrm{w}}{\partial x^0}+c^2\frac{BB_{00}}{\left(1+B_{0\beta}\,\xi^\beta\right)^2}\,. \tag{20.11}$$

At the world-point, eliminating B, we find

$$\frac{c^2}{\left(c^2-\widetilde{\mathrm{w}}\right)_a^2}\left(\frac{^*\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^0}\right)_a=\frac{c^2}{\left(c^2-\mathrm{w}\right)_a^2}\left(\frac{^*\partial\mathrm{w}}{\partial x^0}\right)_a+\frac{c^2}{\left(c^2-\mathrm{w}\right)_a}B_{00}\,. \tag{20.12}$$

We denote

$$Y=\frac{c^2}{(c^2-\mathrm{w})^2}\,\frac{\partial\mathrm{w}}{\partial x^0}\,, \tag{20.13}$$

and then we can write

$$(\widetilde{Y})_a=(Y)_a+\frac{c^2}{\left(c^2-\mathrm{w}\right)_a^2}B_{00}\,. \tag{20.14}$$

Hence,

$$\begin{aligned}\frac{^*\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^i}&=\frac{^*\partial\widetilde{\mathrm{w}}}{\partial x^i}=\frac{\partial\widetilde{\mathrm{w}}}{\partial x^i}+\frac{cv_i}{c^2-\mathrm{w}}\,\frac{\partial\widetilde{\mathrm{w}}}{\partial x^0}=\frac{B}{1+B_{0\beta}\,\xi^\beta}\,\frac{\partial\mathrm{w}}{\partial x^i}+\\&+\frac{(c^2-\mathrm{w})BB_{0i}}{\left(1+B_{0\beta}\,\xi^\beta\right)^2}+\frac{cv_i}{c^2-\mathrm{w}}\,\frac{B}{\left(1+B_{0\beta}\,\xi^\beta\right)}\,\frac{\partial\mathrm{w}}{\partial x^0}+\\&+\frac{cv_iBB_{00}}{\left(1+B_{0\beta}\,\xi^\beta\right)^2}=\frac{B}{1+B_{0\beta}\,\xi^\beta}\left(\frac{\partial\mathrm{w}}{\partial x^i}+\frac{cv_i}{c^2-\mathrm{w}}\,\frac{\partial\mathrm{w}}{\partial x^0}\right)+\\&+\frac{(c^2-\mathrm{w})B}{\left(1+B_{0\beta}\,\xi^\beta\right)^2}\left(B_{0i}+\frac{cv_i}{c^2-\mathrm{w}}B_{00}\right).\end{aligned} \tag{20.15}$$

Because it is evident that

$$\left.\begin{aligned} \frac{c^2}{c^2-\widetilde{\mathrm{w}}}\left(\frac{\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^i}+\frac{c\tilde{v}_i}{c^2-\widetilde{\mathrm{w}}}\frac{\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^0}\right)=\widetilde{F}_i+\frac{c^2}{c^2-\widetilde{\mathrm{w}}}\frac{c\,\partial\tilde{v}_i}{\partial\tilde{x}^0}+c\tilde{v}_i\widetilde{Y} \\ \frac{c^2}{c^2-\mathrm{w}}\left(\frac{\partial\mathrm{w}}{\partial x^i}+\frac{c\,v_i}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^0}\right)=F_i+\frac{c^2}{c^2-\mathrm{w}}\frac{c\,\partial v_i}{\partial x^0}+c\,v_iY \end{aligned}\right\},\tag{20.16}$$

we obtain

$$\begin{gathered}\left(\widetilde{F}_i+\frac{c^2}{c^2-\widetilde{\mathrm{w}}}\frac{c\partial\tilde{v}_i}{\partial\tilde{x}^0}+c\tilde{v}_i\widetilde{Y}\right)_a= \\ =\left(F_i+\frac{c^2}{c^2-\mathrm{w}}\frac{c\partial v_i}{\partial x^0}+c\,v_iY\right)+c^2\left(\frac{c\,v_i}{c^2-\mathrm{w}}\right)_a B_{00}+c^2B_{0i}\,.\end{gathered}\tag{20.17}$$

On the other hand

$$\begin{gathered}\frac{{}^*\partial\tilde{v}_i}{\partial\tilde{x}^0}=\frac{{}^*\partial\tilde{v}_i}{\partial x^0}=\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\tilde{v}_i}{\partial x^0}=\frac{c^2}{c^2-\mathrm{w}}\frac{\partial v_i}{\partial x^0}-\frac{c}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^0}\times \\ \times\frac{B_i+B_{i\gamma}\xi^\gamma}{1+B_{0\beta}\xi^\beta}+c\,\frac{(1+B_{0\beta}\xi^\beta)B_{0i}-B_{00}(B_i+B_{i\gamma}\xi^\gamma)}{(1+B_{0\beta}\xi^\beta)^2}\,.\end{gathered}\tag{20.18}$$

Because it is evident that

$$\left.\begin{aligned} \frac{c^2}{c^2-\widetilde{\mathrm{w}}}\frac{c\,\partial\tilde{v}_i}{\partial\tilde{x}^0}&=-\widetilde{F}_i+\frac{c^2}{c^2-\widetilde{\mathrm{w}}}\frac{\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^i} \\ \frac{c^2}{c^2-\mathrm{w}}\frac{c\,\partial v_i}{\partial x^0}&=-F_i+\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^i} \end{aligned}\right\},\tag{20.19}$$

we obtain

$$\begin{gathered}\left(-\widetilde{F}_i+\frac{c^2}{c^2-\widetilde{\mathrm{w}}}\frac{\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^i}\right)_a=\left(-F_i+\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^i}\right)_a- \\ -\left[\left(\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^0}\right)_a+c^2B_{00}\right]B_i+c^2B_{0i}\end{gathered}\tag{20.20}$$

and, taking (20.10), (20.13), and (20.14) into account, we arrive at

$$\begin{gathered}\left(-\widetilde{F}_i+\frac{c^2}{c^2-\widetilde{\mathrm{w}}}\frac{\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^i}+c\tilde{v}_i\widetilde{Y}\right)_a= \\ =\left(-F_i+\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^i}+c\,v_iY\right)_a+c^2\left(\frac{c\,v_i}{c^2-\mathrm{w}}\right)_a B_{00}+c^2B_{0i}\,.\end{gathered}\tag{20.21}$$

Summarizing (20.17) and (20.21) term-by-term, we obtain

$$\left[\frac{c^2}{c^2-\widetilde{\mathrm{w}}}\left(\frac{\partial\widetilde{\mathrm{w}}}{\partial\tilde{x}^i}+c\,\frac{\partial\tilde{v}_i}{\partial\tilde{x}^0}\right)+2c\tilde{v}_iY\right]_a=\left[\frac{c^2}{c^2-\mathrm{w}}\times\right.$$
$$\left.\times\left(\frac{\partial\mathrm{w}}{\partial x^i}+c\,\frac{\partial v_i}{\partial x^0}\right)+2cv_iY\right]_a+2c^2\left[\left(\frac{cv_i}{c^2-\mathrm{w}}\right)_a B_{00}+B_{0i}\right]. \tag{20.22}$$

We denote

$$\Phi_i=\frac{c^2}{c^2-\mathrm{w}}\left(\frac{\partial\mathrm{w}}{\partial x^i}+c\,\frac{\partial v_i}{\partial x^0}\right)+2cv_iY\,, \tag{20.23}$$

then

$$\left(\widetilde{\Phi}_i\right)_a=\left(\Phi_i\right)_a+2c^2\left[\left(\frac{cv_i}{c^2-\mathrm{w}}\right)_a B_{00}+B_{0i}\right]. \tag{20.24}$$

Taking the chr.inv.-covariant derivative of the vector potential, we obtain

$$\begin{aligned}
{}^*\widetilde{\nabla}_i\,\tilde{v}_k &= {}^*\nabla_i\,\tilde{v}_k=\frac{\partial\tilde{v}_k}{\partial x^i}+\frac{cv_i}{c^2-\mathrm{w}}\frac{\partial\tilde{v}_k}{\partial x^0}-{}^*\Delta^l_{ik}\tilde{v}_l=\\
&=\frac{\partial v_k}{\partial x^i}-\frac{1}{c}\frac{\partial\mathrm{w}}{\partial x^i}\frac{B_k+B_{k\gamma}\xi^\gamma}{1+B_{0\beta}\xi^\beta}+\\
&+\frac{c^2-\mathrm{w}}{c}\frac{(1+B_{0\beta}\xi^\beta)B_{ik}-B_{0i}(B_k+B_{k\gamma}\xi^\gamma)}{\left(1+B_{0\beta}\xi^\beta\right)^2}+\\
&+\frac{cv_i}{c^2-\mathrm{w}}\frac{\partial v_k}{\partial x^0}-\frac{v_i}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^0}\frac{B_k+B_{k\gamma}\xi^\gamma}{1+B_{0\beta}\xi^\beta}+\\
&+v_i\frac{(1+B_{0\beta}\xi^\beta)B_{0k}-B_{00}(B_k+B_{k\gamma}\xi^\gamma)}{\left(1+B_{0\beta}\xi^\beta\right)^2}-\\
&-{}^*\Delta^l_{ik}v_l-\frac{1}{c}\left(c^2-\mathrm{w}\right){}^*\Delta^l_{ik}\frac{B_l+B_{l\gamma}\xi^\gamma}{1+B_{0\beta}\xi^\beta}\,,
\end{aligned} \tag{20.25}$$

and, on the other hand

$$\begin{aligned}
&\left(\frac{\partial\tilde{v}_k}{\partial\tilde{x}^i}+\frac{c\tilde{v}_i}{c^2-\widetilde{\mathrm{w}}}\frac{\partial\tilde{v}_k}{\partial\tilde{x}^0}-{}^*\widetilde{\Delta}^l_{ik}\,\tilde{v}_l\right)_a=\\
&=\left(\frac{\partial v_k}{\partial x^i}+\frac{cv_i}{c^2-\mathrm{w}}\frac{\partial v_k}{\partial x^0}-{}^*\Delta^l_{ik}v_l\right)_a-\frac{(c^2-\mathrm{w})_a}{c^2}\times\\
&\times\left[\frac{c^2}{(c^2-\mathrm{w})_a}\left(\frac{\partial\mathrm{w}}{\partial x^i}+\frac{cv_i}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^0}\right)_a+c^2\left(\frac{cv_i}{c^2-\mathrm{w}}\right)_a B_{00}+\right.\\
&\left.+\,c^2B_{0i}\right]\frac{B_k}{c}+(v_i)_aB_{0k}+\frac{(c^2-\mathrm{w})_a}{c}\left[B_{ik}-\left({}^*\Delta^l_{ik}\right)_aB_l\right].
\end{aligned} \tag{20.26}$$

Owing to the symmetries on both sides

$$
\begin{aligned}
&\left[\frac{1}{2}\left(\frac{\partial \tilde{v}_k}{\partial \tilde{x}^i}+\frac{\partial \tilde{v}_i}{\partial \tilde{x}^k}\right)+\frac{1}{2c^2}\frac{c^2}{c^2-\widetilde{\mathrm{w}}}\left(\tilde{v}_i\frac{c\,\partial \tilde{v}_k}{\partial \tilde{x}^0}+\tilde{v}_k\frac{c\,\partial \tilde{v}_i}{\partial \tilde{x}^0}\right)-\right.\\
&\left.-\,{}^*\widetilde{\Delta}^l_{ik}\,\tilde{v}_l\right]_a=\left[\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}+\frac{\partial v_i}{\partial x^k}\right)+\frac{1}{2c^2}\frac{c^2}{c^2-\mathrm{w}}\times\right.\\
&\times\left.\left(v_i\frac{c\,\partial v_k}{\partial x^0}+v_k\frac{c\,\partial v_i}{\partial x^0}\right)-{}^*\Delta^l_{ik}\,v_l\right]_a-\frac{(c^2-\mathrm{w})_a}{2c^2}\times\\
&\times\left\{\left[\frac{c^2}{(c^2-\mathrm{w})_a}\left(\frac{\partial \mathrm{w}}{\partial x^i}+\frac{c\,v_i}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^0}\right)_a+c^2\left(\frac{c\,v_i}{c^2-\mathrm{w}}\right)_a B_{00}+\right.\right.\\
&\left.+\,c^2B_{0i}\right]\frac{B_k}{c}+\left[\frac{c^2}{(c^2-\mathrm{w})_a}\left(\frac{\partial \mathrm{w}}{\partial x^k}+\frac{c\,v_k}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^0}\right)_a+\right.\\
&\left.\left.+\,c^2\left(\frac{c\,v_k}{c^2-\mathrm{w}}\right)_a B_{00}+c^2B_{0k}\right]\frac{B_i}{c}\right\}+\frac{1}{2}\Big[\,(v_i)_a\,B_{0k}+\\
&+(v_k)_a\,B_{0i}\Big]+\frac{1}{c}\left(c^2-\mathrm{w}\right)_a\Big[B_{ik}-\left({}^*\Delta^l_{ik}\right)_a B_l\Big]\,.
\end{aligned}
\tag{20.27}
$$

Because of (20.10), (20.16), and (20.17), we have

$$
\begin{aligned}
&-\frac{(c^2-\mathrm{w})_a}{2c^2}\left[\frac{c^2}{(c^2-\mathrm{w})_a}\left(\frac{\partial \mathrm{w}}{\partial x^i}+\frac{c\,v_i}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^0}\right)_a+\right.\\
&\left.+\,c^2\left(\frac{c\,v_i}{c^2-\mathrm{w}}\right)_a B_{00}+c^2B_{0i}\right]\frac{B_k}{c}=\\
&=-\frac{1}{2c^2}\,(\tilde{v}_k)_a\left(\widetilde{F}_i+\frac{c^2}{c^2-\widetilde{\mathrm{w}}}\frac{c\,\partial \tilde{v}_i}{\partial \tilde{x}^0}+c\,\tilde{v}_iY\right)_a+\\
&+\frac{1}{2c^2}\,(v_k)_a\left(F_i+\frac{c^2}{c^2-\mathrm{w}}\frac{c\,\partial v_i}{\partial x^0}+c\,v_iY\right)_a+\\
&+\frac{1}{2}\left(\frac{c\,v_iv_k}{c^2-\mathrm{w}}\right)_a B_{00}+\frac{1}{2}\,(v_k)_a\,B_{0i}\,,
\end{aligned}
\tag{20.28}
$$

and so obtain

$$
\begin{aligned}
&\left\{\frac{1}{2}\left(\frac{\partial \tilde{v}_k}{\partial \tilde{x}^i}+\frac{\partial \tilde{v}_i}{\partial \tilde{x}^k}\right)+\frac{1}{2c^2}\Big[\tilde{v}_k\big(\widetilde{\Phi}_i-c\,\tilde{v}_i\widetilde{Y}\big)+\right.\\
&\left.+\,\tilde{v}_i\big(\widetilde{\Phi}_k-c\,\tilde{v}_k\widetilde{Y}\big)\Big]-{}^*\widetilde{\Delta}^l_{ik}\,\tilde{v}_l\right\}_a=\left\{\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}+\frac{\partial v_i}{\partial x^k}\right)+\right.
\end{aligned}
$$

$$+\frac{1}{2c^2}\Big[v_k\big(\Phi_i - c\,v_iY\big) + v_i\big(\Phi_k - c\,v_kY\big)\Big] - {}^*\!\Delta^l_{ik}v_l\Big\}_a +$$
$$+\left(\frac{c\,v_iv_k}{c^2-\mathrm{w}}\right)_a B_{00} + (v_k)_a B_{0i} + (v_i)_a B_{0k} + \qquad (20.29)$$
$$+\frac{(c^2-\mathrm{w})_a}{c}\Big[B_{ik} - \big({}^*\!\Delta^l_{ik}\big)_a B_l\Big],$$

or, in the alternative form

$$\left[\frac{1}{2}\left(\frac{\partial\tilde{v}_k}{\partial\tilde{x}^i} + \frac{\partial\tilde{v}_i}{\partial\tilde{x}^k}\right) + \frac{1}{2c^2}\big(\tilde{v}_k\widetilde{\Phi}_i + \tilde{v}_i\widetilde{\Phi}_k\big) - \right.$$
$$\left. - \frac{1}{c}\,\tilde{v}_i\tilde{v}_k\widetilde{Y} - {}^*\!\widetilde{\Delta}^l_{ik}\,\tilde{v}_l\right]_a = \left[\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} + \frac{\partial v_i}{\partial x^k}\right) + \right.$$
$$\left. + \frac{1}{2c^2}\big(v_k\Phi_i + v_i\Phi_k\big) - \frac{1}{c}\,v_iv_kY - {}^*\!\Delta^l_{ik}\,v_l\right]_a + \qquad (20.30)$$
$$+\left(\frac{c\,v_iv_k}{c^2-\mathrm{w}}\right)_a B_{00} + (v_k)_a B_{0i} + (v_i)_a B_{0k} +$$
$$+\frac{(c^2-\mathrm{w})_a}{c}\Big[B_{ik} - \big({}^*\!\Delta^l_{ik}\big)_a B_l\Big].$$

Denoting

$$X_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} + \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\big(v_i\Phi_k + v_k\Phi_i\big) -$$
$$- \frac{1}{c}\,v_iv_kY - {}^*\!\Delta^l_{ik}\,v_l\,, \qquad (20.31)$$

we can write

$$\big(\widetilde{X}_{ik}\big)_a = \big(X_{ik}\big)_a + \left(\frac{c\,v_iv_k}{c^2-\mathrm{w}}\right)_a B_{00} + (v_k)_a B_{0i} +$$
$$+ (v_i)_a B_{0k} + \frac{1}{c}\,(c^2-\mathrm{w})_a\Big[B_{ik} - \big({}^*\!\Delta^l_{ik}\big)_a B_l\Big]. \qquad (20.32)$$

No numerical values of w, v_i, Y, Φ_i, and X_{ik} at the world-point can prevent selection of the unique set of the coefficients B, B_i, $B_{\beta\gamma}$ giving the quantities $\widetilde{\mathrm{w}}$, $\tilde{v}_i$, $\widetilde{Y}$, $\widetilde{\Phi}_i$, $\widetilde{X}_{ik}$ any preassigned numerical values. Indeed, let us assume that $\widetilde{\mathrm{w}}$ has a numerical value. Then the first of the equalities (20.10) gives B. Assuming v_i, we obtain B_i from the second of the equalities (20.10). Assuming $\widetilde{Y}$, we obtain B_{00} from (20.14). Assuming Φ_i, we obtain B_{0i} from

(20.24) under any numerical value of B_{00}. Assuming X_{ik}, we obtain B_{ik} from (20.32) for any numerical values of B_{00}, B_{0l}, and B_l. Thus, any of our 14 quantities can be set to any numerical value at any given world-point independently of the numerical values of the other quantities. Therefore, we call the 14 quantities the *system of locally-independent quantities*.

It is easy to see that the quantity Y is sub-invariant, the 3 quantities Φ_i constitute a sub-vector and the 6 quantities X_{ik} constitute a symmetric sub-tensor of the 2nd rank.

§ 2.21 The chr.inv.-tensors of the space curvature

Let us select that part of the chr.inv.-Riemann-Christoffel tensor, which is antisymmetric with respect to its inner pair of indices, antisymmetric with respect to the outer pair of indices, and also symmetric with respect to rearrangements of the outer and inner pairs of indices. First, we have

$$H_{kjin} = \frac{1}{2}\left(H_{kjin} - H_{njik}\right) + \frac{1}{2}\left(H_{kjin} + H_{njik}\right), \qquad (21.1)$$

where (see 15.12)

$$\frac{1}{2}\left(H_{kjin} + H_{njik}\right) = \frac{2}{c^2}\,A_{ji}D_{kn}\,, \qquad (21.2)$$

and also

$$\frac{1}{2}\left(H_{kjin} - H_{njik}\right) = \frac{1}{2}\left(\frac{{}^*\partial\,{}^*\Delta_{ik,n}}{\partial x^j} - \frac{{}^*\partial\,{}^*\Delta_{in,k}}{\partial x^j} - \right. \\ \left. - \frac{{}^*\partial\,{}^*\Delta_{jk,n}}{\partial x^i} + \frac{{}^*\partial\,{}^*\Delta_{jn,k}}{\partial x^i}\right) - {}^*\Delta_{ik,l}\,{}^*\Delta^l_{jn} + {}^*\Delta_{jk,l}\,{}^*\Delta^l_{in}\,. \qquad (21.3)$$

Because

$$\frac{1}{2}\left(\frac{{}^*\partial\,{}^*\Delta_{ik,n}}{\partial x^j} - \frac{{}^*\partial\,{}^*\Delta_{in,k}}{\partial x^j} - \frac{{}^*\partial\,{}^*\Delta_{jk,n}}{\partial x^i} + \frac{{}^*\partial\,{}^*\Delta_{jn,k}}{\partial x^i}\right) = \\ = \frac{1}{4}\left(\frac{{}^*\partial^2 h_{kn}}{\partial x^j\partial x^i} + \frac{{}^*\partial^2 h_{in}}{\partial x^j\partial x^k} - \frac{{}^*\partial^2 h_{ik}}{\partial x^j\partial x^n} - \frac{{}^*\partial^2 h_{nk}}{\partial x^j\partial x^i} - \right. \\ - \frac{{}^*\partial^2 h_{ik}}{\partial x^j\partial x^n} + \frac{{}^*\partial^2 h_{in}}{\partial x^j\partial x^k} - \frac{{}^*\partial^2 h_{kn}}{\partial x^i\partial x^j} - \frac{{}^*\partial^2 h_{jn}}{\partial x^i\partial x^k} + \\ \left. + \frac{{}^*\partial^2 h_{jk}}{\partial x^i\partial x^n} + \frac{{}^*\partial^2 h_{nk}}{\partial x^i\partial x^j} + \frac{{}^*\partial^2 h_{jk}}{\partial x^i\partial x^n} - \frac{{}^*\partial^2 h_{jn}}{\partial x^i\partial x^k}\right) = \\ = \frac{1}{2}\left(\frac{{}^*\partial^2 h_{in}}{\partial x^j\partial x^k} + \frac{{}^*\partial^2 h_{jk}}{\partial x^i\partial x^n} - \frac{{}^*\partial^2 h_{ik}}{\partial x^j\partial x^n} - \frac{{}^*\partial^2 h_{jn}}{\partial x^i\partial x^k}\right), \qquad (21.4)$$

we have

$$\frac{1}{2}\left(H_{kjin}-H_{njik}\right)=\frac{1}{2}\left(\frac{{}^*\partial^2 h_{in}}{\partial x^j\partial x^k}+\frac{{}^*\partial^2 h_{jk}}{\partial x^i\partial x^n}-\right.$$
$$\left.-\frac{{}^*\partial^2 h_{ik}}{\partial x^j\partial x^n}-\frac{{}^*\partial^2 h_{jn}}{\partial x^i\partial x^k}\right)-{}^*\Delta_{ik,l}{}^*\Delta^l_{jn}+{}^*\Delta_{jk,l}{}^*\Delta^l_{in}\,. \tag{21.5}$$

Taking that part of the Riemann-Christoffel chr.inv.-Riemann-Christoffel tensor, which is antisymmetric with respect to its inner indices and outer pairs, we select that part which is symmetric with respect to rearrangements of the outer and inner pairs of the indices. As a result, we have

$$\frac{1}{2}\left(H_{kjin}-H_{njik}\right)=\frac{1}{2}\left[\frac{1}{2}\left(H_{kjin}-H_{njik}\right)+\frac{1}{2}\left(H_{jkni}-H_{iknj}\right)\right]+$$
$$+\frac{1}{2}\left[\frac{1}{2}\left(H_{kjin}-H_{njik}\right)-\frac{1}{2}\left(H_{jkni}-H_{iknj}\right)\right], \tag{21.6}$$

and it is easy to see,

$$\frac{1}{2}\left[\frac{1}{2}\left(H_{kjin}-H_{njik}\right)-\frac{1}{2}\left(H_{jkni}-H_{iknj}\right)\right]=$$
$$=\frac{1}{4}\left(\frac{{}^*\partial^2 h_{in}}{\partial x^j\partial x^k}+\frac{{}^*\partial^2 h_{jk}}{\partial x^i\partial x^n}-\frac{{}^*\partial^2 h_{ik}}{\partial x^j\partial x^n}-\frac{{}^*\partial^2 h_{jn}}{\partial x^i\partial x^k}-\right.$$
$$\left.-\frac{{}^*\partial^2 h_{ni}}{\partial x^k\partial x^j}-\frac{{}^*\partial^2 h_{kj}}{\partial x^n\partial x^i}+\frac{{}^*\partial^2 h_{nj}}{\partial x^k\partial x^i}+\frac{{}^*\partial^2 h_{ki}}{\partial x^n\partial x^j}\right)= \tag{21.7}$$
$$=\frac{1}{c^2}\left(A_{jk}D_{in}+A_{in}D_{jk}-A_{ik}D_{jn}-A_{jn}D_{ik}\right),$$

$$\frac{1}{2}\left[\frac{1}{2}\left(H_{kjin}-H_{njik}\right)+\frac{1}{2}\left(H_{jkni}-H_{iknj}\right)\right]=$$
$$=\frac{1}{4}\left(\frac{{}^*\partial^2 h_{in}}{\partial x^j\partial x^k}+\frac{{}^*\partial^2 h_{in}}{\partial x^k\partial x^j}\right)+\frac{1}{4}\left(\frac{{}^*\partial^2 h_{jk}}{\partial x^i\partial x^n}+\frac{{}^*\partial^2 h_{jk}}{\partial x^n\partial x^i}\right)-$$
$$-\frac{1}{4}\left(\frac{{}^*\partial^2 h_{ik}}{\partial x^j\partial x^n}+\frac{{}^*\partial^2 h_{ik}}{\partial x^n\partial x^j}\right)-\frac{1}{4}\left(\frac{{}^*\partial^2 h_{jn}}{\partial x^i\partial x^k}+\frac{{}^*\partial^2 h_{jn}}{\partial x^k\partial x^i}\right)- \tag{21.8}$$
$$-{}^*\Delta_{ik,l}{}^*\Delta^l_{jn}+{}^*\Delta_{jk,l}{}^*\Delta^l_{in}\,.$$

Denoting that part of the chr.inv.-Riemann-Christoffel tensor we have selected by S_{kjin}

$$S_{kjin} = \frac{1}{4}\left(\frac{{}^*\partial^2 h_{in}}{\partial x^j \partial x^k} + \frac{{}^*\partial^2 h_{in}}{\partial x^k \partial x^j}\right) + \\ + \frac{1}{4}\left(\frac{{}^*\partial^2 h_{jk}}{\partial x^i \partial x^n} + \frac{{}^*\partial^2 h_{jk}}{\partial x^n \partial x^i}\right) - \frac{1}{4}\left(\frac{{}^*\partial^2 h_{ik}}{\partial x^j \partial x^n} + \frac{{}^*\partial^2 h_{ik}}{\partial x^n \partial x^j}\right) - \\ - \frac{1}{4}\left(\frac{{}^*\partial^2 h_{jn}}{\partial x^i \partial x^k} + \frac{{}^*\partial^2 h_{jn}}{\partial x^k \partial x^i}\right) + {}^*\Delta^l_{in}\,{}^*\Delta_{jk,l} - {}^*\Delta^l_{ik}\,{}^*\Delta_{jn,l}\,, \tag{21.9}$$

we can write

$$H_{kjin} = S_{kjin} + \frac{1}{c^2}\big(2A_{ji}D_{kn} + A_{in}D_{jk} + \\ + A_{nj}D_{ik} + A_{jk}D_{in} + A_{ki}D_{jn}\big)\,. \tag{21.10}$$

Because H_{kjin}, A_{pq}, and D_{pq} are chr.inv.-tensors, S_{kjin} is also a chr.inv.-tensor. We will refer to S_{kjin} as the *chr.inv.-curvature tensor of the 4th rank*. Thus, the chr.inv.-Riemann-Christoffel tensor differs from the curvature chr.inv.-tensor of the 4th rank, while the Riemann-Christoffel co-tensor and the curvature co-tensor of the 4th rank are merely different names for the same sub-tensor, a formula for which can be written in the form

$$K_{kjin} = \frac{1}{2}\left(\frac{\partial^2 h_{in}}{\partial x^j \partial x^k} + \frac{\partial^2 h_{jk}}{\partial x^i \partial x^n} - \frac{\partial^2 h_{ik}}{\partial x^j \partial x^n} - \frac{\partial^2 h_{jn}}{\partial x^i \partial x^k}\right) + \\ + \Delta^l_{in}\Delta_{jk,l} - \Delta^l_{ik}\,\Delta_{jn,l}\,, \tag{21.11}$$

equivalent to (15.9) and like (21.9). Defining the quantity S_{kjin} as

$$S_{kjin} = -S_{kijn}\,, \tag{21.12}$$

$$S_{kjin} = -S_{njik}\,, \tag{21.13}$$

$$S_{kjin} = S_{jkni}\,, \tag{21.14}$$

we can see that the equalities are true, because of (21.9).

We also introduce the *chr.inv.-curvature tensor of the 2nd rank* (which is different from the chr.inv.-Einstein tensor)

$$S_{kj} = S_{kjin}h^{in} = S_{kjl}^{\cdots l}\,. \tag{21.15}$$

Formula (21.14) leads to that

$$S_{kj} = S_{jk}\,. \tag{21.16}$$

Because

$$S_{kjin} = \frac{1}{2}\left[\frac{1}{2}\left(H_{kjin} - H_{njik}\right) + \frac{1}{2}\left(H_{jkni} - H_{iknj}\right)\right], \tag{21.17}$$

and taking (21.2) into account, we have

$$S_{kjin} = \frac{1}{2}\left(H_{kjin} + H_{jkni}\right) - \frac{1}{c^2}\left(A_{ji}D_{kn} + A_{kn}D_{ji}\right), \tag{21.18}$$

$$S_{kj} = \frac{1}{2}\left(H_{kj} + H_{jk}\right) - \frac{1}{c^2}\left(A_{jl}D^l_k + A_{kl}D^l_j\right). \tag{21.19}$$

Owing to (15.21), we have

$$S_{kj} = H_{kj} - \frac{1}{c^2}\left(A_{jk}D + A_{jl}D^l_k + A_{kl}D^l_j\right), \tag{21.20}$$

$$H_{kj} = S_{kj} + \frac{1}{c^2}\left(A_{jk}D + A_{jl}D^l_k + A_{kl}D^l_j\right), \tag{21.21}$$

that can also be obtained directly from (21.10).

We also introduce the *chr.inv.-curvature invariant*

$$S = h^{kj}S_{kj}\,. \tag{21.22}$$

Formula (21.19) gives

$$S = H. \tag{21.23}$$

Because of the symmetric properties (21.12–21.14), which are analogous to the symmetric properties of the curvature co-tensor of the 4th rank, the number of the different components of S_{kjin} equals the number of the different components of K_{kjin}, i. e. there are only 6 different components. Therefore, by analogy with Ricci's contravariant and covariant tensors (see [8], p. 110)

$$C^{ab} = \frac{1}{4}\,\varepsilon^{aij}\varepsilon^{bkn}K_{kjin}\,, \tag{21.24}$$

$$C_{rs} = h_{ra}h_{sb}C^{ab}, \tag{21.25}$$

we can introduce the chr.inv.-tensors

$$Z^{ab} = \frac{1}{4}\,\varepsilon^{aij}\varepsilon^{bkn}S_{kjin}\,, \tag{21.26}$$

$$Z_{rs} = h_{ra}h_{sb}Z^{ab}, \tag{21.27}$$

which will be referred to as the *chr.inv.-Ricci tensors*. The Ricci co-tensor is linked to the curvature co-tensor of the 2nd rank and the curvature co-invariant by the equation

$$C_{rq} = K_{rq} - \frac{1}{2}\, h_{rq} K\,. \tag{21.28}$$

The same relation holds between the chr.inv.-Ricci tensor, the chr.inv.-curvature tensor of the 2nd rank, and the chr.inv.-curvature invariant

$$Z_{rq} = S_{rq} - \frac{1}{2}\, h_{rq} S\,. \tag{21.29}$$

We have

$$\begin{aligned} \varepsilon_{apq}\,\varepsilon_{brs}\,Z^{ab} &= \frac{1}{4}\,\varepsilon^{aij}\varepsilon_{apq}\,\varepsilon^{bkn}\varepsilon_{brs}S_{kjin} = \\ &= \frac{1}{4}\left(h^i_p h^j_q - h^i_q h^j_p\right)\left(h^k_r h^n_s - h^k_s h^n_r\right) S_{kjin} = \\ &= \frac{1}{4}\left(h^i_p h^j_q - h^i_q h^j_p\right)\left(S_{rjis} - S_{sjir}\right) = \\ &= \frac{1}{2}\left(h^i_p h^j_q - h^i_q h^j_p\right) S_{rjis} = \frac{1}{2}\left(S_{rqps} - S_{rpqs}\right) = S_{rqps}\,, \end{aligned} \tag{21.30}$$

hence

$$S_{rq} = h^{ps}\varepsilon_{apq}\,\varepsilon_{brs}\,Z^{ab}. \tag{21.31}$$

Because of (16.3), we have

$$h^{ps} = \frac{1}{2}\,\varepsilon^{ijp}\varepsilon^{kls}h_{ik}h_{jl}\,, \tag{21.32}$$

hence

$$\begin{aligned} S_{rq} &= \frac{1}{2}\,\varepsilon^{ijp}\varepsilon_{apq}\,\varepsilon^{kls}\varepsilon_{brs}\,h_{ik}h_{jl}Z^{ab} = \\ &= -\frac{1}{2}\left(h^i_a h^j_q - h^i_q h^j_a\right)\left(h^k_b h^l_r - h^k_r h^l_b\right) h_{ik}h_{jl}Z^{ab} = \\ &= -\frac{1}{2}\left(h_{ak}h_{ql} - h_{kq}h_{al}\right)\left(h^l_r Z^{ak} - h^k_r Z^{al}\right) = Z_{rq} - h_{rq}Z\,, \end{aligned} \tag{21.33}$$

$$Z = h^{ik}Z_{ik}\,, \tag{21.34}$$

$$S = -2Z\,. \tag{21.35}$$

Formulae (21.33) and (21.35) lead to (21.29), which is their consequence.

§ 2.22 On the curvature of space

We are going to find the relation between the chr.inv.-curvature tensor of the 4th rank and the curvature co-tensor of the 4th rank. First, we have

$$\begin{gathered}
\frac{{}^*\partial^2 h_{in}}{\partial x^j \partial x^k} = \frac{{}^*\partial}{\partial x^j}\left(\frac{\partial h_{in}}{\partial x^k} + \frac{2}{c^2}\, v_k\, D_{in}\right) = \frac{\partial^2 h_{in}}{\partial x^j \partial x^k} + \\
+ \frac{v_j}{c^2-\mathrm{w}}\frac{\partial^2 h_{in}}{\partial t\, \partial x^k} + \frac{2}{c^2}\left(\frac{\partial v_k}{\partial x^j} + \frac{v_j}{c^2-\mathrm{w}}\frac{\partial v_k}{\partial t}\right) D_{in} + \\
+ \frac{2}{c^2}\, v_k \frac{{}^*\partial D_{in}}{\partial x^j} = \frac{\partial^2 h_{in}}{\partial x^j \partial x^k} - \frac{2}{c^2}\frac{v_j}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^k}\, D_{in} + \\
+ \frac{2}{c^2}\, v_j\left(\frac{{}^*\partial D_{in}}{\partial x^k} - \frac{v_k}{c^2}\frac{{}^*\partial D_{in}}{\partial t}\right) + \frac{2}{c^2}\left(\frac{\partial v_k}{\partial x^j} + \frac{v_j}{c^2-\mathrm{w}}\frac{\partial v_k}{\partial t}\right) D_{in} + \\
+ \frac{2}{c^2}\, v_k \frac{{}^*\partial D_{in}}{\partial x^j} = \frac{\partial^2 h_{in}}{\partial x^j \partial x^k} + \frac{2}{c^2}\left(\frac{\partial v_k}{\partial x^j} - \frac{1}{c^2}\, F_k v_j\right) D_{in} + \\
+ \frac{2}{c^2}\left(v_j \frac{{}^*\partial D_{in}}{\partial x^k} + v_k \frac{{}^*\partial D_{in}}{\partial x^j}\right) - \frac{2}{c^4}\, v_j v_k \frac{{}^*\partial D_{in}}{\partial t}\,,
\end{gathered} \tag{22.1}$$

and also

$$\begin{gathered}
\frac{1}{2}\left(\frac{{}^*\partial^2 h_{in}}{\partial x^j \partial x^k} + \frac{{}^*\partial^2 h_{in}}{\partial x^k \partial x^j}\right) = \frac{\partial^2 h_{in}}{\partial x^j \partial x^k} + \frac{2}{c^2}\, \Psi_{jk}\, D_{in} + \\
+ \frac{2}{c^2}\left(v_j\, {}^*\nabla_k\, D_{in} + v_k\, {}^*\nabla_j\, D_{in}\right) - \frac{2}{c^4}\, v_j v_k \frac{{}^*\partial D_{in}}{\partial t} + \\
+ \frac{2}{c^2}\left({}^*\Delta^l_{jk}\, D_{in}\, v_l + {}^*\Delta^l_{ki}\, D_{nl}\, v_j + {}^*\Delta^l_{kn}\, D_{il}\, v_j + \right. \\
\left. + {}^*\Delta^l_{ji}\, D_{nl}\, v_k + {}^*\Delta^l_{jn}\, D_{il}\, v_k\right),
\end{gathered} \tag{22.2}$$

where we denote

$$\Psi_{jk} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^j} + \frac{\partial v_j}{\partial x^k}\right) - \frac{1}{2c^2}\left(F_k\, v_j + F_j\, v_k\right) - {}^*\Delta^l_{jk}\, v_l\,. \tag{22.3}$$

Therefore, we have

$$\begin{gathered}
\frac{1}{2}\left(\frac{\partial^2 h_{in}}{\partial x^j \partial x^k} + \frac{\partial^2 h_{jk}}{\partial x^i \partial x^n} - \frac{\partial^2 h_{ik}}{\partial x^j \partial x^n} - \frac{\partial^2 h_{jn}}{\partial x^i \partial x^k}\right) = \\
= \frac{1}{4}\left(\frac{{}^*\partial^2 h_{in}}{\partial x^j \partial x^k} + \frac{{}^*\partial^2 h_{in}}{\partial x^k \partial x^j}\right) + \frac{1}{4}\left(\frac{{}^*\partial^2 h_{jk}}{\partial x^i \partial x^n} + \frac{{}^*\partial^2 h_{jk}}{\partial x^n \partial x^i}\right) -
\end{gathered}$$

$$
\begin{aligned}
&-\frac{1}{4}\left(\frac{{}^*\partial^2 h_{ik}}{\partial x^j\partial x^n}+\frac{{}^*\partial^2 h_{ik}}{\partial x^n\partial x^j}\right)-\frac{1}{4}\left(\frac{{}^*\partial^2 h_{jn}}{\partial x^i\partial x^k}+\frac{{}^*\partial^2 h_{jn}}{\partial x^k\partial x^i}\right)-\\
&-\frac{1}{c^2}\left(\Psi_{jk}D_{in}+\Psi_{in}D_{jk}-\Psi_{jn}D_{ik}-\Psi_{ik}D_{jn}\right)-\\
&-\frac{1}{c^2}\left(v_j{}^*\nabla_k D_{in}+v_k{}^*\nabla_j D_{in}+v_i{}^*\nabla_n D_{jk}+v_n{}^*\nabla_i D_{jk}-\right.\\
&\left.-v_j{}^*\nabla_n D_{ik}-v_n{}^*\nabla_j D_{ik}-v_i{}^*\nabla_k D_{jn}-v_k{}^*\nabla_i D_{jn}\right)+\\
&+\frac{1}{c^4}\left(v_jv_k\frac{{}^*\partial D_{in}}{\partial t}+v_iv_n\frac{{}^*\partial D_{jk}}{\partial t}-v_jv_n\frac{{}^*\partial D_{ik}}{\partial t}-v_iv_k\frac{{}^*\partial D_{jn}}{\partial t}\right)-\\
&-\frac{1}{c^2}\left({}^*\Delta^l_{jk}D_{in}v_l+{}^*\Delta^l_{ki}D_{nl}v_j+{}^*\Delta^l_{kn}D_{il}v_j+{}^*\Delta^l_{ji}D_{nl}v_k+\right.\\
&+{}^*\Delta^l_{jn}D_{il}v_k+{}^*\Delta^l_{in}D_{jk}v_l+{}^*\Delta^l_{nj}D_{kl}v_i+{}^*\Delta^l_{nk}D_{jl}v_i+\\
&+{}^*\Delta^l_{ij}D_{kl}v_n+{}^*\Delta^l_{ik}D_{jl}v_n-{}^*\Delta^l_{jn}D_{ik}v_l-{}^*\Delta^l_{ni}D_{kl}v_j-\\
&-{}^*\Delta^l_{nk}D_{il}v_j-{}^*\Delta^l_{ji}D_{kl}v_n-{}^*\Delta^l_{jk}D_{il}v_n-{}^*\Delta^l_{ik}D_{jn}v_l-\\
&\left.-{}^*\Delta^l_{kj}D_{nl}v_i-{}^*\Delta^l_{kn}D_{jl}v_i-{}^*\Delta^l_{ij}D_{nl}v_k-{}^*\Delta^l_{in}D_{jl}v_k\right)=\\
&=\frac{1}{4}\left(\frac{{}^*\partial^2 h_{in}}{\partial x^j\partial x^k}+\frac{{}^*\partial^2 h_{in}}{\partial x^k\partial x^j}\right)+\frac{1}{4}\left(\frac{{}^*\partial^2 h_{jk}}{\partial x^i\partial x^n}+\frac{{}^*\partial^2 h_{jk}}{\partial x^n\partial x^i}\right)-\\
&-\frac{1}{4}\left(\frac{{}^*\partial^2 h_{ik}}{\partial x^j\partial x^n}+\frac{{}^*\partial^2 h_{ik}}{\partial x^n\partial x^j}\right)-\frac{1}{4}\left(\frac{{}^*\partial^2 h_{jn}}{\partial x^i\partial x^k}+\frac{{}^*\partial^2 h_{jn}}{\partial x^k\partial x^i}\right)-\\
&-\frac{1}{c^2}\left(\Psi_{jk}D_{in}+\Psi_{in}D_{jk}-\Psi_{jn}D_{ik}-\Psi_{ik}D_{jn}\right)-\\
&-\frac{1}{c^2}\Big[v_j\left({}^*\nabla_k D_{in}-{}^*\nabla_n D_{ik}\right)+v_k\left({}^*\nabla_j D_{in}-{}^*\nabla_i D_{jn}\right)+\\
&+v_i\left({}^*\nabla_n D_{jk}-{}^*\nabla_k D_{jn}\right)+v_n\left({}^*\nabla_i D_{jk}-{}^*\nabla_j D_{ik}\right)\Big]+\\
&+\frac{1}{c^4}\left(v_jv_k\frac{{}^*\partial D_{in}}{\partial t}+v_iv_n\frac{{}^*\partial D_{jk}}{\partial t}-v_jv_n\frac{{}^*\partial D_{ik}}{\partial t}-v_iv_k\frac{{}^*\partial D_{jn}}{\partial t}\right)+\\
&+\frac{1}{c^2}\Big[{}^*\Delta^l_{jk}\left(D_{il}v_n+D_{nl}v_i-D_{in}v_l\right)+{}^*\Delta^l_{in}\left(D_{jl}v_k+\right.\\
&\left.+D_{kl}v_j-D_{jk}v_l\right)-{}^*\Delta^l_{jn}\left(D_{il}v_k+D_{kl}v_i-D_{ik}v_l\right)-\\
&-{}^*\Delta^l_{ik}\left(D_{jl}v_n+D_{nl}v_j-D_{jn}v_l\right)\Big].
\end{aligned}
\tag{22.4}
$$

We then obtain

$$\Delta^l_{in}\Delta_{jk,l}-\Delta^l_{ik}\Delta_{jn,l}=\Big[{}^*\!\Delta^l_{in}-\frac{1}{c^2}\big(D^l_i v_n+D^l_n v_i-D_{in}v^l\big)\Big]\times$$
$$\times\Big[{}^*\!\Delta_{jk,l}-\frac{1}{c^2}\big(D_{jl}v_k+D_{kl}v_j-D_{jk}v_l\big)\Big]-$$
$$-\Big[{}^*\!\Delta^l_{ik}-\frac{1}{c^2}\big(D^l_i v_k+D^l_k v_i-D_{ik}v^l\big)\Big]\times$$
$$\times\Big[{}^*\!\Delta_{jn,l}-\frac{1}{c^2}\big(D_{jl}v_n+D_{nl}v_j-D_{jn}v_l\big)\Big]={}^*\!\Delta^l_{in}\,{}^*\!\Delta_{jk,l}-$$
$$-{}^*\!\Delta^l_{ik}\,{}^*\!\Delta_{jn,l}-\frac{1}{c^2}\,{}^*\!\Delta_{jk,l}\big(D^l_i v_n+D^l_n v_i-D_{in}v^l\big)- \qquad (22.5)$$
$$-\frac{1}{c^2}\,{}^*\!\Delta^l_{in}\big(D_{jl}v_k+D_{kl}v_j-D_{jk}v_l\big)+\frac{1}{c^2}\,{}^*\!\Delta_{jn,l}\big(D^l_i v_k+$$
$$+D^l_k v_i-D_{ik}v^l\big)+\frac{1}{c^2}\,{}^*\!\Delta^l_{ik}\big(D_{jl}v_n+D_{nl}v_j-D_{jn}v^l\big)+$$
$$+\frac{1}{c^4}\Big[\big(D_{jl}v_k+D_{kl}v_j-D_{jk}v_l\big)\big(D^l_i v_n+D^l_n v_i-D_{in}v^l\big)-$$
$$-\big(D_{jl}v_n+D_{nl}v_j-D_{jn}v_l\big)\big(D^l_i v_k+D^l_k v_i-D_{ik}v^l\big)\Big],$$

hence,

$$K_{kjin}=S_{kjin}-\frac{1}{c^2}\big(\Psi_{jk}D_{in}+\Psi_{in}D_{jk}-$$
$$-\Psi_{jn}D_{ik}-\Psi_{in}D_{jk}\big)-\frac{1}{c^2}\Big[v_j\,({}^*\nabla_k D_{in}-{}^*\nabla_n D_{ik})+$$
$$+v_k\,({}^*\nabla_j D_{in}-{}^*\nabla_i D_{jn})+v_i\,({}^*\nabla_n D_{jk}-{}^*\nabla_k D_{jn})+$$
$$+v_n\,({}^*\nabla_i D_{jk}-{}^*\nabla_j D_{ik})\Big]+\frac{1}{c^4}\left(v_j v_k\frac{{}^*\partial D_{in}}{\partial t}+\right.$$
$$\left.+v_i v_n\frac{{}^*\partial D_{jk}}{\partial t}-v_j v_n\frac{{}^*\partial D_{ik}}{\partial t}-v_i v_k\frac{{}^*\partial D_{jn}}{\partial t}\right)+$$
$$+\frac{1}{c^4}\big(-v_j v_k D_{il}D^l_n-v_i v_n D_{jl}D^l_k+v_j v_n D_{il}D^l_k+ \qquad (22.6)$$
$$+v_i v_k D_{jl}D^l_n\big)+\frac{1}{c^4}\Big[-v^l\big(D_{jl}v_k+D_{kl}v_j-D_{jk}v_l\big)D_{in}-$$
$$-v_l\big(D^l_j v_n+D^l_n v_j-D_{jn}v^l\big)D_{ik}+v^l\big(D_{il}v_n+D_{nl}v_i-$$
$$-D_{in}v_l\big)D_{jk}+v_l\big(D^l_i v_k+D^l_k v_i-D_{ik}v^l\big)D_{jn}\Big]+$$
$$+\frac{1}{c^4}\,v^l v_l\big(-D_{jk}D_{in}+D_{jn}D_{ik}\big).$$

Introducing

$$\Sigma_{jk} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^j} + \frac{\partial v_j}{\partial x^k}\right) - \frac{1}{2c^2}\left(F_k v_j + F_j v_k\right) - \Delta^l_{jk}\, v_l \tag{22.7}$$

instead of Ψ_{jk} , i. e.

$$\Psi_{jk} = \Sigma_{jk} - \frac{1}{c^2}\left(D^l_j v_k + D^l_k v_j - D_{jk} v^l\right) v_l\,, \tag{22.8}$$

and solving equation (22.6) with respect to S_{kjin}, we can finally write

$$\begin{gathered}
S_{kjin} = K_{kjin} + \frac{1}{c^2}\Big[\,\Sigma_{jk} D_{in} + \Sigma_{in} D_{jk} - \Sigma_{jn} D_{ik} - \\
- \Sigma_{ik} D_{jn}\Big] + \frac{1}{c^4}\, v_l v^l \Big[D_{jk} D_{in} - D_{jn} D_{ik}\Big] + \\
+ \frac{1}{c^4}\bigg[v_j v_k\bigg(D_{il} D^l_n - \frac{{}^*\partial D_{in}}{\partial t}\bigg) + v_i v_n\bigg(D_{jl} D^l_k - \frac{{}^*\partial D_{jk}}{\partial t}\bigg) - \\
- v_j v_n\bigg(D_{il} D^l_k - \frac{{}^*\partial D_{ik}}{\partial t}\bigg) - v_i v_k\bigg(D_{jl} D^l_n - \frac{{}^*\partial D_{jn}}{\partial t}\bigg)\bigg] + \\
+ \frac{1}{c^2}\Big[v_k\left({}^*\nabla_j D_{in} - {}^*\nabla_i D_{jn}\right) + v_j\left({}^*\nabla_k D_{in} - {}^*\nabla_n D_{ik}\right) + \\
+ v_i\left({}^*\nabla_n D_{jk} - {}^*\nabla_k D_{jn}\right) + v_n\left({}^*\nabla_i D_{jk} - {}^*\nabla_j D_{ik}\right)\Big].
\end{gathered} \tag{22.9}$$

Comparing formulae (22.7) and (20.31), we verify that the conditions

$$\left.\begin{aligned} \tilde{v}_i &= 0 \\ \widetilde{\Sigma}_{jk} &= 0 \end{aligned}\right\} \tag{22.10}$$

are equivalent to the conditions

$$\left.\begin{aligned} \tilde{v}_i &= 0 \\ \widetilde{X}_{jk} &= 0 \end{aligned}\right\}. \tag{22.11}$$

Hence, there are ∞^5 systems of the numerical values of the coefficients B, B_i, $B_{\beta\gamma}$ in (20.3) which, at the world-point we are considering, permit the conditions (22.10) and also the equality

$$S_{kjin} = \widetilde{K}_{kjin}\,. \tag{22.12}$$

Note that the conditions (22.10) or (22.11) are equivalent to the conditions

$$\left.\begin{aligned} &\tilde{g}_{0i} = 0 \\ &\frac{\partial \tilde{g}_{0k}}{\partial \tilde{x}^j} + \frac{\partial \tilde{g}_{0j}}{\partial \tilde{x}^k} = 0 \end{aligned}\right\}. \tag{22.13}$$

We now have a possibility of finding the geometrical definition of "the space curvature" (see the geometrical definition of space at the beginning of §2.2). We assume a world-point

$$x^\sigma = a^\sigma, \quad \sigma = 0, 1, 2, 3\,, \tag{22.14}$$

at which we take the spatial section

$$x^0 = a^0 \tag{22.15}$$

of the four-dimensional world. We define the metric in this spatial section by the sub-tensor

$$h_{ik} = -g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}}\,. \tag{22.16}$$

It is evident that the curvature of this spatial section will be defined by the sub-tensor K_{kjin}. Transforming the ime coordinate, we obtain another spatial section, fixed through the same world-point. In the new spatial section, the numerical values of h_{ik} at the world-point will be the same, while components of the curvature sub-tensor K_{kjin} will have, generally speaking, different numerical values. We will limit the circle of the spatial sections we are considering. We will consider only those spatial sections which are orthogonal (at this world-point) to the time lines of this reference frame. In other words, we will consider those spatial sections where, at the world-point,

$$\tilde{g}_{0i} = 0\,. \tag{22.17}$$

In this case we have

$$\tilde{v}_i = 0\,, \tag{22.18}$$

$$S_{kjin} = \widetilde{K}_{kjin} + \frac{1}{c^2}\Big[\widetilde{\Sigma}_{jk} D_{in} + \widetilde{\Sigma}_{in} D_{jk} - \widetilde{\Sigma}_{jn} D_{ik} - \widetilde{\Sigma}_{ik} D_{jn}\Big]. \tag{22.19}$$

Since by (22.18) we have

$$\widetilde{\Sigma}_{jk} = \widetilde{X}_{jk}\,, \tag{22.20}$$

hence, at this world-point,

$$S_{kjin} = K_{kjin} + \frac{1}{c^2}\Big[\widetilde{X}_{jk} D_{in} + \widetilde{X}_{in} D_{jk} - \widetilde{X}_{jn} D_{ik} - \widetilde{X}_{ik} D_{jn}\Big]. \qquad (22.21)$$

It is also evident that

$$S = \widetilde{K} + \frac{2}{c^2}\Big[\widetilde{X} D - \widetilde{X}^l_j D^j_l\Big], \qquad (22.22)$$

where

$$\widetilde{X} = \widetilde{X}_{ik} h^{ik}. \qquad (22.23)$$

Changing values of $\widetilde{X}_{ik}$, we change values of $\widetilde{K}_{kjin}$ and $\widetilde{K}$ as well. Hence, different spatial sections, which are orthogonal to time lines at the given world-point, have, generally speaking, different curvatures at the point. Moreover, both their Riemannian curvatures and their scalar (mean) curvatures are different. Now, we will consider only those spatial sections, where the conditions (22.10) are true at the world-point. We will refer to them as the *maximally orthogonal spatial sections*. In all the spatial sections, the conditions (22.10) and (22.12) are true. Therefore their curvatures (both the Riemannian curvature and the mean curvature) are the same.

Taking "space" in the sense of §2.2, we can give the geometric definition of its curvature* as follows:

> We will mean by the "curvature" of the space in a given world-point the curvature of any spatial section which is maximally orthogonal at this world-point.

This definition of the space curvature justifies the terminology we gave, in accordance with which the chr.inv.-tensors S_{kjin} and S_{kj} are the chr.inv.-curvature tensors, the chr.inv.-invariant S is the chr.inv.-curvature invariant, and Z_{kj} is the chr.inv.-Ricci tensor.

Next we assume that U^j is the chr.inv.-unit vector, which is orthogonal (at the given world-point) to the two-dimensional direction we are considering. Having the spatial section in the world-point fixed, we find that its Riemannian curvature $C_R(U)$ along the aforementioned direction is

$$C_R(U) = C_{rq} U^r U^q = K_{rq} U^r U^q - \frac{1}{2} K\,, \qquad (22.24)$$

and because of (21.28),

$$h_{rq} U^r U^q = 1\,. \qquad (22.25)$$

*The Riemannian curvature — along the two-dimensional direction, the scalar curvature — on the average.

Calculating the mean curvature C_N of the spatial section in this world-point, we obtain

$$C_N = \frac{1}{3}\,C = -\frac{1}{6}\,K\,, \tag{22.26}$$

where

$$C = h^{ik}C_{ik}\,. \tag{22.27}$$

So, because of what has been said above, we can write the Riemannian curvature ${}^*C_R$ of the space in the given world-point (along the two-dimensional direction we are considering) and also its mean curvature C_N in the point as follows

$${}^*C_R\,(U) = Z_{rq}\,U^r U^q = S_{rq}\,U^r U^q - \frac{1}{2}\,S \tag{22.28}$$

and

$${}^*C_N = \frac{1}{3}\,Z = -\frac{1}{6}\,S\,. \tag{22.29}$$

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Chapter 3

RELATIVISTIC PHYSICAL EQUATIONS

§ 3.1 The metric tensor

The main task of this section is to put the equations of the General Theory of Relativity into chronometrically invariant (chr.inv.-tensor) form. We will first consider the metric world-tensor.

In Chapter 2, we have obtained the covariant metric tensor equations (2.9), (2.10), and (8.9)

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2, \tag{1.1}$$

$$g_{0i} = -\left(1 - \frac{\mathrm{w}}{c^2}\right)\frac{v_i}{c}\,, \tag{1.2}$$

$$g_{ik} = -h_{ik} + \frac{v_i v_k}{c^2}\,, \tag{1.3}$$

the contravariant metric tensor equations (8.25), (8.24), (8.14)

$$g^{00} = \frac{1}{\left(1 - \frac{\mathrm{w}}{c^2}\right)^2}\left(1 - \frac{v_j v^j}{c^2}\right), \tag{1.4}$$

$$g^{0i} = -\frac{1}{1 - \frac{\mathrm{w}}{c^2}}\frac{v^i}{c}\,, \tag{1.5}$$

$$g^{ik} = -h^{ik}, \tag{1.6}$$

and the fundamental determinant g, equation (8.19)

$$\sqrt{-g} = \left(1 - \frac{\mathrm{w}}{c^2}\right)\sqrt{h}\,. \tag{1.7}$$

§ 3.2 Christoffel's symbols of the 1st kind

Because

$$\frac{\partial g_{00}}{\partial x^0} = -\frac{2}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^0}\,, \tag{2.1}$$

we have

$$\Gamma_{00,0} = -\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial t}\,. \tag{2.2}$$

and so we obtain

$$\frac{\partial g_{00}}{\partial x^i} = -\frac{2}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{2.3}$$

$$\frac{\partial g_{0i}}{\partial x^0} = \frac{1}{c^3}\,v_i\frac{\partial \mathrm{w}}{\partial x^0} - \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial v_i}{\partial x^0} \tag{2.4}$$

and also

$$\begin{gathered}\frac{\partial g_{0i}}{\partial x^0} - \frac{1}{2}\frac{\partial g_{00}}{\partial x^i} = \frac{1}{c^3}\,v_i\frac{\partial \mathrm{w}}{\partial x^0} + \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(\frac{\partial \mathrm{w}}{\partial x^i} - c\,\frac{\partial v_i}{\partial x^0}\right) = \\ = \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F_i + \frac{1}{c^4}\,v_i\frac{\partial \mathrm{w}}{\partial t}\,,\end{gathered} \tag{2.5}$$

because of formula (6.5) of §2.6. Thus, we arrive at

$$\Gamma_{00,i} = \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F_i + \frac{1}{c^4}\,v_i\frac{\partial \mathrm{w}}{\partial t}\,. \tag{2.6}$$

Because

$$\frac{\partial g_{00}}{\partial x^i} = -\frac{2}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{2.7}$$

we have

$$\Gamma_{0i,0} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,. \tag{2.8}$$

and so we obtain

$$\frac{\partial g_{0j}}{\partial x^i} = \frac{1}{c^3}\,v_j\frac{\partial \mathrm{w}}{\partial x^i} - \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial v_j}{\partial x^i}\,, \tag{2.9}$$

$$\frac{\partial g_{ij}}{\partial x^0} = -\frac{2}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)D_{ij} + \frac{1}{c^2}\left(v_j\frac{\partial v_i}{\partial x^0} + v_i\frac{\partial v_j}{\partial x^0}\right) \tag{2.10}$$

and, hence

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^0}+\frac{\partial g_{0j}}{\partial x^i}-\frac{\partial g_{0i}}{\partial x^j}\right)=-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\times\\
&\times\left[D_{ij}+\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}-\frac{\partial v_i}{\partial x^j}\right)\right]+\frac{1}{2c^2}\,v_j\frac{\partial v_i}{\partial x^0}-\frac{1}{2c^3}\,v_j\frac{\partial \mathrm{w}}{\partial x^i}+\\
&+\frac{1}{c^3}\,v_j\frac{\partial \mathrm{w}}{\partial x^i}-\frac{1}{2c^2}\,v_i\frac{\partial v_j}{\partial x^0}+\frac{1}{c^2}\,v_i\frac{\partial v_j}{\partial x^0}+\frac{1}{2c^3}\,v_i\frac{\partial \mathrm{w}}{\partial x^j}-\\
&-\frac{1}{c^3}\,v_i\frac{\partial \mathrm{w}}{\partial x^j}=-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[D_{ij}+\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}-\frac{\partial v_i}{\partial x^j}\right)+\right.\\
&\left.+\frac{1}{2c^2}\left(F_i v_j-F_j v_i\right)+\frac{1}{c^2}F_j v_i\right]+\frac{1}{c^3}\,v_j\frac{\partial \mathrm{w}}{\partial x^i}=\\
&=-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_{ij}+A_{ij}+\frac{1}{c^2}F_j v_i\right)+\frac{1}{c^3}\,v_j\frac{\partial \mathrm{w}}{\partial x^i}\,.
\end{aligned} \tag{2.11}$$

Thus, we arrive at

$$\Gamma_{0i,j}=-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_{ij}+A_{ij}+\frac{1}{c^2}F_j v_i\right)+\frac{1}{c^3}\,v_j\frac{\partial \mathrm{w}}{\partial x^i}\,. \tag{2.12}$$

Because of (2.9) and (2.10), we have

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial g_{0j}}{\partial x^i}+\frac{\partial g_{0i}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^0}\right)=\frac{1}{2c^3}\left(v_j\frac{\partial \mathrm{w}}{\partial x^i}+v_i\frac{\partial \mathrm{w}}{\partial x^j}\right)-\\
&-\frac{1}{2c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)+\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)D_{ij}-\\
&-\frac{1}{2c^2}\left(v_j\frac{\partial v_i}{\partial x^0}+v_i\frac{\partial v_j}{\partial x^0}\right)=\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\times\\
&\times\left[D_{ij}-\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)+\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right],
\end{aligned} \tag{2.13}$$

and hence

$$\Gamma_{ij,0}=\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[D_{ij}-\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)+\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right]. \tag{2.14}$$

Finally, because

$$\frac{\partial g_{ij}}{\partial x^k}=-\frac{\partial h_{ij}}{\partial x^k}+\frac{1}{c^2}\left(v_j\frac{\partial v_i}{\partial x^k}+v_i\frac{\partial v_j}{\partial x^k}\right), \tag{2.15}$$

we have

$$\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)=-\frac{1}{2}\left(\frac{\partial h_{jk}}{\partial x^i}+\frac{\partial h_{ik}}{\partial x^j}-\frac{\partial h_{ij}}{\partial x^k}\right)+ \\ +\frac{1}{2c^2}\left[v_i\left(\frac{\partial v_k}{\partial x^j}-\frac{\partial v_j}{\partial x^k}\right)+v_j\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)+\right. \\ \left.+v_k\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)\right]+\frac{1}{2c^4}\Big[v_i\left(F_j v_k-F_k v_j\right)+ \\ +v_j\left(F_i v_k-F_k v_i\right)-v_k\left(F_i v_j+F_j v_i\right)+2F_k v_i v_j\Big],\ . \tag{2.16}$$

and so

$$\Gamma_{ij,k}=-\Delta_{ij,k}+\frac{1}{c^2}\left[v_i A_{jk}+v_j A_{ik}+\frac{1}{2}\,v_k\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)-\right. \\ \left.-\frac{1}{2c^2}\,v_k\left(F_i v_j+F_j v_i\right)\right]+\frac{1}{c^4}\,F_k v_i v_j\,. \tag{2.17}$$

Formulae (2.2), (2.6), (2.8), (2.12), (2.14), and (2.17) collected in their final forms are

$$\Gamma_{00,0}=-\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial t}\,, \tag{2.18}$$

$$\Gamma_{00,i}=\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F_i+\frac{1}{c^4}\,v_i\frac{\partial \mathrm{w}}{\partial t}\,, \tag{2.19}$$

$$\Gamma_{0i,0}=-\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{2.20}$$

$$\Gamma_{0i,j}=-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_{ij}+A_{ij}+\frac{1}{c^2}F_j v_i\right)+\frac{1}{c^3}\,v_j\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{2.21}$$

$$\Gamma_{ij,0}=\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[D_{ij}-\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)+\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right], \tag{2.22}$$

$$\Gamma_{ij,k}=-\Delta_{ij,k}+\frac{1}{c^2}\left[v_i A_{jk}+v_j A_{ik}+\frac{1}{2}\,v_k\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)-\right. \\ \left.-\frac{1}{2c^2}\,v_k(F_i v_j+F_j v_i)\right]+\frac{1}{c^4}\,F_k v_i v_j\,. \tag{2.23}$$

§3.3 Christoffel's symbols of the 2nd kind

Because of (2.18) and (2.19), we have

$$\begin{gathered}g^{00}\Gamma_{00,0}+g^{0k}\Gamma_{00,k}=-\frac{1}{c^3}\,\frac{c^2}{c^2-\mathrm{w}}\left(1-\frac{v_j v^j}{c^2}\right)\frac{\partial\mathrm{w}}{\partial t}-\\ -\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)v^k F_k-\frac{1}{c^5}\,\frac{c^2}{c^2-\mathrm{w}}\,v_j v^j\,\frac{\partial\mathrm{w}}{\partial t}=\\ =-\frac{1}{c^3}\left[\frac{c^2}{c^2-\mathrm{w}}\,\frac{\partial\mathrm{w}}{\partial t}+\left(1-\frac{\mathrm{w}}{c^2}\right)v_k F^k\right],\end{gathered}\qquad(3.1)$$

hence

$$\Gamma^0_{00}=-\frac{1}{c^3}\left[\frac{c^2}{c^2-\mathrm{w}}\,\frac{\partial\mathrm{w}}{\partial t}+\left(1-\frac{\mathrm{w}}{c^2}\right)v_k F^k\right].\qquad(3.2)$$

Besides these, because

$$g^{k0}\Gamma_{00,0}+g^{kl}\Gamma_{00,l}=\frac{1}{c^4}\,v^k\,\frac{\partial\mathrm{w}}{\partial t}-\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2F^k-\frac{1}{c^4}\,v^k\,\frac{\partial\mathrm{w}}{\partial t}\,,\qquad(3.3)$$

we get

$$\Gamma^k_{00}=-\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2F^k.\qquad(3.4)$$

Because of (2.20) and (2.21)

$$\begin{gathered}g^{00}\Gamma_{0i,0}+g^{0k}\Gamma_{0i,k}=-\frac{1}{c^2}\,\frac{c^2}{c^2-\mathrm{w}}\left(1-\frac{v_j v^j}{c^2}\right)\frac{\partial\mathrm{w}}{\partial x^i}+\\ +\frac{1}{c^2}\,v^k\left(D_{ik}+A_{ik}+\frac{1}{c^2}\,F_k v_i\right)-\frac{1}{c^2}\,\frac{v_k v^k}{c^2-\mathrm{w}}\,\frac{\partial\mathrm{w}}{\partial x^i}=\\ =-\frac{1}{c^2}\,\frac{c^2}{c^2-\mathrm{w}}\,\frac{\partial\mathrm{w}}{\partial x^i}+\frac{1}{c^2}\,v_k\left(D^k_i+A_{i\cdot}^{\cdot k}+\frac{1}{c^2}\,v_i F^k\right),\end{gathered}\qquad(3.5)$$

we obtain

$$\Gamma^0_{0i}=\frac{1}{c^2}\left[-\frac{c^2}{c^2-\mathrm{w}}\,\frac{\partial\mathrm{w}}{\partial x^i}+v_k\left(D^k_i+A_{i\cdot}^{\cdot k}+\frac{1}{c^2}\,v_i F^k\right)\right]\qquad(3.6)$$

and also

$$g^{k0}\Gamma_{0i,0}+g^{kl}\Gamma_{0i,l}=\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D^k_i+A_{i\cdot}^{\cdot k}+\frac{1}{c^2}\,v_i F^k\right),\qquad(3.7)$$

$$\Gamma^k_{0i}=\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D^k_i+A_{i\cdot}^{\cdot k}+\frac{1}{c^2}\,v_i F^k\right).\qquad(3.8)$$

Because of (2.22) and (2.23), we obtain

$$\begin{aligned} & g^{00}\Gamma_{ij,0}+g^{0k}\Gamma_{ij,k}=\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\left(1-\frac{v_k v^k}{c^2}\right)\times \\ & \times\left[D_{ij}-\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)+\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right]+ \\ & +\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\,\Delta_{ij,k}\,v^k-\frac{1}{c^3}\,\frac{c^2}{c^2-\mathrm{w}}\,v^k\left\{v_i A_{jk}+v_j A_{ik}+\right. \\ & \left.+\,v_k\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)-\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right]\right\}- \\ & -\frac{1}{c^5}\,\frac{c^2}{c^2-\mathrm{w}}\,v^k F_k v_i v_j=\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\left(1-\frac{v_k v^k}{c^2}\right)D_{ij}- \\ & -\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\,\Sigma_{ij}-\frac{1}{c^3}\,\frac{c^2}{c^2-\mathrm{w}}\,v^k\left(v_i A_{jk}+v_j A_{ik}\right)- \\ & -\frac{1}{c^5}\,\frac{c^2}{c^2-\mathrm{w}}\,v^k F_k v_i v_j\,, \end{aligned} \tag{3.9}$$

that is

$$\begin{aligned} \Gamma^0_{ij}=-\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\bigg[\Sigma_{ij}-\left(1-\frac{v_k v^k}{c^2}\right)D_{ij}+ \\ +\frac{1}{c^2}\,v_k\left(v_i A_{j\cdot}^{\cdot k}+v_j A_{i\cdot}^{\cdot k}\right)+\frac{1}{c^4}\,v_i v_j v_k F^k\bigg]\,, \end{aligned} \tag{3.10}$$

where Σ_{ij} is defined by formula (22.7), §2.22. Thus, we obtain

$$\begin{aligned} & g^{k0}\Gamma_{ij,0}+g^{kl}\Gamma_{ij,\,l}=-\frac{1}{c^2}\,v^k\left[D_{ij}-\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)+\right. \\ & \left.+\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right]+\Delta^k_{ij}-\frac{1}{c^2}\left\{v_i A_{j\cdot}^{\cdot k}+v_j A_{i\cdot}^{\cdot k}+\right. \\ & \left.+\,v^k\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)-\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right]\right\}-\frac{1}{c^4}\,F^k v_i v_j= \\ & =\Delta^k_{ij}-\frac{1}{c^2}\,v^k D_{ij}-\frac{1}{c^2}\left(v_i A_{j\cdot}^{\cdot k}+v_j A_{i\cdot}^{\cdot k}\right)-\frac{1}{c^2}\,F^k v_i v_j\,, \end{aligned} \tag{3.11}$$

which gives

$$\Gamma^k_{ij}=\Delta^k_{ij}-\frac{1}{c^2}\left(D_{ij}\,v^k+v_i A_{j\cdot}^{\cdot k}+v_j A_{i\cdot}^{\cdot k}+\frac{1}{c^2}\,v_i v_j F^k\right). \tag{3.12}$$

We introduce Ψ_{ij} instead of Σ_{ij} into formula (3.10), and also ${}^*\Delta^k_{ij}$ instead of Δ^k_{ij} into (3.12). Then, according to formula (22.8) of

§2.22, we obtain

$$\Sigma_{ij} - \left(1 - \frac{v_k v^k}{c^2}\right) D_{ij} = \Psi_{ij} - \left(1 - \frac{v_k v^k}{c^2}\right) D_{ij} + \frac{1}{c^2}\left(D_i^k v_j + \right. \\ \left. + D_j^k v_i - D_{ij} v^k\right) v_k = \Psi_{ij} - D_{ij} + \frac{1}{c^2}\, v_k \left(D_j^k v_i + D_i^k v_j\right), \tag{3.13}$$

hence

$$\Gamma_{ij}^0 = -\frac{1}{c}\,\frac{c^2}{c^2 - \mathrm{w}} \left\{\Psi_{ij} - D_{ij} + \frac{1}{c^2}\, v_k \left[\left(D_j^k + A_{j\cdot}^{\cdot k}\right) v_i + \right.\right. \\ \left.\left. + \left(D_i^k + A_{i\cdot}^{\cdot k}\right) v_j + \frac{1}{c^2}\, v_i v_j F^k\right]\right\}. \tag{3.14}$$

Because, in accordance with formula (13.8) of §2.13,

$$\Delta_{ij}^k - \frac{1}{c^2}\, D_{ij} v^k = {}^*\Delta_{ij}^k - \frac{1}{c^2}\left(D_i^k v_j + D_j^k v_i\right), \tag{3.15}$$

we get

$$\Gamma_{ij}^k = {}^*\Delta_{ij}^k - \frac{1}{c^2}\left[v_i \left(D_j^k + A_{j\cdot}^{\cdot k}\right) + v_j \left(D_i^k + A_{i\cdot}^{\cdot k}\right) + \frac{1}{c^2}\, v_i v_j F^k\right]. \tag{3.16}$$

Formulae (3.2), (3.4), (3.6), (3.8), (3.14), and (3.16) collected in their final forms are

$$\Gamma_{00}^0 = -\frac{1}{c^3}\left[\frac{c^2}{c^2 - \mathrm{w}}\,\frac{\partial \mathrm{w}}{\partial t} + \left(1 - \frac{\mathrm{w}}{c^2}\right) v_k F^k\right], \tag{3.17}$$

$$\Gamma_{00}^k = -\frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 F^k, \tag{3.18}$$

$$\Gamma_{0i}^0 = \frac{1}{c^2}\left[-\frac{c^2}{c^2 - \mathrm{w}}\,\frac{\partial \mathrm{w}}{\partial x^i} + v_k \left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}\, v_i F^k\right)\right], \tag{3.19}$$

$$\Gamma_{0i}^k = \frac{1}{c}\left(1 - \frac{\mathrm{w}}{c^2}\right)\left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}\, v_i F^k\right), \tag{3.20}$$

$$\Gamma_{ij}^0 = -\frac{1}{c}\,\frac{c^2}{c^2 - \mathrm{w}} \left\{-D_{ij} + \frac{1}{c^2}\, v_k \left[v_i \left(D_j^k + A_{j\cdot}^{\cdot k}\right) + \right.\right. \\ \left. + v_j \left(D_i^k + A_{i\cdot}^{\cdot k}\right) + \frac{1}{c^2}\, v_i v_j F^k\right] + \frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) - \\ \left. - \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right) - {}^*\Delta_{ij}^k v_k\right\}, \tag{3.21}$$

$$\Gamma_{ij}^k = {}^*\Delta_{ij}^k - \frac{1}{c^2}\left[v_i \left(D_j^k + A_{j\cdot}^{\cdot k}\right) + v_j \left(D_i^k + A_{i\cdot}^{\cdot k}\right) + \frac{1}{c^2}\, v_i v_j F^k\right]. \tag{3.22}$$

We will apply the Christoffel symbols, when we deduce equations for the dynamics of a point-mass in order to find the mechanical sense of the quantities F_i and A_{jk}. However, before we do it that, we need to consider the problem of the square of the velocity of light, and also introduce the mass, energy, and momentum of a test point-body.

§3.4 The speed of light

It easy to see that we can re-write the equality

$$ds^2 = g_{00}\,dx^0dx^0 + 2\,g_{0i}\,dx^0dx^i + g_{ik}\,dx^idx^k \tag{4.1}$$

in the form

$$ds^2 = \left(\sqrt{g_{00}}\,dx^0 + \frac{g_{0i}}{\sqrt{g_{00}}}dx^i\right)^2 + \left(g_{ik} - \frac{g_{0i}\,g_{0k}}{g_{00}}\right)dx^idx^k, \tag{4.2}$$

or, in the alternative form

$$ds^2 = \left(\frac{dx_0}{\sqrt{g_{00}}}\right)^2 - h_{ik}\,dx^idx^k. \tag{4.3}$$

Hence, we have

$$\left(\sqrt{g_{00}}\,\frac{ds}{dx_0}\right)^2 = 1 - h_{ik}\left(\sqrt{g_{00}}\,\frac{dx^i}{dx_0}\right)\left(\sqrt{g_{00}}\,\frac{dx^k}{dx_0}\right) \tag{4.4}$$

and, in accordance with §2.9,

$$\sqrt{g_{00}}\,\frac{dx^i}{dx_0} = \frac{{}^*u^i}{c}\,. \tag{4.5}$$

We therefore obtain

$$\left(\sqrt{g_{00}}\,\frac{ds}{dx_0}\right)^2 = 1 - h_{ik}\frac{{}^*u^i\,{}^*u^k}{c^2} \tag{4.6}$$

or, in the alternative form

$$\left(\sqrt{g_{00}}\,\frac{ds}{dx_0}\right)^2 = 1 - \frac{{}^*u_j\,{}^*u^j}{c^2}\,. \tag{4.7}$$

For light rays we have

$$ds = 0 \tag{4.8}$$

and, hence

$${}^*u_j\,{}^*u^j = c^2. \tag{4.9}$$

So the square of the chr.inv.-vector for the velocity of light equals c^2 in emptiness.

§ 3.5 A point-body. Its mass, energy, and the momentum

Using the transformations (4.9) of §2.4, we can set the potentials to zero at any given world-point. Then, having the potentials zero at the point, we can write equations for the mass m, energy E, and momentum p of a point-mass analogous to the appropriate equations of the Special Theory of Relativity. Namely, we can write

$$m = m_0 \frac{d\tilde{x}^0}{ds} \,, \tag{5.1}$$

$$E = mc^2, \tag{5.2}$$

$$p^i = m_0 \frac{c\, d\tilde{x}^i}{ds} \,. \tag{5.3}$$

At the given world-point in the coordinates $\tilde{x}^\sigma$ ($\sigma = 0, 1, 2, 3$) we have chosen we have

$$\tilde{g}_{00} = 1 \,, \qquad \tilde{g}_{0i} = 0 \tag{5.4}$$

and hence

$$d\tilde{x}^0 = d\tilde{x}_0 = \frac{d\tilde{x}_0}{\sqrt{\tilde{g}_{00}}} \,, \tag{5.5}$$

and so we obtain

$$m = \frac{m_0}{\sqrt{\tilde{g}_{00}}} \frac{d\tilde{x}_0}{ds} \,. \tag{5.6}$$

At the same time the quantity

$$\frac{d\tilde{x}_0}{\sqrt{\tilde{g}_{00}}} = \frac{dx_0}{\sqrt{g_{00}}} \tag{5.7}$$

is a chr.inv.-invariant in any arbitrary system of the spatial coordinates x^σ ($\sigma = 0, 1, 2, 3$) of the given reference frame. For this reason, in general, we have

$$m = \frac{m_0}{\sqrt{g_{00}}} \frac{dx_0}{ds} \tag{5.8}$$

and, because of (4.7),

$$m = \frac{m_0}{\sqrt{1 - \frac{{}^*u_i {}^*u^i}{c^2}}} \,. \tag{5.9}$$

We have thus obtained equations for the chr.inv.-invariant of moving (relativistic) mass. The chr.inv.-invariant of energy can be expressed by formula (5.2). So, with §3.4 as a basis, we can say that the energy of a point-mass equals its dynamic mass, multiplied by the square of the velocity of light.

Because $d\tilde{x}^i$ is a chr.inv.-vector, i. e.

$$d\tilde{x}^i = dx^i, \tag{5.10}$$

in the general case we have

$$p^i = m_0 \frac{c\,dx^i}{ds}\,. \tag{5.11}$$

This is the equation of the chr.inv.-vector of momentum. As we have obtained

$$\frac{dx^i}{ds} = \left(\sqrt{g_{00}}\,\frac{dx^i}{dx_0}\right)\left(\frac{1}{\sqrt{g_{00}}}\frac{dx_0}{ds}\right), \tag{5.12}$$

then, taking (4.5) into account, we arrive at

$$p^i = m^* u^i. \tag{5.13}$$

§3.6 Transformations of the energy and momentum of a point-body

With our results for chr.inv.-derivatives and chr.inv.-velocities a basis, we can say that the operator, invariant with respect to transformations of time the coordinate and also, with the potentials are set to zero, coincident with the operator of the total derivative with respect to the time coordinate

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u^j \frac{\partial}{\partial x^j}\,, \tag{6.1}$$

take the form

$$\frac{^*d}{dt} = \frac{^*\partial}{\partial t} + {}^*u^j \frac{^*\partial}{\partial x^j}\,. \tag{6.2}$$

We will refer to the operator as the *operator of the total derivative with respect to the time coordinate*.

Let us find the relation between the chr.inv.-total derivative with respect to time, on one the hand, and, on the other hand, the derivative with respect to local time and the partial derivative with respect to time

$$\begin{aligned} \frac{^*d}{dt} &= \frac{c^2}{c^2-\mathrm{w}}\frac{\partial}{\partial t} + \frac{c^2 u^j}{c^2-\mathrm{w}-v_k u^k}\frac{\partial}{\partial x^j} + \frac{v_j}{c^2-\mathrm{w}}\,{}^*u^j\frac{\partial}{\partial t} = \\ &= \frac{c^2}{c^2-\mathrm{w}}\left(1+\frac{1}{c^2}\,v_j\,{}^*u^j\right)\frac{\partial}{\partial t} + \frac{c^2}{c^2-\mathrm{w}}\frac{c^2-\mathrm{w}}{c^2-\mathrm{w}-v_k u^k}\,u^j\frac{\partial}{\partial x^j}\,. \end{aligned} \tag{6.3}$$

We know that

$$1+\frac{1}{c^2}\,v_j{}^*u^j = 1+\frac{v_j u^j}{c^2-\mathrm{w}-v_k u^k} = \frac{c^2-\mathrm{w}}{c^2-\mathrm{w}-v_j u^j}\,, \tag{6.4}$$

hence we have

$$\frac{^*d}{dt} = \frac{c^2}{c^2-\mathrm{w}}\left(1+\frac{1}{c^2}\,v_j{}^*u^j\right)\frac{d}{dt} \tag{6.5}$$

or, in the alternative form

$$\frac{^*d}{dt} = \frac{1}{1-\frac{1}{c^2}\left(\mathrm{w}+v_j u^j\right)}\,\frac{d}{dt}\,. \tag{6.6}$$

Because

$$\begin{gathered}\frac{dx_0}{dx^0} = g_{00}+g_{0j}\,\frac{dx^j}{dx^0} = \left(1-\frac{\mathrm{w}}{c^2}\right)\left(1-\frac{\mathrm{w}}{c^2}-\frac{v_j u^j}{c^2}\right) = \\ = \frac{1}{1+\frac{1}{c^2}\,v_j{}^*u^j}\left(1-\frac{\mathrm{w}}{c^2}\right)^2,\end{gathered} \tag{6.7}$$

and taking formula (4.7) into account, we obtain

$$\frac{dx^0}{ds} = \frac{c^2}{c^2-\mathrm{w}}\,\frac{1+\frac{1}{c^2}\,v_j{}^*u^j}{\sqrt{1-\frac{^*u_i{}^*u^i}{c^2}}}\,. \tag{6.8}$$

For this reason, we have

$$\frac{^*d}{dt} = \sqrt{1-\frac{^*u_k{}^*u^k}{c^2}}\;\frac{c\,d}{ds}\,. \tag{6.9}$$

Formula (6.2) leads also to

$$\frac{^*d}{dt} = \left(1+\frac{1}{c^2}\,v_j{}^*u^j\right)\frac{c^2}{c^2-\mathrm{w}}\,\frac{\partial}{\partial t}+{}^*u^j\,\frac{\partial}{\partial x^j}\,. \tag{6.10}$$

Employing the formulae we have obtained for the energy of a point-mass, we, in particular, have

$$\frac{^*dE}{dt} = \sqrt{1-\frac{^*u_k{}^*u^k}{c^2}}\;\frac{c\,dE}{ds}\,. \tag{6.11}$$

Because E is a chr.inv.-invariant, $\frac{{}^*dE}{dt}$ is chr.inv.-invariant as well. The total derivative

$$\frac{dQ^k}{dt} = \frac{\partial Q^k}{\partial t} + u^j \frac{\partial Q^k}{\partial x^j} \tag{6.12}$$

of any sub-vector Q^k is not sub-vector. At the same time, it is possible to introduce the sub-vector

$$\frac{dQ^k}{dt} + \Delta^k_{jl}\, Q^l u^j = \frac{\partial Q^k}{\partial t} + \left(\nabla_j\, Q^k\right) u^j , \tag{6.13}$$

which characterizes the total derivative.

The chr.inv.-total derivative of Q^k, the quantity

$$\frac{{}^*dQ^k}{dt} = \frac{{}^*\partial Q^k}{\partial t} + {}^*u^j \frac{{}^*\partial Q^k}{\partial x^j} , \tag{6.14}$$

is not a sub-vector either. We can introduce the operation of chr. inv.-total differentiation, which, being invariant with respect to the transformations of the time coordinate, coincides with the regular operation of obtaining the sub-vector of the total derivative when the potentials are set to zero.

The sub-vector of the chr.inv.-total derivative of Q^k is different to the regular formula (6.13) only by the presence of asterisks

$$\frac{{}^*dQ^k}{dt} + {}^*\Delta^k_{jl}\, Q^l\, {}^*u^j = \frac{{}^*\partial Q^k}{\partial t} + \left({}^*\nabla_j\, Q^k\right) {}^*u^j . \tag{6.15}$$

Using (6.4–6.6), we obtain

$$\begin{gathered}\frac{{}^*dQ^k}{dt} + {}^*\Delta^k_{jl}\, Q^l\, {}^*u^j = \frac{c^2}{c^2 - \mathrm{w}}\left(1 + \frac{1}{c^2}\, v_n {}^*u^n\right) \times \\ \times \left[\frac{dQ^k}{dt} + \Delta^k_{jl}\, Q^l u^j + \frac{1}{c^2}\left(D^k_j\, v_l + D^k_l\, v_j - D_{jl}\, v^k\right) Q^l u^j\right]\end{gathered} \tag{6.16}$$

or, in the alternative form

$$\begin{gathered}\frac{{}^*dQ^k}{dt} + {}^*\Delta^k_{jl}\, Q^l\, {}^*u^j = \frac{1}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_n u^n\right)} \times \\ \times \left[\frac{dQ^k}{dt} + \Delta^k_{jl}\, Q^l u^j + \frac{1}{c^2}\left(D^k_j\, v_l + D^k_l\, v_j - D_{jl}\, v^k\right) Q^l u^j\right].\end{gathered} \tag{6.17}$$

Comparing (5.11) and (5.13), and taking formula (5.9) into account, we obtain

$$\frac{c\,dx^i}{ds} = \frac{{}^*u^i}{\sqrt{1-\dfrac{{}^*u_k{}^*u^k}{c^2}}}\,. \qquad (6.18)$$

Because of (6.9) and (6.18), we have

$$\frac{{}^*dQ^k}{dt} + {}^*\Delta^k_{jl}\,Q^l\,{}^*u^j = c\sqrt{1-\frac{{}^*u_n{}^*u^n}{c^2}}\left(\frac{dQ^k}{ds} + {}^*\Delta^k_{jl}\,Q^l\frac{dx^j}{ds}\right). \qquad (6.19)$$

Applying the obtained formulae to the momentum of a point-mass, and supposing that its rest-mass m_0 remains unchanged, we can, in particular, write

$$\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{jl}\,p^l\,{}^*u^j = m_0c^2\sqrt{1-\frac{{}^*u_n{}^*u^n}{c^2}}\left(\frac{d^2x^k}{ds^2} + {}^*\Delta^k_{jl}\frac{dx^j}{ds}\frac{dx^l}{ds}\right). \qquad (6.20)$$

Because the sub-vector of momentum is chr.inv.-vector, the sub-vector of the total derivative of momentum with respect to time is also a chr.inv.-vector.

§ 3.7 The time equation of geodesic lines

With the four equations of geodesic lines

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,, \qquad \alpha,\mu,\nu = 0,1,2,3\,, \qquad (7.1)$$

we are going to consider the time equation ($\alpha = 0$)

$$\frac{d^2x^0}{ds^2} + \Gamma^0_{00}\frac{dx^0}{ds}\frac{dx^0}{ds} + 2\Gamma^0_{0i}\frac{dx^0}{ds}\frac{dx^i}{ds} + \Gamma^0_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0\,. \qquad (7.2)$$

Substituting (6.8), (5.9), (5.2), (5.12), (6.9), and (6.10) term-by-term, and supposing that the rest-mass of the point-body we are considering remains unchanged, we have

$$m_0\frac{d^2x^0}{ds^2} = \frac{d}{ds}\left(m_0\frac{dx^0}{ds}\right) = \frac{d}{ds}\left[\frac{mc^2}{c^2-\mathrm{w}}\left(1+\frac{1}{c^2}\,v_j{}^*u^j\right)\right] =$$

$$= \frac{1}{c^2-\mathrm{w}}\left(\frac{dE}{ds} + v_j\frac{dp^j}{ds}\right) + \frac{mc^2}{(c^2-\mathrm{w})^2}\left(1+\frac{1}{c^2}\,v_j{}^*u^j\right)\frac{d\mathrm{w}}{ds} +$$

$$+\frac{m}{c^2-\mathrm{w}}\,{}^*u^j\frac{dv_j}{ds}=\frac{1}{c^3\sqrt{1-\frac{{}^*u_p{}^*u^p}{c^2}}}\left\{\frac{c^2}{c^2-\mathrm{w}}\left(\frac{{}^*dE}{dt}+v_j\frac{dp^j}{dt}\right)+\right.$$
$$+\frac{mc^2}{c^2-\mathrm{w}}\left[\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right){}^*u^j\frac{\partial v_j}{\partial t}+m\,{}^*u^i\,{}^*u^j\frac{\partial v_j}{\partial x^i}\right]+ \qquad (7.3)$$
$$\left.+\frac{mc^4}{(c^2-\mathrm{w})^2}\left(1+\frac{1}{c^2}\,v_j{}^*u^j\right)\left[\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial t}+{}^*u^i\frac{\partial\mathrm{w}}{\partial x^i}\right]\right\}.$$

Because of (3.17), (6.8), and (5.9), we obtain

$$m_0\Gamma^0_{00}\frac{dx^0}{ds}\frac{dx^0}{ds}=-\frac{m_0}{c^3}\frac{c^4}{(c^2-\mathrm{w})^2}\times$$
$$\times\left[\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial t}+\left(1-\frac{\mathrm{w}}{c^2}\right)v_lF^l\right]\frac{\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)^2}{1-\frac{{}^*u_p{}^*u^p}{c^2}}= \qquad (7.4)$$
$$=-\frac{m}{c^3\sqrt{1-\frac{{}^*u_p{}^*u^p}{c^2}}}\frac{c^2}{c^2-\mathrm{w}}\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)^2\left[\frac{c^4}{(c^2-\mathrm{w})^2}\frac{\partial\mathrm{w}}{\partial t}+v_lF^l\right].$$

Next, in the same fashion, because of (3.19), (6.8), (6.18) and (5.9), we obtain

$$2m_0\Gamma^0_{0i}\frac{dx^0}{ds}\frac{dx^i}{ds}=2\,\frac{m_0}{c^2}\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^i}+v_l\left(D^l_i+A_{i\cdot}^{\cdot l}+\right.\right.$$
$$\left.\left.+\frac{1}{c^2}\,v_iF^l\right)\right]\frac{c^2}{c^2-\mathrm{w}}\frac{1+\frac{1}{c^2}\,v_n{}^*u^n}{\sqrt{1-\frac{{}^*u_p{}^*u^p}{c^2}}}\frac{{}^*u^i}{c\sqrt{1-\frac{{}^*u_q{}^*u^q}{c^2}}}=$$
$$=2\,\frac{m}{c^3\sqrt{1-\frac{{}^*u_p{}^*u^p}{c^2}}}\frac{c^2}{c^2-\mathrm{w}}\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\times \qquad (7.5)$$
$$\times\left[-\frac{c^2}{c^2-\mathrm{w}}\,{}^*u^i\frac{\partial\mathrm{w}}{\partial x^i}+v_l\left(D^l_i+A_{i\cdot}^{\cdot l}\right){}^*u^i+\frac{1}{c^2}\,v_i{}^*u^i\left(v_lF^l\right)\right].$$

Finally, because of (3.21), (6.18), (5.9), and also (22.3) of §2.22,

we obtain

$$m_0\Gamma^0_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = -\frac{m_0}{c}\frac{c^2}{c^2-\mathrm{w}}\Big\{\Psi_{ij} - D_{ij} + \\ + \frac{1}{c^2}\,v_l\Big[\left(D^l_j + A^{\cdot l}_{j\cdot}\right)v_i + \left(D^l_i + A^{\cdot l}_{i\cdot}\right)v_j + \frac{1}{c^2}\,v_i v_j F^l\Big]\Big\}\times \\ \times\frac{{}^*u^i\,{}^*u^j}{c^2\left(1-\frac{{}^*u_p{}^*u^p}{c^2}\right)} = -\frac{m}{c^3\sqrt{1-\frac{{}^*u_p{}^*u^p}{c^2}}}\frac{c^2}{c^2-\mathrm{w}}\times \\ \times\Big[{}^*u^i\,{}^*u^j\frac{\partial v_j}{\partial x^i} - D_{ij}{}^*u^i\,{}^*u^j - \frac{1}{c^2}\,v_j{}^*u^j\left(F_i{}^*u^i\right) - {}^*\Delta^n_{ij}{}^*u^i\,{}^*u^j v_n + \\ + \frac{2}{c^2}\,v_j{}^*u^j v_l\left(D^l_i + A^{\cdot l}_{i\cdot}\right){}^*u^i + \frac{1}{c^4}\left(v_j{}^*u^j\right)^2\left(v_l F^l\right)\Big]. \tag{7.6}$$

Hence, we have

$$m_0c^3\left(1-\frac{\mathrm{w}}{c^2}\right)\sqrt{1-\frac{{}^*u_n{}^*u^n}{c^2}}\left(\frac{d^2x^0}{ds^2} + \Gamma^0_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\mu}{ds}\right) = \\ = \frac{{}^*dE}{dt} + v_k\frac{{}^*dp^k}{dt} + m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)^2\left(\frac{c^2}{c^2-\mathrm{w}}\right)^2\frac{\partial\mathrm{w}}{\partial t} + \\ + m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\frac{c^2}{c^2-\mathrm{w}}\,{}^*u^j\frac{\partial\mathrm{w}}{\partial x^j} + m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\times \\ \times\frac{c^2}{c^2-\mathrm{w}}\,{}^*u^j\frac{\partial v_j}{\partial t} - m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)^2\left(\frac{c^2}{c^2-\mathrm{w}}\right)^2\frac{\partial\mathrm{w}}{\partial t} + \\ + m\,{}^*u^i\,{}^*u^j\frac{\partial v_j}{\partial x^i} - m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)^2\left(v_kF^k\right) - \\ - 2m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\frac{c^2}{c^2-\mathrm{w}}\,{}^*u^i\frac{\partial\mathrm{w}}{\partial x^i} + 2m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\times \\ \times\left(D^k_i + A^{\cdot k}_{i\cdot}\right)v_k{}^*u^i + mD_{ij}{}^*u^i\,{}^*u^j - m\frac{\partial v_j}{\partial x^i}\,{}^*u^i\,{}^*u^j + \\ + 2m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\frac{1}{c^2}\,v_m{}^*u^m\left(v_kF^k\right) + \frac{1}{c^2}\,mv_n{}^*u^n\left(F_i{}^*u^i\right) + \\ + mv_k{}^*\Delta^k_{ij}{}^*u^i\,{}^*u^j - 2m\frac{1}{c^2}\,v_n{}^*u^n v_k\left(D^k_i + A^{\cdot k}_{i\cdot}\right){}^*u^i - \\ - \frac{1}{c^4}\,m\left(v_n{}^*u^n\right)^2\left(v_kF^k\right) = \frac{{}^*dE}{dt} + mD_{ij}{}^*u^i\,{}^*u^j - mF_i{}^*u^i + \\ + v_k\left[\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{ij}\,p^i\,{}^*u^j - mF^k + 2m\left(D^k_i + A^{\cdot k}_{i\cdot}\right){}^*u^i\right]. \tag{7.7}$$

Therefore the time equation of geodesics can be written in the form

$$\begin{aligned}&\frac{{}^*dE}{dt} + mD_{ij}\,{}^*u^i\,{}^*u^j - mF_i\,{}^*u^j + \\ &+ v_k\left[\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{ij}\,p^i\,{}^*u^j - mF^k + 2m\left(D^k_i + A^{\cdot k}_{i\cdot}\right){}^*u^i\right] = 0\,.\end{aligned} \tag{7.8}$$

The left side of equation (7.8) is co-invariant, however this formula is unchanged under transformation of the time coordinate, because it is a consequence of one of the four geodesic equations, which have world-tensor nature. For this reason we can use the method to vary the potentials here (see §2.4).

Assuming all the v_k are zero, we obtain

$$\frac{{}^*dE}{dt} + mD_{ij}\,{}^*u^i\,{}^*u^j - mF_j\,{}^*u^j = 0\,. \tag{7.9}$$

Because the left side of this equation is a chr.inv.-invariant, the equation is true under any choice of the time coordinate (in any reference frame). So the following equality holds for any choice of the time coordinate

$$v_k\left[\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{ij}\,p^i\,{}^*u^j - mF^k + 2m\left(D^k_i + A^{\cdot k}_{i\cdot}\right){}^*u^i\right] = 0\,. \tag{7.10}$$

Assuming term-by-term that

$$\left.\begin{aligned} v_1 \neq 0\,, \quad v_2 = v_3 = 0 \\ v_2 \neq 0\,, \quad v_3 = v_1 = 0 \\ v_3 \neq 0\,, \quad v_1 = v_2 = 0 \end{aligned}\right\} \tag{7.11}$$

and taking into account that the quantities

$$\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{ij}\,p^i\,{}^*u^j - mF^k + 2m\left(D^k_i + A^{\cdot k}_{i\cdot}\right){}^*u^i \tag{7.12}$$

are components of a chr.inv.-vector, we see that the equalities

$$\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{ij}\,p^i\,{}^*u^j - mF^k + 2m\left(D^k_i + A^{\cdot k}_{i\cdot}\right){}^*u^i = 0 \tag{7.13}$$

are true for any choice of the time coordinate (in any reference frame).

The four equation we have obtained, namely equation (7.9) and equations (7.13), were deduced from the left side of only one equation — the time equation of geodesics. The other three geodesic equations were not used. At the same time we include the other three geodesic equations in the algebra, because only all the four equations as a whole have world-tensor nature. Therefore we must consider equations (7.9) and (7.13) as consequences of all four geodesic equations, not of only one of them. At the same time the possibility of obtaining equations (7.9) and (7.13) from only one of the geodesic equations demonstrates the utility of the method we used to vary the potentials.

§3.8 The spatial equations of geodesic lines

We are now going to consider the spatial geodesic equations, i. e. equations (7.1) with $\alpha = 1, 2, 3$

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{00}\frac{dx^0}{ds}\frac{dx^0}{ds} + 2\Gamma^k_{0i}\frac{dx^0}{ds}\frac{dx^i}{ds} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0\,. \tag{8.1}$$

After the results we have obtained in §3.7, we expect nothing new from them. For this reason we limit ourselves to their consideration in only a formal way.

Employing (3.27), (6.18), (6.20), and (5.9) term-by-term under the supposition that the rest-mass of the point-body we are considering remains unchanged, we obtain

$$\begin{gathered} m_0\frac{d^2x^k}{ds^2} + m_0\Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = m_0\left(\frac{d^2x^k}{ds^2} + {}^*\!\Delta^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right) - \\ - \frac{m_0}{c^2}\left[\left(D^k_j + A^{\cdot k}_{j\cdot}\right)v_i + \left(D^k_i + A^{\cdot k}_{i\cdot}\right)v_j + \frac{1}{c^2}\,v_i v_j F^k\right]\times \\ \times\frac{{}^*u^i\,{}^*u^j}{c^2\left(1 - \dfrac{{}^*u_p{}^*u^p}{c^2}\right)} = \frac{1}{c^2\sqrt{1 - \dfrac{{}^*u_p{}^*u^p}{c^2}}}\left[\frac{dp^k}{dt} + {}^*\!\Delta^k_{ij}\,p^i\,{}^*u^j - \right. \\ \left. - \frac{2}{c^2}\,m\,v_n{}^*u^n\left(D^k_i + A^{\cdot k}_{i\cdot}\right){}^*u^i - \frac{1}{c^4}\,m\left(v_n{}^*u^n\right)^2 F^k\right]. \end{gathered} \tag{8.2}$$

Because of (3.18), (6.8) and (5.9), we obtain

$$m_0\Gamma^k_{00}\frac{dx^0}{ds}\frac{dx^0}{ds} = -\frac{m_0}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 F^k\left(\frac{c^2}{c^2 - \mathrm{w}}\right)^2 \times$$

$$\times\frac{\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)^2}{1-\frac{{}^*u_p{}^*u^p}{c^2}}=-\frac{m}{c^2\sqrt{1-\frac{{}^*u_p{}^*u^p}{c^2}}}\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)^2F^k. \tag{8.3}$$

In similar fashion, because of (3.20), (6.8), (6.18), and (5.9), we obtain

$$\begin{gathered}2\,m_0\Gamma^k_{0i}\frac{dx^0}{ds}\frac{dx^i}{ds}=2\,\frac{m_0}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D^k_i+A_{i\cdot}^{\cdot k}+\frac{1}{c^2}\,v_iF^k\right)\times\\ \times\frac{c^2}{c^2-\mathrm{w}}\,\frac{1+\frac{1}{c^2}\,v_n{}^*u^n}{c\left(1-\frac{{}^*u_p{}^*u^p}{c^2}\right)}{}^*u^i=\frac{2m}{c^2\sqrt{1-\frac{{}^*u_p{}^*u^p}{c^2}}}\times\\ \times\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\left[\left(D^k_i+A_{i\cdot}^{\cdot k}\right){}^*u^i+\frac{1}{c^2}\,v_m{}^*u^mF^k\right].\end{gathered}\tag{8.4}$$

Hence,

$$\begin{gathered}m_0c^2\sqrt{1-\frac{{}^*u_p{}^*u^p}{c^2}}\left(\frac{d^2x^k}{ds^2}+\Gamma^k_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)=\\ =\frac{{}^*dp^k}{dt}+{}^*\Delta^k_{ij}\,p^i\,{}^*u^j-\frac{2}{c^2}\,m\,v_n{}^*u^n\left(D^k_i+A_{i\cdot}^{\cdot k}\right){}^*u^i-\\ -\frac{1}{c^4}\,m\left(v_n{}^*u^n\right)^2F^k-m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)^2F^k+\\ +2m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\left(D^k_i+A_{i\cdot}^{\cdot k}\right){}^*u^i+\\ +2m\left(1+\frac{1}{c^2}\,v_n{}^*u^n\right)\frac{1}{c^2}\,v_m{}^*u^mF^k=\\ =\frac{{}^*dp^k}{dt}+{}^*\Delta^k_{ij}\,p^i\,{}^*u^j-mF^k+2m\left(D^k_i+A_{i\cdot}^{\cdot k}\right){}^*u^i.\end{gathered}\tag{8.5}$$

Therefore the spatial geodesic equations can be written in the form

$$\frac{{}^*dp^k}{dt}+{}^*\Delta^k_{ij}\,p^i\,{}^*u^j-mF^k+2m\left(D^k_i+A_{i\cdot}^{\cdot k}\right){}^*u^i=0\,, \tag{8.6}$$

which coincides with formula (7.13).

Because of the identity

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\mu}{ds}=1\,, \tag{8.7}$$

only three of the four geodesic equations are independent. Accordingly, only three of the four equations (7.9) and (7.13) are independent. The identity which links equations (7.9) and (7.13) can be obtained from the condition (8.7) or, equivalently, from formula (4.3). As a result, we obtain

$$\left(\frac{m_0 c}{\sqrt{g_{00}}}\frac{dx_0}{ds}\right)^2 - h_{ik}\left(m_0\frac{c\,dx^i}{ds}\right)\left(m_0\frac{c\,dx^k}{ds}\right) = m_0 c^2 \tag{8.8}$$

and, taking (5.8), (5.2), and (5.11) into account,

$$\frac{E^2}{c^2} - h_{ik}\,p^i p^k = m_0^2 c^2 \tag{8.9}$$

or, in the alternative form

$$\frac{E^2}{c^2} - p_j\,p^j = m_0^2 c^2. \tag{8.10}$$

Formula (7.13) can be considered as the main equations for the dynamics of a point-mass, and also as the theorem of momentum. Formula (7.9) can be considered as the theorem of energy, which is a consequence of the main equations (7.13), because of equation (8.9) or equation (8.10).

Let us show how to obtain (7.9) from (7.13), using (8.9) and under the supposition that m_0 remains unchanged. Under this supposition, equation (8.9) leads to

$$\frac{2}{c^2}\,E\,\frac{{}^*dE}{dt} - p^i p^k\,\frac{{}^*dh_{ik}}{dt} - 2h_{ik}\,p^i\,\frac{{}^*dp^k}{dt} = 0\,. \tag{8.11}$$

Because of (6.2) and also (13.11) of §2.13, we have

$$\frac{{}^*dh_{ik}}{dt} = 2D_{ik} + ({}^*\Delta_{ij,k} + {}^*\Delta_{kj,i})\,{}^*u^j. \tag{8.12}$$

For this reason, we obtain

$$\frac{{}^*dE}{dt} = mD_{ij}\,{}^*u^i\,{}^*u^j + \left(\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{ij}\,p^i\,{}^*u^j\right){}^*u^k. \tag{8.13}$$

Finally, considering (8.13) and (7.13) together, we obtain (7.9).

§ 3.9 The mechanical sense of the power quantities

We introduce a chr.inv.-vector Ω_i, defining it by the equality*

$$\Omega_i = \frac{1}{2}\,\varepsilon_{ijk}\,A^{jk}. \tag{9.1}$$

*Compare this definition with formula (16.9) of §2.16.

Then (see formula 16.13 of §2.16) we have

$$A^{jk} = \varepsilon^{ijk}\,\Omega_i\,, \tag{9.2}$$

and we can write the vector product of Ω_i and ${}^*u_j$ as follows

$$\varepsilon^{ijk}\,\Omega_i\,{}^*u_j = A^{jk}\,{}^*u_j = A_{i\cdot}^{\cdot k}\,{}^*u^i. \tag{9.3}$$

Therefore we can write (7.13) in the form

$$\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{ij}\,p^i\,{}^*u^j = mF^k - 2m\,\varepsilon^{ijk}\,\Omega_i\,{}^*u_j - 2mD^k_i\,{}^*u^i. \tag{9.4}$$

Equation (9.4) and also equation (7.9), namely

$$\frac{{}^*dE}{dt} = mF_j\,{}^*u^j - m D_{ij}\,{}^*u^i\,{}^*u^j, \tag{9.5}$$

retain their form in all reference frames, and hence, also in the at any given point locally-stationary reference frame. However the chr.inv.-tensor D_{ik}, by its very definition, is zero in the locally-stationary reference frame, so we have, at the given point of this reference frame

$$\frac{{}^*dp^k}{dt} + {}^*\Delta^k_{ij}\,p^i\,{}^*u^j = mF^k - 2m\,\varepsilon^{ijk}\,\Omega_i\,{}^*u_j\,, \tag{9.6}$$

$$\frac{{}^*dE}{dt} = mF_j\,{}^*u^j. \tag{9.7}$$

Hence, setting the potentials to zero at the world-point we are considering, we have at this point

$$\frac{dp^k}{dt} + \Delta^k_{ij}\,p^i u^j = mF^k - 2m\varepsilon^{ijk}\,\Omega_i\,u_j\,, \tag{9.8}$$

$$\frac{dE}{dt} = mF_j\,u^j. \tag{9.9}$$

On the other hand, the equations in an arbitrary reference frame have the classic form (in curvilinear coordinates)*

$$\frac{dp^k}{dt} + \Delta^k_{ij}\,p^i u^j = \phi^k - 2m\varepsilon^{ijk}\,\psi_i\,u_j\,, \tag{9.10}$$

$$\frac{dE}{dt} = \phi_j\,u^j, \tag{9.11}$$

*As a mater of fact, the relativistic definitions of energy (its varying part) and momentum coincide with the classic definitions only under the approximation $u^k \to 0$.

where ϕ^k and ϕ_j are the contravariant and covariant vectors of a force (the force includes forces of inertia), ψ_i is the covariant vector of the angular velocities of the rotation of the given reference frame. Therefore we can say that F^k and F_j play the part of the *strength of gravitational inertial force fields* or, equivalently, the part of the *gravitational inertial force*, calculated for a unit mass. Thus, Ω_i plays the part of the *momentary chr.inv.-angular velocity of the absolute rotation* of the reference frame at the given point.

§3.10 The energy-momentum tensor

We are going to consider the energy-momentum world-tensor in the coordinates x^σ $(\sigma = 0, 1, 2, 3)$, which set the potentials to zero at the world-point we are considering. At this world-point, in a continuous medium, we have

$$\widetilde{T}^{00} = \tilde{\rho}\,, \tag{10.1}$$

$$\widetilde{T}^{0k} = \frac{1}{c}\,\tilde{J}^k, \tag{10.2}$$

$$\widetilde{T}^{ij} = \frac{1}{c^2}\,\widetilde{U}^{ij}, \tag{10.3}$$

where ρ is the density of the moving substance, J^k is the momentum density (or, equivalently, the density of the mass flux), U^{ij} is the three-dimensional tensor of the kinematic (absolute) stresses, which is the sum of the tensor of the regular (relative) stresses and the tensor of the density of the momentum flux*. The world-point we are considering is characterized by the conditions

$$\tilde{g}_{00} = 1\,, \qquad \tilde{g}_{0i} = 0\,, \tag{10.4}$$

so at this world-point we have

$$\frac{\widetilde{T}_{00}}{\tilde{g}_{00}} = \frac{\tilde{g}_{0\alpha}\tilde{g}_{0\beta}\,\widetilde{T}^{\alpha\beta}}{\tilde{g}_{00}} = \widetilde{T}^{00} \tag{10.5}$$

and also

$$\frac{\widetilde{T}_{00}}{\tilde{g}_{00}} = \tilde{\rho}\,. \tag{10.6}$$

*See [8], p. 231, and also [1], p. 70. Further, we will use ρ to denote the "local density" of matter. In the homogeneous models, regular density and pressure coincide with the local density and pressure.

In similar fashion, and taking (10.4) into account, we obtain

$$\frac{\widetilde{T}_0^k}{\sqrt{\tilde{g}_{00}}} = \frac{\tilde{g}_{0\alpha}\widetilde{T}^{\alpha k}}{\sqrt{\tilde{g}_{00}}} = \widetilde{T}^{0k} \tag{10.7}$$

and hence,

$$\frac{\widetilde{T}_0^k}{\sqrt{\tilde{g}_{00}}} = \frac{1}{c}\,\tilde{J}^k. \tag{10.8}$$

In accordance with §2.3, the quantities on the left sides of (10.6), (10.8), (10.3) are a chr.inv.-invariant, a chr.inv.-vector, and a chr. inv.-tensor, respectively. Therefore, we can write

$$\frac{\widetilde{T}_{00}}{\tilde{g}_{00}} = \frac{T_{00}}{g_{00}}\,. \tag{10.9}$$

$$\frac{\widetilde{T}_0^k}{\sqrt{\tilde{g}_{00}}} = \frac{T_0^k}{\sqrt{g_{00}}}\,, \tag{10.10}$$

$$\widetilde{T}^{ij} = T^{ij}, \tag{10.11}$$

so, in general, we have

$$\frac{T_{00}}{g_{00}} = \rho\,, \tag{10.12}$$

$$\frac{T_0^k}{\sqrt{g_{00}}} = \frac{1}{c}\,J^k, \tag{10.13}$$

$$T^{ij} = \frac{1}{c^2}\,U^{ij}. \tag{10.14}$$

Let us find formulas for the components of the covariant, mixed, and contravariant energy-momentum tensors. Because of (10.2), we obtain

$$T_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 \rho\,. \tag{10.15}$$

Because of (10.13), we obtain

$$T_0^k = \frac{1}{c}\left(1 - \frac{\mathrm{w}}{c^2}\right) J^k. \tag{10.16}$$

Since

$$T_{00} = g_{0\alpha}T_0^\alpha = g_{00}T_0^0 + g_{0k}T_0^k\,, \tag{10.17}$$

and taking (10.15) and (10.16) into account, we obtain

$$\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 \rho = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 T_0^0 - \frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 v_k J^k, \tag{10.18}$$

$$T_0^0 = \rho + \frac{1}{c^2}\, v_j J^j. \tag{10.19}$$

In similar fashion, because

$$T_0^k = g_{0\alpha} T^{\alpha k} = g_{00} T^{0k} + g_{0j} T^{jk}, \tag{10.20}$$

and taking (10.16) and (10.14) into account, we obtain

$$\frac{1}{c}\Big(1 - \frac{\mathrm{w}}{c^2}\Big) J^k = \Big(1 - \frac{\mathrm{w}}{c^2}\Big)^2 T^{0k} - \frac{v_j}{c}\Big(1 - \frac{\mathrm{w}}{c^2}\Big)\frac{1}{c^2}\, U^{jk}, \tag{10.21}$$

$$T^{0k} = \frac{1}{c}\,\frac{c^2}{c^2 - \mathrm{w}}\Big(J^k + \frac{1}{c^2}\, v_j U^{jk}\Big). \tag{10.22}$$

On the other hand, we have

$$T_0^k = g^{k\alpha} T_{\alpha 0} = g^{k0} T_{00} + g^{kj} T_{j0}\,, \tag{10.23}$$

so, taking (10.16) and (10.15) into account, we obtain

$$\frac{1}{c}\Big(1 - \frac{\mathrm{w}}{c^2}\Big) J^k = -\frac{1}{c}\Big(1 - \frac{\mathrm{w}}{c^2}\Big) v^k \rho - h^{kj} T_{j0}\,, \tag{10.24}$$

$$T_{0i} = -\frac{1}{c}\,\Big(1 - \frac{\mathrm{w}}{c^2}\Big)\,(J_i + \rho v_i)\,. \tag{10.25}$$

We are similarly led to

$$T^{ij} = g^{i\alpha} T_\alpha^j = g^{i0} T_0^j + g^{ik} T_k^j\,, \tag{10.26}$$

which, taking (10.14) and (10.16) into account, gives

$$\frac{1}{c^2} U^{ij} = -\frac{1}{c^2}\, v^i J^i - h^{ik} T_k^j\,, \tag{10.27}$$

$$T_i^j = -\frac{1}{c^2}\,\Big(v_i J^j + U_i^j\Big)\,. \tag{10.28}$$

We now use (10.19), (10.25), and (10.28) for deducing the other forms of the energy-momentum tensor. Because

$$T_0^0 = g_{0\alpha} T^{\alpha 0} = g_{00} T^{00} + g_{0k} T^{k0}, \tag{10.29}$$

and taking (10.19) and (10.22) into account, we obtain

$$\rho + \frac{1}{c^2}\, v_j J^j = \Big(1 - \frac{\mathrm{w}}{c^2}\Big)^2 T^{00} - \frac{1}{c^2}\, v_k\Big(J^k + \frac{1}{c^2}\, v_j U^{jk}\Big)\,, \tag{10.30}$$

$$T^{00}=\left(\frac{c^2}{c^2-\mathrm{w}}\right)^2\left(\rho+\frac{2}{c^2}\,v_j J^j+\frac{1}{c^4}\,v_j v_k U^{jk}\right). \tag{10.31}$$

Because

$$T_{0i}=g_{0\alpha}T_i^{\alpha}=g_{00}T_i^0+g_{0j}T_i^j\,, \tag{10.32}$$

and taking (10.25) and (10.28) into account, we obtain

$$\begin{aligned}-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(J_i+\rho v_i\right) &=\\ =\left(1-\frac{\mathrm{w}}{c^2}\right)^2 T_i^0 &+\frac{v_j}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{1}{c^2}\left(v_i J^j+U_i^j\right),\end{aligned} \tag{10.33}$$

$$T_i^0=-\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\left[J_i+\left(\rho+\frac{1}{c^2}\,v_j J^j\right)v_i+\frac{1}{c^2}\,v_j U_i^j\right]. \tag{10.34}$$

Finally, because

$$T_i^j=g^{j\alpha}T_{\alpha i}=g^{j0}T_{0i}+g^{jk}T_{ki}\,, \tag{10.35}$$

and taking (10.28) and (10.25) into account, we obtain

$$-\frac{1}{c^2}\left(v_i J^j+U_i^j\right)=\frac{1}{c^2}\left(J_i+\rho v_i\right)v^j-h^{jk}\,T_{ki}\,, \tag{10.36}$$

$$T_{ij}=\frac{1}{c^2}\left(\rho v_i v_j+v_i J_j+v_j J_i+U_{ij}\right). \tag{10.37}$$

To see the results clearly, we collect formulae (10.15), (10.25), and (10.37)

$$\left.\begin{aligned} T_{00} &=\left(1-\frac{\mathrm{w}}{c^2}\right)^2\rho\\ T_{0i} &=-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(J_i+\rho v_i\right)\\ T_{ij} &=\frac{1}{c^2}\left(\rho v_i v_j+v_i J_j+v_j J_i+U_{ij}\right)\end{aligned}\right\}, \tag{10.38}$$

formulae (10.19), (10.16), (10.34), and (10.28)

$$\left.\begin{aligned} T_0^0 &=\rho+\frac{1}{c^2}\,v_j J^j\\ T_0^k &=\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)J^k\\ T_i^0 &=-\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\left[J_i+\left(\rho+\frac{1}{c^2}\,v_j J^j\right)v_i+\frac{1}{c^2}\,v_j U_i^j\right]\\ T_i^j &=-\frac{1}{c^2}\left(v_i J^j+U_i^j\right)\end{aligned}\right\}, \tag{10.39}$$

and finally, formulae (10.31), (10.22), and (10.14)

$$\left.\begin{aligned} T^{00} &= \left(\frac{c^2}{c^2-\mathrm{w}}\right)^2 \left(\rho + \frac{2}{c^2}\, v_j J^j + \frac{1}{c^4}\, v_j v_k U^{jk}\right) \\ T^{0k} &= \frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}} \left(J^k + \frac{1}{c^2}\, v_j U^{jk}\right) \\ T^{ij} &= \frac{1}{c^2}\, U^{ij} \end{aligned}\right\} . \tag{10.40}$$

Let us deduce a formula for the chr.inv.-invariant

$$\rho_0 = T = g_{\mu\nu} T^{\mu\nu} = g^{\mu\nu} T_{\mu\nu} = T^{\nu}_{\nu} \,. \tag{10.41}$$

Because

$$T^{\nu}_{\nu} = T^0_0 + T^j_j \,, \tag{10.42}$$

and denoting

$$U = U^j_j \,, \tag{10.43}$$

we have

$$T = \rho - \frac{1}{c^2}\, U \tag{10.44}$$

or, in the alternative form

$$\rho_0 = \rho - \frac{1}{c^2}\, U \,. \tag{10.45}$$

§ 3.11 The time equation of the law of energy

We assume the law of energy to be the divergence of the mixed energy-momentum tensor equated to zero

$$\frac{\partial T^{\nu}_{\mu}}{\partial x^{\nu}} - \Gamma^{\sigma}_{\mu\nu}\, T^{\nu}_{\sigma} + \frac{\partial \ln \sqrt{-g}}{\partial x^{\sigma}}\, T^{\sigma}_{\mu} = 0 \,. \tag{11.1}$$

This gives four equations. We consider first the time equation ($\mu = 0$)

$$\frac{\partial T^{\nu}_{0}}{\partial x^{\nu}} - \Gamma^{\sigma}_{0\nu}\, T^{\nu}_{\sigma} + \frac{\partial \ln \sqrt{-g}}{\partial x^{\sigma}}\, T^{\sigma}_{0} = 0 \,. \tag{11.2}$$

Because of (10.39), we have

$$\frac{\partial T^{\nu}_{0}}{\partial x^{\nu}} = \frac{\partial T^{0}_{0}}{\partial x^{0}} + \frac{\partial T^{j}_{0}}{\partial x^{j}} =$$

$$= \frac{1}{c}\frac{\partial \rho}{\partial t} + \frac{1}{c^3}J^j\frac{\partial v_j}{\partial t} + \frac{1}{c^3}v_j\frac{\partial J^j}{\partial t} - \frac{1}{c^3}J^j\frac{\partial \mathrm{w}}{\partial x^j} + \\ + \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial J^j}{\partial x^j} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(\frac{^*\partial \rho}{\partial t} + \frac{^*\partial J^j}{\partial x^j} - \frac{1}{c^2}F_j J^j\right). \tag{11.3}$$

Besides these, using (3.17–3.20), we obtain

$$-\Gamma^{\sigma}_{0\nu}T^{\nu}_{\sigma} = -\Gamma^{0}_{00}T^{0}_{0} - \Gamma^{k}_{00}T^{0}_{k} - \Gamma^{0}_{0i}T^{i}_{0} - \Gamma^{k}_{0i}T^{i}_{k} = \\ = \frac{1}{c^3}\left[\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)v_l F^l\right]\left(\rho + \frac{1}{c^2}v_j F^j\right) - \\ - \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F^k\,\frac{1}{c}\frac{c^2}{c^2-\mathrm{w}}\left[J_k + \left(\rho+\frac{1}{c^2}v_j F^j\right)v_k + \frac{1}{c^2}v_j U^j_k\right] - \\ - \frac{1}{c^2}\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^i} + v_l\left(D^l_i + A^{\cdot l}_{i\cdot} + \frac{1}{c^2}v_i F^l\right)\right]\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)J^i + \\ + \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2}v_i F^k\right)\frac{1}{c^2}\left(v_k J^i + U^i_k\right) = \\ = \frac{1}{c^3}\left[\frac{c^2}{c^2-\mathrm{w}}\left(\rho + \frac{1}{c^2}v_j J^j\right)\frac{\partial \mathrm{w}}{\partial t} - \left(1-\frac{\mathrm{w}}{c^2}\right)F_j J^j + \right. \\ \left. + J^i\frac{\partial \mathrm{w}}{\partial x^i} + \left(1-\frac{\mathrm{w}}{c^2}\right)D_{ik}U^{ik}\right]. \tag{11.4}$$

We finally get

$$\frac{\partial \ln\sqrt{-g}}{\partial x^\sigma}T^{\sigma}_{0} = \frac{\partial \ln\sqrt{-g}}{\partial x^0}T^{0}_{0} + \frac{\partial \ln\sqrt{-g}}{\partial x^j}T^{j}_{0} = \\ = \frac{1}{c}\left[-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)D\right]\left(\rho + \frac{1}{c^2}v_j J^j\right) + \\ + \frac{1}{c}\left(-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^j} + \frac{\partial \ln\sqrt{h}}{\partial x^j}\right)\left(1-\frac{\mathrm{w}}{c^2}\right)J^j = \\ = -\left[\frac{1}{c^3}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t}\left(\rho + \frac{1}{c^2}v_j J^j\right) + J^j\frac{\partial \mathrm{w}}{\partial x^j}\right] + \\ + \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(\rho D + J^j\frac{^*\partial \ln\sqrt{h}}{\partial x^j}\right), \tag{11.5}$$

hence

$$\frac{\partial T_0^\nu}{\partial x^\nu} - \Gamma_{0\nu}^\sigma T_\sigma^\nu + \frac{\partial \ln\sqrt{-g}}{\partial x^\sigma}\, T_0^\sigma = \\ = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(\frac{{}^*\partial\rho}{\partial t} + \rho D + {}^*\nabla_j J^j - \frac{2}{c^2}\, F_j J^j + \frac{1}{c^2}\, D_{ik} U^{ik}\right) \tag{11.6}$$

and (11.2) can be written in the form

$$\frac{{}^*\partial\rho}{\partial t} + \rho D + {}^*\nabla_j J^j - \frac{2}{c^2} F_j J^j + \frac{1}{c^2}\, D_{ik} U^{ik} = 0\,. \tag{11.7}$$

§ 3.12 The spatial equations of the law of energy

We consider next the spatial equations of the law of energy (11.1), i. e. the equations with $\mu = 1, 2, 3$

$$\frac{\partial T_i^\nu}{\partial x^\nu} - \Gamma_{i\nu}^\sigma T_\sigma^\nu + \frac{\partial \ln\sqrt{-g}}{\partial x^\sigma}\, T_i^\sigma = 0\,. \tag{12.1}$$

Following §3.11, we obtain

$$\frac{\partial T_i^\nu}{\partial x^\nu} = \frac{\partial T_i^0}{\partial x^0} + \frac{\partial T_i^j}{\partial x^j} = -\frac{1}{c^2}\frac{c^2}{(c^2-\mathrm{w})^2}\left[J_i + \left(\rho + \frac{1}{c^2}\, v_j J^j\right) v_i + \right. \\ \left. + \frac{1}{c^2}\, v_j U_i^j\right] \frac{\partial \mathrm{w}}{\partial t} - \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\left[\frac{\partial J_i}{\partial t} + \left(\rho + \frac{1}{c^2}\, v_j J^j\right)\frac{\partial v_i}{\partial t} + \right. \\ \left. + v_i\left(\frac{\partial\rho}{\partial t} + \frac{1}{c^2} J^j \frac{\partial v_j}{\partial t} + \frac{1}{c^2}\, v_j \frac{\partial J^j}{\partial t}\right) + \frac{1}{c^2}\, U_i^j \frac{\partial v_j}{\partial t} + \frac{1}{c^2}\, v_j \frac{\partial U_i^j}{\partial t}\right] - \\ - \frac{1}{c^2} J^j \frac{\partial v_i}{\partial x^j} - \frac{1}{c^2}\, v_i \frac{\partial J^j}{\partial x^j} - \frac{1}{c^2}\frac{\partial U_i^j}{\partial x^j} = -\frac{1}{c^2}\frac{c^2}{(c^2-\mathrm{w})^2}\times \\ \times\left[J_i + \left(\rho + \frac{1}{c^2}\, v_j J^j\right) v_i + \frac{1}{c^2}\, v_j U_i^j\right]\frac{\partial \mathrm{w}}{\partial t} - \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\times \\ \times\left[\left(\rho + \frac{1}{c^2}\, v_j J^j\right)\frac{\partial v_i}{\partial t} + \frac{1}{c^2}\, U_i^j \frac{\partial v_j}{\partial t}\right] - \frac{1}{c^2}\frac{{}^*\partial J_i}{\partial t} - \frac{1}{c^2}\frac{{}^*\partial U_i^j}{\partial x^j} - \\ - \frac{1}{c^2}\, v_i\left(\frac{{}^*\partial\rho}{\partial t} + \frac{{}^*\partial J^j}{\partial x^j} + \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\, J^j \frac{{}^*\partial v_j}{\partial t}\right) - \frac{1}{c^2}\, J^j \frac{{}^*\partial v_i}{\partial x^j}\,, \tag{12.2}$$

Hence, we get

$$
\begin{aligned}
&-\Gamma_{i\nu}^{\sigma}\,T_{\sigma}^{\nu} = -\Gamma_{i0}^{0}\,T_{0}^{0} - \Gamma_{i0}^{k}\,T_{k}^{0} - \Gamma_{ij}^{0}\,T_{0}^{j} - \Gamma_{ij}^{k}\,T_{k}^{j} = \\
&= -\frac{1}{c^2}\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^i} + v_l\Big(D_i^l + A_{i\cdot}^{\cdot l} + \frac{1}{c^2}\,v_i F^l\Big)\times\right. \\
&\times\left(\rho + \frac{1}{c^2}\,v_j J^j\right) + \frac{1}{c}\Big(1-\frac{\mathrm{w}}{c^2}\Big)\Big(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}\,v_i F^k\Big)\times \\
&\times\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\left[J_k + \Big(\rho + \frac{1}{c^2}\,v_j J^j\Big)v_k + \frac{1}{c^2}\,v_j U_k^j\right] + \\
&+\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\left\{\Psi_{ij} - D_{ij} + \frac{1}{c^2}\,v_l\Big[\big(D_j^l + A_{j\cdot}^{\cdot l}\big)\,v_i\ + \right. \\
&+\big(D_i^l + A_{i\cdot}^{\cdot l}\big)\,v_j + \frac{1}{c^2}\,v_i v_j F^l\Big]\bigg\}\,\frac{1}{c}\Big(1-\frac{\mathrm{w}}{c^2}\Big)J^j + \bigg\{{}^*\Delta_{ij}^k - \\
&-\frac{1}{c^2}\Big[\big(D_j^k + A_{j\cdot}^{\cdot k}\big)\,v_i + \big(D_i^k + A_{i\cdot}^{\cdot k}\big)\,v_j + \frac{1}{c^2}\,v_i v_j F^k\Big]\bigg\}\times \\
&\times\frac{1}{c^2}\Big(v_k J^j + U_k^j\Big) = \frac{1}{c^2}\,\frac{c^2}{c^2-\mathrm{w}}\Big(\rho + \frac{1}{c^2}\,v_j J^j\Big)\frac{\partial \mathrm{w}}{\partial x^i} + \\
&+\frac{1}{c}\,J^j\frac{\partial v_j}{\partial x^i} + \frac{1}{c^2}\,{}^*\Delta_{ij}^k\,U_k^j - \frac{1}{c^4}\,v_i\,D_{jk}U^{jk}.
\end{aligned}
\tag{12.3}
$$

Finally,

$$
\begin{aligned}
&\frac{\partial\ln\sqrt{-g}}{\partial x^\sigma}\,T_i^\sigma = \frac{\partial\ln\sqrt{-g}}{\partial x^0}\,T_i^0 + \frac{\partial\ln\sqrt{-g}}{\partial x^j}\,T_i^j = \\
&= \frac{1}{c}\left[-\frac{1}{c^2}\,\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t} + \Big(1-\frac{\mathrm{w}}{c^2}\Big)D\right]\frac{1}{c}\,\frac{c^2}{c^2-\mathrm{w}}\times \\
&\times\left[J_i + \Big(\rho + \frac{1}{c^2}\,v_j J^j\Big)v_i + \frac{1}{c^2}\,v_j U_i^j\right] - \\
&-\left(-\frac{1}{c^2}\,\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^j} + \frac{\partial\ln\sqrt{h}}{\partial x^j}\right)\frac{1}{c^2}\Big(v_i J^j + U_i^j\Big) = \\
&= \frac{1}{c^2}\,\frac{c^2}{(c^2-\mathrm{w})^2}\left[J_i + \Big(\rho + \frac{1}{c^2}\,v_j J^j\Big)v_i + \frac{1}{c^2}\,v_j U_i^j\right]\frac{\partial \mathrm{w}}{\partial t} - \\
&-\frac{1}{c^2}\,D J_i - \frac{1}{c^2}\,U_i^j\,\frac{{}^*\partial\ln\sqrt{h}}{\partial x^j} + \frac{1}{c^4}\,\frac{c^2}{c^2-\mathrm{w}}\,U_i^j\,\frac{\partial \mathrm{w}}{\partial x^j} - \\
&-\frac{1}{c^2}\,v_i\left(\rho D + J^j\,\frac{{}^*\partial\ln\sqrt{h}}{\partial x^j} - \frac{1}{c^2}\,\frac{c^2}{c^2-\mathrm{w}}\,J^j\,\frac{\partial \mathrm{w}}{\partial x^j}\right).
\end{aligned}
\tag{12.4}
$$

Hence, we have

$$\frac{\partial T_i^\nu}{\partial x^\nu} - \Gamma_{i\nu}^\sigma T_\sigma^\nu + \frac{\partial \ln\sqrt{-g}}{\partial x^\sigma} T_i^\sigma = \frac{1}{c^2}\left[\left(\rho + \frac{1}{c^2} v_j J^j\right) F_i + \right.$$
$$\left. + \frac{1}{c^2} F_j U_i^j - \frac{{}^*\partial J_i}{\partial t} - DJ_i - {}^*\nabla_j U_i^j + \left(\frac{\partial v_j}{\partial x^i} - \frac{\partial v_i}{\partial x^j}\right) J^j\right] -$$
$$- \frac{1}{c^2} v_i \left[\frac{{}^*\partial \rho}{\partial t} + \rho D + {}^*\nabla_j J^j - \frac{1}{c^2} F_j J^j + \frac{1}{c^2} D_{jk} U^{jk}\right] = \qquad (12.5)$$
$$= -\frac{1}{c^2}\left\{\left[\frac{{}^*\partial J_i}{\partial t} + DJ_i + {}^*\nabla_j U_i^j - 2A_{ij} J^j - F_j\left(\rho h_i^j + U_i^j\right)\right] + \right.$$
$$\left. + v_i \left[\frac{{}^*\partial \rho}{\partial t} + \rho D + {}^*\nabla_j J^j - \frac{2}{c^2} F_j J^j + \frac{1}{c^2} D_{jk} U^{jk}\right]\right\}$$

and (12.1) can be written in the form

$$\left(\frac{{}^*\partial J_i}{\partial t} + DJ_i + {}^*\nabla_j U_i^j - 2A_{ij} J^j - \rho F_i - \frac{1}{c^2} F_j U_i^j\right) +$$
$$+ v_i \left(\frac{{}^*\partial \rho}{\partial t} + \rho D + {}^*\nabla_j J^j - \frac{2}{c^2} F_j J^j + \frac{1}{c^2} D_{jk} U^{jk}\right) = 0\,. \qquad (12.6)$$

With the remaining formula (11.7) unused, we apply the method to vary the potentials.

Equation (12.6) retains its form in transformations of the time coordinate, although its left side is a co-vector, not a chr.inv.-vector. For this reason we can set all values of v_i to zero. Then we have

$$\frac{{}^*\partial J_i}{\partial t} + DJ_i + {}^*\nabla_j U_i^j - 2A_{ij} J^j - \rho F_i - \frac{1}{c^2} F_j U_i^j = 0\,, \qquad (12.7)$$

and this sub-vector equation is valid for any choice of the time coordinate, because its left side is a chr.inv.-vector. Thus, for any choice of the time coordinate, we have

$$v_i \left(\frac{{}^*\partial \rho}{\partial t} + \rho D + {}^*\nabla_j J^j - \frac{2}{c^2} F_j J^j + \frac{1}{c^2} D_{jk} U^{jk}\right) = 0 \qquad (12.8)$$

and hence, we obtain the equation

$$\frac{{}^*\partial \rho}{\partial t} + \rho D + {}^*\nabla_j J^j - \frac{2}{c^2} F_j J^j + \frac{1}{c^2} D_{jk} U^{jk} = 0\,, \qquad (12.9)$$

which coincides with formula (11.7).

So, just as for the geodesic equations, the method of varying the potentials has found four equations from only a subset of them (see the explanation of this fact in §3.7).

Because

$$h^{ik}\frac{{}^*\partial J_i}{\partial t}=\frac{{}^*\partial J^k}{\partial t}-J_i\frac{{}^*\partial h^{ik}}{\partial t}=\frac{{}^*\partial J^k}{\partial t}+2D^{ik}J_i=\frac{{}^*\partial J^k}{\partial t}+2D^k_iJ^i, \quad (12.10)$$

we can write

$$\frac{{}^*\partial J^k}{\partial t}+DJ^k+2\left(D^k_j+A_{j\cdot}^{\cdot k}\right)J^j-\rho F^k-\frac{1}{c^2}F_jU^{jk}+{}^*\nabla_jU^{jk}=0 \quad (12.11)$$

instead of equation (12.7).

§ 3.13 A space element. Its energy and the momentum

Let us take a fixed elementary parallelepiped

$$\Pi^{123}_{abc}=\begin{vmatrix}\delta_ax^1 & \delta_ax^2 & \delta_ax^3\\ \delta_bx^1 & \delta_bx^2 & \delta_bx^3\\ \delta_cx^1 & \delta_cx^2 & \delta_cx^3\end{vmatrix} \qquad \left.\begin{aligned}\delta_ax^i&=const^i_a\\ \delta_bx^i&=const^i_b\\ \delta_cx^i&=const^i_c\end{aligned}\right\} \quad (13.1)$$

in the space we are considering. We can write its volume V, in accordance with formula (12.7) of §2.12, as follows

$$V=\sqrt{h}\,\left|\Pi^{123}_{abc}\right|. \quad (13.2)$$

The energy E and the momentum p^k inside the volume, evidently take the form

$$E=V\rho c^2, \quad (13.3)$$

$$p^k=VJ^k. \quad (13.4)$$

Because this elementary volume is fixed in the space, we can write

$$\left(\frac{{}^*dE}{dt}\right)_{\text{fix}}=\frac{{}^*\partial E}{\partial t}, \quad (13.5)$$

$$\left(\frac{{}^*dp^k}{dt}\right)_{\text{fix}}=\frac{{}^*\partial p^k}{\partial t}. \quad (13.6)$$

On the other hand, because of formula (12.9) of §2.12, we have

$$\frac{{}^*\partial E}{\partial t}=\left(\frac{{}^*\partial\rho}{\partial t}+\rho D\right)c^2V, \quad (13.7)$$

$$\frac{{}^*\partial p^k}{\partial t} = \left(\frac{{}^*\partial J^k}{\partial t} + DJ^k\right) V . \tag{13.8}$$

Hence,

$$\left(\frac{{}^*dE}{dt}\right)_{\text{fix}} = \left(\frac{{}^*\partial\rho}{\partial t} + \rho D\right) c^2 V , \tag{13.9}$$

$$\left(\frac{{}^*dp^k}{dt}\right)_{\text{fix}} = \left(\frac{{}^*\partial J^k}{\partial t} + DJ^k\right) V . \tag{13.10}$$

The volume V is a chr.inv.-invariant, which retains its value in the parallel transfer of the chr.inv.-vectors $\delta_a x^i$, $\delta_b x^i$, $\delta_c x^i$. Naturally, because the chr.inv.-tensor of the 3rd rank Π^{ijk}_{abc} has the same properties as ε_{ijk}, we have

$$V = \frac{1}{6}\left|\varepsilon_{ijk}\,\Pi^{ijk}_{abc}\right| . \tag{13.11}$$

Because of formula (17.4) of §2.17, we have

$${}^*\nabla_p \left(\varepsilon_{ijk}\,\Pi^{ijk}_{abc}\right) = \varepsilon_{ijk}\,{}^*\nabla_p\,\Pi^{ijk}_{abc} . \tag{13.12}$$

The rule for differentiating determinants holds for chr.inv.-differentiation as well as in covariant differentiation. Hence, the equalities

$${}^*\nabla_p \left(\delta_a x^i\right) = 0 , \quad {}^*\nabla_p \left(\delta_b x^i\right) = 0 , \quad {}^*\nabla_p \left(\delta_c x^i\right) = 0 \tag{13.13}$$

lead to the equality

$${}^*\nabla_p\,\Pi^{ijk}_{abc} = 0 , \tag{13.14}$$

so we have

$${}^*\nabla_p V = 0 . \tag{13.15}$$

Because of (13.3), (13.4), (13.9), (13.10), and (13.15), and denoting

$$m = V\rho \tag{13.16}$$

for the moving mass throughout this volume, we can, instead of (12.11) and (12.9) write respectively,

$$\begin{gathered}\left(\frac{{}^*dp^k}{dt}\right)_{\text{fix}} - mF^k + 2\left(D^k_i + A^{\cdot k}_{i\cdot}\right) p^i = \\ = \frac{1}{c^2}\left[\, F_i \left(VU^{ik}\right) - {}^*\nabla_i \left(VU^{ik}\right) c^2 \,\right] ,\end{gathered} \tag{13.17}$$

$$\left(\frac{{}^*dE}{dt}\right)_{\text{fix}} + D_{ij}\left(VU^{ij}\right) - F_j\,p^j = F_j\,p^j - {}^*\nabla_j\,p^j c^2, \tag{13.18}$$

where $p^j c^2$ is the energy flux.

The equations we have obtained here are related to a fixed *volume element*, which is the frame of a flowing continuous matter. This is a very important difference to the equations (7.13) and (7.9) we obtained for a free *mass-particle*, which moves with respect to the given space.

§ 3.14 Einstein's covariant tensor. Its time component

We now consider the Einstein covariant tensor

$$G_{\mu\nu} = -\frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} + \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} + \frac{\partial^2 \ln\sqrt{-g}}{\partial x^\mu \partial x^\nu} - \Gamma^\alpha_{\mu\nu}\frac{\partial \ln\sqrt{-g}}{\partial x^\alpha}\,. \tag{14.1}$$

We first consider its time component, i. e. the component with the indices $\mu,\nu = 0$

$$G_{00} = -\frac{\partial\Gamma^\alpha_{00}}{\partial x^0} + \Gamma^\beta_{0\alpha}\Gamma^\alpha_{0\beta} + \frac{\partial^2 \ln\sqrt{-g}}{\partial x^0 \partial x^0} - \Gamma^\alpha_{00}\frac{\partial \ln\sqrt{-g}}{\partial x^\alpha}\,. \tag{14.2}$$

Because of (3.17) and (3.18), we obtain

$$\begin{aligned}
-\frac{\partial\Gamma^\alpha_{00}}{\partial x^\alpha} &= -\frac{\partial\Gamma^0_{00}}{\partial x^0} - \frac{\partial\Gamma^k_{00}}{\partial x^k} = \frac{1}{c^4}\left[\frac{c^2}{(c^2-\mathrm{w})^2}\left(\frac{\partial \mathrm{w}}{\partial t}\right)^2 + \frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2 \mathrm{w}}{\partial t^2} - \right.\\
&\left. - \frac{1}{c^2}\,v_l F^l \frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)F^l\frac{\partial v_l}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)v_l\frac{\partial F^l}{\partial t}\right] + \\
&+ \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[-\frac{2}{c^2}\,F^k\frac{\partial \mathrm{w}}{\partial x^k} + \left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial F^k}{\partial x^k}\right] = \\
&= \frac{1}{c^4}\left[\frac{c^2}{(c^2-\mathrm{w})^2}\left(\frac{\partial \mathrm{w}}{\partial t}\right)^2 + \frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2 \mathrm{w}}{\partial t^2} - \frac{1}{c^2}\,v_l F^l\frac{\partial \mathrm{w}}{\partial t}\right] - \\
&- \frac{1}{c^4}\left(1-\frac{\mathrm{w}}{c^2}\right)F^k\frac{\partial \mathrm{w}}{\partial x^k} - \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2\left(\frac{1}{c^2}\,F_k F^k + \frac{{}^*\partial F^k}{\partial x^k}\right).
\end{aligned} \tag{14.3}$$

Because of (3.17–3.20), we obtain

$$\begin{aligned}
\Gamma^\beta_{0\alpha}\Gamma^\alpha_{0\beta} &= \Gamma^0_{00}\Gamma^0_{00} + 2\Gamma^k_{00}\Gamma^0_{0k} + \Gamma^k_{0i}\Gamma^i_{0k} = \\
&= \frac{1}{c^6}\left[\frac{c^4}{(c^2-\mathrm{w})^2}\left(\frac{\partial \mathrm{w}}{\partial t}\right)^2 + 2\,v_l F^l\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)^2\left(v_l F^l\right)^2\right] -
\end{aligned}$$

$$-\frac{2}{c^4}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[-F^k\frac{\partial\mathrm{w}}{\partial x^k}+\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_k^l+A_{k\cdot}^{\cdot l}\right)v_l F^k+\right.$$

$$\left.+\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(v_l F^l\right)^2\right]+\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2\times$$

$$\times\left[D_i^k D_k^i+A_{i\cdot}^{\cdot k}A_{k\cdot}^{\cdot i}+\frac{2}{c^2}\left(D_i^k+A_{i\cdot}^{\cdot k}\right)v_k F^i+\frac{1}{c^4}\left(v_k F^k\right)^2\right]= \tag{14.4}$$

$$=\frac{1}{c^4}\left[\frac{c^2}{(c^2-\mathrm{w})^2}\left(\frac{\partial\mathrm{w}}{\partial t}\right)^2+\frac{2}{c^2}\,v_l F^l\frac{\partial\mathrm{w}}{\partial t}\right]+$$

$$+\frac{2}{c^4}\left(1-\frac{\mathrm{w}}{c^2}\right)F^k\frac{\partial\mathrm{w}}{\partial x^k}+\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2\left(D_i^k D_k^i+A_{i\cdot}^{\cdot k}A_{k\cdot}^{\cdot i}\right).$$

Hence, because of (1.7), we obtain

$$\frac{\partial^2\ln\sqrt{-g}}{\partial x^0\partial x^0}=\frac{1}{c^2}\frac{\partial}{\partial t}\left[-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial t}+\left(1-\frac{\mathrm{w}}{c^2}\right)D\right]=$$

$$=-\frac{1}{c^4}\left[\frac{c^2}{(c^2-\mathrm{w})^2}\left(\frac{\partial\mathrm{w}}{\partial t}\right)^2+\frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2\mathrm{w}}{\partial t^2}+\frac{\partial\mathrm{w}}{\partial t}\right]+\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2\frac{{}^*\partial D}{\partial t}, \tag{14.5}$$

and because of (3.17), (3.18), and (1.7), we obtain

$$-\Gamma_{00}^{\alpha}\frac{\partial\ln\sqrt{-g}}{\partial x^{\alpha}}=-\Gamma_{00}^{0}\frac{\partial\ln\sqrt{-g}}{\partial x^0}-\Gamma_{00}^{k}\frac{\partial\ln\sqrt{-g}}{\partial x^k}=$$

$$=\frac{1}{c^4}\left[\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial t}+\left(1-\frac{\mathrm{w}}{c^2}\right)v_l F^l\right]\left[-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial t}+\left(1-\frac{\mathrm{w}}{c^2}\right)D\right]+$$

$$+\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2F^k\left(-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^k}+\frac{\partial\ln\sqrt{h}}{\partial x^k}\right)= \tag{14.6}$$

$$=-\frac{1}{c^4}\left[\frac{c^2}{(c^2-\mathrm{w})^2}\left(\frac{\partial\mathrm{w}}{\partial t}\right)^2+\left(\frac{1}{c^2}\,v_l F^l-D\right)\frac{\partial\mathrm{w}}{\partial t}\right]-$$

$$-\frac{1}{c^4}\left(1-\frac{\mathrm{w}}{c^2}\right)F^k\frac{\partial\mathrm{w}}{\partial x^k}+\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2F^k\frac{{}^*\partial\ln\sqrt{h}}{\partial x^k}.$$

We finally obtain

$$G_{00}=\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2\times$$

$$\times\left(\frac{{}^*\partial D}{\partial t}+D_j^l D_l^j+A_{j\cdot}^{\cdot l}A_{l\cdot}^{\cdot j}+{}^*\nabla_j F^j-\frac{1}{c^2}F_j F^j\right). \tag{14.7}$$

§ 3.15 Einstein's covariant tensor. The mixed components

We now consider mixed (space-time) components of the Einstein tensor (14.1), i. e. its components with the indices $\mu=0$, $\nu=1,2,3$

$$G_{0i} = -\frac{\partial\Gamma^{\alpha}_{0i}}{\partial x^{\alpha}} + \Gamma^{\beta}_{0\alpha}\Gamma^{\alpha}_{i\beta} + \frac{\partial^2 \ln\sqrt{-g}}{\partial x^0 \partial x^i} - \Gamma^{\alpha}_{0i}\frac{\partial \ln\sqrt{-g}}{\partial x^{\alpha}}\,. \tag{15.1}$$

Because of (3.19) and (3.20), we obtain

$$\begin{gathered}
-\frac{\partial\Gamma^{\alpha}_{0i}}{\partial x^{\alpha}} = -\frac{\partial\Gamma^{0}_{0i}}{\partial x^{0}} - \frac{\partial\Gamma^{k}_{0i}}{\partial x^{k}} = -\frac{1}{c^3}\left[-\frac{c^2}{(c^2-\mathrm{w})^2}\frac{\partial \mathrm{w}}{\partial t}\frac{\partial \mathrm{w}}{\partial x^i} - \right.\\
-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2 \mathrm{w}}{\partial t\,\partial x^i} + \left(D^l_i + A^{\cdot l}_{i\cdot} + \frac{1}{c^2}\,v_i F^l\right)\frac{\partial v_l}{\partial t} + \\
\left. + v_l\frac{\partial}{\partial t}\left(D^l_i + A^{\cdot l}_{i\cdot} + \frac{1}{c^2}\,v_i F^l\right)\right] - \frac{1}{c}\left[-\frac{1}{c^2}\left(D^k_i + A^{\cdot k}_{i\cdot} + \right.\right.\\
\left.\left. + \frac{1}{c^2}\,v_i F^k\right)\frac{\partial \mathrm{w}}{\partial x^k} + \left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial}{\partial x^k}\left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2}\,v_i F^k\right)\right] = \\
= \frac{1}{c^3}\frac{c^2}{c^2-\mathrm{w}}\left(\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t}\frac{\partial \mathrm{w}}{\partial x^i} + \frac{\partial^2 \mathrm{w}}{\partial t\,\partial x^i}\right) + \frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)F_k \times \\
\times\left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2}\,v_i F^k\right) - \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{{}^*\partial}{\partial x^k}\left(D^k_i + A^{\cdot k}_{i\cdot} + \right.\\
\left. + \frac{1}{c^2}\,v_i F^k\right) = \frac{1}{c^3}\frac{c^2}{c^2-\mathrm{w}}\left(\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t}\frac{\partial \mathrm{w}}{\partial x^i} + \frac{\partial^2 \mathrm{w}}{\partial t\,\partial x^i}\right) - \\
-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[\frac{{}^*\partial}{\partial x^k}\left(D^k_i + A^{\cdot k}_{i\cdot}\right) - \frac{1}{c^2}\left(D^k_i + A^{\cdot k}_{i\cdot}\right)F_k + \right.\\
\left. + \frac{1}{c^2}F^k\frac{\partial v_i}{\partial x^k} + \frac{1}{c^4}\frac{c^2}{c^2-\mathrm{w}}\,v_k F^k\frac{\partial v_i}{\partial t}\right] - \\
-\frac{1}{c^3}\,v_i\left(1-\frac{\mathrm{w}}{c^2}\right)\left(\frac{{}^*\partial F^k}{\partial x^k} - \frac{1}{c^2}F_k F^k\right).
\end{gathered} \tag{15.2}$$

Because of (3.17–3.22), we obtain

$$\begin{gathered}
\Gamma^{\beta}_{0\alpha}\Gamma^{\alpha}_{i\beta} = \Gamma^{0}_{00}\Gamma^{0}_{i0} + \Gamma^{k}_{00}\Gamma^{0}_{ik} + \Gamma^{0}_{0k}\Gamma^{k}_{i0} + \Gamma^{j}_{0k}\Gamma^{k}_{ij} = \\
-\frac{1}{c^5}\left[\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)v_k F^k\right]\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^i} + \right.\\
\left. + \left(D^l_i + A^{\cdot l}_{i\cdot} + \frac{1}{c^2}\,v_i F^l\right)v_l\right] + \frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)F^k\left\{\Psi_{ik} - D_{ik} + \right.
\end{gathered}$$

$$+\frac{1}{c^2}v_l\Big[\big(D_k^l+A_{k\cdot}^{\cdot l}\big)v_i+\big(D_i^l+A_{i\cdot}^{\cdot l}\big)v_k+\frac{1}{c^2}v_iv_kF^l\Big]\Big\}+$$

$$+\frac{1}{c^3}\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^k}+\left(D_k^l+A_{k\cdot}^{\cdot l}+\frac{1}{c^2}v_kF^l\right)v_l\right]\times$$

$$\times\left(D_i^k+A_{i\cdot}^{\cdot k}+\frac{1}{c^2}v_iF^k\right)\left(1-\frac{\mathrm{w}}{c^2}\right)+\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\times$$

$$\times\left(D_k^j+A_{k\cdot}^{\cdot j}+\frac{1}{c^2}v_kF^j\right)\Big\{{}^*\Delta_{ij}^k-\frac{1}{c^2}\Big[\big(D_j^k+A_{j\cdot}^{\cdot k}\big)v_i+$$

$$+\big(D_i^k+A_{i\cdot}^{\cdot k}\big)v_j+\frac{1}{c^2}v_iv_jF^k\Big]\Big\}=-\frac{1}{c^3}\frac{1}{c^2-\mathrm{w}}\times$$

$$\times\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^i}+\left(D_i^l+A_{i\cdot}^{\cdot l}+\frac{1}{c^2}v_iF^l\right)v_l\right]\frac{\partial\mathrm{w}}{\partial t}+$$

$$+\frac{1}{c^5}v_nF^n\frac{\partial\mathrm{w}}{\partial x^i}-\frac{1}{c^5}\left(1-\frac{\mathrm{w}}{c^2}\right)v_nF^n\big(D_i^l+A_{i\cdot}^{\cdot l}\big)v_l-$$

$$-\frac{1}{c^7}\left(1-\frac{\mathrm{w}}{c^2}\right)\big(v_kF^k\big)^2v_i-\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\Big[D_{ik}F^k-$$

$$-\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}+\frac{\partial v_i}{\partial x^k}\right)F^k+\frac{1}{2c^2}\left(F_kv_i+F_iv_k\right)F^k+$$

$$+{}^*\Delta_{ik}^lv_lF^k-\frac{1}{c^2}v_kF^k\left(D_i^l+A_{i\cdot}^{\cdot l}\right)v_l\Big]+\frac{1}{c^5}v_i\left(1-\frac{\mathrm{w}}{c^2}\right)\times$$

$$\times\Big[\big(D_k^l+A_{k\cdot}^{\cdot l}\big)v_lF^k+\frac{1}{c^2}\big(v_kF^k\big)^2\Big]-\frac{1}{c^3}\big(D_i^k+A_{i\cdot}^{\cdot k}\big)\frac{\partial\mathrm{w}}{\partial x^k}+$$

$$+\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\big(D_i^k+A_{i\cdot}^{\cdot k}\big)\big(D_k^l+A_{k\cdot}^{\cdot l}\big)v_l+\frac{1}{c^5}\left(1-\frac{\mathrm{w}}{c^2}\right)\times$$

$$\times\big(D_i^k+A_{i\cdot}^{\cdot k}\big)v_kv_lF^l+\frac{1}{c^5}v_i\left(1-\frac{\mathrm{w}}{c^2}\right)\Big[-\frac{c^2}{c^2-\mathrm{w}}F^k\frac{\partial\mathrm{w}}{\partial x^k}+$$

$$+\big(D_k^l+A_{k\cdot}^{\cdot l}\big)v_lF^k+\frac{1}{c^2}\big(v_kF^k\big)^2\Big]+\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\times$$

$$\times\left(D_k^j+A_{k\cdot}^{\cdot j}\right){}^*\Delta_{ij}^k+\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right){}^*\Delta_{ij}^kv_kF^j-\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\times$$

$$\times\big(D_i^k+A_{i\cdot}^{\cdot k}\big)\big(D_k^j+A_{k\cdot}^{\cdot j}\big)v_j-\frac{1}{c^5}\left(1-\frac{\mathrm{w}}{c^2}\right)\times$$

$$\times\big(D_i^k+A_{i\cdot}^{\cdot k}\big)v_kv_jF^j-\frac{1}{c^3}v_i\left(1-\frac{\mathrm{w}}{c^2}\right)\Big[D_j^kD_k^j+A_{j\cdot}^{\cdot k}A_{k\cdot}^{\cdot j}+$$

$$+\frac{2}{c^2}\left(D_j^k + A_{j\cdot}^{\cdot k}\right) v_k F^j + \frac{1}{c^4}\left(v_k F^k\right)^2\Bigg] = -\frac{1}{c^3}\frac{1}{c^2-\mathrm{w}}\times$$
$$\times\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^i} + v_l\left(D_i^l + A_{i\cdot}^{\cdot l} + \frac{1}{c^2}v_i F^l\right)\right]\frac{\partial \mathrm{w}}{\partial t} -$$
$$-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\Bigg[-{}^*\Delta_{ij}^k\left(D_k^j + A_{k\cdot}^{\cdot j}\right) - \frac{1}{2c^2}\left(\frac{\partial v_k}{\partial x^i} + \frac{\partial v_i}{\partial x^k}\right)F^k +$$
$$+\frac{1}{2c^4}\left(v_k F^k\right)F_i - \frac{1}{c^4}\frac{c^2}{c^2-\mathrm{w}}v_k F^k\frac{\partial \mathrm{w}}{\partial x^i} + \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\times \tag{15.3}$$
$$\times\left(D_i^k + A_{i\cdot}^{\cdot k}\right)\frac{\partial \mathrm{w}}{\partial x^k} + \frac{1}{c^2}D_{ik}F^k\Bigg] - \frac{1}{c^3}v_i\left(1-\frac{\mathrm{w}}{c^2}\right)\times$$
$$\times\left(\frac{1}{2c^2}F_k F^k + \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}F^k\frac{\partial \mathrm{w}}{\partial x^k} + D_j^k D_k^j + A_{j\cdot}^{\cdot k}A_{k\cdot}^{\cdot j}\right).$$

Because of (1.7), we obtain

$$\frac{\partial^2 \ln\sqrt{-g}}{\partial x^0 \partial x^i} = \frac{1}{c}\frac{\partial}{\partial x^i}\left[-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)D\right] =$$
$$= -\frac{1}{c^3}\frac{c^2}{(c^2-\mathrm{w})^2}\frac{\partial \mathrm{w}}{\partial x^i}\frac{\partial \mathrm{w}}{\partial t} - \frac{1}{c^3}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2 \mathrm{w}}{\partial x^i \partial t} - \tag{15.4}$$
$$-\frac{1}{c^3}D\frac{\partial \mathrm{w}}{\partial x^i} + \left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial D}{\partial x^i}.$$

Because of (3.19), (3.20), and (1.7), we obtain

$$-\Gamma_{0i}^{\alpha}\frac{\partial \ln\sqrt{-g}}{\partial x^\alpha} = -\Gamma_{0i}^{0}\frac{\partial \ln\sqrt{-g}}{\partial x^0} - \Gamma_{0i}^{k}\frac{\partial \ln\sqrt{-g}}{\partial x^k} =$$
$$= -\frac{1}{c^3}\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^i} + v_l\left(D_i^l + A_{i\cdot}^{\cdot l} + \frac{1}{c^2}v_i F^l\right)\right]\times$$
$$\times\left[-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)D\right] - \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\times$$
$$\times\left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}v_i F^k\right)\left(-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^k} + \frac{\partial \ln\sqrt{h}}{\partial x^k}\right) =$$
$$= \frac{1}{c^3}\frac{1}{c^2-\mathrm{w}}\left[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^i} + v_l\left(D_i^l + A_{i\cdot}^{\cdot l} + \frac{1}{c^2}v_i F^l\right)\right]\frac{\partial \mathrm{w}}{\partial t} +$$
$$+\frac{1}{c^3}D\frac{\partial \mathrm{w}}{\partial x^i} - \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\Bigg[\left(D_i^k + A_{i\cdot}^{\cdot k}\right)\frac{\partial \ln\sqrt{h}}{\partial x^k} +$$

$$+\frac{1}{c^2}\,v_k\,D\big(D_i^k + A_{i\cdot}^{\cdot k}\big) - \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\left(D_i^k + A_{i\cdot}^{\cdot k}\right)\frac{\partial \mathrm{w}}{\partial x^k}\Bigg] -$$
$$-\frac{1}{c^3}\,v_i\Big(1-\frac{\mathrm{w}}{c^2}\Big)\bigg(F^k\frac{\partial\ln\sqrt{h}}{\partial x^k} + \frac{1}{c^2}\,v_k DF^k -$$
$$-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}F^k\frac{\partial \mathrm{w}}{\partial x^k}\bigg) = \frac{1}{c^3}\frac{1}{c^2-\mathrm{w}}\bigg[-\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^i} +$$
$$+\,v_l\left(D_i^l + A_{i\cdot}^{\cdot l} + \frac{1}{c^2}\,v_i F^l\right)\bigg]\frac{\partial \mathrm{w}}{\partial t} + \frac{1}{c^3}D\frac{\partial \mathrm{w}}{\partial x^i} - \frac{1}{c}\Big(1-\frac{\mathrm{w}}{c^2}\Big)\times \tag{15.5}$$
$$\times\bigg[\big(D_i^k + A_{i\cdot}^{\cdot k}\big)\frac{{}^*\partial\ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\left(D_i^k + A_{i\cdot}^{\cdot k}\right)\frac{\partial \mathrm{w}}{\partial x^k}\bigg] -$$
$$-\frac{1}{c^3}\,v_i\Big(1-\frac{\mathrm{w}}{c^2}\Big)\bigg(F^k\frac{{}^*\partial\ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}F^k\frac{\partial \mathrm{w}}{\partial x^k}\bigg).$$

This gives

$$G_{0i} = -\frac{1}{c}\Big(1-\frac{\mathrm{w}}{c^2}\Big)\bigg[{}^*\nabla_k\left(D_i^k + A_{i\cdot}^{\cdot k}\right) - \frac{\partial D}{\partial x^i} - \frac{1}{c^2}A_{i\cdot}^{\cdot k}F_k -$$
$$-\frac{1}{2c^2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right)F^k - \frac{1}{2c^4}F_i v_k F^k\bigg] - \frac{1}{c^3}\,v_i\Big(1-\frac{\mathrm{w}}{c^2}\Big)\times \tag{15.6}$$
$$\times\left({}^*\nabla_k F^k - \frac{1}{2c^2}F_kF^k + D_j^k D_k^j + A_{j\cdot}^{\cdot k}A_{k\cdot}^{\cdot j}\right).$$

and so, in the final form, we have

$$G_{0i} = \frac{1}{c}\Big(1-\frac{\mathrm{w}}{c^2}\Big)\bigg[{}^*\nabla_j\big(h_i^j D - D_i^j\big) - {}^*\nabla_j A_{i\cdot}^{\cdot j} + \frac{2}{c^2}A_{ij}F^j -$$
$$-\frac{1}{c^2}\,v_i\left(\frac{{}^*\partial D}{\partial t} + D_j^l D_l^j + A_{j\cdot}^{\cdot l}A_{l\cdot}^{\cdot j} + {}^*\nabla_j F^j - \frac{1}{c^2}F_jF^j\right)\bigg]. \tag{15.7}$$

§ 3.16 Einstein's covariant tensor. The spatial components

We finally consider spatial components of the Einstein tensor (14.1), i. e. its components with $\mu,\nu = 1,2,3$

$$G_{ij} = -\frac{\partial\Gamma_{ij}^{\alpha}}{\partial x^{\alpha}} + \Gamma_{i\alpha}^{\beta}\Gamma_{j\beta}^{\alpha} + \frac{\partial^2\ln\sqrt{-g}}{\partial x^i\partial x^j} - \Gamma_{ij}^{\alpha}\frac{\partial\ln\sqrt{-g}}{\partial x^{\alpha}}\,. \tag{16.1}$$

Because it will entail more difficult algebra than that used for G_{00} and G_{0i}, we will summarize all the necessary terms step-by-step.

First, we have

$$-\frac{\partial\Gamma^{\alpha}_{ij}}{\partial x^{\alpha}} = -\frac{\partial\Gamma^{0}_{ij}}{\partial x^{0}} - \frac{\partial\Gamma^{k}_{ij}}{\partial x^{k}}\,. \tag{16.2}$$

Because of (3.21) and also (22.3) of §2.22, we obtain

$$\begin{gathered}
-\frac{\partial\Gamma^{0}_{ij}}{\partial x^{0}} = \frac{1}{c^2}\frac{c^2}{(c^2-\mathrm{w})^2}\bigg\{\Psi_{ij} - D_{ij} + \\
+\frac{1}{c^2}\,v_l\Big[\big(D^l_j + A^{\cdot l}_{j\cdot}\big)v_i + \big(D^l_i + A^{\cdot l}_{i\cdot}\big)v_j + \frac{1}{c^2}\,v_i v_j F^l\Big]\bigg\}\frac{\partial\mathrm{w}}{\partial t} + \\
+\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\left(\frac{\partial\Psi_{ij}}{\partial t} - \frac{\partial D_{ij}}{\partial t}\right) + \frac{1}{c^4}\frac{c^2 v_l}{c^2-\mathrm{w}}\times \\
\times\left[\big(D^l_j + A^{\cdot l}_{j\cdot}\big)\frac{\partial v_i}{\partial t} + \big(D^l_i + A^{\cdot l}_{i\cdot}\big)\frac{\partial v_j}{\partial t}\right] + \frac{1}{c^4}\frac{c^2 v_i}{c^2-\mathrm{w}}\times \\
\times\left[\big(D^l_j + A^{\cdot l}_{j\cdot}\big)\frac{\partial v_l}{\partial t} + v_l\frac{\partial}{\partial t}\big(D^l_j + A^{\cdot l}_{j\cdot}\big) + \frac{1}{c^2}\,v_l F^l\frac{\partial v_j}{\partial t}\right] + \\
+\frac{1}{c^4}\frac{c^2 v_j}{c^2-\mathrm{w}}\left[\big(D^l_i + A^{\cdot l}_{i\cdot}\big)\frac{\partial v_l}{\partial t} + v_l\frac{\partial}{\partial t}\big(D^l_i + A^{\cdot l}_{i\cdot}\big) + \right. \\
\left. +\frac{1}{c^2}\,v_l F^l\frac{\partial v_i}{\partial t}\right] + \frac{1}{c^6}\frac{c^2 v_i v_j}{c^2-\mathrm{w}}\left(F^l\frac{\partial v_l}{\partial t} + v_l\frac{\partial F^l}{\partial t}\right) = \\
= (\alpha_1)_{ij} + (\beta_1)_{ij} + (\gamma_1)_{ij} + (\delta_1)_{ij} + (\varepsilon_1)_{ij}\,,
\end{gathered} \tag{16.3}$$

where the quantities $(\alpha_1)_{ij}$, $(\beta_1)_{ij}$, $(\gamma_1)_{ij}$, $(\delta_1)_{ij}$, $(\varepsilon_1)_{ij}$, are given by

$$\begin{gathered}
(\alpha_1)_{ij} = \frac{1}{c^2}\frac{c^2}{(c^2-\mathrm{w})^2}\bigg\{\Psi_{ij} - D_{ij} + \frac{1}{c^2}\,v_l\Big[\big(D^l_j + A^{\cdot l}_{j\cdot}\big)v_i + \\
+\big(D^l_i + A^{\cdot l}_{i\cdot}\big)v_j + \frac{1}{c^2}\,v_i v_j F^l\Big]\bigg\}\frac{\partial\mathrm{w}}{\partial t}\,,
\end{gathered} \tag{16.4}$$

$$\begin{gathered}
(\beta_1)_{ij} = \frac{1}{c^2}\left\{-\frac{{}^*\partial D_{ij}}{\partial t} + \frac{{}^*\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) - \right.\right. \\
\left. -\frac{1}{2c^2}\left(F_j v_i + F_i v_j\right)\right] - {}^*\Delta^l_{ij}\,\frac{c^2}{c^2-\mathrm{w}}\frac{\partial v_l}{\partial t} + v_l\left[-\frac{{}^*\partial\,{}^*\Delta^l_{ij}}{\partial t} + \right. \\
\left.\left. +\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\big(D^l_j + A^{\cdot l}_{j\cdot}\big)\frac{\partial v_i}{\partial t} + \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\big(D^l_i + A^{\cdot l}_{i\cdot}\big)\frac{\partial v_j}{\partial t}\right]\right\},
\end{gathered} \tag{16.5}$$

$$(\gamma_1)_{ij}=\frac{1}{c^4}\frac{c^2 v_i}{c^2-\mathrm{w}}\left[\left(D_j^l+A_{j\cdot}^{\cdot l}\right)\frac{\partial v_l}{\partial t}+v_l\frac{\partial}{\partial t}\left(D_j^l+A_{j\cdot}^{\cdot l}\right)+\frac{1}{c^2}v_l F^l\frac{\partial v_j}{\partial t}\right], \tag{16.6}$$

$$(\delta_1)_{ij}=\frac{1}{c^4}\frac{c^2 v_j}{c^2-\mathrm{w}}\left[\left(D_i^l+A_{i\cdot}^{\cdot l}\right)\frac{\partial v_l}{\partial t}+v_l\frac{\partial}{\partial t}\left(D_i^l+A_{i\cdot}^{\cdot l}\right)+\frac{1}{c^2}v_l F^l\frac{\partial v_i}{\partial t}\right], \tag{16.7}$$

$$(\varepsilon_1)_{ij}=\frac{1}{c^6}\frac{c^2 v_i v_j}{c^2-\mathrm{w}}\left(F^l\frac{\partial v_l}{\partial t}+v_l\frac{\partial F^l}{\partial t}\right). \tag{16.8}$$

Because of (3.22), we obtain

$$\begin{aligned}-\frac{\partial\Gamma_{ij}^k}{\partial x^k}&=-\frac{{}^*\partial\,{}^*\Delta_{ij}^k}{\partial x^k}+\frac{1}{c^2}v_l\frac{{}^*\partial\,{}^*\Delta_{ij}^l}{\partial t}+\frac{1}{c^2}\left[\left(D_j^k+A_{j\cdot}^{\cdot k}\right)\frac{\partial v_i}{\partial x^k}+\right.\\&\left.+\left(D_i^k+A_{i\cdot}^{\cdot k}\right)\frac{\partial v_j}{\partial x^k}\right]+\frac{1}{c^2}v_i\left[\frac{\partial}{\partial x^k}\left(D_j^k+A_{j\cdot}^{\cdot k}\right)+\frac{1}{c^2}F^k\frac{\partial v_j}{\partial x^k}\right]+\\&+\frac{1}{c^2}v_j\left[\frac{\partial}{\partial x^k}\left(D_i^k+A_{i\cdot}^{\cdot k}\right)+\frac{1}{c^2}F^k\frac{\partial v_i}{\partial x^k}\right]+\frac{1}{c^4}v_i v_j\frac{\partial F^k}{\partial x^k}=\\&=(\beta_2)_{ij}+(\gamma_2)_{ij}+(\delta_2)_{ij}+(\varepsilon_2)_{ij}\,,\end{aligned} \tag{16.9}$$

where we denote

$$\begin{aligned}(\beta_2)_{ij}&=-\frac{{}^*\partial\,{}^*\Delta_{ij}^k}{\partial x^k}+\frac{1}{c^2}v_l\frac{{}^*\partial\,{}^*\Delta_{ij}^l}{\partial t}+\\&+\frac{1}{c^2}\left[\left(D_j^k+A_{j\cdot}^{\cdot k}\right)\frac{\partial v_i}{\partial x^k}+\left(D_i^k+A_{i\cdot}^{\cdot k}\right)\frac{\partial v_j}{\partial x^k}\right],\end{aligned} \tag{16.10}$$

$$(\gamma_2)_{ij}=\frac{1}{c^2}v_i\left[\frac{\partial}{\partial x^k}\left(D_j^k+A_{j\cdot}^{\cdot k}\right)+\frac{1}{c^2}F^k\frac{\partial v_j}{\partial x^k}\right], \tag{16.11}$$

$$(\delta_2)_{ij}=\frac{1}{c^2}v_j\left[\frac{\partial}{\partial x^k}\left(D_i^k+A_{i\cdot}^{\cdot k}\right)+\frac{1}{c^2}F^k\frac{\partial v_i}{\partial x^k}\right], \tag{16.12}$$

$$(\varepsilon_2)_{ij}=\frac{1}{c^4}v_i v_j\frac{\partial F^k}{\partial x^k}. \tag{16.13}$$

The second term of (16.1) is

$$\Gamma_{i\alpha}^\beta\Gamma_{j\beta}^\alpha=\Gamma_{i0}^0\Gamma_{j0}^0+\Gamma_{i0}^k\Gamma_{jk}^0+\Gamma_{ik}^0\Gamma_{j0}^k+\Gamma_{ik}^l\Gamma_{jl}^k\,. \tag{16.14}$$

Because of (3.19), we obtain

$$\Gamma_{i0}^0\Gamma_{j0}^0=\frac{1}{c^4}\left[\frac{c^2}{(c^2-\mathrm{w})^2}\frac{\partial\mathrm{w}}{\partial x^i}\frac{\partial\mathrm{w}}{\partial x^j}-v_l\left(D_i^l+A_{i\cdot}^{\cdot l}\right)\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial x^j}-\right.$$

$$- \frac{1}{c^2}\, v_i v_l F^l \frac{c^2}{c^2 - \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x^j} - v_l \big(D_j^l + A_{j\cdot}^{\cdot l}\big) \frac{c^2}{c^2 - \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x^i} -$$
$$- \frac{1}{c^2}\, v_j v_l F^l \frac{c^2}{c^2 - \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x^i} + v_k \big(D_i^k + A_{i\cdot}^{\cdot k}\big) v_l \big(D_j^l + A_{j\cdot}^{\cdot l}\big) +$$
$$+ \frac{1}{c^2}\, v_i v_l F^l v_k \big(D_j^k + A_{j\cdot}^{\cdot k}\big) + \frac{1}{c^2}\, v_j v_l F^l v_k \big(D_i^k + A_{i\cdot}^{\cdot k}\big) + \tag{16.15}$$
$$+ \frac{1}{c^4}\, v_i v_j \big(v_l F^l\big)^2 \Bigg] = (\beta_3)_{ij} + (\gamma_3)_{ij} + (\delta_3)_{ij} + (\varepsilon_3)_{ij}\,,$$

where we denote

$$(\beta_3)_{ij} = \frac{1}{c^4} \Bigg[\frac{c^4}{(c^2 - \mathrm{w})^2} \frac{\partial \mathrm{w}}{\partial x^i} \frac{\partial \mathrm{w}}{\partial x^j} - v_l \big(D_i^l + A_{i\cdot}^{\cdot l}\big) \frac{c^2}{c^2 - \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x^j} -$$
$$- v_l \big(D_j^l + A_{j\cdot}^{\cdot l}\big) \frac{c^2}{c^2 - \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x^i} + v_k \big(D_i^k + A_{i\cdot}^{\cdot k}\big) v_l \big(D_j^l + A_{j\cdot}^{\cdot l}\big) \Bigg], \tag{16.16}$$

$$(\gamma_3)_{ij} = \frac{1}{c^6}\, v_i v_l F^l \left[- \frac{c^2}{c^2 - \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x^j} + v_k \big(D_j^k + A_{j\cdot}^{\cdot k}\big) \right], \tag{16.17}$$

$$(\delta_3)_{ij} = \frac{1}{c^6}\, v_j v_l F^l \left[- \frac{c^2}{c^2 - \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x^i} + v_k \big(D_i^k + A_{i\cdot}^{\cdot k}\big) \right], \tag{16.18}$$

$$(\varepsilon_3)_{ij} = \frac{1}{c^8}\, v_i v_j \big(v_l F^l\big)^2. \tag{16.19}$$

Because of (3.20), (3.21), and also (22.3) of §2.22, we obtain

$$\Gamma_{i0}^k \Gamma_{jk}^0 + \Gamma_{ik}^0 \Gamma_{j0}^k = - \frac{1}{c^2} \Bigg[\big(D_i^k + A_{i\cdot}^{\cdot k}\big)\big(\Psi_{jk} - D_{jk}\big) +$$
$$+ \frac{1}{c^2}\, v_i \left(\Psi_{jk} - D_{jk}\right) F^k + \frac{1}{c^2}\, v_j v_l \big(D_i^k + A_{i\cdot}^{\cdot k}\big)\big(D_k^l + A_{k\cdot}^{\cdot l}\big) +$$
$$+ \frac{1}{c^4}\, v_j v_l F^l v_k \big(D_i^k + A_{i\cdot}^{\cdot k}\big) + \frac{1}{c^2}\, v_k v_l \big(D_i^k + A_{i\cdot}^{\cdot k}\big) \big(D_j^l + A_{j\cdot}^{\cdot l}\big) +$$
$$+ \frac{1}{c^4}\, v_i v_j v_l \big(D_k^l + A_{k\cdot}^{\cdot l}\big) F^k + \frac{1}{c^6}\, v_i v_j \big(v_l F^l\big)^2 +$$
$$+ \frac{1}{c^4}\, v_i v_l F^l v_k \big(D_j^k + A_{j\cdot}^{\cdot k}\big) + \big(D_j^k + A_{j\cdot}^{\cdot k}\big)\left(\Psi_{ik} - D_{ik}\right) +$$
$$+ \frac{1}{c^2}\, v_j \left(\Psi_{ik} - D_{ik}\right) F^k + \frac{1}{c^2}\, v_i v_l \big(D_j^k + A_{j\cdot}^{\cdot k}\big)\big(D_k^l + A_{k\cdot}^{\cdot l}\big) +$$
$$+ \frac{1}{c^4}\, v_i v_l F^l v_k \big(D_j^k + A_{j\cdot}^{\cdot k}\big) + \frac{1}{c^2}\, v_k v_l \big(D_j^k + A_{j\cdot}^{\cdot k}\big)\big(D_i^l + A_{i\cdot}^{\cdot l}\big) +$$

$$+\frac{1}{c^4}\,v_i v_j v_l\big(D_k^l + A_{k\cdot}^{\cdot l}\big)F^k + \frac{1}{c^6}\,v_i v_j\big(v_l F^l\big)^2 + \\ + \frac{1}{c^4}\,v_j v_l F^l v_k\big(D_i^k + A_{i\cdot}^{\cdot k}\big)\Big] = (\beta_4)_{ij} + (\gamma_4)_{ij} + (\delta_4)_{ij} + (\varepsilon_4)_{ij}\,, \tag{16.20}$$

where we denote

$$(\beta_4)_{ij} = \frac{1}{c^2}\Big\{2D_i^k D_{jk} + A_{i\cdot}^{\cdot k}D_{jk} + A_{j\cdot}^{\cdot k}D_{ik} - \\ -\big(D_i^k + A_{i\cdot}^{\cdot k}\big)\Big[\frac{1}{2}\Big(\frac{\partial v_k}{\partial x^j} + \frac{\partial v_j}{\partial x^k}\Big) - \frac{1}{2c^2}(F_j v_k + F_k v_j)\Big] - \\ -\big(D_j^k + A_{j\cdot}^{\cdot k}\big)\Big[\frac{1}{2}\Big(\frac{\partial v_k}{\partial x^i} + \frac{\partial v_i}{\partial x^k}\Big) - \frac{1}{2c^2}(F_i v_k + F_k v_i)\Big] + \\ + {}^*\Delta_{jk}^l v_l\big(D_i^k + A_{i\cdot}^{\cdot k}\big) + {}^*\Delta_{ik}^l v_l\big(D_j^k + A_{j\cdot}^{\cdot k}\big) - \\ - \frac{2}{c^2}\,v_k v_l\big(D_i^k + A_{i\cdot}^{\cdot k}\big)\big(D_j^l + A_{j\cdot}^{\cdot l}\big)\Big\}, \tag{16.21}$$

$$(\gamma_4)_{ij} = \frac{1}{c^4}\,v_i\,\Big\{D_{jk}F^k - \\ -\Big[\frac{1}{2}\Big(\frac{\partial v_k}{\partial x^j} + \frac{\partial v_j}{\partial x^k}\Big) - \frac{1}{2c^2}(F_j v_k + F_k v_j)\Big]F^k + {}^*\Delta_{jk}^l v_l F^k - \\ - \frac{2}{c^2}\,v_k F^k v_l\big(D_j^l + A_{j\cdot}^{\cdot l}\big) - v_l\big(D_j^k + A_{j\cdot}^{\cdot k}\big)\big(D_k^l + A_{k\cdot}^{\cdot l}\big)\Big\}, \tag{16.22}$$

$$(\delta_4)_{ij} = \frac{1}{c^4}\,v_j\,\Big\{D_{ik}F^k - \\ -\Big[\frac{1}{2}\Big(\frac{\partial v_k}{\partial x^i} + \frac{\partial v_i}{\partial x^k}\Big) - \frac{1}{2c^2}(F_i v_k + F_k v_i)\Big]F^k + {}^*\Delta_{ik}^l v_l F^k - \\ - \frac{2}{c^2}\,v_k F^k v_l\big(D_i^l + A_{i\cdot}^{\cdot l}\big) - v_l\big(D_i^k + A_{i\cdot}^{\cdot k}\big)\big(D_k^l + A_{k\cdot}^{\cdot l}\big)\Big\}, \tag{16.23}$$

$$(\varepsilon_4)_{ij} = \frac{1}{c^6}\,v_i v_j\Big[-2v_l\big(D_k^l + A_{k\cdot}^{\cdot l}\big)F^k - \frac{2}{c^2}\big(v_l F^l\big)^2\Big]. \tag{16.24}$$

Because of (3.22), we obtain

$$\Gamma_{ik}^l \Gamma_{jl}^k = {}^*\Delta_{ik}^l\,{}^*\Delta_{jl}^k - \\ - \frac{1}{c^2}\,{}^*\Delta_{ik}^l\Big[\big(D_l^k + A_{l\cdot}^{\cdot k}\big)v_j + \big(D_j^k + A_{j\cdot}^{\cdot k}\big)v_l + \frac{1}{c^2}\,v_j v_l F^k\Big] -$$

$$- \frac{1}{c^2}\,{}^*\Delta^k_{jl}\left[\left(D^l_k + A^{\cdot l}_{k\cdot}\right)v_i + \left(D^l_i + A^{\cdot l}_{i\cdot}\right)v_k + \frac{1}{c^2}\,v_i v_k F^l\right] +$$
$$+ \frac{1}{c^4}\left[\left(D^l_k D^k_l + A^{\cdot l}_{k\cdot} A^{\cdot k}_{l\cdot}\right)v_i v_j + v_i v_l\left(D^k_j + A^{\cdot k}_{j\cdot}\right)\left(D^l_k + A^{\cdot l}_{k\cdot}\right) + \right.$$
$$+ \frac{1}{c^2}\,v_i v_j v_l\left(D^l_k + A^{\cdot l}_{k\cdot}\right)F^k + v_j v_k\left(D^l_i + A^{\cdot l}_{i\cdot}\right)\left(D^k_l + A^{\cdot k}_{l\cdot}\right) +$$
$$+ v_l v_k\left(D^l_i + A^{\cdot l}_{i\cdot}\right)\left(D^k_j + A^{\cdot k}_{j\cdot}\right) + \frac{1}{c^2}\,v_j v_k F^k v_l\left(D^l_i + A^{\cdot l}_{i\cdot}\right) + \tag{16.25}$$
$$+ \frac{1}{c^2}\,v_i v_j v_k\left(D^k_l + A^{\cdot k}_{l\cdot}\right)F^l + \frac{1}{c^2}\,v_i v_l F^l v_k\left(D^k_j + A^{\cdot k}_{j\cdot}\right) +$$
$$\left. + \frac{1}{c^4}\,v_i v_j\left(v_l F^l\right)^2\right] = (\beta_5)_{ij} + (\gamma_5)_{ij} + (\delta_5)_{ij} + (\varepsilon_5)_{ij}\,,$$

where we denote

$$(\beta_5)_{ij} = {}^*\Delta^l_{ik}\,{}^*\Delta^k_{jl} - \frac{1}{c^2}\,{}^*\Delta^l_{ik}\,v_l\left(D^k_j + A^{\cdot k}_{j\cdot}\right) -$$
$$- \frac{1}{c^2}\,{}^*\Delta^k_{jl}\,v_k\left(D^l_i + A^{\cdot l}_{i\cdot}\right) + \frac{1}{c^4}\,v_l v_k\left(D^l_i + A^{\cdot l}_{i\cdot}\right)\left(D^k_j + A^{\cdot k}_{j\cdot}\right), \tag{16.26}$$

$$(\gamma_5)_{ij} = \frac{1}{c^5}\,v_i\left[-{}^*\Delta^k_{jl}\left(D^l_k + A^{\cdot l}_{k\cdot}\right) - \frac{1}{c^2}\,{}^*\Delta^k_{jl}\,v_k F^l + \right.$$
$$\left. + \frac{1}{c^2}\,v_l\left(D^k_j + A^{\cdot k}_{j\cdot}\right)\left(D^l_k + A^{\cdot l}_{k\cdot}\right) + \frac{1}{c^4}\,v_l F^l v_k\left(D^k_j + A^{\cdot k}_{j\cdot}\right)\right], \tag{16.27}$$

$$(\delta_5)_{ij} = \frac{1}{c^2}\,v_j\left[-{}^*\Delta^l_{ik}\left(D^k_l + A^{\cdot k}_{l\cdot}\right) - \frac{1}{c^2}\,{}^*\Delta^l_{ik}\,v_l F^k + \right.$$
$$\left. + \frac{1}{c^2}\,v_k\left(D^l_i + A^{\cdot l}_{i\cdot}\right)\left(D^k_l + A^{\cdot k}_{l\cdot}\right) + \frac{1}{c^4}\,v_k F^k v_l\left(D^l_i + A^{\cdot l}_{i\cdot}\right)\right], \tag{16.28}$$

$$(\varepsilon_5)_{ij} = \frac{1}{c^4}\,v_i v_j \times$$
$$\times\left[D^l_k D^k_l + A^{\cdot l}_{k\cdot} A^{\cdot k}_{l\cdot} + \frac{2}{c^2}\,v_l\left(D^l_k + A^{\cdot l}_{k\cdot}\right)F^k + \frac{1}{c^4}\left(v_l F^l\right)^2\right]. \tag{16.29}$$

Because of (1.7), we obtain

$$\frac{\partial^2 \ln\sqrt{-g}}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i}\left(-\frac{1}{c^2}\,\frac{c^2}{c^2 - \mathrm{w}}\,\frac{\partial \mathrm{w}}{\partial x^j} + \frac{\partial \ln\sqrt{h}}{\partial x^j}\right) =$$
$$= -\frac{1}{c^2}\,\frac{c^2}{(c^2 - \mathrm{w})^2}\,\frac{\partial \mathrm{w}}{\partial x^i}\,\frac{\partial \mathrm{w}}{\partial x^j} - \frac{1}{c^2}\,\frac{c^2}{c^2 - \mathrm{w}}\,\frac{\partial^2 \mathrm{w}}{\partial x^i \partial x^j} + \frac{\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^j}\,. \tag{16.30}$$

On the other hand, the last term is

$$\frac{{}^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^j} = \frac{{}^*\partial}{\partial x^i}\left(\frac{{}^*\partial \ln\sqrt{h}}{\partial x^j}\right) = \frac{{}^*\partial}{\partial x^i}\left(\frac{\partial \ln\sqrt{h}}{\partial x^j} + \frac{1}{c}\, v_j\, D\right) =$$
$$= \frac{\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^j} + \frac{1}{c^2 - \mathrm{w}}\, v_j \frac{\partial}{\partial x^i}\left[\left(1 - \frac{\mathrm{w}}{c^2}\right) D\right] +$$
$$+ \frac{1}{c^2}\left(\frac{\partial v_i}{\partial x^j} + \frac{v_j}{c^2 - \mathrm{w}}\frac{\partial v_i}{\partial t}\right) D + \frac{1}{c^2}\, v_i \frac{{}^*\partial D}{\partial x^j} = \frac{\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^j} -$$
$$- \frac{1}{c^4}\, v_j \frac{c^2}{c^2 - \mathrm{w}}\, D \frac{\partial \mathrm{w}}{\partial x^i} + \frac{1}{c^2}\, v_j \frac{\partial D}{\partial x^i} + \frac{1}{c^2}\, D \frac{\partial v_i}{\partial x^j} +$$
$$+ \frac{1}{c^4}\, v_j \frac{c^2}{c^2 - \mathrm{w}}\, D \frac{\partial v_i}{\partial t} + \frac{1}{c^2}\, v_i \frac{{}^*\partial D}{\partial x^j} = \frac{\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^j} +$$
$$+ \frac{1}{c^2} D\left(\frac{\partial v_i}{\partial x^j} - \frac{1}{c^2} F_i v_j\right) + \frac{1}{c^2}\left(v_i \frac{{}^*\partial D}{\partial x^j} + v_j \frac{{}^*\partial D}{\partial x^i}\right) - \frac{1}{c^4}\, v_i v_j \frac{{}^*\partial D}{\partial t}\,. \qquad (16.31)$$

For this reason, we have

$$\frac{\partial^2 \ln\sqrt{-g}}{\partial x^i \partial x^j} = -\frac{1}{c^2}\frac{c^2}{(c^2 - \mathrm{w})^2}\frac{\partial \mathrm{w}}{\partial x^i}\frac{\partial \mathrm{w}}{\partial x^j} - \frac{1}{c^2}\frac{c^2}{c^2 - \mathrm{w}}\frac{\partial^2 \mathrm{w}}{\partial x^i \partial x^j} +$$
$$+ \frac{{}^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^j} - \frac{1}{c^2} D\left(\frac{\partial v_i}{\partial x^j} - \frac{1}{c^2} F_i v_j\right) - \frac{1}{c^2}\left(v_i \frac{{}^*\partial D}{\partial x^j} + v_j \frac{{}^*\partial D}{\partial x^i}\right) + \qquad (16.32)$$
$$+ \frac{1}{c^4}\, v_i v_j \frac{{}^*\partial D}{\partial t} = (\beta_6)_{ij} + (\gamma_6)_{ij} + (\delta_6)_{ij} + (\varepsilon_6)_{ij}\,,$$

where we denote

$$(\beta_6)_{ij} = \frac{{}^*\partial^2 \ln\sqrt{h}}{\partial x^j \partial x^i} - \frac{1}{c^2}\left[\frac{c^2}{(c^2 - \mathrm{w})^2}\frac{\partial \mathrm{w}}{\partial x^i}\frac{\partial \mathrm{w}}{\partial x^j} + \right.$$
$$\left. + \frac{c^2}{c^2 - \mathrm{w}}\frac{\partial^2 \mathrm{w}}{\partial x^i \partial x^j} + D\left(\frac{\partial v_i}{\partial x^j} - \frac{1}{c^2} F_i v_j\right)\right], \qquad (16.33)$$

$$(\gamma_6)_{ij} = -\frac{1}{c^2}\, v_i \frac{{}^*\partial D}{\partial x^j}\,, \qquad (16.34)$$

$$(\delta_6)_{ij} = -\frac{1}{c^2}\, v_j \frac{{}^*\partial D}{\partial x^i}\,, \qquad (16.35)$$

$$(\varepsilon_6)_{ij} = \frac{1}{c^4}\, v_i v_j \frac{{}^*\partial D}{\partial t}\,. \qquad (16.36)$$

Finally, we have the term

$$-\Gamma^{\alpha}_{ij}\frac{\partial\ln\sqrt{-g}}{\partial x^{\alpha}} = -\Gamma^{0}_{ij}\frac{\partial\ln\sqrt{-g}}{\partial x^{0}} - \Gamma^{k}_{ij}\frac{\partial\ln\sqrt{-g}}{\partial x^{k}}\,. \qquad (16.37)$$

Because of (3.21), (1.7), and also (22.3) of §3.22, we obtain

$$-\Gamma^{0}_{ij}\frac{\partial\ln\sqrt{-g}}{\partial x^{0}} = \frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\left\{\Psi_{ij} - D_{ij} + \frac{1}{c^2}\,v_l\Big[\big(D^l_j + A_{j\cdot}^{\cdot l}\big)v_i + \right.$$
$$\left. + \big(D^l_i + A_{i\cdot}^{\cdot l}\big)v_j + \frac{1}{c^2}\,v_i v_j F^l\Big]\right\}\left[-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial t} + \Big(1-\frac{\mathrm{w}}{c^2}\Big)D\right] = \qquad (16.38)$$
$$= (\alpha_7)_{ij} + (\beta_7)_{ij} + (\gamma_7)_{ij} + (\delta_7)_{ij} + (\varepsilon_7)_{ij}\,,$$

where we denote

$$(\alpha_7)_{ij} = -\frac{1}{c^2}\frac{c^2}{(c^2-\mathrm{w})^2}\left\{\Psi_{ij} - D_{ij} + \right.$$
$$\left. + \frac{1}{c^2}\,v_l\Big[\big(D^l_j + A_{j\cdot}^{\cdot l}\big)v_i + \big(D^l_i + A_{i\cdot}^{\cdot l}\big)v_j + \frac{1}{c^2}\,v_i v_j F^l\Big]\right\}\frac{\partial \mathrm{w}}{\partial t}\,, \qquad (16.39)$$

$$(\beta_7)_{ij} = \frac{1}{c^2}\,D\,\times$$
$$\times\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) - D_{ij} - \frac{1}{2c^2}\big(F_i v_j + F_j v_i\big) - {}^*\!\Delta^l_{ij}\,v_l\right], \qquad (16.40)$$

$$(\gamma_7)_{ij} = \frac{1}{c^4}\,v_i\,D\,v_l\big(D^l_j + A_{j\cdot}^{\cdot l}\big)\,, \qquad (16.41)$$

$$(\delta_7)_{ij} = \frac{1}{c^4}\,v_j\,D\,v_l\big(D^l_i + A_{i\cdot}^{\cdot l}\big)\,, \qquad (16.42)$$

$$(\varepsilon_7)_{ij} = \frac{1}{c^6}\,v_i v_j\,D\,v_l F^l. \qquad (16.43)$$

Because of (3.22) and (1.7), we obtain

$$-\Gamma^{k}_{ij}\frac{\partial\ln\sqrt{-g}}{\partial x^{k}} = -\left\{{}^*\!\Delta^k_{ij} - \frac{1}{c^2}\Big[\big(D^k_j + A_{j\cdot}^{\cdot k}\big)\,v_i + \right.$$
$$\left. + \big(D^k_i + A_{i\cdot}^{\cdot k}\big)v_j + \frac{1}{c^2}\,v_i v_j F^k\Big]\right\}\left(-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^k} + \frac{\partial\ln\sqrt{h}}{\partial x^k}\right) = \qquad (16.44)$$
$$= (\beta_8)_{ij} + (\gamma_8)_{ij} + (\delta_8)_{ij} + (\varepsilon_8)_{ij}\,,$$

where we denote

$$(\beta_8)_{ij} = \frac{1}{c^2}\,{}^*\!\Delta^k_{ij}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^k} - {}^*\!\Delta^k_{ij}\frac{{}^*\partial \ln\sqrt{h}}{\partial x^k} + \frac{1}{c^2}\,D\,{}^*\!\Delta^k_{ij}\,v_k\,, \tag{16.45}$$

$$(\gamma_8)_{ij} = \frac{1}{c^2}\,v_i\left[-\frac{1}{c^2}\left(D^k_j + A_{j\cdot}^{\cdot k}\right)\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^k} + \left(D^k_j + A_{j\cdot}^{\cdot k}\right)\frac{\partial \ln\sqrt{h}}{\partial x^k}\right], \tag{16.46}$$

$$(\delta_8)_{ij} = \frac{1}{c^2}\,v_j\left[-\frac{1}{c^2}\left(D^k_i + A_{i\cdot}^{\cdot k}\right)\frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial x^k} + \left(D^k_i + A_{i\cdot}^{\cdot k}\right)\frac{\partial \ln\sqrt{h}}{\partial x^k}\right], \tag{16.47}$$

$$(\varepsilon_8)_{ij} = \frac{1}{c^4}\,v_i v_j\left(-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}F^k\frac{\partial \mathrm{w}}{\partial x^k} + F^k\frac{\partial \ln\sqrt{h}}{\partial x^k}\right). \tag{16.48}$$

Introducing the notation

$$\begin{gathered}(\alpha)_{ij} = (\alpha_1)_{ij} + (\alpha_7)_{ij}\,, \quad (\beta)_{ij} = \sum_{n=1}^{8}(\beta_n)_{ij}\,, \\ (\gamma)_{ij} = \sum_{n=1}^{8}(\gamma_n)_{ij}\,, \quad (\delta)_{ij} = \sum_{n=1}^{8}(\delta_n)_{ij}\,, \quad (\varepsilon)_{ij} = \sum_{n=1}^{8}(\varepsilon_n)_{ij}\,,\end{gathered} \tag{16.49}$$

we can write

$$G_{ij} = (\alpha)_{ij} + (\beta)_{ij} + (\gamma)_{ij} + (\delta)_{ij} + (\varepsilon)_{ij}\,. \tag{16.50}$$

Comparing $(\alpha_1)_{ij}$ and $(\alpha_7)_{ij}$, we see that

$$(\alpha)_{ij} = 0\,. \tag{16.51}$$

We now deduce $(\beta)_{ij}$, $(\gamma)_{ij}$, $(\delta)_{ij}$, and $(\varepsilon)_{ij}$. Because of (16.5), (16.10), (16.16), (16.21), (16.26), (16.33), (16.40), and (16.45), we obtain the first of the quantities, namely

$$\begin{aligned}\sum_{n=1}^{8}(\beta_n)_{ij} &= -\frac{1}{c^2}\frac{{}^*\partial D_{ij}}{\partial t} + \frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) - \right. \\ &\left. - \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right)\right] + \frac{1}{c^2}\,{}^*\!\Delta^k_{ij}F_k - \frac{1}{c^4}\,v_l\left(D^l_j + A_{j\cdot}^{\cdot l}\right)F_i - \\ &- \frac{1}{c^4}\,v_l\left(D^l_i + A_{i\cdot}^{\cdot l}\right)F_j + H_{ij} + \frac{1}{c^2}\left(D^k_j + A_{j\cdot}^{\cdot k}\right)\frac{\partial v_i}{\partial x^k} +\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{c^2}\left(D_i^k+A_{i\cdot}^{\cdot k}\right)\frac{\partial v_j}{\partial x^k}+\frac{2}{c^2}\,D_i^k D_{jk}+\frac{1}{c^2}A_{i\cdot}^{\cdot k}D_{jk}\,+\\
&+\frac{1}{c^2}A_{j\cdot}^{\cdot k}D_{ik}-\frac{1}{c^2}\left(D_i^k+A_{i\cdot}^{\cdot k}\right)\left[\frac{1}{2}\left(\frac{\partial v_k}{\partial x^j}+\frac{\partial v_j}{\partial x^k}\right)-\right.\\
&\left.-\frac{1}{2c^2}\left(F_j v_k+F_k v_j\right)\right]-\frac{1}{c^2}\left(D_j^k+A_{j\cdot}^{\cdot k}\right)\left[\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}+\frac{\partial v_i}{\partial x^k}\right)-\right.\\
&\left.-\frac{1}{2c^2}\left(F_i v_k+F_k v_i\right)\right]-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2\mathrm{w}}{\partial x^i\partial x^j}-\frac{1}{c^2}\,D\,\times\\
&\times\left(\frac{\partial v_i}{\partial x^j}-\frac{1}{c^2}F_i v_j\right)-\frac{1}{c^2}DD_{ij}+\frac{1}{c^2}D\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)-\right.\\
&\left.-\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right]=H_{ij}+\frac{1}{c^2}A_{ij}D-\frac{2}{c^2}A_{i\cdot}^{\cdot k}A_{jk}\,-\\
&-\frac{1}{c^2}\left(\frac{{}^*\partial D_{ij}}{\partial t}-2D_i^k D_{jk}+DD_{ij}\right)-\frac{1}{c^2}\left\{\frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2\mathrm{w}}{\partial x^i\partial x^j}\,-\right.\\
&\left.-\frac{{}^*\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)-\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right]-\frac{1}{c^2}\,{}^*\Delta_{ij}^k F_k\right\}.
\end{aligned}
\tag{16.52}
$$

On the other hand,

$$
\begin{aligned}
&{}^*\nabla_i F_j-\frac{1}{c^2}F_iF_j=\frac{\partial F_j}{\partial x^i}+\frac{v_i}{c^2-\mathrm{w}}\frac{\partial F_j}{\partial t}-{}^*\Delta_{ij}^k F_k-\frac{1}{c^2}F_iF_j=\\
&=\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}F_j\frac{\partial\mathrm{w}}{\partial x^i}+\frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2\mathrm{w}}{\partial x^i\partial x^j}-\frac{{}^*\partial}{\partial t}\left(\frac{\partial v_j}{\partial x^i}\right)+\\
&+\frac{1}{c^2}v_i\frac{{}^*\partial F_j}{\partial t}-\frac{1}{c^2}\frac{c^2}{c^2-\mathrm{w}}F_j\frac{\partial\mathrm{w}}{\partial x^i}+\frac{1}{c^2}F_j\frac{{}^*\partial v_i}{\partial t}-{}^*\Delta_{ij}^k F_k=\\
&=\frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2\mathrm{w}}{\partial x^i\partial x^j}-\frac{{}^*\partial}{\partial t}\left(\frac{\partial v_j}{\partial x^i}-\frac{1}{c^2}F_j v_i\right)-{}^*\Delta_{ij}^k F_k\,,
\end{aligned}
\tag{16.53}
$$

so we have

$$
\begin{aligned}
&\left({}^*\nabla_i F_j+{}^*\nabla_j F_i\right)-\frac{1}{c^2}F_iF_j=\frac{c^2}{c^2-\mathrm{w}}\frac{\partial^2\mathrm{w}}{\partial x^i\partial x^j}\,-\\
&-\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)-\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right]-{}^*\Delta_{ij}^k F_k\,.
\end{aligned}
\tag{16.54}
$$

Besides these (see 21.21 of §2.21), we have

$$
H_{ij}=S_{ij}+\frac{1}{c^2}\left(A_{ji}D+A_{jk}D_i^k+A_{ik}D_j^k\right).
\tag{16.55}
$$

Therefore, we get

$$\begin{aligned}(\beta)_{ij} = S_{ij} - \frac{1}{c^2}\left[\frac{{}^*\partial D_{ij}}{\partial t} - 2D_i^k D_{jk} + DD_{ij} - D_i^k A_{jk} - \right.\\ \left. - D_j^k A_{ik} + 2A_{i\cdot}^{\cdot k} A_{jk} + \frac{1}{2}\left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2} F_i F_j\right].\end{aligned} \tag{16.56}$$

Because of (16.6), (16.11), (16.17), (16.22), (16.27), (16.34), (16.41), and (16.46), we obtain

$$\begin{aligned}\sum_{n=1}^{8}(\gamma_n)_{ij} = -\frac{1}{c^2}\, v_i \left\{-\frac{1}{c^2} F_k \left(D_j^k + A_{j\cdot}^{\cdot k}\right) + \right.\\ + \frac{{}^*\partial}{\partial x^k}\left(D_j^k + A_{j\cdot}^{\cdot k}\right) - \frac{1}{c^4} F_j v_l F^l + \frac{1}{c^2} F^k \frac{\partial v_j}{\partial x^k} + \frac{1}{c^2} D_{jk} F^k - \\ - \frac{1}{c^2}\left[\frac{1}{2}\left(\frac{\partial v_k}{\partial x^j} + \frac{\partial v_j}{\partial x^k}\right) - \frac{1}{2c^2}\left(F_j v_k + F_k v_j\right)\right] F^k - \\ \left. - {}^*\Delta_{jl}^k \left(D_k^l + A_{k\cdot}^{\cdot l}\right) - \frac{{}^*\partial D}{\partial x^j} + \left(D_j^k + A_{j\cdot}^{\cdot k}\right)\frac{{}^*\partial \ln\sqrt{h}}{\partial x^k}\right\} = \\ = \frac{1}{c^2}\, v_i \left[{}^*\nabla_k \left(D_j^k + A_{j\cdot}^{\cdot k}\right) - \frac{{}^*\partial D}{\partial x^j} - \frac{2}{c^2} A_{jk} F^k\right]\end{aligned} \tag{16.57}$$

or, finally,

$$(\gamma)_{ij} = -\frac{1}{c^2}\, v_i \left[{}^*\nabla_k \left(h_j^k D - D_j^k\right) - {}^*\nabla_k A_{j\cdot}^{\cdot k} + \frac{2}{c^2} A_{jk} F^k\right]. \tag{16.58}$$

In the same way, using formulae (16.7), (16.12), (16.18), (16.23), (16.28), (16.35), (16.42), and (16.47), we arrive to

$$(\delta)_{ij} = -\frac{1}{c^2}\, v_j \left[{}^*\nabla_k \left(h_i^k D - D_i^k\right) - {}^*\nabla_k A_{i\cdot}^{\cdot k} + \frac{2}{c^2} A_{ik} F^k\right]. \tag{16.59}$$

Formulae (16.8), (16.13), (16.19), (16.24), (16.29), (16.36), (16.43), and (16.48) give

$$\begin{aligned}\sum_{n=1}^{8}(\varepsilon_n)_{ij} = \frac{1}{c^4}\, v_i v_j \left(-\frac{1}{c^2} F_k F^k + \right.\\ \left. + \frac{{}^*\partial F^k}{\partial x^k} + F^k \frac{{}^*\partial \ln\sqrt{h}}{\partial x^k} + D_k^l D_l^k + A_{k\cdot}^{\cdot l} A_{l\cdot}^{\cdot k} + \frac{{}^*\partial D}{\partial t}\right),\end{aligned} \tag{16.60}$$

so we obtain

$$(\varepsilon_n)_{ij} = \frac{1}{c^4}\, v_i v_j \left(\frac{{}^*\partial D}{\partial t} + D_k^l D_l^k + A_{k\cdot}^{\cdot l} A_{l\cdot}^{\cdot k} + {}^*\nabla_k F^k - \frac{1}{c^2} F_k F^k\right). \tag{16.61}$$

As the final result, we have

$$\begin{gathered} G_{ij} = S_{ij} - \frac{1}{c^2}\left[\frac{{}^*\partial D_{ij}}{\partial t} - 2D_i^k D_{jk} + D D_{ij} + D_i^k A_{kj} + \right. \\ \left. + D_j^k A_{ki} - 2A_{i\cdot}^{\cdot k} A_{kj} + \frac{1}{2}\left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2} F_i F_j \right] - \\ - \frac{1}{c^2}\, v_i \left[{}^*\nabla_k \left(h_j^k D - D_j^k\right) - {}^*\nabla_k A_{j\cdot}^{\cdot k} + \frac{2}{c^2} A_{jk} F^k \right] - \\ - \frac{1}{c^2}\, v_j \left[{}^*\nabla_k \left(h_i^k D - D_i^k\right) - {}^*\nabla_k A_{i\cdot}^{\cdot k} + \frac{2}{c^2} A_{ik} F^k \right] + \\ + \frac{1}{c^4}\, v_i v_j \left(\frac{{}^*\partial D}{\partial t} + D_k^l D_l^k + A_{k\cdot}^{\cdot l} A_{l\cdot}^{\cdot k} + {}^*\nabla_k F^k - \frac{1}{c^2} F_k F^k\right), \end{gathered} \tag{16.62}$$

because

$$A_{kj} = -A_{jk}\,, \qquad A_{ki} = -A_{ik}\,. \tag{16.63}$$

§3.17 The Einstein tensor and chr.inv.-quantities

According to §2.3, the quantities

$$G = g^{\mu\nu} G_{\mu\nu} = g_{\mu\nu} G^{\mu\nu} = G_\nu^\nu\,, \tag{17.1}$$

$$L = c^2 \frac{G_{00}}{g_{00}}\,, \tag{17.2}$$

$$M^k = -c\, \frac{G_0^k}{\sqrt{g_{00}}}\,, \tag{17.3}$$

$$N^{jk} = -c^2 G^{jk} \tag{17.4}$$

are chr.inv.-invariants, a chr.inv.-vector, and a chr.inv.-tensor of the 2nd rank, respectively. Using formulae (14.7), (15.7), and (16.62), we are going to deduce detailed formulae for the quantities, and also for the Einstein covariant tensor and the invariant G, expressed through the quantities.

Because of (17.2) and (14.7), we obtain

$$L=\frac{{}^*\partial D}{\partial t}+D_j^l D_l^j+A_{j\cdot}^{\cdot l}A_{l\cdot}^{\cdot j}+{}^*\nabla_j F^j-\frac{1}{c^2}F_jF^j. \tag{17.5}$$

Because of (15.7) and (17.5), we obtain

$$\begin{gathered}G_{0i}=\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\times\\ \times\left[{}^*\nabla_j\left(h_i^j D-D_i^j\right)-{}^*\nabla_j A_{i\cdot}^{\cdot j}+\frac{2}{c^2}A_{ij}F^j-\frac{1}{c^2}v_iL\right].\end{gathered} \tag{17.6}$$

It is evident that

$$\begin{gathered}G_0^k=g^{k0}G_{00}+g^{ki}G_{0i}=-\frac{1}{c^3}v^k\left(1-\frac{\mathrm{w}}{c^2}\right)L-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\times\\ \times\left[{}^*\nabla_j\left(h^{kj}D-D^{kj}\right)-{}^*\nabla_j A^{kj}+\frac{2}{c^2}A^{kj}F_j-\frac{1}{c^2}v^kL\right]=\\ =-\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[{}^*\nabla_j\left(h^{kj}D-D^{kj}\right)-{}^*\nabla_j A^{kj}+\frac{2}{c^2}A^{kj}F_j\right]\end{gathered} \tag{17.7}$$

and, hence (see formula 17.3)

$$M^k={}^*\nabla_j\left(h^{kj}D-D^{kj}\right)-{}^*\nabla_j A^{kj}+\frac{2}{c^2}A^{kj}F_j\,. \tag{17.8}$$

Comparing (17.6) and (17.8), we have

$$G_{0i}=\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(M_i-\frac{1}{c^2}v_iL\right). \tag{17.9}$$

Comparing (16.62) to (17.8) and (17.5), we have

$$\begin{gathered}G_{ij}=S_{ij}-\frac{1}{c^2}\left[\frac{{}^*\partial D_{ij}}{\partial t}-2D_i^kD_{jk}+DD_{ij}+D_i^kA_{kj}+\right.\\ \left.+D_j^kA_{ki}-2A_{i\cdot}^{\cdot k}A_{kj}+\frac{1}{2}\left({}^*\nabla_i F_j+{}^*\nabla_j F_i\right)-\frac{1}{c^2}F_iF_j\right]-\\ -\frac{1}{c^2}v_iM_j-\frac{1}{c^2}v_jM_i+\frac{1}{c^4}v_iv_jL\,.\end{gathered} \tag{17.10}$$

Because

$$\begin{gathered}h^{jp}h^{kq}\frac{{}^*\partial D_{pq}}{\partial t}=\frac{{}^*\partial D^{jk}}{\partial t}-D_{pq}\frac{{}^*\partial\left(h^{jp}h^{kq}\right)}{\partial t}=\\ =\frac{{}^*\partial D^{jk}}{\partial t}+2D^{jp}D_p^k+2D^{kq}D_q^j=\frac{{}^*\partial D^{jk}}{\partial t}+4D^{jl}D_l^k\,,\end{gathered} \tag{17.11}$$

we obtain

$$
\begin{aligned}
&G^{jk} = g^{j\alpha} g^{k\beta} G_{\alpha\beta} = g^{j0} g^{k0} G_{00} + g^{j0} g^{kq} G_{0q} + g^{jp} g^{k0} G_{p0} + \\
&+ g^{jp} g^{kq} G_{pq} = \frac{1}{c^4}\, v^j v^k L + \frac{1}{c^2}\, v^j \Big(M^k - \frac{1}{c^2}\, v^k L \Big) + \\
&+ \frac{1}{c^2}\, v^k \Big(M^j - \frac{1}{c^2}\, v^j L \Big) + S^{jk} - \frac{1}{c^2} \bigg[\frac{{}^*\partial D^{jk}}{\partial t} + \\
&+ 2 D^{jl} D^k_l + D D^{jk} + D^{jl} A_{l\cdot}^{\cdot k} + D^{kl} A_{l\cdot}^{\cdot j} - 2 A^{jl} A_{l\cdot}^{\cdot k} + \\
&+ \frac{1}{2} \left({}^*\nabla^j F^k + {}^*\nabla^k F^j \right) - \frac{1}{c^2}\, F^j F^k \bigg] - \frac{1}{c^2}\, v^j M^k - \\
&- \frac{1}{c^2}\, v^k M^j + \frac{1}{c^4}\, v^j v^k L = S^{jk} - \frac{1}{c^2} \bigg[\frac{{}^*\partial D^{jk}}{\partial t} + \\
&+ 2 D^{jl} D^k_l + D D^{jk} + D^{jl} A_{l\cdot}^{\cdot k} + D^{kl} A_{l\cdot}^{\cdot j} - 2 A^{jl} A_{l\cdot}^{\cdot k} + \\
&+ \frac{1}{2} \left({}^*\nabla^j F^k + {}^*\nabla^k F^j \right) - \frac{1}{c^2}\, F^j F^k \bigg]
\end{aligned}
\tag{17.12}
$$

and, hence

$$
\begin{aligned}
N^{jk} = -c^2 S^{jk} &+ \frac{{}^*\partial D^{jk}}{\partial t} + 2 D^{jl} D^k_l + D D^{jk} + D^{jl} A_{l\cdot}^{\cdot k} + \\
&+ D^{kl} A_{l\cdot}^{\cdot j} - 2 A^{jl} A_{l\cdot}^{\cdot k} + \frac{1}{2} \left({}^*\nabla^j F^k + {}^*\nabla^k F^j \right) - \frac{1}{c^2}\, F^j F^k .
\end{aligned}
\tag{17.13}
$$

Comparing (17.10) and (17.13), we have

$$
G_{ij} = -\frac{1}{c^2} \Big(N_{ij} + v_i M_j + v_j M_i - \frac{1}{c^2}\, v_i v_j L \Big) . \tag{17.14}
$$

Because of (17.2), (17.6), and (17.4), we obtain

$$
G = \frac{1}{c^2} \left(L + N \right) , \tag{17.15}
$$

where the chr.inv.-invariant N is defined as follows

$$
N = h^{ij} N_{ij} = h_{ij} N^{ij} = N^j_j \,. \tag{17.16}
$$

Because

$$
h_{ij} \frac{{}^*\partial D^{jk}}{\partial t} = \frac{{}^*\partial D^k_i}{\partial t} - D^{jk} \frac{{}^*\partial h_{ij}}{\partial t} = \frac{{}^*\partial D^k_i}{\partial t} - 2 D_{ij} D^{jk} , \tag{17.17}
$$

we obtain

$$\begin{aligned} N_i^k = -c^2 S_i^k + \frac{{}^*\partial D_i^k}{\partial t} + D D_i^k + D_i^l A_{l\cdot}^{\cdot k} + D_l^k A_{\cdot i}^{l\cdot} - \\ - 2A_{i\cdot}^{\cdot l} A_{l\cdot}^{\cdot k} + \frac{1}{2}\left({}^*\nabla_i F^k + {}^*\nabla^k F_i\right) - \frac{1}{c^2} F_i F^k . \end{aligned} \tag{17.18}$$

Because of (17.15), (17.5), and (17.19), we obtain

$$N = -c^2 S + \frac{{}^*\partial D}{\partial t} + D^2 - 2A_{j\cdot}^{\cdot l} A_{l\cdot}^{\cdot j} + {}^*\nabla_j F^j - \frac{1}{c^2} F_j F^j , \tag{17.19}$$

$$G = -S + \frac{1}{c^2}\left(2\frac{{}^*\partial D}{\partial t} + D^2 + D_j^l D_l^j - A_{j\cdot}^{\cdot l} A_{l\cdot}^{\cdot j} + 2\,{}^*\nabla_j F^j - \frac{2}{c^2} F_j F^j\right). \tag{17.20}$$

We now collect the formulae we have obtained. Using formula (20.18) of §2.20, and introducing the concise notation

$${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2} F_i \,, \qquad {}^*\widetilde{\nabla}^i = {}^*\nabla^i - \frac{1}{c^2} F^i , \tag{17.21}$$

we have*

$$\left.\begin{aligned} L &= \frac{{}^*\partial D}{\partial t} + D_j^l D_l^j + A_{j\cdot}^{\cdot l} A_{l\cdot}^{\cdot j} + {}^*\widetilde{\nabla}_j F^j \\ M^k &= {}^*\nabla_j\left(h^{kj} D - D^{kj}\right) - {}^*\nabla_j A^{kj} + \frac{2}{c^2} A^{kj} F_j \\ N_i^k &= -c^2 S_i^k + \frac{{}^*\partial D_i^k}{\partial t} + D D_i^k + D_i^l A_{l\cdot}^{\cdot k} + D_l^k A_{\cdot i}^{l\cdot} - \\ &\qquad - 2A_{i\cdot}^{\cdot l} A_{l\cdot}^{\cdot k} + \frac{1}{2}\left({}^*\widetilde{\nabla}_i F^k + {}^*\widetilde{\nabla}^k F_i\right) \end{aligned}\right\} , \tag{17.22}$$

$$N = -c^2 S + \frac{{}^*\partial D}{\partial t} + D^2 - 2A_{j\cdot}^{\cdot l} A_{l\cdot}^{\cdot j} + {}^*\widetilde{\nabla}_j F^j , \tag{17.23}$$

$$G = -S + \frac{1}{c^2}\left(2\frac{{}^*\partial D}{\partial t} + D^2 + D_j^l D_l^j - A_{j\cdot}^{\cdot l} A_{l\cdot}^{\cdot j} + 2\,{}^*\widetilde{\nabla}_j F^j\right), \tag{17.24}$$

---

*Zelmanov regularly called tensor quantities, contracted with the chr.inv.-operator ${}^*\widetilde{\nabla}$ (17.21) in one index, their *physical chronometrically invariant divergence*. As a matter of fact, the physical chr.inv.-divergence and regular chr.inv.-divergence (the contraction with ${}^*\nabla_i$ in one index, see §2.14) are not the same. For instance, ${}^*\nabla_i Q^i$ and ${}^*\nabla_i Q^{ij}$ are the regular chr.inv.-divergences of the chr.inv.-vector Q^i and the chr.inv.-tensor of the 2nd rank Q^{ij}, while ${}^*\widetilde{\nabla}_i Q^i$ and ${}^*\widetilde{\nabla}_i Q^{ij}$ are the physical chr.inv.-divergences of the same quantities. — Editor's comment. D. R.

$$\left.\begin{aligned} G_{00} &= \frac{1}{c^2}\Big(1-\frac{\mathrm{w}}{c^2}\Big)^2 L \\ G_{0i} &= \frac{1}{c}\Big(1-\frac{\mathrm{w}}{c^2}\Big)\Big(M_i-\frac{1}{c^2}\,v_i L\Big) \\ G_{ij} &= -\frac{1}{c^2}\Big(N_{ij}+v_i M_j+v_j M_i-\frac{1}{c^2}\,v_i v_j L\Big) \end{aligned}\right\}, \qquad (17.25)$$

$$G = \frac{1}{c^2}\,(L+N)\;. \qquad (17.26)$$

§3.18 The law of gravitation. Its time covariant equation

Let us take the equations of gravitation in the form

$$G_{\mu\nu} = -\kappa\left(T_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}\,T\right)+\Lambda\,g_{\mu\nu} \qquad (18.1)$$

and consider the equations with $\mu,\nu=0$, i. e. the time equation

$$G_{00} = -\kappa\left(T_{00}-\frac{1}{2}\,g_{00}\,T\right)+\Lambda\,g_{00}\,. \qquad (18.2)$$

Because of (10.38) and (10.44), we obtain

$$T_{00}-\frac{1}{2}\,g_{00}\,T = \frac{1}{2}\Big(1-\frac{\mathrm{w}}{c^2}\Big)^2\Big(\rho+\frac{1}{c^2}\,U\Big)\,. \qquad (18.3)$$

Because of (17.25), the initial equation (18.2) becomes

$$L = -\frac{\kappa}{2}\left(\rho c^2+U\right)+\Lambda\,c^2. \qquad (18.4)$$

§3.19 The law of gravitation. The mixed covariant equations

We next consider the mixed (space-time) equations of gravitation, which are the equations (18.1) with $\mu=0$ and $\nu=1,2,3$

$$G_{0i} = -\kappa\left(T_{0i}-\frac{1}{2}\,g_{0i}\,T\right)+\Lambda\,g_{0i}\,. \qquad (19.1)$$

Because of (10.38) and (10.44), we have

$$T_{0i}-\frac{1}{2}\,g_{0i}\,T = -\frac{1}{c}\Big(1-\frac{\mathrm{w}}{c^2}\Big)\Big[J_i+\frac{v_i}{2c^2}\big(\rho c^2+U\big)\Big]\,, \qquad (19.2)$$

then, taking (17.25) into account, the equations (19.1) give

$$M_i-\frac{1}{c^2}\,v_i\,L = \kappa\Big[J_i+\frac{v_i}{2c^2}\big(\rho c^2+U\big)\Big]-\Lambda\,v_i \qquad (19.3)$$

or, in the other form

$$M_i - \kappa J_i = \frac{1}{c^2}\, v_i \Big[L + \frac{\kappa}{2}\big(\rho c^2 + U\big) - \Lambda c^2\Big]\,. \tag{19.4}$$

Because formula (19.4) holds for any choice of the time coordinate, we can apply the method to vary the potentials. Setting all values of v_i to zero, we obtain

$$M_i = \kappa J_i\,. \tag{19.5}$$

Formula (19.5) retains its form for any choice of the time coordinate, because of its chr.inv.-tensor (namely — chr.inv.-vector) nature. Therefore (19.4) leads to, besides (19.5), the equality

$$L = -\frac{\kappa}{2}\big(\rho c^2 + U\big) + \Lambda c^2, \tag{19.6}$$

which coincides with early obtained formula (18.4) and is a chr.inv.-tensor as well.

§3.20 The law of gravitation. The spatial covariant equations

We now consider the spatial equations of gravitation, which are equations (18.1) with $\mu, \nu = 1, 2, 3$

$$G_{ij} = -\kappa\Big(T_{ij} - \frac{1}{2}\, g_{ij}\, T\Big) + \Lambda\, g_{ij}\,. \tag{20.1}$$

Because of (10.38) and (10.44), we obtain

$$\begin{gathered} T_{ij} - \frac{1}{2}\, g_{ij}\, T = \frac{1}{c^2}\,\Big[U_{ij} - \frac{1}{2}\, h_{ij} U + \frac{1}{2}\, h_{ij}\,\rho c^2 + \\ + v_i J_j + v_j J_i + \frac{1}{2c^2}\, v_i v_j \big(\rho c^2 + U\big)\Big]\,. \end{gathered} \tag{20.2}$$

Because of (17.25) and (20.2), the initial formula (20.1) can be written as follows

$$\begin{gathered} N_{ij} + v_i M_j + v_j M_i - \frac{1}{c^2}\, v_i v_j L = \kappa\Big[U_{ij} - \frac{1}{2} h_{ij} U + \frac{1}{2} h_{ij}\,\rho c^2 + \\ + v_i J_j + v_j J_i + \frac{1}{2c^2}\, v_i v_j\,\big(\rho c^2 + U\big)\Big] + \Lambda c^2\Big(h_{ij} - \frac{1}{c^2}\, v_i v_j\Big) \end{gathered} \tag{20.3}$$

or, in the alternative form

$$\begin{aligned} & N_{ij}-\kappa\left(U_{ij}-\frac{1}{2}h_{ij}U+\frac{1}{2}h_{ij}\rho c^2\right)-\Lambda c^2 h_{ij}+v_i\left(M_j-\kappa J_j\right)+ \\ & +v_j\left(M_i-\kappa J_i\right)-\frac{1}{c^2}v_i v_j\left[L+\frac{\kappa}{2}\left(\rho c^2+U\right)-\Lambda c^2\right]=0\,. \end{aligned} \tag{20.4}$$

Because (20.4) is true for any choice of the time coordinate, we can use once again the method to vary the potentials. Setting all the v_k to zero, we obtain

$$N_{ij}=\kappa\left(U_{ij}-\frac{1}{2}h_{ij}U+\frac{1}{2}h_{ij}\rho c^2\right)+\Lambda c^2 h_{ij}\,. \tag{20.5}$$

The equality we have obtained, as well as any chr.inv.-tensor equality, is valid for any choice of the time coordinate. Hence

$$v_i\left(M_j-\kappa J_j\right)+v_j\left(M_i-\kappa J_i\right)-\frac{1}{c^2}v_i v_j\left[L+\frac{\kappa}{2}\left(\rho c^2+U\right)-\Lambda c^2\right]=0 \tag{20.6}$$

is also true for any choice of the time coordinate. Next, we suppose that

$$v_1\neq 0\,,\qquad v_2=v_3=0\,, \tag{20.7}$$

then (20.6) takes the form (after v_1 has been excluded)

$$\left.\begin{aligned} & 2\left(M_1-\kappa J_1\right)-\frac{1}{c^2}\left[L+\frac{\kappa}{2}\left(\rho c^2+U\right)-\Lambda c^2\right]=0 \\ & M_2-\kappa J_2=0\,,\qquad M_3-\kappa J_3=0 \end{aligned}\right\}. \tag{20.8}$$

Because (20.8) is true for any $v_1\neq 0$, we obtain the equalities

$$M_i=\kappa J_i\,, \tag{20.9}$$

$$L=-\frac{\kappa}{2}\left(\rho c^2+U\right)+\Lambda c^2. \tag{20.10}$$

The equalities retain their form for any choice of the time coordinate because of their chr.inv.-tensor nature. It is evident that the equalities coincide with formulae (19.5) and (18.4).

§ 3.21 The scalar equation of gravitation

We finally consider the scalar equation

$$G=\kappa T+4\Lambda\,, \tag{21.1}$$

which is a consequence of the tensor equations of gravitation. So, because of (17.26) and (10.44), equation (21.1) takes the form

$$L+N=\kappa\left(\rho c^2-U\right)+4\Lambda c^2 \tag{21.2}$$

or, in the alternative form

$$L+N=\kappa\rho_0 c^2+4\Lambda c^2. \tag{21.3}$$

§ 3.22 The first chr.inv.-tensor form of the equations of gravitation

We have obtained the system of equations (18.4), (19.5), and (20.5). Actually, this system is a chr.inv.-tensor form for the equations of gravitation. We will refer to the system as the *first chr.inv.-form of the equations of gravitation*. Let us write the system in the form

$$\left.\begin{aligned} & L=-\frac{\kappa}{2}\left(\rho c^2+U\right)+\Lambda c^2 \\ & M^k=\kappa J^k \\ & N^{ik}=\kappa\left(U^{ik}-\frac{1}{2}\,h^{ik}U+\frac{1}{2}\,h^{ik}\rho c^2\right)+\Lambda c^2 h^{ik} \end{aligned}\right\}. \tag{22.1}$$

It is easy to see that the first of the equations (the chr.inv.-invariant) is equivalent to the equation

$$G_{00}=-\kappa\left(T_{00}-\frac{1}{2}\,g_{00}\,T\right)+\Lambda\,g_{00}\,, \tag{22.2}$$

the second (chr.inv.-vector) equation is equivalent to

$$G_0^k=-\kappa\left(T_0^k-\frac{1}{2}\,g_0^k\,T\right)+\Lambda\,g_0^k\,, \tag{22.3}$$

which coincides with the equations

$$G_0^k=-\kappa\,T_0^k\,, \tag{22.4}$$

and finally, the third (chr.inv.-tensor) equation is equivalent to the equations

$$G^{ik}=-\kappa\left(T^{ik}-\frac{1}{2}\,g^{ik}\,T\right)+\Lambda\,g^{ik}. \tag{22.5}$$

We re-write the system (22.1) in component form, having lowered the index i in the tensor equation, and also using (17.22). So, we have

$$\frac{{}^*\partial D}{\partial t}+D_j^l\,D_l^j+A_{j\cdot}^{\cdot l}\,A_{l\cdot}^{\cdot j}+{}^*\widetilde{\nabla}_j F^j=-\frac{\kappa}{2}\left(\rho c^2+U\right)+\Lambda c^2, \tag{22.6}$$

$$ {}^*\nabla_j\left(h^{kj}D - D^{kj}\right) - {}^*\nabla_j A^{kj} + \frac{2}{c^2}A^{kj}F_j = \kappa J^k, \tag{22.7} $$

$$ \begin{gathered} -c^2 S_i^k + \frac{{}^*\partial D_i^k}{\partial t} + DD_i^k + D_i^l A_{l\cdot}^{\cdot k} + D_l^k A_{\cdot i}^{l\cdot} - 2A_{i\cdot}^{\cdot l}A_{l\cdot}^{\cdot k} + \\ + \frac{1}{2}\left({}^*\widetilde{\nabla}_i F^k + {}^*\widetilde{\nabla}^k F_i\right) = \kappa\left(U_i^k - \frac{1}{2}h_i^k U + \frac{1}{2}h_i^k \rho c^2\right) + \Lambda c^2 h_i^k. \end{gathered} \tag{22.8} $$

Because of (9.2) and also (16.3) of §2.16, we obtain

$$ A_{il} = \varepsilon_{jil}\,\Omega^j, \qquad A^{lk} = \varepsilon^{qlk}\,\Omega_q, \tag{22.9} $$

$$ \begin{gathered} A_{i\cdot}^{\cdot l}A_{l\cdot}^{\cdot k} = A_{il}A^{lk} = \varepsilon_{jil}\,\varepsilon^{qlk}\,\Omega^j\Omega_q = -\varepsilon_{jil}\,\varepsilon^{qkl}\,\Omega_q\,\Omega^j = \\ = -\left(h_j^q h_i^k - h_i^q h_j^k\right)\Omega_q\,\Omega^j = -h_i^k\,\Omega_j\,\Omega^j + \Omega_i\Omega^k, \end{gathered} \tag{22.10} $$

$$ A_{j\cdot}^{\cdot l}A_{l\cdot}^{\cdot j} = -2\Omega_j\Omega^j. \tag{22.11} $$

Because of (9.2) and also (16.4), (14.18), (16.15) obtained in Chapter 2, we get

$$ {}^*\nabla_j A^{kj} = {}^*\nabla_j\left(\varepsilon^{pkj}\,\Omega_p\right) = \varepsilon^{pkj}\,{}^*\nabla_j\,\Omega_p = \varepsilon^{kjp}\,{}^*\nabla_j\,\Omega_p = {}^*r^k(\Omega). \tag{22.12} $$

Hence, because of (9.2), we obtain

$$ A^{kj}F_j = \varepsilon^{pkj}\,\Omega_p F_j = -\varepsilon^{pjk}\,\Omega_p F_j = -\varepsilon^{jlk}\,\Omega_j F_l, \tag{22.13} $$

$$ D_i^l A_{l\cdot}^{\cdot k} + D_l^k A_{\cdot i}^{l\cdot} = D_{il}A^{lk} + D^{lk}A_{li} = \varepsilon^{jlk}D_{il}\Omega_j + \varepsilon_{jli}D^{lk}\Omega^j. \tag{22.14} $$

Using (22.10–22.14) and also (19.18), (19.19) of §2.19, we can re-write equations (22.6), (22.7), and (22.8), respectively, in the forms

$$ \frac{{}^*\partial D}{\partial t} + D_j^l D_l^j - 2\Omega_j\Omega^j + {}^*\widetilde{\nabla}_j F^j = -\frac{\kappa}{2}\left(\rho c^2 + U\right) + \Lambda c^2, \tag{22.15} $$

$$ R^k({}^*\omega) = {}^*r^k(\Omega) - \frac{2}{c^2}\,\varepsilon^{jlk}\,\Omega_j F_l + \kappa J^k, \tag{22.16} $$

$$ \begin{gathered} -c^2 S_i^k + \frac{{}^*\partial D_i^k}{\partial t} + DD_i^k + \varepsilon^{jlk}D_{il}\,\Omega_j + \varepsilon_{jli}\,D^{lk}\,\Omega^j - \\ -2\left(\Omega_i\Omega^k - h_i^k\,\Omega_j\Omega^j\right) + \frac{1}{2}\left({}^*\widetilde{\nabla}_i F^k + {}^*\widetilde{\nabla}^k F_i\right) = \\ = \kappa\left(U_i^k - \frac{1}{2}h_i^k U + \frac{1}{2}h_i^k\rho c^2\right) + \Lambda c^2 h_i^k. \end{gathered} \tag{22.17} $$

Equation (21.1) can be (because of 17.24 and 10.44) re-written as follows

$$-c^2S+2\frac{{}^*\partial D}{\partial t}+D^2+D_j^lD_l^j-A_{j\cdot}^{\cdot l}A_{l\cdot}^{\cdot j}+2\,{}^*\widetilde{\nabla}_jF^j=\kappa\left(\rho c^2-U\right)+4\Lambda c^2. \quad (22.18)$$

Using (22.11) again, we put equation (22.18) into the form

$$-c^2S+2\frac{{}^*\partial D}{\partial t}+D^2+D_j^lD_l^j+2\Omega_j\Omega^j+2\,{}^*\widetilde{\nabla}_jF^j=\kappa\left(\rho c^2-U\right)+4\Lambda c^2. \quad (22.19)$$

§ 3.23 The second chr.inv.-tensor form of the equations of gravitation

We consider next the system of equations

$$G_{00}-\frac{1}{2}\,g_{00}\,G=-\kappa\,T_{00}-\Lambda\,g_{00}\,, \quad (23.1)$$

$$G_0^k-\frac{1}{2}\,g_0^k\,G=-\kappa\,T_0^k-\Lambda\,g_0^k\,, \quad (23.2)$$

where the second coincides with (22.4)

$$G_0^k=-\kappa\,T_0^k\,, \quad (23.3)$$

and also the equation

$$G^{jk}-\frac{1}{2}\,g^{jk}\,G=-\kappa\,T^{jk}-\Lambda\,g^{jk}. \quad (23.4)$$

Because of (17.2) and (17.26), we have

$$G_{00}-\frac{1}{2}\,g_{00}\,G=\frac{1}{2c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2(L-N)\,. \quad (23.5)$$

For this reason, and because of (10.38), the equation (23.1) is equivalent to

$$\frac{1}{2}\,(N-L)=\kappa\rho c^2+\Lambda c^2. \quad (23.6)$$

As a result of (17.4) and (17.26), we have

$$G^{jk}-\frac{1}{2}\,g^{jk}\,G=\frac{1}{c^2}\left[\frac{1}{2}\,h^{jk}(L+N)-N^{jk}\right], \quad (23.7)$$

and the third equation (23.4) is, as a result of (10.40), equivalent to

$$N^{jk}-\frac{1}{2}\,h^{jk}\,(L+N)=\kappa\,U^{jk}-\Lambda c^2h^{jk}. \quad (23.8)$$

As a result, it is easy to see that the system of equations (23.1), (23.2), and (23.4) is equivalent to the system of equations (22.2), (22.3), and (22.5).

We can see directly that the system of equations

$$\left.\begin{aligned} &\frac{1}{2}\,(N-L)=\kappa\rho c^2+\Lambda c^2\\ &M^k=\kappa J^k\\ &N^{jk}-\frac{1}{2}\,h^{jk}\,(L+N)=\kappa\,U^{jk}-\Lambda c^2 h^{jk}\end{aligned}\right\} \tag{23.9}$$

is equivalent to the system (22.1), which is the first chr.inv.-tensor form of the equations of gravitation. For this reason, we will refer to the system (23.9) as the *second chr.inv.-tensor form of the equations of gravitation*.

Using (17.22) and (17.23), we re-write equations (23.6) and (23.8) of the system (23.9) in component form (having lowered the index i in 23.8). As a result we obtain

$$\frac{1}{2}\left(-c^2S+D^2-D_j^lD_l^j-3A_{j\cdot}^{\cdot l}A_{l\cdot}^{\cdot j}\right)=\kappa\rho c^2+\Lambda c^2, \tag{23.10}$$

$$\begin{gathered}-c^2\left(S_i^k-\frac{1}{2}\,h_i^kS\right)+\frac{{}^*\partial D_i^k}{\partial t}+DD_i^k+D_i^lA_{l\cdot}^{\cdot k}+D_l^kA_{\cdot i}^{l\cdot}-\\ -2A_{i\cdot}^{\cdot l}A_{l\cdot}^{\cdot k}+\frac{1}{2}\left({}^*\widetilde{\nabla}_i\,F^k+{}^*\widetilde{\nabla}^kF_i\right)-h_i^k\left[\frac{{}^*\partial D}{\partial t}+\right.\\ \left.+\frac{1}{2}\left(D^2+D_j^lD_l^j-A_{j\cdot}^{\cdot l}A_{l\cdot}^{\cdot j}\right)+{}^*\widetilde{\nabla}_j\,F^j\right]=\kappa\,U_i^k-\Lambda c^2h_i^k\,.\end{gathered} \tag{23.11}$$

Using (22.11), (22.14), (22.10), and also (21.29), (21.35) of §2.21, we put equations (23.10) and (23.11), respectively, into the forms

$$c^2Z+\frac{1}{2}\left(D^2-D_j^lD_l^j\right)+3\Omega_j\Omega^j=\kappa\rho c^2+\Lambda c^2, \tag{23.12}$$

$$\begin{gathered}-c^2Z_i^k+\frac{{}^*\partial D_i^k}{\partial t}+DD_i^k+\varepsilon^{jlk}D_{il}\Omega_j+\varepsilon_{jli}\,D^{lk}\Omega^j-\\ -2\Omega_i\Omega^k+\frac{1}{2}\left({}^*\widetilde{\nabla}_i\,F^k+{}^*\widetilde{\nabla}^kF_i\right)-h_i^k\left[\frac{{}^*\partial D}{\partial t}+\right.\\ \left.+\frac{1}{2}\left(D^2+D_j^lD_l^j\right)-\Omega_j\Omega^j+{}^*\widetilde{\nabla}_j\,F^j\right]=\kappa\,U_i^k-\Lambda c^2h_i^k\,.\end{gathered} \tag{23.13}$$

We also re-write (22.19), using (21.35) of §2.21. So, we obtain

$$c^2Z+\frac{{}^*\partial D}{\partial t}+\frac{1}{2}\left(D^2+D_j^lD_l^j\right)+\Omega_j\Omega^j+{}^*\widetilde{\nabla}_jF^j=\frac{\kappa}{2}\left(\rho c^2-U\right)+2\Lambda c^2. \tag{23.14}$$

§ 3.24 The structure of the equations of gravitation

Examining the chr.inv.-tensor forms of the equations of gravitation, we see that some quantities appear only in combination with other quantities. A peculiarity of the combinations is that in eliminating one of the quantities from the equations, we eliminate the entire combination. The combinations are

$$\varphi_i^k = \frac{{}^*\partial D_i^k}{\partial t} + \frac{1}{2}\left({}^*\widetilde{\nabla}_i\, F^k + {}^*\widetilde{\nabla}^k F_i\right) - \kappa\left(U_i^k - \frac{1}{2}\, h_i^k\, U\right), \tag{24.1}$$

$$\varphi = \frac{{}^*\partial D}{\partial t} + {}^*\widetilde{\nabla}_j\, F^j + \frac{\kappa}{2}\, U\,, \tag{24.2}$$

$$\sigma_i^k = -c^2 S_i^k + D D_i^k\,, \tag{24.3}$$

$$\sigma = -c^2 S + D^2, \tag{24.4}$$

$$\omega_i^k = D_i^l A_{l\cdot}^{\cdot k} + D_l^k A_{\cdot i}^{l\cdot} = \varepsilon^{jlk} D_{il}\,\Omega_j + \varepsilon_{jli}\, D^{lk}\Omega^j. \tag{24.5}$$

We introduce the notation*

$$\alpha_i^k = -A_{i\cdot}^{\cdot l} A_{l\cdot}^{\cdot k} = -\Omega_i\Omega^k + h_i^k\,\Omega_j\Omega^j\,, \tag{24.6}$$

$$\alpha = -A_{j\cdot}^{\cdot l} A_{l\cdot}^{\cdot j} = 2\Omega_j\Omega^j\,, \tag{24.7}$$

$$\delta = D_j^l D_l^j\,, \tag{24.8}$$

$$\begin{aligned} \theta^k &= {}^*\nabla_j\left(h^{kj} D - D^{kj}\right) - {}^*\nabla_j A^{kj} + \frac{2}{c^2} A^{kj} F_j - \kappa J^k = \\ &= R^k\left({}^*\omega\right) - {}^*r^k\left(\Omega\right) + \frac{2}{c^2}\,\varepsilon^{jlk}\,\Omega_j F_l - \kappa J^k. \end{aligned} \tag{24.9}$$

It is also evident that

$$\varphi_j^j = \varphi\,, \quad \sigma_j^j = \sigma\,, \quad \omega_j^j = 0\,, \quad \alpha_j^j = \alpha\,. \tag{24.10}$$

The equations of gravitation take the following form. The first chr.inv.-form is

$$\left.\begin{aligned} &\alpha + \Lambda c^2 = \delta + \varphi + \frac{\kappa}{2}\,\rho c^2 \\ &\theta^k = 0 \\ &2\alpha_i^k + \sigma_i^k + \varphi_i^k + \omega_i^k = \frac{\kappa}{2}\, h_i^k\, \rho c^2 + \Lambda c^2 h_i^k \end{aligned}\right\}, \tag{24.11}$$

*See the definition of chr.inv.rotor ${}^*r^k(\Omega) = \varepsilon^{qpk}\,{}^*\nabla_q\,\Omega_p = {}^*\nabla_j\, a^{kj}$ in §2.16. — Editor's comment. D. R.

the second chr.inv.-form is

$$\left.\begin{aligned} & 3\alpha+\sigma=\delta+2\kappa\rho c^2+2\Lambda c^2 \\ & \theta^k=0 \\ & \left(2\alpha_i^k-\frac{1}{2}\,h_i^k\alpha\right)+\left(\sigma_i^k-\frac{1}{2}\,h_i^k\sigma\right)+\left(\varphi_i^k-h_i^k\varphi\right)+ \\ & +\,\omega_i^k+\Lambda c^2h_i^k=\frac{1}{2}\,h_i^k\delta \end{aligned}\right\}, \qquad (24.12)$$

the scalar equation is

$$\alpha+\delta+\sigma+2\varphi=\kappa\rho c^2+4\Lambda c^2. \qquad (24.13)$$

Let us consider the variable chr.inv.-invariants ρ, α, δ, φ, σ, which are contained in the equations of gravitation. From the physical perspective, we assume that

$$\rho\geqslant 0\,. \qquad (24.14)$$

We next introduce spatial coordinates that are locally Cartesian and orthogonal at the given world-point. Then we have

$$\Omega_j=\Omega^j, \qquad D_i^k=D_k^i \qquad (24.15)$$

and, hence, it is easy to see that,

$$\alpha\geqslant 0\,, \qquad (24.16)$$

$$\delta\geqslant 0\,. \qquad (24.17)$$

Because α and δ are sub-invariants, the conditions (24.16) and (24.17) hold in any coordinates.

Inspecting the first equation of the system (24.11), we see that, generally speaking,

$$\varphi \gtreqless 0\,. \qquad (24.18)$$

Similarly, inspection of the first equation of the system (24.12), we see that, generally speaking,

$$\sigma \gtreqless 0\,. \qquad (24.19)$$

◇

Chapter 4

NUMEROUS COSMOLOGICAL CONSEQUENCES

§4.1 Locally-accompanying frames of reference

We are going to consider substance as continuous media. We can introduce a *locally accompanying reference frame* at any point A of a substance (the point A must not be a break-point for the substance's velocity with respect to the initial reference frame we have chosen arbitrarily). We introduce the reference frame as follows:

> A reference frame, which locally accompanies a substance at a given point A, is such that at this point A of the substance for the time interval t we are considering, (1) the frame is locally-stationary, and (2) the frame does not rotate with respect to the space.

We assume that $^*u^i$ is the chr.inv.-velocity of the substance with respect to the reference frame which accompanies the substance locally at the point A. Then, at the point A, we have

$$\left(^*u^i\right)_A \equiv 0\,, \tag{1.1}$$

$$\left(^*\omega_k\right)_A \equiv 0\,. \tag{1.2}$$

Because formula (10.6) of Chapter 2 gives (10.8), (17.12), and (17.13), taking (1.1) and (1.2) into account, we obtain

$$\left(u^i\right)_A \equiv 0\,, \tag{1.3}$$

$$\left(\omega_k\right)_A \equiv 0\,. \tag{1.4}$$

§4.2 Accompanying frames of reference

In accordance with §1.10 and §1.19, we will refer to the reference frame which moves in company with the substance at any world-point of the given four-dimensional volume, as the *accompanying* *reference frame*. If $^*u^i$ is the chr.inv.-velocity of the substance with respect to the given reference frame, then, in this accompanying reference frame, we have

$$^*u^i \equiv 0 \tag{2.1}$$

and, hence

$$u^i \equiv 0\,. \tag{2.2}$$

It is clear that in any volume, where the substance's velocity has no break-points (with respect to an arbitrary reference frame), we can introduce the accompanying reference frame. Naturally, let us assume that components $\tilde{u}^i$ of the substance's velocity are finite, simple, and continuous functions* of their arguments $\tilde{x}^0$, $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$ in a four-dimensional volume Q of a coordinate frame $\widetilde{S}$. Then the system of functions

$$x^{i'} = x^{i'}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)\,, \quad i = 1, 2, 3 \tag{2.3}$$

exist, where the functions are finite, simple, continuous, and differentiable in the volume Q, and satisfy the equations

$$\frac{\partial x^{i'}}{\partial \tilde{x}^0} + \frac{\partial x^{i'}}{\partial \tilde{x}^j}\,\tilde{u}^j = 0 \tag{2.4}$$

in this volume. We introduce a coordinate frame S', which is linked to the coordinate frame $\widetilde{S}$ by the transformations

$$\left.\begin{aligned} x^{0'} &= \tilde{x}^0 \\ x^{i'} &= x^{i'}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \end{aligned}\right\}. \tag{2.5}$$

Let us find the velocity $\tilde{\mathrm{v}}^i$ of the motion of the system S' in the system $\widetilde{S}$. Because

$$\left.\begin{aligned} d\tilde{x}^0 &= dx^{0'} \\ d\tilde{x}^i &= \frac{\partial \tilde{x}^i}{\partial x^{0'}}dx^{0'} + \frac{\partial \tilde{x}^i}{\partial x^{j'}}dx^{j'} \end{aligned}\right\}, \tag{2.6}$$

we have

$$\tilde{\mathrm{v}}^i = \frac{\partial \tilde{x}^i}{\partial x^{0'}}\,. \tag{2.7}$$

*In the §4.1 and §4.2 above, we did not mention that the velocity is finite and unique, because the properties were implicit from physical considerations.

On the other hand, because

$$\frac{\partial x^{i'}}{\partial x^{0'}} \equiv 0\,, \tag{2.8}$$

we have

$$\frac{\partial x^{i'}}{\partial \tilde{x}^{0}} + \frac{\partial x^{i'}}{\partial \tilde{x}^{j}}\frac{\partial \tilde{x}^{j}}{\partial x^{0'}} = 0\,. \tag{2.9}$$

Therefore, we obtain

$$\frac{\partial x^{i'}}{\partial \tilde{x}^{0}} + \frac{\partial x^{i'}}{\partial \tilde{x}^{j}}\,\tilde{\mathrm{v}}^{j} = 0\,. \tag{2.10}$$

Subtracting (2.4) from (2.10) term-by-term, we obtain

$$\frac{\partial x^{i'}}{\partial \tilde{x}^{j}}\left(\tilde{\mathrm{v}}^{j} - \tilde{u}^{j}\right) = 0 \tag{2.11}$$

and because

$$\frac{\partial\left(x^{1'}, x^{2'}, x^{3'}\right)}{\partial\left(\tilde{x}^{1}, \tilde{x}^{2}, \tilde{x}^{3}\right)} \not\equiv 0\,, \tag{2.12}$$

we get

$$\tilde{\mathrm{v}}^{i} \equiv \tilde{u}^{i}\,. \tag{2.13}$$

So the coordinate frame S' moves in company with the substance, and hence, S' is of the reference frame which accompanies the volume Q.

§4.3 The kinds of matter. Their motions

We assume that the four-dimensional volume Q we are considering is filled with matter which satisfies the following conditions:

1. The matter is a continuously distributed substance — a continuous medium, which has no pressure or stresses, is free of heat flux, and has positive density;
2. At any world-point of the volume Q, the components u^i of the substance's velocity with respect to an arbitrary reference frame (x^0,x^1,x^2,x^3), real numerical values of coordinates of which cover the neighbourhood of this world-point, are finite, simple, and differentiable functions of the time coordinate and the spatial coordinates.

The first condition implies that the matter can be considered as an ideal fluid, free of heat flux, of positive density ρ_{00} and zero pressure p_0. So, we have

$$T^{\mu\nu} = \left(\rho_{00} + \frac{p_0}{c^2}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{p_0}{c^2}\,g^{\mu\nu}, \quad \rho_{00} > 0\,, \quad p_0 = 0 \tag{3.1}$$

or, in the alternative form

$$T^{\mu\nu} = \rho_{00}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\,, \quad \rho_{00} > 0\,. \tag{3.2}$$

It is evident that

$$T_{00} = \rho_{00}\frac{dx_0}{ds}\frac{dx_0}{ds}\,, \tag{3.3}$$

$$T_0^i = \rho_{00}\frac{dx_0}{ds}\frac{dx^i}{ds}\,, \tag{3.4}$$

$$T^{ik} = \rho_{00}\frac{dx^i}{ds}\frac{dx^k}{ds}\,. \tag{3.5}$$

The second condition, taking the results of §4.2 into account, implies that we can cover the volume Q point-by-point by the accompanying coordinate frames. In other words, we can introduce the accompanying reference frame throughout the volume Q.

§4.4 Matter in the accompanying frame of reference

We assume that matter* is linked to the accompanying reference frame. For any point-mass of the substance in this reference frame

$$dx^i = 0\,, \qquad i = 1, 2, 3\,, \tag{4.1}$$

is true, hence

$$dx_0 = g_{00}\,dx^0, \tag{4.2}$$

$$ds^2 = g_{00}\,dx^0 dx^0. \tag{4.3}$$

Therefore, because of (3.3) and (3.5), we obtain

$$\rho = \rho_{00}\,, \tag{4.4}$$

$$J^i = 0\,, \tag{4.5}$$

$$U^{ik} = 0\,. \tag{4.6}$$

It is evident that the equalities (4.4–4.6) are essentially linked to our suppositions about the properties of matter. On the contrary,

*A substance and radiations. — Editor's comment. D. R.

speculations in this section are independent of the suppositions given for the 1st condition of §4.3; the speculations are linked only to the 2nd condition, containing only suppositions about motions of the substance. The substance is at rest with respect to the accompanying reference frame by definition of such reference frames. In the neighbourhood of every point of the accompanying space, the substance, generally speaking, moves with respect to the reference frame locally accompanying the substance at the point. At every point of the accompanying space we can introduce: (1) the chr.inv.-tensors (covariant, mixed and contravariant) and the chr.inv.-invariant of the rate of deformations of the substance in respect of the reference frame, which is locally accompanying in this point; (2) the chr.inv.-rotor of the chr.inv.-vector of the angular velocities of rotations of the substance with respect to the same locally accompanying reference frame. Because the accompanying space moves with the substance, the aforementioned quantities coincide with the chr.inv.-tensors and the chr.inv.-invariant of the rate of deformations of the accompanying space and, respectively, the chr. inv.-rotor of the chr.inv.-vector of the angular velocities of the accompanying space with respect to the reference frame locally accompanying at this point. At the same time the locally accompanying reference frame is of the locally-stationary reference frames. For this reason, we can identify the aforementioned quantities with the quantities D_{ik}, D_i^k, D^{ik}, D, and $R^i({}^*\omega) = {}^*\nabla_j\left(h^{ij}D - D^{ij}\right)$, respectively*.

Thus, the quantities D_{ik}, D_i^k, D^{ik}, D, and $R^i({}^*\omega)$, characterizing the rate of deformations and the velocities of relative rotations of elements of the accompanying space, actually characterize the same for an element of the substance. In particular, the chr.inv.-invariant of the rate of relative expansions of an elementary volume of the space, namely — the quantity D, gives the rate of relative expansions of the element of the substance at any given point.

In addition to the above, we can also say that, in the accompanying reference frame, relative motions of elements of the substance characterize themselves by the fact that the quantities h_{ik} (hence,

*As mentioned above, it can be a set of locally-stationary reference frames at any given point. We saw that the arbitrary rule for choosing the locally-stationary reference systems does not affect the aforementioned equations containing the quantities D_{ik}, D_i^k, D^{ik}, D, and $R^i({}^*\omega)$. Now, as seen from this section, we specialize this choice of locally-stationary reference frames, taking locally accompanying reference frames instead of them.

also the quantities h^{ik} and h) are dependent on the time coordinate. In particular, transformations of the volume of the substance's element are described by a function of $\sqrt{h}$ from t (see formula 12.7 of §2.12).

§4.5 The cosmological equations of gravitation

We are going to consider the system of the equations of gravitation. This system in the case (3.2) can be considered as a system of 10 equations with respect to 10 unknowns: one quantity ρ, three u^k and 6 of 10 independent quantities $g_{\mu\nu}$.* At the same time, the equations can be considered as equations that define ρ from 9 of the components $g_{\mu\nu}$ when u^k and one other component of $g_{\mu\nu}$ are given. Introducing the accompanying reference frame, we apply this approach to the equations of gravitation. Therefore, the system of the equations of gravitation in the accompanying reference frame can be considered, for instance, as a system, which define the value of ρ, the 3 value of v_i, and the 6 values of h_{ik} (when w is given by a special choice of the time coordinate).

So, we can write the equations of gravitation in the accompanying reference frame as formulae (24.11–24.13) of §3.24: the first chr.inv.-form

$$\left.\begin{aligned} &\alpha+\Lambda c^2=\delta+\varphi+\frac{\kappa}{2}\,\rho c^2\\ &\theta^k=0\\ &2\alpha_i^k+\sigma_i^k+\varphi_i^k+\omega_i^k=\frac{\kappa}{2}\,h_i^k\rho c^2+\Lambda c^2h_i^k \end{aligned}\right\}, \tag{5.1}$$

the second chr.inv.-form

$$\left.\begin{aligned} &3\alpha+\sigma=\delta+2\kappa\rho c^2+2\Lambda c^2\\ &\theta^k=0\\ &\left(2\alpha_i^k-\frac{1}{2}\,h_i^k\alpha\right)+\left(\sigma_i^k-\frac{1}{2}\,h_i^k\sigma\right)+\left(\varphi_i^k-h_i^k\varphi\right)+\\ &+\omega_i^k+\Lambda c^2h_i^k=\frac{1}{2}\,h_i^k\delta \end{aligned}\right\}, \tag{5.2}$$

the scalar equation of gravitation

$$\alpha+\delta+\sigma+2\varphi=\kappa\rho c^2+4\Lambda c^2. \tag{5.3}$$

*We can define the other 4 components of $g_{\mu\nu}$ by a specific choice of coordinate frame. For instance, see [64], p. 237.

Besides these, formulae (24.3–24.8) of §3.24 remain unchanged, while formulae (24.1), (24.2), and (24.9) will be simplified (because of formulae 4.5 and 4.6 of §4.4) as follows

$$\varphi_i^k = \frac{{}^*\partial D_i^k}{\partial t} + \frac{1}{2}\left({}^*\widetilde{\nabla}_i F^k + {}^*\widetilde{\nabla}^k F_i\right), \tag{5.4}$$

$$\varphi = \frac{{}^*\partial D}{\partial t} + {}^*\widetilde{\nabla}_j F^j, \tag{5.5}$$

$$\sigma_i^k = -c^2 S_i^k + D D_i^k\,, \tag{5.6}$$

$$\sigma = -c^2 S + D^2, \tag{5.7}$$

$$\omega_i^k = D_i^l A_{l\cdot}^{\cdot k} + D_l^k A_{\cdot i}^{l\cdot} = \varepsilon^{jlk} D_{il}\Omega_j + \varepsilon_{jli} D^{lk}\Omega^j, \tag{5.8}$$

$$\alpha_i^k = -A_{i\cdot}^{\cdot l} A_{l\cdot}^{\cdot k} = -\Omega_i\Omega^k + h_i^k\Omega_j\Omega^j, \tag{5.9}$$

$$\alpha = -A_{j\cdot}^{\cdot l} A_{l\cdot}^{\cdot j} = 2\Omega_j\Omega^j, \tag{5.10}$$

$$\delta = D_j^l D_l^j\,, \tag{5.11}$$

$$\begin{gathered}\theta^k = {}^*\nabla_j\left(h^{kj}D - D^{kj}\right) - {}^*\nabla_j A^{kj} + \frac{2}{c^2}A^{kj}F_j = \\ = R^k({}^*\omega) - {}^*r^k(\Omega) + \frac{2}{c^2}\varepsilon^{jlk}\Omega_j F_l\,.\end{gathered} \tag{5.12}$$

It is also evident that (24.15) and (24.17–24.19) of §3.24 retain their power, whilst the equality sign in (24.14) disappears, so

$$\rho > 0\,, \tag{5.13}$$

$$\alpha \geqslant 0\,, \tag{5.14}$$

$$\delta \geqslant 0\,, \tag{5.15}$$

$$\varphi \gtreqless 0\,, \tag{5.16}$$

$$\sigma \gtreqless 0\,. \tag{5.17}$$

Thus, we see that the equations (5.1–5.3), because of (5.4–5.12), create a link between the following quantities:

- Quantities which characterize the state of the matter;
- Quantities which characterize the evolution of the accompanying space;
- Quantities which characterize the geometric properties of the space.

In other words, equations (5.1–5.3) are *cosmological equations*. In accordance with §1.17, they are the *cosmological equations of gravitation*. If we use the equations to find ρ, v_i, and h_{ik}, we will then need to add 18 other equations to them. The additional equations define D_{ik} (6 components), Z_{ik} (6 components), Ω_i (3 components), and F_i (3 components) as functions of w, v_i, and h_{ik}. So we will need to consider the whole system of 28 equations with respect to the same number of unknown functions.

§4.6 The cosmological equations of energy

The equations of the law of energy (see formulae 12.9 and 12.11 of §3.12), because of (4.3) and (4.6), take the form

$$\frac{{}^*\partial\rho}{\partial t} + \rho D = 0\,, \tag{6.1}$$

$$F^k \rho = 0\,. \tag{6.2}$$

Formula (6.2), because of (5.13), leads to

$$F_i = 0\,. \tag{6.3}$$

In accordance with §1.17, equations (6.1) and (6.2)* are the *cosmological equations of energy*.

It is known that the equations of the law of energy are consequences of the equations of the law of gravitation, in the sense that they are derived from the four identities between the left sides of the equations of gravitation. For this reason we can use the equations of energy instead of the four aforementioned identities.

§4.7 The main form of the cosmological equations

Because of the third of equations (5.1), we obtain

$$2\alpha + \sigma + \varphi = \frac{3\kappa}{2}\,\rho c^2 + 3\Lambda c^2 \tag{7.1}$$

and so

$$2\left(\alpha_i^k - \frac{1}{3}\,h_i^k\alpha\right) + \left(\sigma_i^k - \frac{1}{3}\,h_i^k\sigma\right) + \left(\varphi_i^k - \frac{1}{3}\,h_i^k\varphi\right) + \omega_i^k = 0\,. \tag{7.2}$$

Adding the first of equations (5.1) and the equation (7.1) term-

*Or formula (6.3) instead of (6.2).

by-term, we obtain the first of equations (5.2). The last, in its connection with the first of equations (5.1), gives (7.1). Hence, equation (7.1), in connection with (7.2), leads to the third of equations (5.1). Therefore, the system

$$\left.\begin{aligned} &\alpha + \Lambda c^2 = \delta + \varphi + \frac{\kappa}{2}\,\rho c^2 \\ &3\alpha + \sigma = \delta + 2\kappa\rho c^2 + 2\Lambda c^2 \\ &2\left(\alpha_i^k - \frac{1}{3}\,h_i^k\alpha\right) + \left(\sigma_i^k - \frac{1}{3}\,h_i^k\sigma\right) + \left(\varphi_i^k - \frac{1}{3}\,h_i^k\varphi\right) + \omega_i^k = 0 \\ &\theta^k = 0 \end{aligned}\right\} \tag{7.3}$$

is equivalent to the system (5.1) and therefore to the system (5.2). It easy to see that the third of equations (7.3) gives only 5 independent relations, because of its symmetry and becoming zero under its reduction.

Because

$$\varepsilon_{j}^{\cdot lk} h_{il} + \varepsilon_{jli}\, h^{lk} = \varepsilon_{ji\cdot}^{\cdot\cdot k} + \varepsilon_{j\cdot i}^{\cdot k\cdot} = h^{kq}\left(\varepsilon_{jiq} + \varepsilon_{jqi}\right) = 0\,, \tag{7.4}$$

we can write (5.8) in the form

$$\omega_i^k = \varepsilon^{jlk}\,\Omega_j\left(D_{il} - \frac{1}{3}\,h_{il}D\right) + \varepsilon_{jli}\,\Omega^j\left(D^{lk} - \frac{1}{3}\,h^{lk}D\right). \tag{7.5}$$

Using (21.33) and (21.35) of §2.21, instead of (5.6) and (5.7), we can write

$$\sigma_i^k = -c^2\left(Z_i^k - h_i^k Z\right) + DD_i^k\,, \tag{7.6}$$

$$\sigma = 2c^2 Z + D^2. \tag{7.7}$$

Introducing the notation

$$\Pi = D_j^l D_l^j - \frac{1}{3}\,D^2, \tag{7.8}$$

we re-write (5.11) in the form

$$\delta = \frac{1}{3}\,D^2 + \Pi\,. \tag{7.9}$$

Using equation (6.3), equalities (5.4), (5.5), (5.12) become the more simple

$$\varphi_i^k = \frac{{}^*\partial D_i^k}{\partial t}\,, \tag{7.10}$$

$$\varphi = \frac{{}^*\partial D}{\partial t}\,, \tag{7.11}$$

$$\theta^k = {}^*\nabla_j\left(h^{kj}D - D^{kj}\right) - {}^*\nabla_j A^{kj} = R^k({}^*\omega) - {}^*r^k(\Omega)\,. \tag{5.12}$$

Let us write equations (6.1), (6.3) and the system (7.3), taking equalities (5.9), (5.10), (7.5–7.7), (7.9–7.12) into account,

$$\frac{{}^*\partial\rho}{\partial t} + \rho D = 0\,, \tag{7.13}$$

$$F_i = 0\,, \tag{7.14}$$

$$\frac{{}^*\partial D}{\partial t} + \frac{1}{3}D^2 + \Pi - 2\Omega_j\Omega^j = -\frac{\kappa}{2}\rho c^2 + \Lambda c^2, \tag{7.15}$$

$$\frac{1}{3}D^2 - \frac{1}{2}\Pi + 3\Omega_j\Omega^j + c^2 Z = \kappa\rho c^2 + \Lambda c^2, \tag{7.16}$$

$$\begin{gathered}\frac{{}^*\partial}{\partial t}\left(D_i^k - \frac{1}{3}h_i^k D\right) + D\left(D_i^k - \frac{1}{3}h_i^k D\right) + \\ + \varepsilon^{jlk}\Omega_j\left(D_{il} - \frac{1}{3}h_{il}D\right) + \varepsilon_{jli}\Omega^j\left(D^{lk} - \frac{1}{3}h^{lk}D\right) = \\ = 2\left(\Omega_i\Omega^k - \frac{1}{3}h_i^k\Omega_j\Omega^j\right) + c^2\left(Z_i^k - \frac{1}{3}h_i^k Z\right),\end{gathered} \tag{7.17}$$

$$R^k\left({}^*\omega\right) = {}^*r^k\left(\Omega\right), \tag{7.18}$$

where the equality (7.18) can be written in the form

$${}^*\nabla_j\left(h^{kj}D - D^{kj}\right) = {}^*\nabla_j A^{kj}. \tag{7.19}$$

We take equations (7.13–7.17) and (7.18) as the *main chr.inv.-form of the cosmological equations*.

§4.8 Free fall. The primary coordinate of time

Equality (7.14) implies that the substance falls freely in the field of gravitational inertial forces. This is self-evident because there are no other forces under the conditions (4.5) and (4.6). Equality (7.14) leads to numerous consequences*, one of which will be considered now — that there is a possibility of choosing the primary coordinate of time at any point of the space.

*We used the equality, numbered as (6.3), earlier in §4.7.

We assume that condition (7.14) holds everywhere in a four-dimensional volume Q. We introduce a coordinate of time such that the condition

$$\mathrm{w}=0 \tag{8.1}$$

holds throughout the volume. Then, because of (7.14), we obtain

$$\frac{\partial v_i}{\partial t}=0\,. \tag{8.2}$$

The formulae (8.1) and (8.2) lead to

$$Y=0\,, \tag{8.3}$$

$$\Phi_i=0 \tag{8.4}$$

in the volume Q (see formulae 23.18 and 23.28 of §2.23). So, numerical values of 5 quantities ($\widetilde{w}$, Y, Φ_i) of the 14 locally independent quantities we considered in §2.20 are defined throughout the volume Q. Next, we need to find numerical values of the other 9 quantities (v_i and X_{ik}) at any point of the volume Q. We assume that, at a space point A at a moment $t=t_0$, we have

$$v_i=0\,, \tag{8.5}$$

$$X_{ik}=0\,. \tag{8.6}$$

Because (8.2) holds throughout the volume Q, the condition (8.5) remains unchanged at the point A for all numerical values of t in the said four-dimensional volume.

Owing to (8.5), formula (8.6) can be written as follows (see formula 23.31 of §2.23)

$$\frac{\partial v_k}{\partial x^i}+\frac{\partial v_i}{\partial x^k}=0\,. \tag{8.7}$$

Because we have

$$\frac{\partial}{\partial t}\left(\frac{\partial v_q}{\partial x^p}\right)=\frac{\partial}{\partial x^p}\left(\frac{\partial v_q}{\partial t}\right) \tag{8.8}$$

and (8.2) holds throughout the volume Q, we have everywhere in that volume

$$\frac{\partial}{\partial t}\left(\frac{\partial v_k}{\partial x^i}+\frac{\partial v_i}{\partial x^k}\right)=0\,. \tag{8.9}$$

Hence, formula (8.7) remains unchanged at the point A for all numerical values of t in the volume Q. It is evident that (8.6) also remains unchanged at the point A for all values of t in the volume Q.

So, if the conditions (7.14) are true in the volume Q, then we have the possibility of choosing, at any point A of this volume, time coordinates which permit conditions (8.1) and (8.3–8.6) at the point A for all t inside the four-dimensional volume. In accordance with §2.22, the spatial section, corresponding to this choice of time coordinates, is a maximally orthogonal one (at the point A for all values of t in the volume Q), so that

$$K_{kjin} = S_{kjin} \tag{8.10}$$

and hence

$$H_{ik} = S_{ik}\,, \qquad C_{ik} = Z_{ik}\,. \tag{8.11}$$

If the condition

$$A_{ik} = 0\,, \tag{8.12}$$

aside of the condition (8.1), is also true in the volume Q, then we can, in accordance with §2.7, introduce the cosmic universal time (see §1.2) — time coordinates by which the equalities (8.1), (8.5), and hence (8.3), (8.4), (8.6), (8.10), (8.11) hold everywhere in the volume Q.

§4.9 Characteristics of anisotropy

We consider next the problem of anisotropy of volume elements, in both the mechanical and the geometrical senses. Let us first consider the problem of the mathematical characteristics of the anisotropy of that mechanical geometrical factor, which is described by a symmetric sub-tensor of the 2nd rank B_{ik}. If this factor is spatially isotropic at a world-point, then, in locally Cartesian spatial coordinates at this point, we have

$$B_{ik} = B_i^k = B^{ik} = \left\{ \begin{array}{ll} \frac{1}{3}B\,, & i = k \\ 0\,, & i \neq k \end{array} \right\}, \tag{9.1}$$

where we denote

$$B = B_j^j\,. \tag{9.2}$$

In arbitrary coordinates, we have

$$B_{ik} = \frac{1}{3}\,h_{ik}B\,, \qquad B_i^k = \frac{1}{3}\,h_i^k B\,, \qquad B^{ik} = \frac{1}{3}\,h^{ik}B\,. \tag{9.3}$$

Next, we introduce the sub-invariant

$$\Gamma = \left(B_j^l - \frac{1}{3}h_j^l B\right)\left(B_l^j - \frac{1}{3}h_l^j B\right) = B_j^l B_l^j - \frac{1}{3}B^2 . \qquad (9.4)$$

In locally Cartesian coordinates the sub-invariant is equal to the sum of the squares of the quantities $B_l^j - \frac{1}{3}h_l^j B$. So the sub-invariant is zero under the isotropy conditions (9.3), and it is positive if conditions (9.3) are violated. For these reasons, we can identify the introduced sub-invariant Γ as the *quantity of the spatial anisotropy*.

It follows from what has been said that, in particular, the chr. inv.-invariant $\Pi = D_j^l D_l^j - \frac{1}{3}D^2$ (7.8) has a nonnegative numerical value, and the chr.inv.-invariant characterizes the *anisotropy of deformations* of the volume element (for instance, compare this result with [58], p. 612).

Let us consider the case of

$$B_{ik} = \beta_i \beta_k \,, \qquad (9.5)$$

where β_j is a sub-vector. Then

$$B = \beta_j \beta^j, \qquad \Gamma = \frac{2}{3}\left(\beta_j \beta^j\right)^2 = \frac{2}{3}B^2 \qquad (9.6)$$

and so the spatial isotropy manifests if and only if the sub-vector β_j is zero. From this, in particular, we can see that the chr.inv.-tensor $\Omega_i \Omega^k - \frac{1}{3}h_i^k \Omega_j \Omega^j$ becomes zero only with the chr.inv.-vector Ω_j.

It is possible to say that the chr.inv.-invariants $\Pi = D_j^l D_l^j - \frac{1}{3}D^2$ and $\Omega_j \Omega^j$, the chr.inv.-vector Ω_i, and also the chr.inv.-tensors $D_i^k - \frac{1}{3}h_i^k D$ and $\Omega_i \Omega^k - \frac{1}{3}h_i^k \Omega_j \Omega^j$ characterize the *mechanical anisotropy* of the volume element (its kinematic anisotropy and its dynamic anisotropy), while the chr.inv.-tensor $Z_i^k - \frac{1}{3}h_i^k Z$ characterizes its *geometrical anisotropy*. We shall consider the chr.inv.-vector Ω_i first.

§4.10 Absolute rotations

In §3.9 we saw that in relativistic equations of dynamics Ω_i plays a part akin to momentary angular velocity of the absolute rotations of the reference frame in classic equations of dynamics. For this reason we can call the chr.inv.-vectors Ω_i and Ω^k the *chr.inv.-vectors of the angular velocities of the dynamic absolute rotation*.

It is evident that they are the angular velocities, which can be found from experiments in mechanics. If we consider instead of the Metagalaxy, the regular test-bodies in an Earth-based laboratory, then the velocity can be found from a Foucault pendulum experiment, for instance. It is known (for instance, see [7], p. 183), that this velocity is different from the angular velocity of the kinematic "absolute" rotation by a quantity, which is, generally speaking, a function of a point. It is evident that the angular velocity of the kinematic absolute rotation can be characterized (to within a chr.inv.-vector, the chr.inv.-derivative of which disappears) by the chr.inv.-vector $^*\omega$ in the left side of the equation $R^k(^*\omega) = {}^*r^k(\Omega)$ (7.18). Therefore it is possible to say that the equation (7.18) links the dynamic absolute rotations to the kinematic absolute rotations*. In particular, this equation implies that the difference between the chr.inv.-vectors of the angular velocities of the dynamic absolute rotation and the kinematic absolute rotation is an irrotational chr. inv.-vector.

Now let us consider the angular velocity of the dynamic absolute rotation. We will use the condition $F_i = 0$ (7.14). Throughout the four-dimensional volume, where this condition is true, we have:

1. There is
$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right); \tag{10.1}$$

2. Choice of the time coordinate in such way as to make equality (8.2) true throughout the said volume. So, because of (8.2), (8.8), and (10.1), we obtain
$$\frac{\partial A_{ik}}{\partial t} = 0 \tag{10.2}$$
and hence
$$\frac{^*\partial A_{ik}}{\partial t} = 0\,. \tag{10.3}$$

The latter equality and also (10.2), equivalent to it, because of their chr.inv.-tensor nature, are true for any choice of the time coordinate.

*We can see more clearly the geometrical sense of equation (7.18), which links the dynamic absolute rotations to the kinematic absolute rotations, from its "detailed" form $^*\nabla_j\,(h^{kj}D - D^{kj}) = {}^*\nabla_j A^{kj}$ (7.19). This detailed form can be obtained after substituting $R^i(^*\omega) = {}^*\nabla_j(h^{ij}D - D^{ij})$ and $^*r^k(\Omega) = {}^*\nabla_j A^{kj}$ into (7.18). Zelmanov uses $R^k(^*\omega)$ here as an alternative to denote the chr.inv.-rotor $^*r^k(^*\omega)$ of the chr.inv.-vector $^*\omega$ (see formula 17.15 of §2.17 for the details). — Editor's comment. D. R.

Because of (9.1) of §3.9, we have

$$\Omega^i = \frac{1}{2}\,\varepsilon^{ijk}A_{jk}\,. \tag{10.4}$$

Because

$$\varepsilon^{ijk} = 0\,, \tag{10.5}$$

or, in other word,

$$\varepsilon^{ijk} = \pm\frac{1}{\sqrt{h}}\,, \tag{10.6}$$

we obtain

$$\frac{^*\partial\varepsilon^{ijk}}{\partial t} = 0\,, \tag{10.7}$$

or, in the alternative form

$$\frac{^*\partial\varepsilon^{ijk}}{\partial t} = \mp\frac{1}{h}\frac{^*\partial\sqrt{h}}{\partial t} = -D\left(\pm\frac{1}{\sqrt{h}}\right). \tag{10.8}$$

Hence, in general, we have

$$\frac{^*\partial\varepsilon^{ijk}}{\partial t} = -\varepsilon^{ijk}D\,. \tag{10.9}$$

Therefore, formula (10.4), taking (10.3) into account, gives

$$\frac{^*\partial\Omega^i}{\partial t} = -\Omega^i D\,. \tag{10.10}$$

Because of (10.10), we obtain

$$\frac{^*\partial}{\partial t}\big(\sqrt{h}\,\Omega^i\big) = \Omega^i\frac{^*\partial\sqrt{h}}{\partial t} + \sqrt{h}\,\frac{^*\partial\Omega^i}{\partial t} = \sqrt{h}\,\Omega^i D - \sqrt{h}\,\Omega^i D\,. \tag{10.11}$$

So, we arrive at

$$\frac{^*\partial}{\partial t}\big(\sqrt{h}\,\Omega^i\big) = 0\,, \tag{10.12}$$

and, because of (13.2) of §3.13,

$$\frac{^*\partial}{\partial t}\big(V\Omega^i\big) = 0\,, \tag{10.13}$$

where V is the volume of the element of the accompanying space. Formula (10.13) implies that, in every element of the accompanying space, the chr.inv.-vector Ω^i retains its direction and the non-zero component of the vector is inversely proportionally to the volume of the element.

Because of (10.12), we can write

$$\Omega^i = \frac{\xi^i}{\sqrt{h}}\,, \qquad \xi^i \not\parallel t\,. \tag{10.14}$$

Because of (10.10), the square of the chr.inv.-vector Ω^i is

$$\frac{{}^*\partial}{\partial t}\left(h_{jl}\,\Omega^j\Omega^l\right) = 2\left(D_{jl}\,\Omega^j\Omega^l - Dh_{jl}\,\Omega^j\Omega^l\right). \tag{10.15}$$

Let us take coordinate axes at the given point in such a way that at a given moment of time we have

$$\Omega^2 = \Omega^3 = 0\,. \tag{10.16}$$

Because of (10.10), this equality will hold for all moments of time. Hence, at this point, we have

$$D_{jl}\,\Omega^j\Omega^l = h_{11}\left(\Omega^1\right)^2, \tag{10.17}$$

$$\frac{{}^*\partial}{\partial t}\Big[h_{11}\big(\Omega^1\big)^2\Big] = 2\Big[D_{11}\big(\Omega^1\big)^2 - Dh_{11}\big(\Omega^1\big)^2\Big] = 2\left(\frac{D_{11}}{h_{11}} - D\right)h_{11}\big(\Omega^1\big)^2. \tag{10.18}$$

We direct the axes x^2 and x^3 in those directions so that, at the moment of time we are considering, the axes are orthogonal to x^1 at this point.

Then, at that moment of time, we have

$$h_{ik} = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & h_{23} \\ 0 & h_{32} & h_{33} \end{pmatrix} \tag{10.19}$$

and hence

$$D_{11} = \left(h_{11}\right)^2 D^{11}, \tag{10.20}$$

$$h_{11} = \frac{1}{h^{11}}\,, \tag{10.21}$$

so that

$$\frac{D_{11}}{h_{11}} = \frac{D^{11}}{h^{11}}\,. \tag{10.22}$$

We can re-write (10.18) in the form

$$\frac{1}{\sqrt{h_{11}\left(\Omega^1\right)^2}}\,\frac{{}^*\partial\sqrt{h_{11}\left(\Omega^1\right)^2}}{\partial t} = -\left(D - \frac{D^{11}}{h^{11}}\right) \tag{10.23}$$

and with §2.12 as a basis, interpret it as follows:

> At any point at any moment of time the velocity of relative changes of the ch.inv.-vector of the angular velocities Ω^k (or Ω_i) of the absolute dynamic rotation is equal in its modulus and converse in its sign to the velocity of relative changes of the square of the surface element, which is orthogonal to the direction of the vector Ω^k.

The equality (10.23) is an analogue of the known theorem of Classic Mechanics, according to which the strength of a vortex remains unchanged in time (for instance, see [58], p. 187).

§4.11 Mechanical anisotropy and geometrical anisotropy

In §4.10, we saw that the cosmological equation (7.14) provides a means to select one of the anisotropy factors for its studying, namely — the dynamic absolute rotation. At the same time, it is impossible to divorce the *deformation anisotropy* and the *curvature anisotropy*. This is true, in particular, because it is impossible to divorce quantities S_i^k and DD_i^k (see §3.24). Their relation to one another, and also their relation to the dynamic absolute rotation, give the cosmological equation (7.17). We shall now consider its consequences.

A. We assume that at a moment of time at a given point we have

$$D_i^k - \frac{1}{3}\,h_i^k D = 0\,, \tag{11.1}$$

then, generally speaking,

$$\frac{{}^*\partial}{\partial t}\Big(D_i^k - \frac{1}{3}\,h_i^k D\Big) \neq 0\,. \tag{11.2}$$

So, in contrast to the dynamical absolute rotation, which can not appear or disappear, the deformation anisotropy under the presence of the absolute rotation or the curvature anisotropy (or, under both factors) can disappear for a moment and then appear again.

B. If at the point we have

$$\Omega_j = 0\,, \tag{11.3}$$

$$Z_i^k - \frac{1}{3}\,h_i^k Z \equiv 0\,, \tag{11.4}$$

then at this point,

$$\frac{{}^*\partial}{\partial t}\Big(D_i^k - \frac{1}{3}\,h_i^k D\Big) + D\Big(D_i^k - \frac{1}{3}\,h_i^k D\Big) \equiv 0\,, \tag{11.5}$$

or, because of (12.9) of §2.12,

$$\frac{{}^*\partial}{\partial t}\left[V\Big(D_i^k-\frac{1}{3}\,h_i^k D\Big)\right]\equiv 0\,. \qquad (11.6)$$

Therefore, when the dynamical absolute rotation is absent and the curvature anisotropy remains unchanged, the deformation anisotropy decreases with expansion of the volume element, and it increases with contraction of the volume.

C. If at this point condition (11.3) and

$$D_i^k-\frac{1}{3}\,h_i^k D\equiv 0 \qquad (11.7)$$

are true, then (11.4) are true there as well. So, when the dynamical absolute rotation is absent and the deformation isotropy remains unchanged, the curvature isotropy manifests. In other words, mechanical isotropy leads to geometrical isotropy.

D. If at the point conditions (11.4) and (11.7) hold, then the condition (11.3) is true in the point. So, if the space deformation and the space curvature retain their isotropy unchanged, then the dynamic absolute rotation is absent.

§4.12 Isotropy and homogeneity

We suppose that conditions (11.3), (11.4), and (11.7) hold in a four-dimensional volume. Then we have

$$\Pi\equiv 0\,, \qquad (12.1)$$

$$A^{kj}\equiv 0\,, \qquad {}^*\nabla_j\, A^{kj}\equiv 0\,, \qquad (12.2)$$

so the cosmological equations (7.15), (7.16), and (7.19) take the forms, respectively

$$\frac{{}^*\partial D}{\partial t}+\frac{1}{3}\,D^2=-\frac{\kappa}{2}\,\rho c^2+\Lambda c^2, \qquad (12.3)$$

$$\frac{1}{3}\,D^2+c^2 Z=\kappa\rho c^2+\Lambda c^2, \qquad (12.4)$$

$$\frac{{}^*\partial D}{\partial x^i}=0\,, \qquad (12.5)$$

while the cosmological equation (7.17) becomes an identity. Because of the first of the equalities (12.2), we can introduce the following coordinates

$$\left.\begin{aligned} \tilde{x}^0 &= \tilde{x}^0(x^0, x^1, x^2, x^3) \\ \tilde{x}^i &= x^i\,, \qquad i = 1, 2, 3 \end{aligned}\right\} \tag{12.6}$$

so that throughout the aforementioned volume, we have

$$\tilde{v}_i \equiv 0\,, \tag{12.7}$$

and so

$$\widetilde{\Sigma}_{jk} \equiv 0\,. \tag{12.8}$$

In this case, in accordance with §2.22, we have

$$\widetilde{C}_i^k \equiv Z_i^k \tag{12.9}$$

and, because of (11.4),

$$\widetilde{C}_i^k - \frac{1}{3}\,\tilde{h}_i^k \widetilde{C} \equiv 0\,. \tag{12.10}$$

Employing Shur's theorem, we obtain

$$\frac{\partial \widetilde{C}}{\partial \tilde{x}^i} \equiv 0\,. \tag{12.11}$$

Hence, going over to arbitrary coordinates and taking (12.9) into account, throughout this volume we have

$$\frac{{}^*\partial Z}{\partial x^i} = 0\,. \tag{12.12}$$

Formula (12.4), because of (12.5) and (12.2), leads to

$$\frac{{}^*\partial \rho}{\partial x^i} = 0\,. \tag{12.13}$$

We have now obtained equations for a homogeneous model. We can transform the equations to the form (17.1) and (17.2) of §1.17, using the cosmic universal time coordinates (such time coordinates can be introduced, because of 7.14 and 11.3). Then we obtain

$$\frac{{}^*\partial}{\partial t} = \frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} \tag{12.14}$$

and hence (see 7.13)

$$\frac{\partial D}{\partial t}+\frac{1}{3}D^2=-\frac{\varkappa}{2}\rho c^2+\Lambda c^2, \tag{12.15}$$

$$\frac{1}{3}D^2+c^2C=\varkappa\rho c^2+\Lambda c^2, \tag{12.16}$$

$$\frac{\partial D}{\partial x^i}=0\,,\qquad \frac{\partial C}{\partial x^i}=0\,,\qquad \frac{\partial \rho}{\partial x^i}=0\,, \tag{12.7}$$

$$\frac{\partial \rho}{\partial t}+D\rho=0\,. \tag{12.18}$$

So, if the conditions (4.4–4.6), linked to our suppositions about the properties of matter, are true, then the isotropy of the finite four-dimensional volume implies that this volume is homogeneous.

§4.13 Stationarity in a finite volume

We assume that a substance throughout a finite four-dimensional volume, undergoes no deformations

$$D_{ik}\equiv 0\,, \tag{13.1}$$

so, as it is evidently, we have

$$h_{ik}\not\parallel t\,. \tag{13.2}$$

Because of (7.13) and (10.10), we obtain, respectively

$$\rho\not\parallel t\,,\qquad \Omega^k\not\parallel t\,. \tag{13.3}$$

Thus, this is a statistical case. The cosmological equations of gravitation in this case can be written as follows

$$\frac{\varkappa}{2}\rho c^2=2\Omega_j\Omega^j+\Lambda c^2, \tag{13.4}$$

$$3\Omega_j\Omega^j+c^2Z=\varkappa\rho c^2+\Lambda c^2, \tag{13.5}$$

$$2\Big(\Omega_i\Omega^k-\frac{1}{3}h_i^k\Omega_j\Omega^j\Big)+\Big(Z_i^k-\frac{1}{3}h_i^kZ\Big)=0\,, \tag{13.6}$$

$${}^*r^k\big(\Omega\big)=0\,. \tag{13.7}$$

We also assume that everywhere in the volume we are considering the condition

$$\Omega^k = 0 \tag{13.8}$$

is true. Then equation (13.7) becomes an identity, while equations (13.4), (13.5), and (13.6) take the forms, respectively

$$\frac{\kappa}{2}\,\rho = \Lambda\,, \tag{13.9}$$

$$Z = \kappa\rho + \Lambda\,, \tag{13.10}$$

$$Z_i^k - \frac{1}{3}\,h_i^k Z = 0\,. \tag{13.11}$$

Because of (13.8), we can introduce coordinates (12.6), where the conditions (12.7) are true. Then the conditions (12.8) and (12.9) will be true as well, because of (13.11), the equalities (12.10–12.13)*. Using cosmic universal time, we can express Z and Z_i^k in (13.10) and (13.11) through C and C_i^k. Inspecting equations (7.1) and (7.2) of §1.7 and comparing them with the foregoing, we see that the volume we are considering can be interpreted as the Einstein model.

As a result, if the conditions (4.4–4.6), linked to suppositions about properties of matter, are true, and the absolute dynamical rotation (13.8) is absent, then the condition of stability leads to the Einstein model. So, if non-empty static models, different from the Einstein model, can exist under the conditions (4.4–4.6), then they are models with dynamic absolute rotation.

§4.14 Transformations of the mean curvature of space

Before we consider transformations of the volume of any space element in time, we are going to deduce numerous consequences of the cosmological equations (7.13), (7.15), (7.16), here and in §4.15. Eliminating the cosmological constant from the last two equations, we can write

$$-\frac{^*\partial D}{\partial t} + c^2 Z - \frac{1}{2}\,\Pi + 3\Omega_j\Omega^j = \Pi - 2\Omega_j\Omega^j + \frac{3\kappa}{2}\,\rho c^2. \tag{14.1}$$

Differentiating (7.16) term-by-term, we obtain

$$\frac{2}{3}\,D\frac{^*\partial D}{\partial t} + \frac{^*\partial}{\partial t}\Big(c^2 Z - \frac{1}{2}\,\Pi + 3\Omega_j\Omega^j\Big) = \kappa c^2\,\frac{^*\partial\rho}{\partial t}\,. \tag{14.2}$$

As a result, multiplying (14.1) by $\frac{2}{3}\,D$ term-by-term and sum-

*The last two of the equalities are direct consequences of (13.9) and (13.10).

ming with (14.2), after taking (7.13) into account, we obtain

$$\left(\frac{{}^*\partial}{\partial t}+\frac{2}{3}D\right)\left(c^2Z-\frac{1}{2}\,\Pi+3\Omega_j\Omega^j\right)=\frac{2}{3}D\left(\Pi-2\Omega_j\Omega^j\right). \tag{14.3}$$

If, at a given point, we have

$$\Pi-2\Omega_j\Omega^j\equiv 0\,, \tag{14.4}$$

then it is evident that at this point we obtain

$$\left(\frac{{}^*\partial}{\partial t}+\frac{2}{3}D\right)\left(c^2Z-\frac{1}{2}\,\Pi+3\Omega_j\Omega^j\right)=0\,, \tag{14.5}$$

$$\frac{{}^*\partial}{\partial t}\left[\sqrt[3]{h}\left(c^2Z-\frac{1}{2}\,\Pi+3\Omega_j\Omega^j\right)\right]=0\,, \tag{14.6}$$

or, after the primary time coordinate has been used (see §4.8),

$$\frac{\partial}{\partial t}\left[\sqrt[3]{h}\left(c^2C-\frac{1}{2}\,\Pi+3\Omega_j\Omega^j\right)\right]=0\,. \tag{14.7}$$

In the case where (14.4) is a consequence of the equalities

$$\Pi\equiv 0\,,\qquad \Omega_j\Omega^j\equiv 0\,, \tag{14.8}$$

which imply that the space is isotropic at this point (see §3.11), equations (14.6) and (14.7) take the forms, respectively

$$\frac{{}^*\partial}{\partial t}\left(\sqrt[3]{h}\,Z\right)=0\,, \tag{14.9}$$

$$\frac{\partial}{\partial t}\left(\sqrt[3]{h}\,C\right)=0\,. \tag{14.10}$$

Thus we see that the equalities (14.5) and (14.6) are generalizations of the equalities (17.4) and (17.6) of §1.17. So (17.4) and (17.6) can be realized at any given point, if the space is isotropic at this point and the isotropy remains unchanged.

§4.15 When the mass of an element and its energy remain unchanged

Formula (7.13) leads to

$$\frac{{}^*\partial}{\partial t}\left(\sqrt{h}\,\rho\right)=0 \tag{15.1}$$

and, because of (12.7) and (12.8) of §2.12, we obtain

$$\frac{{}^*\partial M}{\partial t} = 0\,, \tag{15.2}$$

where M is the mass of a volume element of the accompanying space, or equivalently, the mass of an element of the substance in the space.

Because this space accompanies the substance (in other words, this is the accompanying space (see formula 6.2 of §3.6), we have

$$\frac{{}^*dM}{dt} = \frac{{}^*\partial M}{\partial t}\,, \tag{15.3}$$

so that

$$\frac{{}^*dM}{dt} = 0\,. \tag{15.4}$$

Because the energy of the element is

$$E = Mc^2, \tag{15.5}$$

we have

$$\frac{{}^*dE}{dt} = 0\,. \tag{15.6}$$

Instead of the equality (7.13), which is a particular case of (12.9) in §3.12, we can use the equality

$$\left(\frac{{}^*dE}{dt}\right)_{\text{fix}} = 0\,, \tag{15.7}$$

which is a particular case of (13.18) in §3.13*. Because we consider all in the accompanying space, we shall drop the suffix "fix".

Formulae (15.4) and (15.6) lead to the equalities

$$\frac{dM}{dt} = 0\,, \tag{15.8}$$

$$\frac{dE}{dt} = 0 \tag{15.9}$$

for any choice of the time coordinate, so the mass and energy of the element of the substance remain unchanged.

Finally, we can write the density of the element at this point, because of (13.2) and (13.3) of §3.13, in the form

$$\rho = \frac{\mu}{\sqrt{h}}\,, \qquad \mu \not\parallel t\,. \tag{15.10}$$

*Formula (13.18) is a consequence of (12.9) in §3.12 in the same Chapter 3.

§4.16 Criteria of the extreme volumes

Looking at formula (12.7) of §2.12 we can see that the value of the space volume element is different from $\sqrt{h}$ by a multiplier, which is independent of the time coordinate. For this reason, in studying the evolution of the value of a volume element, it would be sufficient to consider changes of $\sqrt{h}$ in time.

First, let us consider criteria for the extreme states* of the volume of any given element during its transformations in time. Having chosen spatial coordinates so that the condition

$$\sqrt{h} \not\equiv 0 \tag{16.1}$$

is true at the point we are considering, we can write criteria for the extrema as follows: the necessary criteria of the minimum

$$\frac{\partial\sqrt{h}}{\partial t} = 0\,, \qquad \frac{\partial^2\sqrt{h}}{\partial t^2} \geqslant 0\,, \tag{16.2}$$

the sufficient criteria of the minimum

$$\frac{\partial\sqrt{h}}{\partial t} = 0\,, \qquad \frac{\partial^2\sqrt{h}}{\partial t^2} > 0\,, \tag{16.3}$$

the necessary criteria of the maximum

$$\frac{\partial\sqrt{h}}{\partial t} = 0\,, \qquad \frac{\partial^2\sqrt{h}}{\partial t^2} \leqslant 0\,, \tag{16.4}$$

the sufficient criteria of the maximum

$$\frac{\partial\sqrt{h}}{\partial t} = 0\,, \qquad \frac{\partial^2\sqrt{h}}{\partial t^2} < 0\,. \tag{16.5}$$

On the one hand, we have

$$\begin{gathered}\frac{{}^*\partial\sqrt{h}}{\partial t} = \frac{c^2}{c^2-\mathrm{w}}\frac{\partial\sqrt{h}}{\partial t}\,,\\ \frac{{}^*\partial^2\sqrt{h}}{\partial t^2} = \left(\frac{c^2}{c^2-\mathrm{w}}\right)^2\left(\frac{c^2}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial t}\frac{\partial\sqrt{h}}{\partial t} + \frac{\partial^2\sqrt{h}}{\partial t^2}\right),\end{gathered} \tag{16.6}$$

and on the other hand

$$\frac{{}^*\partial\sqrt{h}}{\partial t} = \sqrt{h}\,D\,, \qquad \frac{{}^*\partial^2\sqrt{h}}{\partial t^2} = \sqrt{h}\left(\frac{{}^*\partial D}{\partial t} + D^2\right), \tag{16.7}$$

*Here and in what follows, where we do not say the inverse, we mean regular extrema.

so we obtain

$$\begin{gathered}\frac{\partial\sqrt{h}}{\partial t}=\left(1-\frac{\mathrm{w}}{c^2}\right)\sqrt{h}\,D\,,\\ \frac{\partial^2\sqrt{h}}{\partial t^2}=\sqrt{h}\left[\left(1-\frac{\mathrm{w}}{c^2}\right)^2\left(\frac{{}^*\partial D}{\partial t}+D^2\right)-D\,\frac{\partial\mathrm{w}}{\partial t}\right].\end{gathered}\tag{16.8}$$

Limiting our tasks by the extreme states of the *finite volume* of the element (hence, its density is finite as well), we can see that the conditions (16.2–16.5) are equivalent to the conditions, respectively

$$D=0\,,\qquad \frac{{}^*\partial D}{\partial t}\geqslant 0\,,\tag{16.9}$$

$$D=0\,,\qquad \frac{{}^*\partial D}{\partial t}>0\,,\tag{16.10}$$

$$D=0\,,\qquad \frac{{}^*\partial D}{\partial t}\leqslant 0\,,\tag{16.11}$$

$$D=0\,,\qquad \frac{{}^*\partial D}{\partial t}<0\,.\tag{16.12}$$

The criteria, and hence the "maxima" and "minima" are independent of our choice of time coordinate. Note that the criterion for non-accelerated transformations of the volume (this is the necessary criterion for the inflection point of the function $\sqrt{h}$) is

$$\frac{\partial^2\sqrt{h}}{\partial t^2}=0\tag{16.13}$$

or, in the alternative form

$$\frac{{}^*\partial D}{\partial t}+D^2-\left(\frac{c^2}{c^2-\mathrm{w}}\right)^2 D\,\frac{\partial\mathrm{w}}{\partial t}=0\,,\tag{16.14}$$

so the accelerated and non-accelerated transformations are dependent on our choice of time coordinate. Taking a time coordinate so that

$$\mathrm{w}=0\,,\tag{16.15}$$

we transform (16.14) into the chr.inv.-condition

$$\frac{{}^*\partial D}{\partial t}+D^2=0\,,\tag{16.16}$$

where, as well as for the extrema, we suppose $\sqrt{h}$ finite.

§4.17 The evolution of the volume of an element

Considering the cosmological equations, we see that the function $\sqrt{h}$ of t is clearly linked to only ρ, Π, $\Omega_j\Omega^j$, and Z by equations (7.13), (7.15), and (7.16). In this section and we shall find the limitations the equations impose on transformations of the volume element when the spatial coordinates remain unchanged and the time coordinate changes arbitrarily. In this study, we consider the evolution not of the quantity $\sqrt{h}$, but the quantity*

$$\eta = \sqrt[6]{h}\,. \tag{17.1}$$

Denoting chr.inv.-differentiation with respect to time by an asterisk, we can write

$$D = \frac{{}^*\partial \ln\sqrt{h}}{\partial t} = 3\,\frac{\overset{*}{\eta}}{\eta}\,, \tag{17.2}$$

$$\frac{{}^*\partial D}{\partial t} = 3\left(\frac{\overset{**}{\eta}}{\eta} - \frac{\overset{*}{\eta}{}^2}{\eta^2}\right). \tag{17.3}$$

It is not difficult to deduce, with $\sqrt{h}$ (and, hence η) finite, that the minimum or the maximum of one of the quantities leads to the minimum or the maximum of the other. So conditions (16.9–16.12) are equivalent to, respectively

$$\overset{*}{\eta} = 0\,, \qquad \overset{**}{\eta} \geqslant 0\,, \tag{17.4}$$

$$\overset{*}{\eta} = 0\,, \qquad \overset{**}{\eta} > 0\,, \tag{17.5}$$

$$\overset{*}{\eta} = 0\,, \qquad \overset{**}{\eta} \leqslant 0\,, \tag{17.6}$$

$$\overset{*}{\eta} = 0\,, \qquad \overset{**}{\eta} < 0\,. \tag{17.7}$$

Let us introduce the auxiliary quantities τ and ξ (we will refer to the latter as the *criterion of the curvature*) so that

$$\Pi - 2\Omega_j\Omega^j = \frac{3}{2}\frac{\tau}{\eta}\,, \tag{17.8}$$

$$c^2 Z - \frac{1}{2}\,\Pi + 3\Omega_j\Omega^j = 3\,\frac{\xi}{\eta^2}\,. \tag{17.9}$$

*Compare it with R in the regular equations for homogeneous models.

Then equations (7.13), (7.15), (7.16) take the forms, respectively*

$$\overset{*}{\rho} + 3\frac{\overset{*}{\eta}}{\eta}\rho = 0\,, \tag{17.10}$$

$$3\frac{\overset{**}{\eta}}{c^2\eta} + \frac{3}{2c^2}\frac{\tau}{\eta} = -\frac{\kappa}{2}\rho + \Lambda\,, \tag{17.11}$$

$$3\frac{\overset{*}{\eta}{}^2}{c^2\eta^2} + 3\frac{\xi}{c^2\eta^2} = \kappa\rho + \Lambda\,, \tag{17.12}$$

and the equation (14.3), after its multiplication by η^2 term-by-term, takes the form

$$\overset{*}{\xi} = \tau\overset{*}{\eta}\,. \tag{17.13}$$

Any of the four equations (17.10–17.13) can be obtained as a consequence of the other three. So equation (17.12), because of equations (17.10) and (17.13), is the first integral of equation (17.11). Therefore we can, in particular, take (17.10), (17.12), (17.13) as the initial equations. Instead of the latter, we can also take

$$\xi = \int \tau\dot{\eta}\,dt\,, \tag{17.14}$$

where the dot implies regular differentiation with respect to an arbitrary time coordinate. Eliminating the curvature criterion, we arrive at two cosmological equations, namely

$$\left.\begin{aligned} &\overset{*}{\rho} + 3\frac{\overset{*}{\eta}}{\eta}\rho = 0 \\ &3\frac{\overset{*}{\eta}{}^2}{c^2\eta^2} + \frac{3}{c^2\eta^2}\int \tau\dot{\eta}\,dt = \kappa\rho + \Lambda \end{aligned}\right\}\,, \tag{17.15}$$

which are analogous to (6.1) of §1.6. Next, eliminating ρ, we obtain one equation with respect to η (see 15.10)

$$3\frac{\overset{*}{\eta}{}^2}{c^2\eta^2} + \frac{3}{c^2\eta^2}\int \tau\dot{\eta}\,dt = \frac{\kappa\mu}{\eta^3} + \Lambda\,, \tag{17.16}$$

which leads to

$$\rho = \frac{\mu}{\eta^3}\,, \qquad \mu \not\parallel t\,. \tag{17.17}$$

*Compare the equations with equations (4.2), (5.1), (5.4) for homogeneous models (see §1.4 and §1.5).

To solve this equation we need to know τ as a function of t or η. At the same time, presupposing something about the quantities τ and ξ, we can obtain something about the evolution of η. This will be one of our tasks. Moreover, in that study we will suppose that η does not remain fixed during the finite time interval we are considering.

Let us split the interval of changes of the time coordinate t (which can be finite or infinite), where the essential positive function

$$\eta = \eta(t) \tag{17.18}$$

is defined, into the minimum number of time intervals where this function is monotonic. Then at one of the boundaries of each the intervals (the intervals can be finite or infinite) we have the minimum numerical value of the function, and at the other boundary, its maximum numerical value. We suppose this function continuous everywhere and its derivative continuous under any non-zero numerical value of the function (under any finite numerical value of the density, in other words). Then 6 kinds of the minimum numerical values of the function $\eta = \eta(t)$ can result. We label the kinds a, m, p, q, r, s. The maximum numerical values can be of 3 kinds, and we call them A, D, M. In this terminology, we specify:

- a — the non-zero asymptotic numerical value which the function approaches from above (under time coordinate changes in its positive or negative direction);
- m — the non-zero minimum of the function;
- p — its zero asymptotic value;
- q — zero value of the function when its derivative is zero;
- r — zero value of the function when its derivative is non-zero and finite;
- s — zero value of the function when its derivative becomes infinite;
- A — the finite asymptotic value the function approaches from below (under time coordinate changes in its positive or negative direction);
- D — the infinite value of the function;
- M — the finite maximum of the function.

In accordance with this terminology, we will refer to states of the volume element as follows. States of finite density:

D — the limit state of the infinite rarefaction;
M — the state of the maximum volume;
A — a state, where the volume remains unchanged;
a — a state, where the volume remains unchanged;
m — the state of the minimum volume.

States of infinite density (p, q, r, and s):

p — the asymptotic state of infinite density;
q — the minimum state of infinite density;
r — the collapsed state of infinite density;
s — the special state of infinite density;

All the states can appear in the theory of non-empty homogeneous models (see §1.7) except the states p, q, and r.

§4.18 Changes of the criterion of the curvature

We fix the spatial coordinates, so we consider x^i, τ, and η as functions of only the time coordinate. At the same time, the time coordinate is linked to the function η by a mutually-unique relation in every interval, where this function undergoes monotonic changes. Therefore, in each of the aforementioned intervals, we can consider ξ and τ as functions of η. Inspecting (17.13), we see that

$$\frac{\partial \xi}{\partial \eta} = \tau \,, \qquad \xi = \int \tau d\eta \,. \tag{18.1}$$

We use the equations

$$\frac{3}{c^2}\overset{**}{\eta} + \frac{3}{2c^2}\,\tau = -\frac{\kappa}{2}\frac{\mu}{\eta^2} + \Lambda\eta \,, \tag{18.2}$$

$$\frac{3}{c^2}\overset{*}{\eta}{}^2 + \frac{3}{c^2}\,\xi = \kappa\frac{\mu}{\eta} + \Lambda\eta^2, \tag{18.3}$$

which have the same power (for finite numerical values of η) as those equations which can be deduced by eliminating ρ from (17.10–17.12). Because of (18.1), equation (18.3) is the first integral of equation (18.2), so we could limit our tasks by considering only equation (18.3). However some consequences could be obtained more easily from (18.2), so we also consider this equation.

The previous section described some limitations on the function η and its derivative. Taking the limitations, and because of (17.12), (18.3), and since $\mathrm{w} < c^2$ is continuous, we deduce that, for $\eta \neq 0$, the quantity ξ is a continuous function of t. As a result, the curvature criterion is a continuous function of η in any finite interval of η, where a zero-point is absent. So we will consider those intervals where η is a monotonic function of t. Let us consider different cases of this function.

Case I:

$$\tau \equiv 0\,. \tag{18.4}$$

This case manifests, in particular, under the conditions

$$\Pi \equiv 0\,, \qquad \Omega_j \Omega^j \equiv 0\,, \tag{18.5}$$

i. e. when the mechanical isotropy, and hence the geometrical isotropy, remain unchanged at the given point we are considering (see §4.11). Because of (18.4), and taking (18.1) into account, we obtain

$$\xi = const \gtreqless 0\,. \tag{18.6}$$

So, in this case, we have $\tau \equiv 0$. In other words, three kinds of the curvature criterion are possible under conditions (18.5):

(1) the curvature criterion is positive;

(2) the curvature criterion equals zero;

(3) the curvature criterion is negative.

We introduce notation for the cases: I_1, I_2, I_3. In addition, we introduce

$$R = R\,(t) \tag{18.7}$$

so that, at the point we are considering, we have

$$\frac{\overset{*}{\eta}}{\eta} = \frac{\overset{*}{R}}{R}\,, \qquad \frac{\xi}{c^2\eta^2} = \frac{k}{R^2}\,, \qquad k = 0; \pm 1\,. \tag{18.8}$$

Besides these, choosing time coordinates so that

$$\mathrm{w} \equiv 0 \tag{18.9}$$

at this point, we bring the cosmological equations (7.10–7.12) for this volume element into their known form in the theory of a homogeneous universe.

Case II:

$$\tau \geqslant 0\,. \tag{18.10}$$

This case manifests, in particular, under the conditions

$$\Pi \not\equiv 0\,, \qquad \Omega_j \Omega^j = 0\,, \tag{18.11}$$

i. e. under anisotropic (generally speaking) deformations in the absence of the dynamical absolute rotation at the point we are considering.

Because of (18.10), and taking (18.1) into account, we conclude that the curvature criterion does not decrease if η increases, and also that the curvature criterion does not increase if η decreases. It is evident that the following cases are possible:

(1) the curvature criterion is positive for the minimum numerical value of η, so the curvature criterion is positive for all numerical values of η in the interval we are considering;

(2) the curvature criterion equals zero at one of the boundaries, or inside the interval. In particular, when η transits from smaller numerical values to the larger ones, the negative curvature criterion can become positive;

(3) the curvature criterion is negative for the maximum numerical value of η, and hence the curvature criterion is also negative for all numerical values of η in the interval.

We will refer to the cases as II_1, II_2, and II_3.

Case III (the general case):

$$\tau \gtreqless 0\,. \tag{18.12}$$

We also have three different cases here. So we will refer to the cases as III_1, III_2, III_3:

(1) the curvature criterion is positive throughout the interval of η, the boundaries included;

(2) the curvature criterion takes numerical values, which can be zero. In particular, the curvature criterion can change its sign;

(3) the curvature criterion remains negative throughout the interval of numerical values of η, the boundaries included.

§4.19 The states of the infinite density

When a volume element we are considering approaches the asymptotic state of infinite density, we have

$$\eta \rightarrow \eta_p = 0\,, \qquad \overset{*}{\eta} \rightarrow \overset{*}{\eta}_p = 0\,, \qquad \overset{**}{\eta} \rightarrow \overset{**}{\eta}_p = 0\,. \tag{19.1}$$

When the volume element approaches the minimum state of infinite density, we have

$$\eta \to \eta_q = 0\,, \qquad \overset{*}{\eta} \to \overset{*}{\eta}_q = 0\,, \qquad \overset{**}{\eta} \to \overset{**}{\eta}_q \geqslant 0\,. \tag{19.2}$$

In both cases, because of (18.2) and (18.3), we obtain

$$\tau \to -\infty\,, \tag{19.3}$$

$$\xi \to +\infty\,. \tag{19.4}$$

It is easy to see that the formula (19.4) leads to (19.3), because the curvature criterion is finite and continuous when $\eta \neq 0$.

When the element approaches the collapsed state of infinite density, we have

$$\eta \to \eta_r = 0\,, \qquad \overset{*}{\eta} \to \overset{*}{\eta}_r \neq 0 \tag{19.5}$$

and (18.3) leads to (19.4) and, hence (19.3). From the foregoing we conclude that: in the cases I_1, I_2, I_3, II_1, II_2, II_3, III_3 the asymptotic, minimum, and collapsed states of the infinite density are impossible; in the cases III_1 and III_2 the aforementioned states are possible only under the specific evolution of the curvature criterion. On the possibility of states which are not prohibited by the foregoing results, see §4.25.

When the volume element approaches the special state of infinite density, we have

$$\eta \to 0\,, \qquad \overset{*}{\eta} \to \pm\infty\,. \tag{19.6}$$

It is evident that under the specific sequence by which $\overset{*}{\eta}$ approaches infinity, the special states of infinite density are conceivable in all the cases of §4.18 (see §4.25 for the details).

§4.20 The ultimate states of the infinite rarefaction

We assume that

$$\eta \to \infty\,. \tag{20.1}$$

Let us consider those cases where the cosmological constant is positive, zero, and negative.

If the cosmological constant is positive, then (18.2) and (18.3) under the condition (20.1) give, respectively

$$\left(\overset{**}{\eta} + \frac{1}{2}\,\tau\right) \to +\infty\,, \tag{20.2}$$

$$\left(\overset{*}{\eta}^2 + \xi\right) \to +\infty\,. \tag{20.2}$$

Hence, under the specific changes of $\overset{*}{\eta}$ and $\overset{**}{\eta}$, the element can approach the ultimate state of infinite rarefaction in all 9 cases we have considered in §4.18 (see also §4.25 and §4.26).

If the cosmological constant equals zero, then (18.2) and (18.3), under the condition (20.1), give

$$\left(\overset{**}{\eta} + \frac{1}{2}\,\tau\right) \to 0\,, \tag{20.4}$$

$$\left(\overset{*}{\eta}^2 + \xi\right) \to 0\,. \tag{20.5}$$

So the curvature criterion approaches a nonpositive numerical value, while the derivative of the curvature criterion, depending on the changes of $\overset{**}{\eta}$, can evolve in different ways. Hence, the element can not approach the ultimate state of infinite rarefaction in the cases I_1, II_1, III_1.

Finally, if the cosmological constant is negative, then (18.2) and (18.3), under the condition (20.1), respectively lead to

$$\left(\overset{**}{\eta} + \frac{1}{2}\,\tau\right) \to -\infty\,, \tag{20.6}$$

$$\left(\overset{*}{\eta}^2 + \xi\right) \to -\infty\,. \tag{20.7}$$

From (20.7) we obtain

$$\xi \to -\infty\,, \tag{20.8}$$

so τ cannot remain nonnegative. So the element cannot approach the ultimate state of infinite rarefaction in the cases I_1, I_2, I_3, II_1, II_2, II_3, III_1.

§4.21 The states of unchanged volume and of the minimum volume

When a volume element approaches a state of unchanged volume, we have

$$\overset{*}{\eta} \to 0\,, \qquad \overset{**}{\eta} \to 0\,, \tag{21.1}$$

so formulae (18.2) and (18.3) give, in their limits,

$$\frac{3}{2c^2}\,\tau = -\frac{\kappa\mu}{\eta^2} + \Lambda\eta\,, \tag{21.2}$$

$$\frac{3}{c^2}\,\xi = \frac{\kappa\mu}{\eta} + \Lambda\eta^2. \tag{21.3}$$

In the state of minimum volume, we have

$$\overset{*}{\eta} = 0\,, \qquad \overset{**}{\eta} \geqslant 0 \tag{21.4}$$

and hence (18.2) transforms into

$$\frac{3}{2c^2}\,\tau \leqslant -\frac{\kappa\mu}{\eta^2} + \Lambda\eta\,, \tag{21.5}$$

while (18.3) transforms into (21.3). So, both in an unchanged volume and in minimum volume we have

$$\frac{3}{2c^2}\,\tau < \Lambda\eta\,, \tag{21.6}$$

$$\frac{3}{c^2}\,\xi > \Lambda\eta^2. \tag{21.7}$$

Let us consider the cases where the cosmological constant is positive, zero, or negative, just as we did in §4.20.

With the cosmological constant positive, equation (21.7) implies that the transit through the state of minimum volume and also the asymptotic approach to the state of unchanged volume from above are impossible in the cases I_2, I_3, II_2, II_3, III_3, while the asymptotic approach from below is impossible in the cases I_2, I_3, II_3, III_3.

If the cosmological constant is zero, equations (21.6) and (21.7) imply that the transit through the state of minimum volume and also the asymptotic approach to the state of unchanged volume (from above or below) are impossible in the cases I_1, I_2, I_3, II_1, II_2, II_3, III_3.

Finally, if the cosmological constant is negative, equations (21.6) and (21.7) imply that the transit through the state of the minimum volume and also the asymptotic approach to the state of unchanged volume are impossible in the cases I_1, I_2, I_3, II_1, II_2, II_3.

§4.22 The states of the maximum volume

In the state of maximal volume, we have

$$\overset{*}{\eta} = 0\,, \qquad \overset{**}{\eta} \leqslant 0 \tag{22.1}$$

so (18.2) and (18.3) give

$$\frac{3}{2c^2}\,\tau \geqslant -\frac{\kappa\mu}{\eta^2} + \Lambda\eta\,, \tag{22.2}$$

$$\frac{3}{c^2}\,\xi = \frac{\kappa\mu}{\eta} + \Lambda\eta^2, \tag{22.3}$$

so that

$$\frac{3}{c^2}\,\tau \gtreqless \Lambda\eta\,, \tag{22.4}$$

$$\frac{3}{c^2}\,\xi > \Lambda\eta^2. \tag{22.5}$$

The result (22.5) implies that, with the cosmological constant nonpositive, the state of the maximum volume is impossible in the cases I_2, I_3, II_3, III_3. If the cosmological constant is strictly negative, then equations (18.2) and (18.3) permit the state of maximum volume in all 9 cases we have considered in §4.18 (see also §4.25 and §4.26).

§4.23 The transformations, limited from above and below

We will now consider cases where monotonic changes of η are limited from above and below.

The lower boundary is the finite asymptotic or the finite minimum numerical value η_1, while the upper boundary is the finite asymptotic or the maximum numerical value η_2. Then it is evident that

$$\eta_1 < \eta_2\,, \tag{23.1}$$

$$\overset{*}{\eta}_1 = 0 = \overset{*}{\eta}_2\,, \tag{23.2}$$

$$\overset{**}{\eta}_1 \geqslant 0 \geqslant \overset{**}{\eta}_2\,. \tag{23.3}$$

If the cosmological constant is nonnegative, then equation (18.2) gives

$$\tau_1 < \tau_2\,. \tag{23.4}$$

If the cosmological constant is nonpositive, then equation (18.3) gives

$$\xi_1 > \xi_2\,. \tag{23.5}$$

Because of (23.4) and (23.5), the kinds of changes of η we are considering are impossible in the cases I_1, I_2, I_3 for a strictly positive cosmological constant, and in the cases I_1, I_2, I_3, II_1, II_2, II_3 for a nonpositive cosmological constant.

§4.24 The area, where deformations of an element are real

In §4.19–§4.23 we ascertained numerous prohibitions and limitations on the evolution of the volume element, due to equations (18.1) and (18.3) or equivalently, by equations (7.13), (7.15), (7.16). So the problem we have stated in §4.17 has been solved. It is easy to see the prohibitions are generalizations of those prohibitions on the evolution of any element of a homogeneous universe, the cosmological equations resulted when the pressure becomes zero.

The prohibitions can also be found by considering an area of real deformations of the element in the plane η, ξ, the area, where

$$\eta \geqslant 0\,, \tag{24.1}$$

$$\overset{*}{\eta}{}^2 \geqslant 0\,. \tag{24.2}$$

Because of (18.3), equation (24.2) leads to

$$\xi \leqslant \frac{c^2}{3}\left(\frac{\kappa\mu}{\eta} + \Lambda\eta^2\right). \tag{24.3}$$

So the area of real deformations is bounded by the ordinate axis and the ultimate curve

$$\xi = \frac{c^2}{3}\left(\frac{\kappa\mu}{\eta} + \Lambda\eta^2\right), \qquad \eta \geqslant 0\,. \tag{24.4}$$

Let us consider this curve in detail. If the cosmological constant is positive, then the whole curve is located in the first quadrant, is convex everywhere with respect to abscissa axis, and it has a minimum when

$$\eta = \sqrt[3]{\frac{\kappa\mu}{2\Lambda}}\,. \tag{24.5}$$

If η approaches zero (the ordinate axis is the asymptote), then we have

$$\xi \to +\infty\,, \qquad \frac{\partial\xi}{\partial\eta} \to -\infty\,, \tag{24.6}$$

and if η approaches infinity, then

$$\xi \to +\infty\,, \qquad \frac{\partial\xi}{\partial\eta} \to +\infty\,. \tag{24.7}$$

If the cosmological constant is zero, then the whole ultimate curve lies in the first quadrant and is convex with respect to the abscissa axis. The curve is a branch of an equilateral hyperbola, the asymptotes of which are the coordinate axes, so that: (1) if η approaches zero, then conditions (24.6) are true, and (2) if η approaches infinity, then

$$\xi \to 0\,, \qquad \frac{\partial \xi}{\partial \eta} \to 0\,. \tag{24.8}$$

Finally, if the cosmological constant is negative, then the ultimate curve is monotonic (like the previous case) decreasing function. In this case the curve is convex with respect to the abscissa axis and intersects the axis at the point

$$\eta = -\sqrt[3]{\frac{\kappa\mu}{\Lambda}}\,, \tag{24.9}$$

which is, hence, the inflection point. If η approaches zero, we have, just as in the previous cases, the conditions (24.6), so the ordinate axis is the asymptote. If η approaches infinity, then

$$\xi \to -\infty\,, \qquad \frac{\partial \xi}{\partial \eta} \to -\infty\,. \tag{24.10}$$

In the case of the positive cosmological constant, the real states of infinite rarefaction are points at infinity on the straight line

$$\eta = +\infty\,. \tag{24.11}$$

In the case where the cosmological constant is zero, the states are points on the straight half-line

$$\eta = +\infty\,, \qquad \xi \leqslant 0\,. \tag{24.12}$$

Finally, in the case of the negative cosmological constant, the states are the points

$$\eta = +\infty\,, \qquad \xi = -\infty\,. \tag{24.13}$$

The special states of infinite density are all points on the ordinate axis. The asymptotic, minimum, and collapsed states of infinite density are the points

$$\eta = 0\,, \qquad \xi = +\infty \tag{24.14}$$

on this axis.

The states of unchanged volume, the minimum volume, and the maximum volume are points on the ultimate curve. The asymptotic approach to a state of unchanged volume is, it is easy to understand, the approach to the point of this state along a curve, a tangential line to which at this point is the same as the tangential line to the ultimate curve.

§4.25 The kinds of monotonic transformations of a volume

We will label each kind of evolution of the volume element by various letters, denoting the states the element evolves through (the states are boundaries of the monotonic change of the volume). In this terminology, the kinds of evolution we consider in the theory of a non-empty homogeneous universe will be denoted as follows:

A_1 — sA (expansion) or As (contraction);
A_2 — aD (expansion) or Da (contraction);
M_1 — sD (expansion) or Ds (contraction);
M_2 — DmD;
O_1 — sMs;
O_2 — $\dots mMmM \dots$

It is evident that the number of kinds of evolution of the element which are conceivable inside the interval of its monotonic transformations is 18, if the volume expands, and the same number if the volume contracts (6 kinds of minimum value and 3 kinds of the maximum value). Let us make a list of the kinds of evolution when the element expands (by changing the subscript letters we obtain the corresponding kinds of contraction).

1. Transformations of the volume, limited by its finite value only from below: aD, mD.
2. Transformations of the volume, limited by its finite value from below and above: aA, aM, mA, mM.
3. Transformations of the volume, limited by its finite value only from above: pA, pM, qA, qM, rA, rM, sA, sM.
4. Unlimited transformations: pD, qD, rD, sD.

The kinds aD, mD and the corresponding kinds of contraction are impossible in the cases (see §4.20 and §4.21)

$$\left.\begin{array}{llll} \Lambda > 0: & \mathrm{I}_2, \mathrm{I}_3; & \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_3 \\ \Lambda = 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; & \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_1, \mathrm{III}_3 \\ \Lambda < 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; & \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_1, \end{array}\right\} . \qquad (25.1)$$

The kinds *aA*, *aM*, *mA*, *mM* and the corresponding kinds of contraction are impossible in the cases (see §4.21, §4.22, §4.23)

$$\left.\begin{array}{lll}\Lambda > 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_3 \\ \Lambda = 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_3 \\ \Lambda < 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3 & \end{array}\right\} . \tag{25.2}$$

The kinds *pA*, *pM*, *qA*, *qM*, *rA*, *rM* and the corresponding kinds of contraction are impossible in the cases (see §4.19, §4.21, §4.22)

$$\Lambda \gtreqless 0: \quad \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3; \quad \mathrm{III}_3 \ . \tag{25.3}$$

The kind *sA* and the corresponding kind of contraction are impossible in the cases (see §4.19 and §4.21)

$$\left.\begin{array}{lll}\Lambda > 0: & \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_3; & \mathrm{III}_3 \\ \Lambda = 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_3 \\ \Lambda < 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3 & \end{array}\right\} . \tag{25.4}$$

The kind *sM* and the corresponding kind of contraction are impossible in the cases (see §4.19 and §4.22)

$$\Lambda \geqslant 0: \quad \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_3; \quad \mathrm{III}_3 \ . \tag{25.5}$$

The kinds *pD*, *qD*, *rD* and the corresponding kinds of contraction are impossible in the cases (see §4.19 and §4.20)

$$\left.\begin{array}{lll}\Lambda > 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_3 \\ \Lambda \leqslant 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; \quad \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_1, \mathrm{III}_3 \end{array}\right\} . \tag{25.6}$$

The kind *sD* and the corresponding kind of contraction are impossible in the cases (see §4.19 and §4.20)

$$\left.\begin{array}{llll}\Lambda = 0: & \mathrm{I}_1; & \mathrm{II}_1; & \mathrm{III}_1 \\ \Lambda < 0: & \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3; & \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3; & \mathrm{III}_1 \end{array}\right\} . \tag{25.7}$$

We have listed those cases for each type of evolution prohibited by the limitations we have obtained from equations (7.13), (7.15), (7.16) in §4.19–§4.23. Considering the ultimate curve in an area of real deformations (see §4.24), we see that each kind is possible (more exactly, permitted by equations 7.13, 7.15, and 7.16) in all cases where the type is not prohibited by the aforementioned limitations.

We give two tables in which each case is specified with the type of evolution (for brevity, we include only the kinds of expansion), which are permitted by equations (7.13), (7.15), (7.16) or equivalently, by equations (18.1) and (18.3). The meanings of the underlined and the bracketed terms are given in §4.27.

§4.26 The possibility of an arbitrary evolution of an element of volume

We have considered the limitations that equations (7.13), (7.15), (7.16) place upon the evolution of η, if the time coordinate changes when the spatial coordinates are fixed. Let us recall those considerations, whereby we conclude that the other cosmological equations do not have additional limitations on the evolution of η at the given point.

It is known [59] that we can always* introduce coordinates $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$, where the conditions

$$\frac{\partial\left(\tilde{g}^{\mu\nu}\sqrt{-\tilde{g}}\,\right)}{\partial\tilde{x}^{\nu}} = 0 \tag{26.1}$$

hold (the harmonic coordinates). The equations of gravitation in these coordinates can be written in the form

$$\frac{1}{2}\,\widetilde{\Box}\,\tilde{g}^{\mu\nu} - \widetilde{\Gamma}^{\mu}_{\alpha\beta}\,\widetilde{\Gamma}^{\nu}_{\varepsilon\zeta}\;\tilde{g}^{\alpha\varepsilon}\tilde{g}^{\beta\zeta} + \Lambda\,\tilde{g}^{\mu\nu} = \kappa\left(\widetilde{T}^{\mu\nu} - \frac{1}{2}\,\tilde{g}^{\mu\nu}\widetilde{T}\right), \tag{26.2}$$

where we denote

$$\Box \equiv g^{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}}\,, \tag{26.3}$$

and hence the equations of gravitation can be transformed into their regular form

$$\begin{aligned}\frac{\partial^2\tilde{g}^{\mu\nu}}{\partial\tilde{x}^3\partial\tilde{x}^3} = F^{\mu\nu}\Bigg(&\tilde{x}^0,\, \tilde{x}^1,\, \tilde{x}^2,\, \tilde{x}^3; \quad \tilde{g}^{00},\, \tilde{g}^{01},\, \ldots,\, \tilde{g}^{33};\\ &\frac{\partial\tilde{g}^{00}}{\partial\tilde{x}^0}, \frac{\partial\tilde{g}^{00}}{\partial\tilde{x}^1}, \ldots, \frac{\partial\tilde{g}^{33}}{\partial\tilde{x}^3}; \quad \frac{\partial^2\tilde{g}^{\mu\nu}}{\partial\tilde{x}^0\partial\tilde{x}^0}, \frac{\partial^2\tilde{g}^{\mu\nu}}{\partial\tilde{x}^0\partial\tilde{x}^1}, \ldots, \frac{\partial^2\tilde{g}^{\mu\nu}}{\partial\tilde{x}^3\partial\tilde{x}^1}\Bigg).\end{aligned} \tag{26.4}$$

*However, generally speaking, it is not possible in every reference frame.

	I_1	I_2	I_3
$\Lambda > 0$	$\underline{aD}\,(A_2)$, $mD\,[M_2]$ $\underline{sA}\,(A_1)$ $sM\,[O_1]$ $\underline{sD}\,(M_1)$	$\underline{sD}\,(M_1)$	$\underline{sD}\,(M_1)$
$\Lambda = 0$	$sM\,[O_1]$	$\underline{sD}\,(M_1)$	$\underline{sD}\,(M_1)$
$\Lambda < 0$	$sM\,[O_1]$	$sM\,[O_1]$	$sM\,[O_1]$

Table 4.1 Kinds of evolution of the volume element in Case I, i. e. for fixed mechanical isotropy and fixed geometrical isotropy.

	II_1	II_2	II_3
$\Lambda > 0$	$\underline{aD}\,(A_2)$, mD $\underline{aA}, aM, mA, mM$ $\underline{sA}\,(A_1)$ sM $\underline{sD}\,(M_1)$	$\underline{sA}\,(A_1)$ $sM\,[O_1]$ $\underline{sD}\,(M_1)$	$\underline{sD}\,(M_1)$
$\Lambda = 0$	$sM\,[O_1]$	$sM\,[O_1]$ $\underline{sD}\,(M_1)$	$\underline{sD}\,(M_1)$
$\Lambda < 0$	$sM\,[O_1]$	$sM\,[O_1]$	$sM\,[O_1]$

Table 4.2 Kinds of evolution of the volume element in Case II, i. e. for anisotropic deformations in the absence of dynamical absolute rotation.

	III_1	III_2	III_3
$\Lambda > 0$	$\underline{aD}\,(\mathrm{A}_2),\ mD$ $\underline{aA}, aM, mA, mM$ $\underline{pA}, pM, \underline{qA}, qM, \underline{rA}, rM$ $\underline{sA}\,(\mathrm{A}_1)$ sM $\underline{pD}, \underline{qD}, \underline{rD}$ $\underline{sD}\,(\mathrm{M}_1)$	$\underline{aD}\,(\mathrm{A}_2),\ mD$ $\underline{aA}, aM, mA, mM$ $\underline{pA}, pM, \underline{qA}, qM, \underline{rA}, rM$ $\underline{sA}\,(\mathrm{A}_1)$ sM $\underline{pD}, \underline{qD}, \underline{rD}$ $\underline{sD}\,(\mathrm{M}_1)$	$\underline{sD}\,(\mathrm{M}_1)$
$\Lambda = 0$	$\underline{aA}, aM, mA, mM$ $\underline{pA}, pM, \underline{qA}, qM, \underline{rA}, rM$ $\underline{sA}\,(\mathrm{A}_1)$ sM	$\underline{aD}\,(\mathrm{A}_2),\ mD$ $\underline{aA}, aM, mA, mM$ $\underline{pA}, pM, \underline{qA}, qM, \underline{rA}, rM$ $\underline{sA}\,(\mathrm{A}_1)$ sM $\underline{pD}, \underline{qD}, \underline{rD}$ $\underline{sD}\,(\mathrm{M}_1)$	$\underline{sD}\,(\mathrm{M}_1)$
$\Lambda < 0$	$\underline{aA}, aM, mA, mM$ $\underline{pA}, pM, \underline{qA}, qM, \underline{rA}, rM$ $\underline{sA}\,(\mathrm{A}_1)$ sM	$\underline{aD}\,(\mathrm{A}_2),\ mD$ $\underline{aA}, aM, mA, mM$ $\underline{pA}, pM, \underline{qA}, qM, \underline{rA}, rM$ $\underline{sA}\,(\mathrm{A}_1)$ sM $\underline{pD}, \underline{qD}, \underline{rD}$ $\underline{sD}\,(\mathrm{M}_1)$	$\underline{aD}\,(A_2), mD$ $\underline{aA}, aM, mA, mM$ $\underline{sA}\,(\mathrm{A}_1)$ sM $\underline{sD}\,(\mathrm{M}_1)$

Table 4.3 Kinds of evolution of the element in Case III (the general case).

In such a case* we can use the general theorem concerning integrals, which satisfy the given initial conditions for

$$\tilde{x}^3 = \tilde{a}^3 = const\,, \tag{26.5}$$

$$\tilde{g}^{\mu\nu} = \tilde{f}^{\mu\nu}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2)\,, \qquad \frac{\partial \tilde{g}^{\mu\nu}}{\partial \tilde{x}^3} = \tilde{f}_3^{\mu\nu}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2)\,. \tag{26.6}$$

All 20 functions on the right sides of equalities (26.6) must satisfy† only 4 of the harmonic conditions (26.1). Thus, 16 of the functions and, hence, no less than 6 of the 0 functions $\tilde{f}^{\mu\nu}$ can be given by our arbitrary choice.

Let us link the accompanying coordinate frame (x^0, x^1, x^2, x^3) we are considering to a harmonic coordinate frame by the general transformations

$$\left.\begin{aligned} x^0 &= x^0\,(\tilde{x}^0,\, \tilde{x}^1,\, \tilde{x}^2,\, \tilde{x}^3) \\ x^i &= x^i\,(\tilde{x}^0,\, \tilde{x}^1,\, \tilde{x}^2,\, \tilde{x}^3) \end{aligned}\right\}, \tag{26.7}$$

and we choose coordinates in the harmonic coordinate frame and a point in the accompanying coordinate frame

$$x^i = a^i = const^i, \tag{26.8}$$

so that this point would always be in the surface (26.5). Then at this point, we have

$$g^{\alpha\beta} = f^{\alpha\beta}(x^0)\,, \tag{26.9}$$

where we denote‡

$$f^{\alpha\beta} = \left(\tilde{f}^{\mu\nu}\,\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\,\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\right)_a. \tag{26.10}$$

Thus, 10 values of $f^{\alpha\beta}$ are linear functions of 10 values of $\left(\tilde{f}^{\mu\nu}\right)_a$, 6 of which can be given arbitrarily. For this reason, we can preassign the 6 values of $f^{\alpha\beta}$ instead of $\left(\tilde{f}^{\mu\nu}\right)_a$.

The foregoing implies that at a given point of the space, preassigning numerical values of w, v^1, v^2, v^3 as functions of the time

*We suppose that all the necessary requirements of continuity and differentiability are realized.

†Beside the regular requirements of continuity and that their continuous first derivatives must exist.

‡The index a here implies that we consider all the functions, which have been contained by the brackets, along the world-line of the point (26.8) of the accompanying space in the harmonic coordinate frame.

coordinate, we can also preassign no less than two values of h^{ik} or h_{ik} as functions of the time coordinate at this point. Alternatively, we can preassign two functions of the h^{ik} or h_{ik}, where we can choose for one of the functions the value of η. In this case we have:

- A meaning for the limitations the cosmological equations (7.13), (7.15), (7.16) place upon the evolution of the numerical value of η — setting of a relation between the evolution of η and the evolution of the curvature criterion;
- The other cosmological equations cannot give additional relations between η and ξ, because in the opposite case the quantities would be defined by no less than two independent equations, so it would be impossible to preassign $\eta(t)$ arbitrarily;
- When η, during its evolution, transits from one interval of its monotonic transformations into another, any type of the monotonic transformations can be replaced by any other, which is permitted for the given cosmological constant and the preassigned evolution of the curvature criterion.

The foregoing implies that, generally speaking, it is impossible to preassign η for all elements of a finite volume.

In conclusion, let us make a list of data, which can be obtained from the cosmological equations we considered as the equations for the given point.

Let us suppose that at the given point we preassign

$$\mathrm{w} = 0\,, \tag{26.11}$$

$$\frac{\partial \mathrm{w}}{\partial x^i} = 0\,, \tag{26.12}$$

$$v_i = 0\,, \tag{26.13}$$

numerical values of μ, ν^1, ν^2, ν^3, and also 5 of 6 numerical values of h^{ik} as functions of the time coordinate (in other words, we know 5 functions f^{ik} for the given point). Then we have:

- Equation (7.14) will be satisfied, because of (26.11) and (26.13);
- Equations (7.13) and (7.15) establish the 6th value* of all numerical values of h^{ik} and ρ;
- Equation (7.15) establishes the value of Z;

*So we see that to presuppose all 6 values of f^{ik} when (26.11) has been presupposed, is prohibited by the cosmological equations for the given point.

- Equation (7.17), because of (10.14), determines 5 values of all numerical values of $Z_i^k - \frac{1}{3} h_i^k Z$;
- Equation (7.19) links the evolution of the given element to the evolution of the neighbouring elements, if the relation between values of ν^i and the spatial coordinates we use is known.

§4.27 The kinds of evolution of the element

We are now going to consider the kinds of evolution of the volume element throughout the interval of changes of η, where its numerical value and the substance's density remain finite. It is evident that the limiting states in this case can be all kinds, aside of m and M, so only 10 of 18 kinds of the monotonic expansion (or contraction, respectively) are evolution types throughout the interval we are considering (the kinds are underlined in Table 4.1, Table 4.2, and Table 4.3). Examining the types, we see that 7 of the 10 kinds are impossible for a homogeneous universe (these are aA, pA, qA, rA, pD, qD, rD), while the other 3 kinds aA, sA, sD coincide, respectively, with kinds A_2, A_1, M_1, which are possible for a homogeneous universe (the kinds have been contained by parentheses in the Tables).

It is conceivable that cases can exist where the volume element can transit from the states m or M into only one of the various states of infinite density, unchanged volume, or infinite rarefaction (as a result of the monotonic transformation of the volume). In such cases, the kind of monotonic transformations of the volume element defines the kind of its evolution throughout the interval we have an interest in. For instance, if the volume element can transit from the state m into only the state D, then the presence of the kind mD implies the presence of the kind DmD, which coincides with kind M_2. If the element can transit from the state M into only the kind s, then the presence of the kind sM implies the presence of the kind sMs, which coincides with kind O_1, as shown in Table 4.1 and Table 4.2 by the terms in square brackets.

The foregoing considerations are sufficient for determining all the kinds of evolution:

- In the cases I_1, I_2, I_3, II_2, II_3 under $\Lambda \lesseqqgtr 0$;
- In the case II_1 under $\Lambda \leqslant 0$;
- In the case III_3 under $\Lambda \geqslant 0$.

To ascertain the kinds of evolution:

- In the case II_1 under $\Lambda > 0$,
- In the cases III_1 and III_2 under $\Lambda \gtreqless 0$,
- In the case III_3 under $\Lambda < 0$

we need to use our findings on the possibility of combining the kinds of monotonic transformations of the volume (see §4.26). This enables us to conclude that:

- In the cases II_1, $\Lambda > 0$ and III_3, $\Lambda < 0$, the volume element can transit from any of the states a, m, s into any of the states A, M, D and back;
- In the cases III_1, $\Lambda > 0$ and III_2, $\Lambda \gtreqless 0$, the volume element can transit from any of the states a, m, p, q, r, s into any of the states A, M, D and back;
- In the case III_1, $\Lambda \leqslant 0$, the volume element can transit from any of the states a, m, p, q, r, s into any of the states A, M and back.

Note that the kind $\dots mMmMmM\dots$ (i. e. the kind O_2) is possible in all cases.

§4.28 The rôles of absolute dynamic rotation and deformation anisotropy

To clarify the effects of dynamic absolute rotation and deformation anisotropy, let us consider the following cases: (1) dynamic absolute rotation is absent when the deformations are isotropic; (2) dynamic absolute rotation is absent when the deformations are anisotropic; (3) dynamic absolute rotation is present when the deformations are anisotropic.

The dynamic absolute rotation is absent when the deformations are isotropic

We assume that the mechanical isotropy and, hence, the geometrical isotropy remain unchanged at the given point we are considering. It easy to see from §4.11 that the necessary and sufficient conditions for this conservation can be written in the form

$$\Pi \equiv 0\,, \qquad \Omega_j \Omega^j = 0\,. \tag{28.1}$$

Then, on one hand, we have

$$\tau \equiv 0\,, \qquad \xi = const\,, \tag{28.2}$$

and on the other hand

$$c^2 Z = 3\,\frac{\xi}{\eta^2}\,. \tag{28.3}$$

In §4.14 we saw (it is also seen from formula 28.3) that, if the isotropy is at the point and remains unchanged, then the mean curvature at the point transforms in company with transformations of the volume of the element just as for a homogeneous universe. Considering formula (28.2), we obtain (see Table 4.1, taking 28.3 into account) that the kinds of evolution of the volume element, possible for a given cosmological constant and mean curvature (in the sense of its sign or if it is zero), are the same as for a homogeneous universe. Moreover, we saw in §4.18 that equations (7.13), (7.15), (7.16), under the conditions (28.2) at the given point, can be transformed into their regular form for a homogeneous universe.

The dynamic absolute rotation is absent when the deformations are anisotropic

We assume that the dynamic absolute rotation at this point is again absent, while the deformation anisotropy is present

$$\Pi \geqslant 0\,, \qquad \Omega_j \Omega^j = 0\,. \tag{28.4}$$

Then, as it is easy to see, we have

$$\tau \geqslant 0\,, \tag{28.5}$$

$$c^2 Z \geqslant 3\,\frac{\xi}{\eta^2}\,. \tag{28.6}$$

The deformation anisotropy, generally speaking, complicates the function of the evolution of the mean curvature from the evolution of the volume of the element (see §4.14). In particular, it makes changing of the sign of the mean curvature possible. If the dynamic absolute rotation is absent, then the deformation anisotropy results in new kinds of evolution of the element (see Table 4.2), which are absent for a homogeneous universe. For a positive cosmological constant and always for the positive curvature criterion*, the limited from above and below monotonic transformations of the volume of the element (the kinds aA, aM, mA, mM) are possible. So the kind O_2 is also possible under the conditions. Note also that for any numerical value of the cosmological constant and an always

*Hence, because of (28.6), for the always positive mean curvature.

nonpositive mean curvature, only those kinds of evolution are possible which are also possible for a homogeneous universe for the same cosmological constant and the same nonpositive curvature.

The presence of the dynamic absolute rotation when the deformations are anisotropic

We assume that the deformation anisotropy and the dynamical absolute rotation are present at the point we are considering. Let us consider the general case, where Π can be greater than, equal to, or less than $2\Omega_j\Omega^j$. Then it is evident that we have

$$\tau \gtreqless 0\,, \tag{28.7}$$

$$c^2 Z \gtreqless 3\,\frac{\xi}{\eta^2}\,. \tag{28.8}$$

Because of (28.7) (see Table 4.3), the kinds aA, aM, mA, mM and consequently the kind O_2 are possible not only for a positive cosmological constant, but also when the constant becomes zero or negative. Moreover, in the case of a nonnegative cosmological constant, the curvature criterion does not always remain positive. This limitation is not present in the case of a strictly negative cosmological constant*.

If dynamic absolute rotation is present, then the new states p, q, r and the associated types pA, qA, rA, pM, qM, rM, pD, qD, rD will be possible.

Such are the various consequences of deformation anisotropy and dynamic absolute rotation.

◇

*Whatever the sign of the curvature criterion, condition (28.8) permits a positive, zero, or negative mean curvature.

Bibliography

1. **Tolman R. C.** Relativity, thermodynamics, and cosmology. Clarendon Press, Oxford, 1934.
2. **Milne E. A.** Relativity, gravitation, and world-structure. Clarendon Press, Oxford, 1935.
3. **Robertson H. P.** *Proc. Nat. Acad. Sci. USA*, 15, 822, 1929.
4. **Einstein A.** *Berlinere Berichte*, 142, 1917
5. **Friedmann A.** *Zeitschrift für Physik*, 10, 377, 1922.
6. **Einstein A.** *Berlinere Berichte*, 235, 1931.
7. **Eddington A. S.** The mathematical theory of relativity. 3rd exp. ed., GTTI, Moscow, 1934.
8. **Levi-Civita T.** Das absolute Differentialkalkül. Springer, Berlin, 1928.
9. **de Sitter W.** *Kon. Ned. Acad. Amsterdam Proc.*, 19, 1217, 1917.
10. **de Sitter W.** *Monthly Notices*, 78, 3, 1917.
11. **Lemaître G.** *Journal of Mathematical Physics*, 4, 188, 1925.
12. **Friedmann A.** *Zeitschrift für Physik*, 21, 326, 1924.
13. **Lanczos K.** *Physikalische Zeitschrift*, 23, 539, 1922.
14. **Weyl H.** *Physikalische Zeitschrift*, 24, 230, 1923.
15. **Lemaître G.** *Ann. Soc. Sci. Bruxelles*, 47A, 49, 1927; *Monthly Notices*, 91, 483, 1931.
16. **Lemaître G.** *Bull. Astron. Inst. Netherlands*, 5, 273, 1930.
17. **Heckmann O.** *Göttingene Nachrichten*, 97, 1932.
18. **Robertson H. P.** *Review of Modern Physics*, 5, 62, 1933.
19. **Eddington A. S.** *Monthly Notices*, 90, 668, 1930.
20. **Tolman R. C.** *Proc. Nat. Acad. Sci. USA*, 14, 268, 1928.
21. **Tolman R. C.** *Proc. Nat. Acad. Sci. USA*, 14, 701, 1928.
22. **Tolman R. C.** *Physical Review*, 37, 1639, 1931.
23. **Tolman R. C.** *Physical Review*, 38, 797, 1931.
24. **Tolman R. C.** *Physical Review*, 39, 320, 1932.
25. **Milne E. A.** *Quart. Journal of Math.*, 5, 64, 1934.
26. **McCrea W. H. and Milne E. A.** *Quart. Journal of Math.*, 5, 73, 1934.
27. **Eigenson M. S.** *Zeitschrift für Astrophysik*, 4, 224, 1932.
28. **Eddington A. S.** Relativity theory of protons and electrons. Cambridge Univ. Press, Cambridge, 1936.
29. **Einstein A. and de Sitter W.** *Proc. Nat. Ac. Sci. USA*, 18, 213, 1932.

30. **Tolman R. C.** *Physical Review*, 38, 1758, 1931.
31. **Lanczos K.** *Zeitschrift für Physik*, 17, 168, 1923.
32. **Hubble E. P.** *Astrophysical Journal*, 64, 321, 1926.
33. **Hubble E. P.** *Proc. Nat. Acad. Sci. USA*, 15, 168, 1929.
34. **Hubble E. P. and Humason M. L.** *Astrophysical Journal*, 74, 43, 1931.
35. **Shapley H. and Ames A.** *Harvard Annals*, 88, 43, 1932.
36. **Hubble E. P.** *Astrophysical Journal*, 79, 8, 1934.
37. **Hubble E. P.** *Astrophysical Journal*, 84, 517, 1936.
38. **Hubble E. P. and Tolman R. C.** *Astrophysical Journal*, 82, 302, 1935.
39. **McVittie G. C.** *Monthly Notices*, 97, 163, 1937.
40. **McVittie G. C.** Cosmological theory. Ch. IV. Methuen Monographs, London, 1937.
41. **McVittie G. C.** *Monthly Notices*, 98, 384, 1938.
42. **McVittie G. C.** *Zeitschrift für Astrophysik*, 14, 274, 1937.
43. **McCrea W. H.** *Zeitschrift für Astrophysik*, 9, 290, 1935.
44. **Eddington A. S.** *Monthly Notices*, 97, 156, 1937.
45. **Shapley H.** *Proc. Nat. Acad. Sci. USA*, 24, 148, 1938.
46. **Shapley H.** *Proc. Nat. Acad. Sci. USA*, 24, 527, 1938.
47. **Shapley H.** *Monthly Notices*, 94, 791, 1934.
48. **Shapley H.** *Proc. Nat. Acad. Sci. USA*, 21, 587, 1935.
49. **Shapley H.** *Proc. Nat. Acad. Sci. USA*, 19, 389, 1933.
50. **Shapley H.** *Proc. Nat. Acad. Sci. USA*, 24, 282, 1938.
51. **McCrea W. H.** *Zeitschrift für Astrophysik*, 9, 98, 1939.
52. **Mason W. R.** *Philosophical Magazine*, 14, 386, 1932.
53. **Fesenkov V. G.** *Doklady Acad. Nauk USSR*, new ser., 15, 125, 1937.
54. **Fesenkov V. G.** *Soviet Astron. Journal*, 14, 413, 1937.
55. **Krat V. A.** *Proc. Phys.–Math. Soc., Kiev Univ.*, ser. 3, 1936–37.
56. **Eigenson M. S.** *Doklady Acad. Nauk USSR*, new ser., 26, 759, 1940.
57. **Cartan E.-J.** La géométrie des espaces de Riemann (The geometry of Riemannian spaces). ONTI, Moscow–Leningrad, 1936.
58. **Lamb H.** Hydrodynamics. 5th ed., Cambridge Univ. Press, Cambridge, 1930.
59. **Lanczos K.** *Physikalische Zeitschrift*, 23, 537, 1922.
60. **Fock V. A.** *Soviet Physics JETP–USSR*, 9, 375, 1939.
61. **Eisenhart L. P.** *Trans. Amer. Math. Soc.*, 26, 205, 1924.
62. **Kochin N. E.** Vector calculus and the foundations of tensor analysis. 6th ed., GONTI, Moscow–Leningrad, 1938.
63. **Cartan E.-J.** Integral invariants. GITTL, Moscow–Leningrad, 1940.
64. **Landau L. D. and Lifshitz E. M.** The classical theory of fields. GITTL, Moscow–Leningrad, 1939.

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