ANNUAL ISSUE 2005

PROGRESS IN PHYSICS

"All scientists shall have the right to present their scientific research results, in whole or in part, at relevant scientific conferences, and to publish the same in printed scientific journals, electronic archives, and any other media." — Declaration of Academic Freedom, Article 8

ISSN 1555-5534

The Journal on Advanced Studies in Theoretical and Experimental Physics, including Related Themes from Mathematics

PROGRESS IN PHYSICS

A quarterly issue scientific journal, registered with the Library of Congress (DC).
This journal is peer reviewed and included in the abstracting and indexing coverage of:
Mathematical Reviews and MathSciNet (AMS, USA), DOAJ of Lund University (Sweden),
Referativnyi Zhurnal VINITI (Russia), etc.

ISSN: 1555-5534 (print)
ISSN: 1555-5615 (online)
4 issues per annum

Electronic version of this journal:
http://www.geocities.com/ptep_online

To order printed issues of this journal, contact the Editor in Chief.

This journal is powered by BaKoMa-TEX

A variety of books can be downloaded free from the E-library of Science:
http://www.gallup.unm.edu/∼smarandache

ISSN: 1555-5534 (print)
ISSN: 1555-5615 (online)

Standard Address Number: 297-5092
Printed in the United States of America

APRIL 2005 **VOLUME 1**

CONTENTS

Information for Authors and Subscribers

Progress in Physics has been created for publications on advanced studies in theoretical and experimental physics, including related themes from mathematics. All submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

All submissions should be designed in LaTeX format using *Progress in Physics* template. This template can be downloaded from *Progress in Physics* home page http://www.geocities.com/ptep_online. Abstract and the necessary information about author(s) should be included into the papers. To submit a paper, mail the file(s) to Chief Editor.

All submitted papers should be as brief as possible. Short articles are preferable.

All that has been accepted for the online issue of *Progress in Physics* is printed in the paper version of the journal. To order printed issues, contact Chief Editor.

This journal is non-commercial, academic edition. It is printed from private donations.

A New Method to Measure the Speed of Gravitation

Dmitri Rabounski
E-mail: rabounski@yahoo.com

According to the standard viewpoint the speed of gravitation is the speed of weak waves of the metrics. This study proposes a new approach, defining the speed as the speed of travelling waves in the field of gravitational inertial force. D'Alembert's equations of the field show that this speed is equal to the velocity of light corrected by gravitational potential. The approach leads to a new experiment to measure the speed of gravitation, which, using "detectors" such as planets and their satellites, is not linked to deviation of geodesic lines and quadrupole mass-detectors with their specific technical problems.

1 Introduction

Herein we use a pseudo-Riemannian space with the signature $(+---)$, where time is real and spatial coordinates are imaginary, because the projection of a four-dimensional impulse on the spatial section of any given observer is positive in this case. We also denote space-time indices in Greek, while spatial indices are Roman. Hence the time term in d'Alembert's operator $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ will be positive, while the spatial part (Laplace's operator) will be negative $\Delta = -g^{ik}\nabla_i\nabla_k$.

By applying the d'Alembert operator to a tensor field, we obtain the d'Alembert equations of the field. The non-zero elements are the d'Alembert equations containing the field-inducing sources. The zero elements are the equations without the sources. If there are no sources the field is free, giving a free wave. There is the time term $\frac{1}{a^2}\frac{\partial^2}{\partial t^2}$ containing the linear velocity a of the wave. For this reason, in the case of gravitational fields, the d'Alembert equations give rise to a possibility of calculating the speed of propagation of gravitational attraction (the speed of gravitation). At the same time the result may be different according to the way we define the speed as the velocity of waves of the metric, or something else.

The usual approach sets forth the speed of gravitation as follows [1, 5]. One considers the space-time metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$, composed of a Galilean metric $g^{(0)}_{\alpha\beta}$ (wherein $g^{(0)}_{00} = 1$, $g^{(0)}_{0i} = 0$, $g^{(0)}_{ik} = -\delta_{ik}$) and tiny corrections $\zeta_{\alpha\beta}$ defining a weak gravitational field. Because the $\zeta_{\alpha\beta}$ are tiny, we can raise and lower indices with the Galilean metric tensor $g^{(0)}_{\alpha\beta}$. The quantities $\zeta^{\alpha\beta}$ are defined by the main property of the fundamental metric tensor $g_{\alpha\sigma}g^{\sigma\beta} = \delta^\beta_\alpha$ as follows: $(g^{(0)}_{\alpha\sigma} + \zeta_{\alpha\sigma})\, g^{\sigma\beta} = \delta^\beta_\alpha$. Besides this approach defines $g^{\alpha\beta}$ and $g = \det\|g_{\alpha\beta}\|$ to within higher order terms withheld as $g^{\alpha\beta} = g^{(0)\alpha\beta} - \zeta^{\alpha\beta}$ and $g = g^{(0)}(1+\zeta)$, where $\zeta = \zeta^\sigma_\sigma$. Because $\zeta_{\alpha\beta}$ are tiny we can take Ricci's tensor $R_{\alpha\beta} = R^{\cdots\sigma}_{\alpha\sigma\beta}$ (the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ contracted on two indices) to within higher order terms withheld. Then the Ricci tensor for the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$ is

$$R_{\alpha\beta} = \frac{1}{2}\, g^{(0)\mu\nu}\frac{\partial^2\zeta_{\alpha\beta}}{\partial x^\mu \partial x^\nu} = \frac{1}{2}\Box\zeta_{\alpha\beta}\,,$$

which simplifies Einstein's field equations $R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = -\kappa\, T_{\alpha\beta} + \lambda g_{\alpha\beta}$, where in this case $R = g^{(0)\mu\nu}R_{\mu\nu}$. In the absence of matter and λ-fields ($T_{\alpha\beta} = 0$, $\lambda = 0$), that is, in emptiness, the Einstein equations for the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$ become

$$\Box\zeta^\beta_\alpha = 0\,.$$

Actually, these are the d'Alembert equations of the corrections $\zeta_{\alpha\beta}$ to the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$ (weak waves of the metric). Taking the flat wave travelling in the direction $x^1 = x$, we see

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\zeta^\beta_\alpha = 0\,,$$

so weak waves of the metric travel at the velocity of light in empty space.

This approach leads to an experiment, based on the principle that geodesic lines of two infinitesimally close test-particles will deviate in a field of waves of the metric. A system of two real particles connected by a spring (a quadrupole mass-detector) should react to the waves. Most of these experiments have since 1968 been linked to Weber's detector. The experiments have not been technically decisive until now, because of problems with precision of measurement and other technical problems [3] and some purely theoretical problems [4, 5].

Is the approach given above the best? Really, the resulting d'Alembert equations are derived from that form of the Ricci tensor obtained under the substantial simplifications of higher order terms withheld (i.e. to first order). Eddington [1] wrote that a source of this approximation is a specific reference frame which differs from Galilean reference frames by the tiny corrections $\zeta_{\alpha\beta}$, the origin of which could be very different from gravitation. This argument leads, as Eddington remarked, to a "vicious circle". So the standard approach has inherent drawbacks, as follows:

(1) The approach gives the Ricci tensor and hence the d'Alembert equations of the metric to within higher order terms withheld, so the velocity of waves of the metric calculated from the equations is not an exact theoretical result;

(2) A source of this approximation are the tiny corrections $\zeta_{\alpha\beta}$ to a Galilean metric, the origin of which may be very different: not only gravitation;

(3) Two bodies attract one another because of the transfer of gravitational force. A wave travelling in the field of gravitational force is not the same as a wave of the metric — these are different tensor fields. When a quadrupole mass-detector registers a signal, the detector reacts to a wave of the metric in accordance with this theory. Therefore it is concluded that quadrupole mass detectors would be the means by to discovery of waves of the metric. However, the experiment is only incidental to the measurement of the speed of gravitation.

For these reasons we are lead to consider gravitational waves as waves travelling in the field of gravitational force, which provides two important advantages:

(1) The mathematical apparatus of chronometric invariants (physical observable quantities in the General Theory of Relativity) defines gravitational inertial force F_i without the Riemann-Christoffel curvature tensor [1, 2]. Using this method, we can deduce the exact d'Alembert equations for the force field, giving an exact formula for the velocity of waves of the force;

(2) Experiments to register waves of the force field, using "detectors" such as planets or their satellites, does not involve a quadrupole mass-detector and its specific technical problems.

2 The new approach

The basis here is the mathematical apparatus of chronometric invariants, created by Zelmanov in the 1940's [1, 2]. Its essence is that if an observer accompanies his reference body, his observable quantities (chronometric invariants) are projections of four-dimensional quantities on his time line and the spatial section, made by projecting operators $b^\alpha = \frac{dx^\alpha}{ds}$ and $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$, which fully define his real reference space. Thus, chr.inv.-projections of a world-vector Q^α are $b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}}$ and $h^i_\alpha Q^\alpha = Q^i$, while chr.inv.-projections of a world-tensor of the 2nd rank $Q^{\alpha\beta}$ are $b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}$, $h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}$, $h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}$. Physical observable properties of the space are derived from the fact that the chr. inv.-differential operators $\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2} v_i \frac{^*\partial}{\partial t}$ are non-commutative. They are the chr.inv.-vector of gravitational inertial force F_i, the chr.inv.-tensor of angular velocities of the space rotation A_{ik}, and the chr.inv.-tensor of rates of the space deformations D_{ik}, namely

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right),$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right),$$

$$v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,,$$

$$D_{ik} = \frac{1}{2}\frac{^*\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{^*\partial h^{ik}}{\partial t}\,, \quad D = D^k_k = \frac{^*\partial \ln\sqrt{h}}{\partial t}\,,$$

where w is gravitational potential, v_i is the linear velocity of the space rotation, $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ is the chr.inv.-metric tensor, and also $h = \det\|h_{ik}\|$, $\sqrt{-g} = \sqrt{h}\sqrt{g_{00}}$, $g = \det\|g_{\alpha\beta}\|$. Observable non-uniformity of the space is set up by the chr.inv.-Christoffel symbols $\Delta^i_{jk} = h^{im}\Delta_{jk,m}$, which are built just like Christoffel's usual symbols $\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$, using h_{ik} instead of $g_{\alpha\beta}$.

The four-dimensional generalization of the chr.inv.-quantities F_i, A_{ik}, and D_{ik} had been obtained by Zelmanov [8] as $F_\alpha = -2c^2 b^\beta a_{\beta\alpha}$, $A_{\alpha\beta} = ch^\mu_\alpha h^\nu_\beta a_{\mu\nu}$, $D_{\alpha\beta} = ch^\mu_\alpha h^\nu_\beta d_{\mu\nu}$, where $a_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha b_\beta - \nabla_\beta b_\alpha\right)$, $d_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha b_\beta + \nabla_\beta b_\alpha\right)$.

Following the study [9], we consider a field of the gravitational inertial force $F_\alpha = -2c^2 b^\beta a_{\beta\alpha}$, the chr.inv.-spatial projection of which is F^i, so that $F_i = h_{ik}F^k$. The d'Alembert equations of the vector field $F^\alpha = -2c^2 b^\beta a_{\beta\cdot}^{\cdot\alpha}$ in the absence of sources are

$$\Box F^\alpha = 0\,.$$

Their chr.inv.-projections (referred to as the chr.inv.-d'Alembert equations) can be deduced as follows

$$b_\sigma\, g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta F^\sigma = 0\,, \qquad h^i_\sigma\, g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta F^\sigma = 0\,.$$

After some algebra we obtain the chr.inv.-d'Alembert equations for the field of the gravitational inertial force $F^\alpha = -2c^2 b^\beta a_{\beta\cdot}^{\cdot\alpha}$ in their final form. They are

$$\frac{1}{c^2}\frac{^*\partial}{\partial t}\left(F_k F^k\right) + \frac{1}{c^2}F_i\frac{^*\partial F^i}{\partial t} + D^k_m\frac{^*\partial F^m}{\partial x^k} + h^{ik}\frac{^*\partial}{\partial x^i}\left[(D_{kn} + A_{kn})F^n\right] - \frac{2}{c^2}A_{ik}F^iF^k + \frac{1}{c^2}F_mF^mD + \Delta^m_{kn}D^k_mF^n - h^{ik}\Delta^m_{ik}\left(D_{mn} + A_{mn}\right)F^n = 0\,,$$

$$\frac{1}{c^2}\frac{{}^*\partial^2 F^i}{\partial t^2} - h^{km}\frac{{}^*\partial^2 F^i}{\partial x^k \partial x^m} + \frac{1}{c^2}\left(D_k^i + A_{k\cdot}^{\cdot i}\right)\frac{{}^*\partial F^k}{\partial t} + $$
$$+ \frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left[\left(D_k^i + A_{k\cdot}^{\cdot i}\right)F^k\right] + \frac{1}{c^2}D\frac{{}^*\partial F^i}{\partial t} + \frac{1}{c^2}F^k\frac{{}^*\partial F^i}{\partial x^k} +$$
$$+ \frac{1}{c^2}\left(D_n^i + A_{n\cdot}^{\cdot i}\right)F^n D + \frac{1}{c^4}F_k F^k F^i + \frac{1}{c^2}\Delta_{km}^i F^k F^m -$$
$$- h^{km}\left\{\frac{{}^*\partial}{\partial x^k}\left(\Delta_{mn}^i F^n\right) + \left(\Delta_{kn}^i \Delta_{mp}^n - \Delta_{km}^n \Delta_{np}^i\right)F^p + \right.$$
$$\left. + \Delta_{kn}^i \frac{{}^*\partial F^n}{\partial x^m} - \Delta_{km}^n \frac{{}^*\partial F^i}{\partial x^n}\right\} = 0\,.$$

Calling upon the formulae for chr.inv.-derivatives, we transform the first term in the chr.inv.-d'Alembert vector equations into the form

$$\frac{1}{c^2}\frac{{}^*\partial^2 F^i}{\partial t^2} = \frac{1}{c^2 g_{00}}\frac{\partial^2 F^i}{\partial t^2} + \frac{1}{c^4\sqrt{g_{00}}}\frac{{}^*\partial \mathrm{w}}{\partial t}\frac{{}^*\partial F^i}{\partial t}\,,$$

so waves of gravitational inertial force travel at a velocity u^k, the square of which is $u_k u^k = c^2 g_{00}$ and the modulus

$$u = \sqrt{u_k u^k} = c\left(1 - \frac{\mathrm{w}}{c^2}\right).$$

Because waves of the field of gravitational inertial force transfer gravitational interaction, this wave speed is the speed of gravitation as well. The speed depends on the scalar potential w of the field itself, which leads us to the following conclusions:

(1) In a weak gravitational field, the potential w of which is negligible but its gradient F_i is non-zero, the speed of gravitation equals the velocity of light;

(2) According to this formula, the speed of gravitation will be less than the velocity of light near bulky bodies like stars or planets, where gravitational potential is perceptible. On the Earth's surface slowing gravitation will be slower than light by 21 cm/sec. Gravitation near the Sun will be about 6.3×10^4 cm/sec slower than light;

(3) Under gravitational collapse ($\mathrm{w} = c^2$) the speed of gravitation becomes zero.

Let us turn now from theory to experiment. An idea as to how to measure the speed of gravitation as the speed to transfer of the attracting force between space bodies had been proposed by the mathematician Dombrowski [10] in conversation with me more than a decade ago. But in the absence of theory the idea had not developed to experiment in that time. Now we have an exact formula for the speed of waves travelling in the field of gravitational inertial force, so we can propose an experiment to measure the speed (a Weber detector reacts to weak waves of the metric, so it does not apply to this experiment).

The Moon attracts the Earth's surface, causing the flow "hump" in the ocean surface that follows the moving Moon, producing ebbs and flows. An analogous "hump" follows the Sun: its magnitude is more less. A satellite in an Earth orbit has the same ebb and flow oscillations — its orbit rises and falls a little, following the Moon and the Sun as well. A satellite in space experiences no friction, contrary of the viscous waters of the oceans. A satellite is a perfect system, which reacts instantly to the flow. If the speed of gravitation is limited, the moment of the satellite's maximum flow rise should be later than the lunar/solar upper transit by the amount of time taken by waves of the gravitational force field to travel from the Moon/Sun to the satellite.

The Earth's gravitational field is not absolutely symmetric, because of the imperfect form of the terrestrial globe. A real satellite reacts to the field defects during its orbital flight around the Earth — the height of its orbit oscillates in decimetres, giving rise to substantial noise in the experiment. For this reason a geostationary satellite would be best. Such a satellite, having an equatorial orbit, requires an angular velocity the same as that of the Earth. As a result, the height of a geostationary satellite above the Earth does not depend on non-uniformities of the Earth's gravitational field. The height could be measured with high precision by a laser range-finder, almost without interruption, providing a possibility of registering the moment of the maximum flow rise of the satellite, perfectly.

In accordance with our formula the speed of gravitation near the Earth is 21 cm/sec less than the velocity of light. In this case the maximum of the lunar flow wave in a satellite orbit will be about 1 sec later than the lunar upper culmination. The lateness of the flow wave of the Sun will be about 500 sec after the upper transit of the Sun. The question is how precisely could the moment of the maximum flow rise of a satellite in its orbit be determined, because the real maximum can be "fuzzy" in time.

3 Effect of the curvature

If a space is homogeneous ($\Delta_{km}^i = 0$) and it is free of rotation and deformation ($A_{ik} = 0$, $D_{ik} = 0$), then the chr.inv.-d'Alembert equations for the field of gravitational inertial force take the form

$$\frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left(F_k F^k\right) + \frac{1}{c^2}F_i\frac{{}^*\partial F^i}{\partial t} = 0\,,$$

$$\frac{1}{c^2}\frac{{}^*\partial^2 F^i}{\partial t^2} - h^{km}\frac{{}^*\partial^2 F^i}{\partial x^k \partial x^m} + \frac{1}{c^2}F^k\frac{{}^*\partial F^i}{\partial x^k} + \frac{1}{c^4}F_k F^k F^i = 0\,,$$

so waves of gravitational inertial force are permitted even in this very simple case.

Are waves of the metric possible in this case or not?

As it is known, waves of the metric are linked to the space-time curvature derived from the Riemann-Christoffel curvature tensor. If the first derivatives of the metric (the space deformations) are zero, then its second derivatives (the curvature) are zero too. Therefore waves of the metric have no place in a non-deforming space, while waves of gravitational inertial force are possible there.

In connection with this fact, following the study [9], another question arises. By how much does the curvature affect waves of gravitational inertial force?

To answer the question let us recall that Zelmanov, following the same procedure by which the Riemann-Christoffel tensor was introduced, after considering non-commutativity of the chr.inv.-second derivatives of a vector ${}^*\nabla_i {}^*\nabla_k Q_l - {}^*\nabla_k {}^*\nabla_i Q_l = \frac{2A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t} + H_{lki}^{\cdots j} Q_j$, had obtained the chr. inv.-tensor $H_{lki}^{\cdots j}$ like Schouten's tensor [11]. Its generalization gives the chr.inv.-curvature tensor $C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right)$, which has all the properties of the Riemann-Christoffel tensor in the observer's spatial section. So the chr.inv.-spatial projection $Z^{iklj} = -c^2 R^{iklj}$ of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$, after contraction twice by h_{ik}, is $Z = h^{il} Z_{il} = D_{ik} D^{ik} - D^2 - A_{ik} A^{ik} - c^2 C$, where $C = C_j^j = h^{lj} C_{lj}$ and $C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im} C_{kimj}$ [1].

At the same time, as Synge's well-known book [12] shows, in a space of constant four-dimensional curvature, $K = \text{const}$, we have $R_{\alpha\beta\gamma\delta} = K\left(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}\right)$, $R_{\alpha\beta} = -3K g_{\alpha\beta}$, $R = -12K$. With these formulae as a basis, after calculation of the chr.inv.-spatial projection of the Riemann-Christoffel tensor, we deduce that in a constant curvature space $Z = 6c^2 K$. Equating this to the same quantity in an arbitrary curvature space, we obtain a correlation between the four-dimensional curvature K and the observable three-dimensional curvature in the constant curvature space

$$6c^2 K = D_{ik} D^{ik} - D^2 - A_{ik} A^{ik} - c^2 C\,.$$

If the four-dimensional curvature is zero ($K = 0$), and the space does no deformations ($D_{ik} = 0$ — its metric is stationary, $h_{ik} = \text{const}$), then no waves of the metric are possible. In such a space the observable three-dimensional curvature is

$$C = -\frac{1}{c^2} A_{ik} A^{ik},$$

which is non-zero ($C \neq 0$), only if the space rotates ($A_{ik} \neq 0$). If aside of these factors, the space does not rotate, then its observable curvature also becomes zero; $C = 0$. Even in this case the chr.inv.- d'Alembert equations show the presence of waves of gravitational inertial force.

What does this imply? As a matter of fact, gravitational attraction is an everyday reality in our world, so waves of gravitational inertial force transferring the attraction shall be incontrovertible. Therefore we adduce the alternatives:

(1) Waves of gravitational inertial force depend on a curvature of space — then the real space-time is not a space of constant curvature, or,

(2) Waves of gravitational inertial force do not depend on the curvature.

Acknowledgements

My late acknowledgements to Prof. Kyril Stanyukovich (1916–1989) and Dr. Abraham Zelmanov (1913–1987) for all my experience in General Relativity given by them. Special thanks go to Dr. Kyril Dombrowski for our friendly conversations initiated this research many years ago. I am also very grateful to my closest colleagues: Dr. Larissa Borissova for her checking of the calculations, and Stephen J. Crothers for his editing this paper.

References

1. Eddington A. S. The mathematical theory of relativity. Cambridge University Press, Cambridge, 1924 (ref. with the 3rd exp. edition, GTTI, Moscow, 1934).
2. Landau L. D. and Lifshitz E. M. The classical theory of fields. GITTL, Moscow, 1939 (ref. with the 4th final exp. edition, Butterworth–Heinemann, 1980).
3. Rudenko V. N. Relativistic experiments in gravitational field. *Uspekhi Fizicheskikh Nauk*, 1978, v. 126 (3), 361–401.
4. Borissova L. B. Relative oscillations of test-particles in accompanying reference frames. *Doklady Acad. Nauk USSR*, 1975, v. 225 (4), 786–789.
5. Borissova L. B. Quadrupole mass-detector in a field of weak flat gravitational waves. *Izvestia VUZov*, *Physica*, 1978 (10), 109–114.
6. Zelmanov A. L. Chronometric invariants. Dissertation, 1944. First published: CERN, EXT-2004-117, 236 pages.
7. Zelmanov A. L. Chronometric invariants and co-moving coordinates in the general relativity theory. *Doklady Acad. Nauk USSR*, 1956, v. 107 (6), 815–818.
8. Zelmanov A. L. Orthometric form of monad formalism and its relations to chronometric and kinemetric invariants. *Doklady Acad. Nauk USSR*, 1976, v. 227 (1), 78–81.
9. Rabounski D. D. The new aspects of General Relativity. CERN, EXT-2004-025, 117 pages.
10. Dombrowski K. I. Private communications, 1990.
11. Schouten J. A. und Struik D. J. Einführung in die neuren Methoden der Differentialgeometrie. *Zentralblatt für Mathematik*, 1935, Bd. 11 und Bd. 19.
12. Synge J. L. Relativity: the General Theory. North Holland, Amsterdam, 1960 (referred with the 2nd edition, Foreign Literature, Moscow, 1963, 432 pages).

Quantum Quasi-Paradoxes and Quantum Sorites Paradoxes

Florentin Smarandache

Dept. of Mathematics, University of New Mexico, 200 College Road, Gallup, NM 87301, USA

E-mail: fsmarandache@yahoo.com; smarand@unm.edu

There can be generated many paradoxes or quasi-paradoxes that may occur from the combination of quantum and non-quantum worlds in physics. Even the passage from the micro-cosmos to the macro-cosmos, and reciprocally, can generate unsolved questions or counter-intuitive ideas. We define a quasi-paradox as a statement which has a *prima facie* self-contradictory support or an explicit contradiction, but which is not completely proven as a paradox. We present herein four elementary quantum quasi-paradoxes and their corresponding quantum Sorites paradoxes, which form a class of quantum quasi-paradoxes.

1 Introduction

According to the Dictionary of Mathematics (Borowski and Borwein, 1991 [1]), the **paradox** is "an apparently absurd or self-contradictory statement for which there is *prima facie* support, or an explicit contradiction derived from apparently unexceptionable premises". Some paradoxes require the revision of their intuitive conception (Russell's paradox, Cantor's paradox), others depend on the inadmissibility of their description (Grelling's paradox), others show counter-intuitive features of formal theories (Material implication paradox, Skolem Paradox), others are self-contradictory — Smarandache Paradox: "All is <A> the <Non-A> too!", where <A> is an attribute and <Non-A> its opposite; for example "All is possible the impossible too!" (Weisstein, 1998 [2]).

Paradoxes are normally true and false in the same time.

The **Sorites paradoxes** are associated with Eubulides of Miletus (fourth century B. C.) and they say that there is not a clear frontier between visible and invisible matter, determinist and indeterminist principle, stable and unstable matter, long time living and short time living matter.

Generally, between <A> and <Non-A> there is no clear distinction, no exact frontier. Where does <A> really end and <Non-A> begin? One extends Zadeh's "fuzzy set" concept to the "neutrosophic set" concept.

Let's now introduce the notion of quasi-paradox:

A **quasi-paradox** is a statement which has a *prima facia* self-contradictory support or an explicit contradiction, but which is not completely proven as a paradox. A quasi-paradox is an *informal* contradictory statement, while a paradox is a *formal* contradictory statement.

Some of the below quantum quasi-paradoxes can later be proven as real quantum paradoxes.

2 Quantum Quasi-Paradoxes and Quantum Sorites Paradoxes

The below quasi-paradoxes and Sorites paradoxes are based on the antinomies: visible/invisible, determinist/indeterminist, stable/unstable, long time living/short time living, as well as on the fact that there is not a clear separation between these pairs of antinomies.

2.1.1 **Invisible Quasi-Paradox:** Our visible world is composed of a totality of invisible particles.

2.1.2 **Invisible Sorites Paradox:** There is not a clear frontier between visible matter and invisible matter.

(a) An invisible particle does not form a visible object, nor do two invisible particles, three invisible particles, etc. However, at some point, the collection of invisible particles becomes large enough to form a visible object, but there is apparently no definite point where this occurs.

(b) A similar paradox is developed in an opposite direction. It is always possible to remove a particle from an object in such a way that what is left is still a visible object. However, repeating and repeating this process, at some point, the visible object is decomposed so that the left part becomes invisible, but there is no definite point where this occurs.

2.2.1 **Uncertainty Quasi-Paradox:** Large matter, which is at some degree under the "determinist principle", is formed by a totality of elementary particles, which are under Heisenberg's "indeterminacy principle".

2.2.2 **Uncertainty Sorites Paradox:** Similarly, there is not a clear frontier between the matter under the "determinist principle" and the matter under "indeterminist principle".

2.3.1 **Unstable Quasi-Paradox:** "Stable" matter is formed by "unstable" elementary particles (elementary particles decay when free).

2.3.2 **Unstable Sorites Paradox:** Similarly, there is not a clear frontier between the "stable matter" and the "unstable matter".

2.4.1 **Short-Time-Living Quasi-Paradox:** "Long-time-living" matter is formed by very "short-time-living" elementary particles.

2.4.2 **Short-Time-Living Sorites Paradox:** Similarly, there is not a clear frontier between the "long-time-living" matter and the "short-time-living" matter.

3 Conclusion

"More such quantum quasi-paradoxes and paradoxes can be designed, all of them forming a class of Smarandache quantum quasi-paradoxes." (Dr. M. Khoshnevisan, Griffith University, Gold Coast, Queensland, Australia [3])

References

1. Borowski E. J. and Borwein J. M. The Harper Collins Dictionary of Mathematics. Harper Perennial, A Division of Harper Collins Publishers, New York, 1991.
2. Weisstein E. W. Smarandache Paradox. *CRC Concise Encyclopaedia of Mathematics*, CRC Press, Boca Raton, Florida, 1998, 1661, (see the e-print version of this article in http://mathworld.wolfram.com/SmarandacheParadox.html).
3. Khoshnevisan M. Private communications, 1997.
4. Motta L. A Look at the Smarandache Sorites Paradox. *Proceedings of the Second International Conference on Smarandache Type Notions In Mathematics and Quantum Physics*, University of Craiova, Craiova, Romania, December 21–24, 2000 (see the e-print version in the web site at York University, Canada, http://at.yorku.ca/cgi-bin/amca/caft-04).
5. Niculescu G. On Quantum Smarandache Paradoxes. *Proceedings of the Second International Conference on Smarandache Type Notions In Mathematics and Quantum Physics*, University of Craiova, Craiova, Romania, December 21–24, 2000 (see the e-print version in the web site at York University, Canada, http://at.yorku.ca/cgi-bin/amca/caft-20).
6. Smarandache F. Invisible paradox. *Neutrosophy, Neutrosophic Probability, Set, and Logic*, American Research Press, Rehoboth, 1998 (see the third e-print edition of the book in the web site http://www.gallup.unm.edu/∼smarandache).
7. Smarandache F. Sorites paradoxes. *Definitions, Solved and Unsolved Problems, Conjectures, and Theorems in Number Theory and Geometry* (edited by M. L. Perez), Xiquan Publishing House, Phoenix, 2000.

A New Form of Matter — Unmatter, Composed of Particles and Anti-Particles

Florentin Smarandache
Dept. of Mathematics, University of New Mexico, 200 College Road, Gallup, NM 87301, USA
E-mail: fsmarandache@yahoo.com; smarand@unm.edu

Besides *matter* and *antimatter* there must exist *unmatter* (as a new form of matter) in accordance with the neutrosophy theory that between an entity <A> and its opposite <AntiA> there exist intermediate entities <NeutA>. Unmatter is neither matter nor antimatter, but something in between. An atom of unmatter is formed either by (1): electrons, protons, and antineutrons, or by (2): antielectrons, antiprotons, and neutrons. At CERN it will be possible to test the production of unmatter. The existence of unmatter in the universe has a similar chance to that of the antimatter, and its production also difficult for present technologies.

1 Introduction

This article is an improved version of an old manuscript [1]. This is a theoretical assumption about the possible existence of a new form of matter. Up to day the unmatter was not checked in the lab.

According to the neutrosophy theory in philosophy [2], between an entity <A> and its opposite <AntiA> there exist intermediate entities <NeutA> which are neither <A> nor <AntiA>.

Thus, between "matter" and "antimatter" there must exist something which is neither matter nor antimatter, let's call it UNMATTER.

In neutrosophy, <NonA> is what is not <A>, i. e. <NonA> = <AntiA> ∪ <NeutA>. Then, in physics, NONMATTER is what is not matter, i. e. nonmatter means antimatter together with unmatter.

2 Classification

A. Matter is made out of electrons, protons, and neutrons.

Each matter atom has electrons, protons, and neutrons, except the atom of ordinary hydrogen which has no neutron.

The number of electrons is equal to the number of protons, and thus the matter atom is neutral.

B. Oppositely, the **antimatter** is made out of antielectrons, antiprotons, and antineutrons.

Each antimatter atom has antielectrons (positrons), antiprotons, and antineutrons, except the antiatom of ordinary hydrogen which has no antineutron.

The number of antielectrons is equal to the number of antiprotons, and thus the antimatter atom is neutral.

C. Unmatter means neither matter nor antimatter, but in between, an entity which has common parts from both of them.

Etymologically "un-matter" comes from [ME $<$ OE, akin to Gr. *an-*, *a-*, Latin *in-*, and to the negative elements in *no*, *not*, *nor*] and [ME *matière* $<$ OFr $<$ Latin *material*] matter (see [3]), signifying no/without/off the matter.

There are two types of unmatter atoms, that we call unatoms:

u1. The first type is derived from matter; and a such unmatter atom is formed by electrons, protons, and antineutrons;

u2. The second type is derived from antimatter, and a such unmatter atom is formed by antielectrons, antiprotons, and neutrons.

One unmatter type is oppositely charged with respect to the other, so when they meet they annihilate.

The unmatter nucleus, called **unnucleus**, is formed either by protons and antineutrons in the first type, or by antiprotons and neutrons in the second type.

The charge of unmatter should be neutral, as that of matter or antimatter.

The charge of un-isotopes will also be neutral, as that of isotopes and anti-isotopes. But, if we are interested in a negative or positive charge of un-matter, we can consider an un-ion. For example an anion is negative, then its corresponding unmatter of type 1 will also be negative. While taking a cation, which is positive, its corresponding unmatter of type 1 will also be positive.

Sure, it might be the question of how much *stable* the unmatter is, as J. Murphy pointed out in a private e-mail. But Dirac also theoretically supposed the existence of antimatter in 1928 which resulted from Dirac's mathematical equation, and finally the antimatter was discovered/produced in large accelerators in 1996 when it was created the first atom of antihydrogen which lasted for 37 nanoseconds only.

There does not exist an unmatter atom of ordinary hydrogen, neither an unnucleus of ordinary hydrogen since the ordinary hydrogen has no neutron. Yet, two isotopes of the hydrogen, *deuterium* (^{2}H) which has one neutron, and artificially made *tritium* (^{3}H) which has two neutrons have corresponding unmatter atoms of both types, *un-deuterium* and *un-tritium* respectively. The isotopes of an element X differ in the number of neutrons, thus their nuclear mass is different, but their nuclear charges are the same.

For all other matter atom X, there is corresponding an antimatter atom and two unmatter atoms

The unmatter atoms are also neutral for the same reason that either the number of electrons is equal to the number of protons in the first type, or the number of antielectrons is equal to the number of antiprotons in the second type.

If antimatter exists then a higher probability would be for the unmatter to exist, and reciprocally.

Unmatter atoms of the same type stick together form an **unmatter molecule** (we call it **unmolecule**), and so on. Similarly one has two types of unmatter molecules.

The *isotopes* of an atom or element X have the same atomic number (same number of protons in the nucleus) but different atomic masses because the different number of neutrons.

Therefore, similarly the **un-isotopes of type 1** of X will be formed by electrons, protons, and antineutrons, while the **un-isotopes of type 2** of X will be formed by antielectrons, antiprotons, and neutrons.

An *ion* is an atom (or group of atoms) X which has last one or more electrons (and as a consequence carries a negative charge, called *anion*, or has gained one or more electrons (and as a consequence carries a positive charge, called *cation*).

Similarly to isotopes, the **un-ion of type 1** (also called **un-anion 1** or **un-cation 1** if resulted from a negatively or respectively positive charge ion) of X will be formed by electrons, protons, and antineutrons, while the **un-ion of type 2** of X (also called **un-anion 2** or **un-cation 2** if resulted from a negatively or respectively positive charge ion) will be formed by antielectrons, antiprotons, and neutrons.

The ion and the un-ion of type 1 have the same charges, while the ion and un-ion of type 2 have opposite charges.

D. Nonmatter means what is not matter, therefore nonmatter actually comprises antimatter and unmatter. Similarly one defines a nonnucleus.

3 Unmatter propulsion

We think (as a prediction or supposition) it could be possible at using unmatter as fuel for space rockets or for weapons platforms because, in a similar way as antimatter is presupposed to do [4, 5], its mass converted into energy will be fuel for propulsion.

It seems to be a little easier to build unmatter than antimatter because we need say antielectrons and antiprotons only (no need for antineutrons), but the resulting energy might be less than in matter-antimatter collision.

We can collide unmatter 1 with unmatter 2, or unmatter 1 with antimatter, or unmatter 2 with matter.

When two, three, or four of them (unmatter 1, unmatter 2, matter, antimatter) collide together, they annihilate and turn into energy which can materialize at high energy into new particles and antiparticles.

4 Existence of unmatter

The existence of unmatter in the universe has a similar chance to that of the antimatter, and its production also difficult for present technologies. At CERN it will be possible to test the production of unmatter.

If antimatter exists then a higher probability would be for the unmatter to exist, and reciprocally.

The 1998 Alpha Magnetic Spectrometer (AMS) flown on the International Space Station orbiting the Earth would be able to detect, besides cosmic antimatter, unmatter if any.

5 Experiments

Besides colliding electrons, or protons, would be interesting in colliding neutrons. Also, colliding a neutron with an antineutron in accelerators.

We think it might be easier to produce in an experiment an unmatter atom of deuterium (we can call it un-deuterium of type 1). The deuterium, which is an isotope of the ordinary hydrogen, has an electron, a proton, and a neutron. The idea would be to convert/transform in a deuterium atom the neutron into an antineutron, then study the properties of the resulting un-deuterium 1.

Or, similarly for un-deuterium 2, to convert/transform in a deuterium atom the electron into an antielectron, and the proton into an antiproton (we can call it un-deuterium of type 2).

Or maybe choose another chemical element for which any of the previous conversions/transformations might be possible.

6 Neutrons and antineutrons

Hadrons consist of baryons and mesons and interact via strong force.

Protons, neutrons, and many other hadrons are composed from quarks, which are a class of fermions that possess a fractional electric charge. For each type of quark there exists a corresponding antiquark. Quarks are characterized by properties such as *flavor* (up, down, charm, strange, top, or bottom) and *color* (red, blue, or green).

A neutron is made up of quarks, while an antineutron is made up of antiquarks.

A neutron (see [9]) has one Up quark (with the charge of $+\frac{2}{3}\times 1.606\times 10^{19}$ C) and two Down quarks (each with the charge of $-\frac{1}{3}\times 1.606\times 10^{19}$ C), while an antineutron has one anti Up quark (with the charge of $-\frac{2}{3}\times 1.606\times 10^{19}$ C) and two anti Down quarks (each with the charge of $+\frac{1}{3}\times 1.606\times$ $\times 10^{19}$ C).

An antineutron has also a neutral charge, through it is opposite to a neutron, and they annihilate each other when meeting.

Both, the neutron and the antineutron, are neither attracted to nor repelling from charges particles.

7 Characteristics of unmatter

Unmatter should look identical to antimatter and matter, also the gravitation should similarly act on all three of them. Unmatter may have, analogously to antimatter, utility in medicine and may be stored in vacuum in traps which have the required configuration of electric and magnetic fields for several months.

8 Open Questions

8.a Can a matter atom and an unmatter atom of first type stick together to form a molecule?

8.b Can an antimatter atom and an unmatter atom of second type stick together to form a molecule?

8.c There might be not only a You and an anti-You, but some versions of an un-You in between You and anti-You. There might exist un-planets, un-stars, un-galaxies? There might be, besides our universe, an anti-universe, and more un-universes?

8.d Could this unmatter explain why we see such an imbalance between matter and antimatter in our corner of the universe? (Jeff Farinacci)

8.e If matter is thought to create gravity, is there any way that antimatter or unmatter can create antigravity or ungravity? (Mike Shafer from Cornell University)

I assume that since the magnetic field or the gravitons generate gravitation for the matter, then for antimatter and unmatter the corresponding magnetic fields or gravitons would look different since the charges of subatomic particles are different. . .

I wonder how would the universal law of attraction be for antimmater and unmatter?

References

1. Smarandache F. Unmatter, mss., 1980, Archives Vâlcea.
2. Smarandache F. A unifying field in logics: neutrosophic logic. Neutrosophy, Neutrosophic Probability, Set, and Logic. American Research Press, Rehoboth, 2002, 144 p. (see it in e-print: http://www.gallup.unm.edu/∼smarandache).
3. Webster's New World Dictionary. Third College Edition, Simon and Schuster Inc., 1988.
4. Mondardini R. The history of antimatter. CERN Laboratory, Genève, on-line http://livefromcern.web.cern.ch/livefromcern/antimatter/history/AM-history00.html.
5. De Rújula A. and Landua R. Antimatter — frequently asked questions. CERN Laboratory, Genève, http://livefromcern.web.cern.ch/livefromcern/antimatter/FAQ1.html.
6. Pompos A. Inquiring minds — questions about physics. Fermilab, see on-line http://www.fnal.gov/pub/inquiring/questions/antineuron.html.

Fractality Field in the Theory of Scale Relativity

Laurent Nottale

Centre National de la Recherche Scientifique, Laboratoire "Univers et Théories", Observatoire de Paris-Meudon, F-92195 Meudon Cedex, France

E-mail: laurent.nottale@obspm.fr

In the theory of scale relativity, space-time is considered to be a continuum that is not only curved, but also non-differentiable, and, as a consequence, fractal. The equation of geodesics in such a space-time can be integrated in terms of quantum mechanical equations. We show in this paper that the quantum potential is a manifestation of such a fractality of space-time (in analogy with Newton's potential being a manifestation of curvature in the framework of general relativity).

1 Introduction

The theory of scale relativity aims at describing a non-differentiable continuous manifold by the building of new tools that implement Einstein's general relativity concepts in the new context (in particular, covariant derivative and geodesics equations). We refer the reader to Refs. [1, 2, 3, 4] for a detailed description of the construction of these tools. In the present short research note, we want to address a specific point of the theory, namely, the emergence of an additional potential energy which manifests the fractal and nondifferentiable geometry.

2 Non relativistic quantum mechanics

2.1 Quantum potential

In the scale relativity approach, one decomposes the velocity field on the geodesics bundle of a nondifferentiable space-time in terms of a classical, differentiable part, $\mathcal{V}$, and of a fractal, divergent, nondifferentiable part $\mathcal{W}$ of zero mean. Both velocity fields are complex due to a fundamental two-valuedness of the classical (differentiable) velocity issued from the nondifferentiability [1]. Then one builds a complex covariant total derivative that reads in the simplest case (spinless particle, nonrelativistic velocities and no external field) [1, 2, 3]

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}\cdot\nabla - i\,\mathcal{D}\Delta\,. \tag{1}$$

The constant $2\mathcal{D} = \langle d\xi^2\rangle/dt$ ($=\hbar/m$ in standard quantum mechanics) measures the amplitude of the fractal fluctuations. Note that it is possible to have a more complete construction in which the full velocity field $\mathcal{V}+\mathcal{W}$ intervenes in the covariant derivative [6]. In the same way as in general relativity, the geodesics equation can therefore be written, using this covariant derivative, in terms of a free, inertial motion-like equation,

$$\frac{\widehat{d}\mathcal{V}}{dt} = 0\,. \tag{2}$$

Let us explicitly introduce the real and imaginary parts of the complex velocity $\mathcal{V} = V - iU$,

$$\frac{\widehat{d}\mathcal{V}}{dt} = \left(\left\{\frac{\partial}{\partial t} + V\cdot\nabla\right\} - i\,\{U\cdot\nabla + \mathcal{D}\Delta\}\right)(V - iU) = 0\,. \tag{3}$$

We see in this expression that the real part of the covariant derivative, $\widehat{d}_R/dt = \partial/\partial t + V\cdot\nabla$, is the standard total derivative expressed in terms of partial derivatives, while the new terms are included in the imaginary part, $\widehat{d}_I/dt = -(U\cdot\nabla + \mathcal{D}\Delta)$. The field will find its origin in the consequences of these additional terms on the imaginary part of the velocity $-U$. Indeed, by separating the real and imaginary parts, equation (3) reads:

$$\begin{aligned}&\left\{\left(\frac{\partial}{\partial t} + V\cdot\nabla\right)V - (U\cdot\nabla + \mathcal{D}\Delta)U\right\} - \\ &- i\left\{(U\cdot\nabla + \mathcal{D}\Delta)V + \left(\frac{\partial}{\partial t} + V\cdot\nabla\right)U\right\} = 0\,.\end{aligned} \tag{4}$$

Therefore the real part of this equation takes the form of an Euler-Newton equation of dynamics

$$\left(\frac{\partial}{\partial t} + V\cdot\nabla\right)V = (U\cdot\nabla + \mathcal{D}\Delta)\,U\,, \tag{5}$$

i. e.,

$$\frac{dV}{dt} = \frac{F}{m}\,, \tag{6}$$

where the total derivative of the velocity field V takes its standard form $dV/dt = (\partial/\partial t + V\nabla)V$ and where the force F is given by $F = m(U\cdot\nabla U + \mathcal{D}\Delta U)$.

Recall that, after one has introduced the wave function ψ from the complex action $\mathcal{S} = S_R + iS_I$, namely, $\psi = \exp(i\mathcal{S}/2m\mathcal{D}) = \sqrt{P}\exp(iS_R/2m\mathcal{D})$, equation (2) and its generalization including a scalar field, $m\,\widehat{d}\mathcal{V}/dt = -\nabla\phi$ can be integrated under the form of a Schrödinger equation [1]

$$\mathcal{D}^2\Delta\psi + i\,\mathcal{D}\frac{\partial\psi}{\partial t} - \frac{\phi}{2m}\psi = 0\,. \tag{7}$$

Let us now show that the additional force derives from a potential. Indeed, the imaginary part of the complex velocity field is given, in terms of the modulus of ψ, by the expression:

$$U = \mathcal{D}\nabla \ln P \,. \tag{8}$$

The force becomes

$$F = m\mathcal{D}^2 \left[(\nabla \ln P \cdot \nabla)(\nabla \ln P) + \Delta(\nabla \ln P) \right]. \tag{9}$$

Now, by introducing $\sqrt{P}$ in this expression, one makes explicitly appear the remarkable identity that is already at the heart of the proof of the Schrödinger equation ([1], p. 151), namely,

$$F = 2m\mathcal{D}^2 \Big[2(\nabla \ln \sqrt{P} \cdot \nabla)(\nabla \ln \sqrt{P}) + \\ + \Delta(\nabla \ln \sqrt{P}) \Big] = 2m\mathcal{D}^2 \nabla \left(\frac{\Delta \sqrt{P}}{\sqrt{P}} \right). \tag{10}$$

Therefore the force F derives from a potential energy

$$Q = -2m\mathcal{D}^2 \frac{\Delta \sqrt{P}}{\sqrt{P}} \,, \tag{11}$$

which is nothing but the standard "quantum potential", but here established as a mere manifestation of the nondifferentiable and fractal geometry instead of being deduced from a postulated Schrödinger equation.

The real part of the motion equation finally takes the standard form of the equation of dynamics in presence of a scalar potential,

$$\frac{dV}{dt} = \left(\frac{\partial}{\partial t} + V \cdot \nabla \right) V = -\frac{\nabla Q}{m} \,, \tag{12}$$

while the imaginary part is the equation of continuity $\partial P / \partial t + \mathrm{div}(PV) = 0$. The fact that the field equation is derived from the same remarkable identity that gives rise to the Schrödinger equation is also manifest in the similarity of its form with the free stationary Schrödinger equation, namely,

$$\mathcal{D}^2 \Delta \sqrt{P} + \frac{Q}{2m} \sqrt{P} = 0 \quad \longleftrightarrow \quad \mathcal{D}^2 \Delta \psi + \frac{E}{2m} \psi = 0 \,. \tag{13}$$

Now, the form (11) of the field equation means that the field can be known only after having solved the Schrödinger equation for the wave function. This is a situation somewhat different from that of general relativity, where, at least for test-particles, the description is reversed: given the energy-momentum tensor, one solves the Einstein field (i. e. space-time geometry) equations for the metric potentials, then one writes the geodesics equation in the space-time so determined and solve it for the motion of the particle. However, even in general relativity this case is an ideally simplified situation, since already in the two-body problem the motion of the bodies should be injected in the energy-momentum tensor, so that this is a looped system which has no exact analytical solution.

In the case of a quantum mechanical particle considered in scale relativity, the loop between the motion (geodesics) equation and the field equation is even more tight. Indeed, here the concept of test-particle loses its meaning. Even in the case of only one "particle", the space-time geometry is determined by the particle itself and by its motion, so that the field equation and the geodesics equation now participate of the same level of description. This explains why the motion/geodesics equation, in its Hamilton-Jacobi form that takes the form of the Schrödinger equation, is obtained without having first written the field equation in an explicit way. Actually, the potential Q is implicitly contained in the Schrödinger form of the equations, and it is made explicit only when coming back to a fluid-like Euler-Newton representation. In the end, the particle is described by a wave function (which is constructed, in the scale relativity theory, from the geodesics), of which only the square of the modulus P is observable. Therefore one expects the "field" to be given by a function of P, which is exactly what is found.

2.2 Invariants and energy balance

Let us now make explicit the energy balance by accounting for this additional potential energy. This question has already been discussed in [7, 8] and in [9], but we propose here a different presentation. We shall express the energy equation in terms of the various equivalent variables which we use in scale relativity, namely, the wave function ψ, the complex velocity $\mathcal{V}$ or its real and imaginary parts V and $-U$.

The first and main form of the energy equation is the Schrödinger equation itself, that we have derived as a prime integral of the geodesics equation. The Schrödinger equation is therefore the quantum equivalent of the metric form (i. e., of the equation of conservation of the energy). It may be written in the free case under the form

$$\mathcal{D}^2 \, \frac{\Delta \psi}{\psi} = -i\mathcal{D} \, \frac{\partial \ln \psi}{\partial t} \,. \tag{14}$$

In the stationary case with given energy E, it becomes:

$$E = -2m\mathcal{D}^2 \, \frac{\Delta \psi}{\psi} \,. \tag{15}$$

Now we can use the fundamental remarkable identity $\Delta\psi/\psi = (\nabla \ln \psi)^2 + \Delta \ln \psi$. Re-introducing the complex velocity field $\mathcal{V} = -2i\mathcal{D}\nabla \ln \psi$ in this expression we finally obtain the correspondence:

$$E = -2m\mathcal{D}^2 \, \frac{\Delta \psi}{\psi} = \frac{1}{2} \, m \left(\mathcal{V}^2 - 2i\mathcal{D}\nabla \cdot \mathcal{V} \right). \tag{16}$$

Note that when a potential term is present, all these relations remain true by replacing E by $E - \phi$.

This is the non-relativistic equivalent of Pissondes' relation [8] in the relativistic case, $\mathcal{V}^\mu \mathcal{V}_\mu + i\lambda\partial^\mu \mathcal{V}_\mu = 1$ (see also hereafter). Therefore the form of the energy $E = (1/2)mV^2$ is not conserved: this is precisely due to the existence of the additional potential energy of geometric origin. Let us prove this statement.

From equation (16) we know that the imaginary part of $(\mathcal{V}^2 - 2i\mathcal{D}\nabla\cdot\mathcal{V})$ is zero. By writing its real part in terms of the real velocities U and V, we find:

$$E = \frac{1}{2}\,m\left(\mathcal{V}^2 - 2i\mathcal{D}\nabla\cdot\mathcal{V}\right) = \\ = \frac{1}{2}\,m\left(V^2 - U^2 - 2\mathcal{D}\nabla\cdot U\right). \tag{17}$$

Now we can express the potential energy Q given in equation (11) in terms of the velocity field U:

$$Q = -\frac{1}{2}\,m\left(U^2 - 2\mathcal{D}\nabla\cdot U\right), \tag{18}$$

so that we finally write the energy balance under the three equivalent forms:

$$E = -2m\mathcal{D}^2\,\frac{\Delta\psi}{\psi} = \\ = \frac{1}{2}\,m\left(\mathcal{V}^2 - 2i\mathcal{D}\nabla\cdot\mathcal{V}\right) = \frac{1}{2}\,mV^2 + Q\,. \tag{19}$$

More generally, in presence of an external potential energy ϕ and in the non-stationary case, it reads:

$$-\frac{\partial S_R}{\partial t} = \frac{1}{2}\,mV^2 + Q + \phi\,, \tag{20}$$

where S_R is the real part of the complex action (i.e., $S_R/2m\mathcal{D}$ is the phase of the wave function).

3 Relativistic quantum mechanics

3.1 Quantum potential

All the above description can be directly generalized to relativistic QM and the Klein-Gordon equation [10, 2, 3]. The geodesics equation still reads in this case:

$$\frac{\hat{d}\mathcal{V}_\alpha}{ds} = 0\,, \tag{21}$$

where the total derivative is given by [10, 3]

$$\frac{\hat{d}}{ds} = \left(\mathcal{V}^\mu + i\,\frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu\,. \tag{22}$$

The complex velocity field $\mathcal{V}_\alpha$ reads in terms of the wave function

$$\mathcal{V}_\alpha = i\lambda\,\partial_\alpha \ln\psi\,. \tag{23}$$

The relation between the non-relativistic fractal parameter $\mathcal{D}$ and the relativistic one λ is simply $2\mathcal{D} = \lambda c$. In particular, in the standard QM case, λ is the Compton length of the particle, $\lambda = \hbar/mc$, and we recover $\mathcal{D} = \hbar/2m$.

The calculations are similar to the non-relativistic case. We decompose the complex velocity in terms of its real and imaginary parts, $\mathcal{V}_\alpha = V_\alpha - i\,U_\alpha$, so that the geodesics equation becomes

$$\left\{V^\mu - i\left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\right\}\partial_\mu\left(V_\alpha - i\,U_\alpha\right) = 0\,, \tag{24}$$

i.e.,

$$\left\{V^\mu\partial_\mu V_\alpha - \left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu U_\alpha\right\} - \\ - i\left\{\left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu V_\alpha + V^\mu\partial_\mu U_\alpha\right\} = 0\,. \tag{25}$$

The real part of this equation takes the form of a relativistic Euler-Newton equation of dynamics:

$$\frac{dV_\alpha}{ds} = V^\mu\partial_\mu V_\alpha = \left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu U_\alpha\,. \tag{26}$$

Therefore the relativistic case is similar to the non-relativistic one, since a generalized force also appears in the right-hand side of this equation. Let us now prove that it also derives from a potential. Using the expression for U_α in terms of the modulus $\sqrt{P}$ of the wave function,

$$U_\alpha = -\lambda\,\partial_\alpha \ln\sqrt{P}\,, \tag{27}$$

we may write the force under the form

$$\frac{F_\alpha}{m} = -\lambda\,\partial^\mu \ln\sqrt{P}\,\partial_\mu(-\lambda\,\partial_\alpha \ln\sqrt{P}) + \\ + \frac{\lambda^2}{2}\,\partial^\mu\partial_\mu\partial_\alpha \ln\sqrt{P} = \\ = \lambda^2\left(\partial^\mu \ln\sqrt{P}\,\partial_\mu\partial_\alpha \ln\sqrt{P} + \frac{1}{2}\partial^\mu\partial_\mu\partial_\alpha \ln\sqrt{P}\right). \tag{28}$$

Since $\partial^\mu\partial_\mu\partial_\alpha = \partial_\alpha\partial^\mu\partial_\mu$ commutes and since $\partial_\alpha(\partial^\mu \ln f\,\partial_\mu \ln f) = 2\,\partial^\mu \ln f\,\partial_\alpha\partial^\mu \ln f$, we obtain

$$\frac{F_\alpha}{m} = \frac{1}{2}\,\lambda^2\,\partial_\alpha\left(\partial^\mu \ln\sqrt{P}\,\partial_\mu \ln\sqrt{P} + \partial^\mu\partial_\mu \ln\sqrt{P}\right). \tag{29}$$

We can now make use of the remarkable identity (that generalizes to four dimensions the one which is also at the heart of the non-relativistic case)

$$\partial^\mu \ln\sqrt{P}\,\partial_\mu \ln\sqrt{P} + \partial^\mu\partial_\mu \ln\sqrt{P} = \frac{\partial^\mu\partial_\mu\sqrt{P}}{\sqrt{P}}\,, \tag{30}$$

and we finally obtain

$$\frac{dV_\alpha}{ds} = \frac{1}{2}\,\lambda^2\,\partial_\alpha\left(\frac{\partial^\mu\partial_\mu\sqrt{P}}{\sqrt{P}}\right). \tag{31}$$

Therefore, as in the non-relativistic case, the force derives from a potential energy

$$Q_R = \frac{1}{2} mc^2 \, \lambda^2 \, \frac{\partial^\mu \partial_\mu \sqrt{P}}{\sqrt{P}} \,, \tag{32}$$

that can also be expressed in terms of the velocity field U as

$$Q_R = \frac{1}{2} mc^2 \left(U^\mu U_\mu - \lambda \, \partial^\mu U_\mu \right) . \tag{33}$$

At the non-relativistic limit ($c \to \infty$), the Dalembertian $\partial^\mu \partial_\mu = (\partial^2/c^2 \partial t^2 - \Delta)$ is reduced to $-\Delta$, and since $\lambda = 2\mathcal{D}/c$, we recover the nonrelativistic potential energy $Q = -2m\mathcal{D}^2 \Delta \sqrt{P}/\sqrt{P}$. Note the correction to the potential introduced by Pissondes [7] which is twice this potential and therefore cannot agree with the nonrelativistic limit.

3.2 Invariants and energy balance

As shown by Pissondes [7, 8], the four-dimensional energy equation $u^\mu u_\mu = 1$ is generalized in terms of the complex velocity under the form $\mathcal{V}^\mu \mathcal{V}_\mu + i\lambda \partial^\mu \mathcal{V}_\mu = 1$. Let us show that the additional term is a manifestation of the new scalar field Q which takes its origin in the fractal and nondifferentiable geometry. Start with the geodesics equation

$$\frac{\hat{d} \mathcal{V}_\alpha}{ds} = \left(\mathcal{V}^\mu + i \, \frac{\lambda}{2} \, \partial^\mu \right) \partial_\mu \, \mathcal{V}_\alpha = 0 \,. \tag{34}$$

Then, after introducing the wave function by using the relation $\mathcal{V}_\alpha = i \lambda \, \partial_\alpha \ln \psi$, after calculations similar to the above ones (now on the full function ψ instead of only its modulus $\sqrt{P}$), the geodesics equation becomes:

$$\begin{aligned} \frac{\hat{d} \mathcal{V}_\alpha}{ds} &= -\frac{\lambda^2}{2} \, \partial_\alpha \left(\partial^\mu \ln \psi \, \partial_\mu \ln \psi + \partial^\mu \partial_\mu \ln \psi \right) = \\ &= \frac{1}{2} \, \partial_\alpha \left(-\lambda^2 \, \frac{\partial^\mu \partial_\mu \psi}{\psi} \right) = 0 \,. \end{aligned} \tag{35}$$

Under its right-hand form, this equation is integrated in terms of the Klein-Gordon equation,

$$\lambda^2 \, \partial^\mu \partial_\mu \psi + \psi = 0 \,. \tag{36}$$

Under its left hand form, the integral writes

$$-\lambda^2 (\partial^\mu \ln \psi \, \partial_\mu \ln \psi + \partial^\mu \partial_\mu \ln \psi) = 1 \,. \tag{37}$$

It becomes in terms of the complex velocity [8]

$$\mathcal{V}^\mu \mathcal{V}_\mu + i\lambda \partial^\mu \mathcal{V}_\mu = 1 \,, \tag{38}$$

which is therefore but another form taken by the KG equation (as expected from the fact that the KG equation is the quantum equivalent of the Hamilton-Jacobi equation). Let us now separate the real and imaginary parts of this equation. One obtains:

$$\begin{aligned} & V^\mu V_\mu - (U^\mu U_\mu - \lambda \, \partial^\mu U_\mu) = 1 \,, \\ & 2 \, V^\mu U_\mu - \lambda \, \partial^\mu V_\mu = 0 \,. \end{aligned} \tag{39}$$

Then the energy balance writes, in terms of the additional potential energy Q_R

$$V^\mu V_\mu = 1 + 2 \, \frac{Q_R}{mc^2} \,. \tag{40}$$

Let us show that we actually expect such a relation for the quadratic invariant in presence of an external potential ϕ. The energy relation writes in this case $(E - \phi)^2 = p^2 c^2 + m^2 c^4$, i. e. $E^2 - p^2 c^2 = m^2 c^4 + 2E\phi - \phi^2$. Introducing the rest frame energy by writing $E = mc^2 + E'$, we obtain

$$\begin{aligned} V^\mu V_\mu &= \frac{E^2 - p^2 c^2}{m^2 c^4} = \\ &= 1 + 2 \, \frac{\phi}{mc^2} + \left[2 \, \frac{E'}{mc^2} \, \frac{\phi}{mc^2} - \frac{\phi^2}{m^2 c^4} \right] . \end{aligned} \tag{41}$$

This justifies the relativistic factor 2 in equation (40) and supports the interpretation of Q_R in terms of a potential, at least at the level of the leading terms.

Now, concerning the additional terms, it should remain clear that this is only an approximate description in terms of field theory of what are ultimately (in this framework) the manifestations of the fractal and nondifferentiable geometry of space-time. Therefore we expect the field theory description to be a first order approximation in the same manner as, in general relativity, the description in terms of Newtonian potential.

In particular, in the non-relativistic limit $c \to \infty$ the last two terms of equation (41) vanish and we recover the energy equation (19) which is therefore exact in this case.

4 Conclusion

Placing ourselves in the framework of the scale-relativity theory, we have shown in a detailed way that the quantum potential, whose origin remained mysterious in standard quantum mechanics, is a manifestation of the nondifferentiability and fractality of space-time in the new approach.

This result is expected to have many applications, as well in physics as in other sciences, including biology [4]. It has been used, in particular, to suggest a new solution to the problem of "dark matter" in cosmology [11, 5], based on the proposal that chaotic gravitational system can be described on long time scales (longer than their horizon of predictibility) by the scale-relativistic equations and therefore by a macroscopic Schrödinger equation [12]. In this case there would be no need for additional non baryonic dark matter, since the various observed non-Newtonian dynamical effects (that the hypothesis of dark matter wants to explain despite the check of all attempts of detection) would be readily accounted for by the new scalar field that manifests the fractality of space.

References

1. Nottale L. Fractal space-time and microphysics: towards a Theory of Scale Relativity. World Scientific, Singapore, 1993.
2. Nottale L. *Chaos, Solitons & Fractals*, v. 7, 1996, 877.
3. Célérier M. N. and Nottale L. *J. Phys. A: Math. Gen.*, v. 37, 2004, 931.
4. Nottale L. *Computing Anticipatory Systems. CASYS'03 — Sixth International Conference*, Liège, Belgique, 11–16 August 2003, Ed. by Daniel M. Dubois, American Institute of Physics Conference Proceedings, v. 718, 2004, p. 68.
5. da Rocha D. and Nottale L. *Chaos Solitons & Fractals*, v. 16, 2003, 565.
6. Nottale L. *Chaos, Solitons & Fractals*, v. 10, 1999, 459.
7. Pissondes J. C. *Chaos, Solitons & Fractals*, v. 10, 1999, 513.
8. Pissondes J. C. *J. Phys. A: Math. Gen.*, v. 32, 1999, 2871.
9. Nottale L. *Chaos, Solitons & Fractals*, v. 9, 1998, 1051.
10. Nottale L. *Relativity in General*, 1993 Spanish Relativity Meeting, Salas, Ed. by J. Diaz Alonso and M. Lorente Paramo (Frontières), 1994, p. 121.
11. Nottale L. *Frontiers of Fundamental Physics*, Proceedings of Birla Science Center Fourth International Symposium, 11–13 Dec. 2000, Hyderabad, India, Ed. by B. G. Sidharth and M. V. Altaisky, Kluwer Academic, p. 65.
12. Nottale L. *Astron. Astrophys.*, v. 327, 1997, 867.

On the Possibility of Instant Displacements in the Space-Time of General Relativity

Larissa Borissova and Dmitri Rabounski
E-mail: lborissova@yahoo.com; rabounski@yahoo.com

Employing the mathematical apparatus of chronometric invariants (physical observable quantities), this study finds a theoretical possibility for the instant displacement of particles in the space-time of the General Theory of Relativity. This is to date the sole theoretical explanation of the well-known phenomenon of photon teleportation, given by the purely geometrical methods of Einstein's theory.

As it is known, the basic space-time of the General Theory of Relativity is a four-dimensional pseudo-Riemannian space, which is, in general, inhomogeneous, curved, rotating, and deformed. There the square of the space-time interval $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$, being expressed in the terms of physical observable quantities — chronometric invariants [1, 2], takes the form

$$ds^2 = c^2 d\tau^2 - d\sigma^2 .$$

Here the quantity

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i$$

is an interval of physical observable time, $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ is the gravitational potential, $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation, $d\sigma^2 = h_{ik}\,dx^i dx^k$ is the square of a spatial observable interval, $h_{ik} = -g_{ik} + \frac{1}{c^2}\, v_i v_k$ is the metric observable tensor, g_{ik} are spatial components of the fundamental metric tensor $g_{\alpha\beta}$ (space-time indices are Greek $\alpha, \beta = 0, 1, 2, 3$, while spatial indices — Roman $i, k = 1, 2, 3$).

Following this form we consider a particle displaced by ds in the space-time. We write ds^2 as follows

$$ds^2 = c^2 d\tau^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right),$$

where $\mathrm{v}^2 = h_{ik}\,\mathrm{v}^i \mathrm{v}^k$, and $\mathrm{v}^i = \frac{dx^i}{d\tau}$ is the three-dimensional observable velocity of the particle. So ds is: (1) a substantial quantity under $\mathrm{v} < c$; (2) a zero quantity under $\mathrm{v} = c$; (3) an imaginary quantity under $\mathrm{v} > c$.

Particles of non-zero rest-masses $m_0 \neq 0$ (substance) can be moved: (1) along real world-trajectories $cd\tau > d\sigma$, having real relativistic masses $m = \frac{m_0}{\sqrt{1 - \mathrm{v}^2/c^2}}$; (2) along imaginary world-trajectories $cd\tau < d\sigma$, having imaginary relativistic masses $m = \frac{im_0}{\sqrt{\mathrm{v}^2/c^2 - 1}}$ (tachyons). World-lines of both kinds are known as *non-isotropic trajectories*.

Particles of zero rest-masses $m_0 = 0$ (massless particles), having non-zero relativistic masses $m \neq 0$, move along world-trajectories of zero four-dimensional lengths $cd\tau = d\sigma$ at the velocity of light. They are known as *isotropic trajectories*. Massless particles are related to light-like particles — quanta of electromagnetic fields (photons).

A condition under which a particle may realize an instant displacement (*teleportation*) is equality to zero of the observable time interval $d\tau = 0$ so that the *teleportation condition* is

$$\mathrm{w} + v_i u^i = c^2,$$

where $u^i = \frac{dx^i}{dt}$ is its three-dimensional coordinate velocity. From this the square of that space-time interval by which this particle is instantly displaced takes the form

$$ds^2 = -d\sigma^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\,dx^i dx^k,$$

where $1 - \frac{\mathrm{w}}{c^2} = \frac{v_i u^i}{c^2}$ in this case, because $d\tau = 0$.

Actually, the signature $(+---)$ in the space-time area of a regular observer becomes $(-+++)$ in that space-time area where particles may be teleported. So the terms "time" and "three-dimensional space" are interchanged in that area. "Time" of teleporting particles is "space" of the regular observer, and vice versa "space" of teleporting particles is "time" of the regular observer.

Let us first consider substantial particles. As it easy to see, instant displacements (teleportation) of such particles manifests along world-trajectories in which $ds^2 = -d\sigma^2 \neq 0$ is true. So the trajectories represented in the terms of observable quantities are purely spatial lines of imaginary three-dimensional lengths $d\sigma$, although being taken in ideal world-coordinates t and x^i the trajectories are four-dimensional. In a particular case, where the space is free of rotation ($v_i = 0$) or its rotation velocity v_i is orthogonal to the particle's coordinate velocity u^i (so that $v_i u^i = |v_i||u^i| \cos(v_i; u^i) = 0$), substantial particles may be teleported only if gravitational collapse occurs ($\mathrm{w} = c^2$). In this case world-trajectories of teleportation taken in ideal world-coordinates become also purely spatial $ds^2 = g_{ik}\,dx^i dx^k$.

Second, massless light-like particles (photons) may be teleported along world-trajectories located in a space of the metric

$$ds^2 = -d\sigma^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\,dx^i dx^k = 0\,,$$

because for photons $ds^2 = 0$ by definition. So the space of photon teleportation characterizes itself by the conditions $ds^2 = 0$ and $d\sigma^2 = c^2 d\tau^2 = 0$.

The obtained equation is like the "light cone" equation $c^2 d\tau^2 - d\sigma^2 = 0$ ($d\sigma \neq 0$, $d\tau \neq 0$), elements of which are world-trajectories of light-like particles. But, in contrast to the light cone equation, the obtained equation is built by ideal world-coordinates t and x^i — not this equation in the terms of observable quantities. So teleporting photons move along trajectories which are elements of the world-cone (like the light cone) in that space-time area where substantial particles may be teleported (the metric inside that area has been obtained above).

Considering the photon teleportation cone equation from the viewpoint of a regular observer, we can see that the spatial observable metric $d\sigma^2 = h_{ik}\, dx^i dx^k$ becomes degenerate, $h = \det\,||h_{ik}|| = 0$, in the space-time area of that cone. Taking the relationship $g = -h\, g_{00}$ [1, 2] into account, we conclude that the four-dimensional metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ degenerates there as well $g = \det\,||g_{\alpha\beta}|| = 0$. The last fact implies that signature conditions defining pseudo-Riemannian spaces are broken, so that photon teleportation manifests outside the basic space-time of the General Theory of Relativity. Such a fully degenerate space was considered in [3, 4], where it was referred to as a *zero-space* because, from viewpoint of a regular observer, all spatial intervals and time intervals are zero there.

When $d\tau = 0$ and $d\sigma = 0$ observable relativistic mass m and the frequency ω become zero. Thus, from the viewpoint of a regular observer, all particles located in zero-space (in particular, teleporting photons) having zero rest-masses $m_0 = 0$ appear as zero relativistic masses $m = 0$ and the frequencies $\omega = 0$. Therefore particles of this kind may be assumed to be the ultimate case of massless light-like particles.

We will refer to all particles located in zero-space as *zero-particles*.

In the frames of the particle-wave concept each particle is given by its own wave world-vector $K_\alpha = \frac{\partial\psi}{\partial x^\alpha}$, where ψ is the wave phase (eikonal). The eikonal equation $K_\alpha K^\alpha = 0$ [5], setting forth that the length of the wave vector K^α remains unchanged*, for regular massless light-like particles (regular photons), becomes a travelling wave equation

$$\frac{1}{c^2}\left(\frac{{}^*\partial\psi}{\partial t}\right)^2 - h^{ik}\,\frac{{}^*\partial\psi}{\partial x^i}\,\frac{{}^*\partial\psi}{\partial x^k} = 0\,,$$

that may be obtained after taking $K_\alpha K^\alpha = g^{\alpha\beta}\,\frac{\partial\psi}{\partial x^\alpha}\,\frac{\partial\psi}{\partial x^\beta} = 0$ in the terms of physical observable quantities [1, 2], where we formulate regular derivatives through chronometrically invariant (physical observable) derivatives $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\,\frac{\partial}{\partial t}$ and $\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\,v_i\,\frac{{}^*\partial}{\partial t}$ and we use $g^{00} = \frac{1}{g_{00}}\left(1 - \frac{1}{c^2}\,v_i v^i\right)$, $v_k = h_{ik} v^i$, $v^i = -c g^{0i}\sqrt{g_{00}}$, $g^{ik} = -h^{ik}$.

The eikonal equation in zero-space takes the form

$$h^{ik}\,\frac{{}^*\partial\psi}{\partial x^i}\,\frac{{}^*\partial\psi}{\partial x^k} = 0$$

because there $\omega = \frac{{}^*\partial\psi}{\partial t} = 0$, putting the equation's time term into zero. It is a standing wave equation. So, from the viewpoint of a regular observer, in the frames of the particle-wave concept, all particles located in zero-space manifest as *standing light-like waves*, so that all zero-space appears filled with a system of light-like standing waves — a light-like hologram. This implies that an experiment for discovering non-quantum teleportation of photons should be linked to stationary light.

There is no problem in photon teleportation being realised along fully degenerate world-trajectories ($g = 0$) outside the basic pseudo-Riemannian space ($g < 0$), while teleportation trajectories of substantial particles are strictly non-degenerate ($g < 0$) so the lines are located in the pseudo-Riemannian space†. It presents no problem because at any point of the pseudo-Riemannian space we can place a tangential space of $g \leqslant 0$ consisting of the regular pseudo-Riemannian space ($g < 0$) and the zero-space ($g = 0$) as two different areas of the same manifold. A space of $g \leqslant 0$ is a natural generalization of the basic space-time of the General Theory of Relativity, permitting non-quantum ways for teleportation of both photons and substantial particles (previously achieved only in quantum fashion — quantum teleportation of photons in 1998 [6] and of atoms in 2004 [7, 8]).

Until now teleportation has had an explanation given only by Quantum Mechanics [9]. Now the situation changes: with our theory we can find physical conditions for the realisation of teleportation of both photons and substantial particles in a non-quantum way.

The only difference is that from the viewpoint of a regular observer the square of any parallely transported vector remains unchanged. It is also an "observable truth" for vectors in zero-space, because the observer reasons standards of his pseudo-Riemannian space anyway. The eikonal equation in zero-space, expressed in his observable world-coordinates, is $K_\alpha K^\alpha = 0$. But in ideal world-coordinates t and x^i the metric inside zero-space, $ds^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k = 0$, degenerates into a three-dimensional $d\mu^2$ which, depending

*According to Levi-Civita's rule, in a Riemannian space of n dimensions the length of any n-dimensional vector Q^α remains unchanged in parallel transport, so $Q_\alpha Q^\alpha = \text{const}$. So it is also true for the four-dimensional wave vector K^α in a four-dimensional pseudo-Riemannian space — the basic space-time of the General Theory of Relativity. Since $ds = 0$ is true along isotropic trajectories (because $cd\tau = d\sigma$), the length of any isotropic vector is zero, so that we have $K_\alpha K^\alpha = 0$.

†Any space of Riemannian geometry has the strictly non-degenerate metric feature $g < 0$ by definition. Pseudo-Riemannian spaces are a particular case of Riemannian spaces, where the metric is sign-alternating. So the four-dimensional pseudo-Riemannian space of the signature $(+---)$ or $(-+++)$ on which Einstein based the General Theory of Relativity is also a strictly non-degenerated metric ($g < 0$).

on gravitational potential w uncompensated by something else, is not invariant, $d\mu^2 = g_{ik}\,dx^i dx^k = \left(1-\frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 \neq$ $\neq$ inv. As a result, within zero-space, the square of a transported vector, a four-dimensional coordinate velocity vector U^α for instance, being degenerated into a spatial U^i, does not remain unchanged

$$U_i U^i = g_{ik}\,U^i U^k = \left(1-\frac{\mathrm{w}}{c^2}\right)^2 c^2 \neq \text{const},$$

so that although the geometry is Riemannian for a regular observer, the real geometry of zero-space within the space itself is non-Riemannian.

We conclude from this brief study that instant displacements of particles are naturally permitted in the space-time of the General Theory of Relativity. As it was shown, teleportation of substantial particles and photons realizes itself in different space-time areas. But it would be a mistake to think that teleportation requires the acceleration of a substantial particle to super-light speeds (the tachyons area), while a photon needs to be accelerated to infinite speed. No — as it is easy to see from the teleportation condition $\mathrm{w} + v_i u^i = c^2$, if gravitational potential is essential and the space rotates at a speed close to the velocity of light, substantial particles may be teleported at regular sub-light speeds. Photons can reach the teleportation condition easier, because they move at the velocity of light. From the viewpoint of a regular observer, as soon as the teleportation condition is realised in the neighbourhood of a moving particle, such particle “disappears” although it continues its motion at a sub-light coordinate velocity u^i (or at the velocity of light) in another space-time area invisible to us. Then, having its velocity reduced, or by the breaking of the teleportation condition by something else (lowering gravitational potential or the space rotation speed), it “appears” at the same observable moment at another point of our observable space at that distance and in the direction which it obtained by u^i there.

In connection with the results, it is important to remember the “Infinity Relativity Principle”, introduced by Abraham Zelmanov (1913–1987), a prominent cosmologist. Proceeding from his cosmological studies [1], he concluded that “. . . in homogeneous isotropic cosmological models spatial infinity of the Universe depends on our choice of that reference frame from which we observe the Universe (the observer’s reference frame). If the three-dimensional space of the Universe, being observed in one reference frame, is infinite, it may be finite in another reference frame. The same is as well true for the time during which the Universe evolves.”

We have come to the “Finite Relativity Principle” here. As it was shown, because of a difference between physical observable world-coordinates and ideal ones, the same space-time areas may be very different, being defined in each of the frames. Thus, in observable world-coordinates, zero-space is a point ($d\tau = 0$, $d\sigma = 0$), while $d\tau = 0$ and $d\sigma = 0$ taken in ideal world-coordinates become $-\left(1-\frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 +$ $+\, g_{ik}\,dx^i dx^k = 0$, which is a four-dimensional cone equation like the light cone. Actually here is the “Finite Relativity Principle” for observed objects — an observed point is the whole space taken in ideal coordinates.

Conclusions

This research currently is the sole explanation of virtual particles and virtual interaction given by the purely geometrical methods of Einstein’s theory. It is possible that this method will establish a link between Quantum Electrodynamics and the General Theory of Relativity.

Moreover, this research is currently the sole theoretical explanation of the observed phenomenon of teleportation [6, 7, 8] given by the General Theory of Relativity.

This paper was read at the conference “Today’s Take on Einstein’s Relativity” (Tucson, Arizona), February 18, 2005. The authors are grateful to Prof. Florentin Smarandache for his assistance and Stephen J. Crothers for some editing.

References

1. Zelmanov A. L. Chronometric invariants. Dissertation, 1944. First published: CERN, EXT-2004-117, 236 pages.
2. Zelmanov A. L. Chronometric invariants and co-moving coordinates in the general relativity theory. *Doklady Acad. Nauk USSR*, 1956, v. 107 (6), 815–818.
3. Rabounski D. D. and Borissova L. B. Particles here and beyond the Mirror. Editorial URSS, Moscow, 2001, 84 pages.
4. Borissova L. B. and Rabounski D. D. Fields, vacuum, and the mirror Universe. Editorial URSS, Moscow, 2001, 272 pages (the 2nd revised ed.: CERN, EXT-2003-025).
5. Landau L. D. and Lifshitz E. M. The classical theory of fields. GITTL, Moscow, 1939 (ref. with the 4th final exp. edition, Butterworth–Heinemann, 1980).
6. Boschi D., Branca S., De Martini F., Hardy L., and Popescu S. Experimental realization of teleporting an unknown pure quantum state via dual classical and Einstein-Podolsky-Rosen Channels. *Phys. Rev. Lett.*, 1998, v. 80, 1121–1125.
7. Riebe M., Häffner H., Roos C. F., Hänsel W., Benhelm J., Lancaster G. P. T., Korber T. W., Becher C., Schmidt-Kaler F., James D. F. V., and Blatt R. Deterministic quantum teleportation with atoms. *Nature*, 2004, v. 429, 734–736.
8. Barrett M. D., Chiaverini J., Schaetz T., Britton J., Itano W. M., Jost J. D., Knill E., Langer C., Leibfried D., Ozeri R., Wineland D. J. Deterministic quantum teleportation of atomic qubits. *Nature*, 2004, v. 429, 737–739.
9. Bennett C. H., Brassard G., Crepeau C., Jozsa R., Peres A., and Wootters W. K. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett.*, 1993, v. 70, 1895–1899.

On Dual Phase-Space Relativity, the Machian Principle and Modified Newtonian Dynamics

Carlos Castro
Center for Theor. Studies of Physical Systems, Clark Atlanta University, Atlanta, USA
E-mail: czarlosromanov@yahoo.com; castro@ctsps.cau.edu

We investigate the consequences of the Mach's principle of inertia within the context of the Dual Phase Space Relativity which is compatible with the Eddington-Dirac large numbers coincidences and may provide with a physical reason behind the observed anomalous Pioneer acceleration and a solution to the riddle of the cosmological constant problem. The cosmological implications of Non-Archimedean Geometry by assigning an upper impossible scale in Nature and the cosmological variations of the fundamental constants are also discussed. We study the corrections to Newtonian dynamics resulting from the Dual Phase Space Relativity by analyzing the behavior of a test particle in a *modified* Schwarzschild geometry (due to the the effects of the maximal acceleration) that leads in the weak-field approximation to essential *modifications* of the Newtonian dynamics and to violations of the equivalence principle. Finally we follow another avenue and find modified Newtonian dynamics induced by the Yang's Noncommutative Spacetime algebra involving a lower and upper scale in Nature.

1 Introduction

In recent years we have argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein's general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature. A scale relativistic theory involving spacetime *resolutions* was developed long ago by Nottale where the Planck scale was postulated as the minimum observer independent invariant resolution in Nature [2]. Since "points" cannot be observed physically with an ultimate resolution, they are fuzzy and smeared out into fuzzy balls of Planck radius of arbitrary dimension. For this reason one must construct a theory that includes all dimensions (and signatures) on the equal footing. Because the notion of dimension is a topological invariant, and the concept of a fixed dimension is lost due to the fuzzy nature of points, dimensions are resolution-dependent, one must also include a theory with *all* topologies as well. It turned out that Clifford algebras contained the appropriate algebro-geometric features to implement this principle of polydimensional transformations that reshuffle a five-brane history for a membrane history, for example. For an extensive review of this Extended Relativity Theory in Clifford Spaces that encompasses the unified dynamics of all p-branes, for different values of the dimensions of the extended objects, and numerous physical consequences, see [1].

A Clifford-space dynamical derivation of the stringy-minimal length uncertainty relations [11] was furnished in [45]. The dynamical consequences of the minimal-length in Newtonian dynamics have been recently reviewed by [44]. The idea of minimal length (the Planck scale L_P) can be incorporated within the context of the maximal acceleration Relativity principle [68] $a_{max} = c^2/L_P$ in Finsler Geometries [56] and [14]. A different approach than the one based on Finsler Geometries is the pseudo-complex Lorentz group description by Schuller [61] related to the effects of maximal acceleration in Born-Infeld models that also maintains Lorentz invariance, in contrast to the approaches of Double Special Relativity (DSR) [70] where the Lorentz symmetry is deformed. Quantum group deformations of the Poincaré symmetry and of Gravity have been analyzed by [69] where the deformation parameter q could be interpreted in terms of an upper and lower scale as $q = e^{L_P/R}$ such that the undeformed limit $q = 1$ can be attained when $L_P \to 0$ and/or when $R \to \infty$ [68]. For a discussions on the open problems of Double Special Relativity theories based on kappa-deformed Poincaré symmetries [63] and motivated by the anomalous Lorentz-violating dispersion relations in the ultra high energy cosmic rays [71, 73], we refer to [70].

An upper limit on the maximal acceleration of particles was proposed long ago by Caianiello [52]. This idea is a direct consequence of a suggestion made years earlier by Max Born on a Dual Relativity principle operating in Phase Spaces [49], [74] where there is an upper bound on the four-force (maximal string tension or tidal forces in strings) acting on a particle as well as an upper bound in the particle's velocity given by the speed of light. For a recent status of the geometries behind maximal-acceleration see [73]; its relation to the Double Special Relativity programs was studied by [55] and the possibility that Moyal deformations of Poincaré algebras could be related to the kappa-deformed Poincaré algebras was raised in [68]. A thorough study of Finsler geometry and Clifford algebras has been undertaken by Vacaru [81] where Clifford/spinor structures were defined with respect to Nonlinear connections associated with certain non-holonomic modifications of Riemann-Cartan gravity. The study of non-holonomic Clifford-Structures in the construction of a Noncommutative Riemann-Finsler Geometry has recently been advanced by [81].

Other implications of the maximal acceleration principle in Nature, like neutrino oscillations and other phenomena, have been studied by [54], [67], [22]. Recently, the variations of the fine structure constant α [64] with the cosmological accelerated expansion of the Universe was recast as a renormalization group-like equation governing the cosmological red shift (Universe scale) variations of α based on this maximal acceleration principle in Nature [68]. The fine structure constant was smaller in the past. Pushing the cutoff scale to the minimum Planck scale led to the intriguing result that the fine structure constant could have been extremely small (zero) in the early Universe and that all matter in the Universe could have emerged via the Unruh-Rindler-Hawking effect (creation of radiation/matter) due to the acceleration w. r. t the vacuum frame of reference. For reviews on the alleged variations of the fundamental constants in Nature see [65].

The outline of this work goes as follows. In section 2 we review the Dual Phase Space Relativity and show why the Planck areas are invariant under acceleration-boosts transformations.

In 3.1 we investigate the consequences of the Mach's principle of inertia within the context of the Dual Phase Space Relativity Principle which is compatible with the Eddington-Dirac large numbers coincidence and may provide with a very plausible physical reason behind the observed anomalous Pioneer acceleration due to the fact that the universe is in accelerated motion (a non-inertial frame of reference) w. r. t the vacuum. Our proposal shares similarities with the previous work of [6], [3]. To our knowledge, the first person who *predicted* the Pioneer anomaly in 1978 was P. LaViolette [5], from an entirely different approach based on the novel theory of sub-quantum kinetics to explain the vacuum fluctuations, two years *prior* to the Anderson et al observations [7]. The cosmological implications of Non-Archimedean Geometry [94] by assigning an upper impassible scale in Nature [2] and the cosmological variations of the fundamental constants are also discussed.

In 3.2 the crucial modifications to Newtonian dynamics resulting from the Dual Phase Space Relativity are analyzed further. In particular, the physical consequences of an upper and lower bounds in the acceleration and an upper and lower bounds in the angular velocity. We study the particular behavior of a test particle living in a *modified* Schwarzschild geometry (due to the the effects of the principle of maximal acceleration) that leads in the weak-field approximation to essential *modifications* of the Newtonian dynamics and to violations of the equivalence principle. For violations of the equivalence principle in neutrino oscillations see [42], [54].

Finally, in 4 we study another interesting avenue for the origins of modified Newtonian dynamics based on Yang's Noncommutative Spacetime algebra involving a lower and upper scale [136] that has been revisited recently by us [134] in the context of holography and area-quantization in C-spaces (Clifford spaces); in the physics of D-branes and covariant Matrix models by [137] and within the context of Lie algebra stability by [48]. A different algebra with two length scales has been studied by [43] in order to account for modifications of Newtonian dynamics (that also violates the equivalence principle).

2 Dual Phase-Space Relativity

In this section we will review in detail the Born's Dual Phase Space Relativity and the principle of Maximal-acceleration Relativity [68] from the perspective of $8D$ Phase Spaces and the role of the invariance $U(1,3)$ Group. We will focus for simplicity on a *flat* $8D$ Phase Space. A *curved* case scenario has been analyzed by Brandt [56] within the context of the Finsler geometry of the $8D$ tangent bundle of spacetime and written the generalized $8D$ gravitational equations that reduce to the ordinary Einstein-Riemannian gravitational equations in the *infinite* acceleration limit. Vacaru [81] has constructed the Riemann-Finsler geometries endowed with non-holonomic structures induced by *nonlinear* connections and developed the formalism to build a Noncommutative Riemann-Finsler Geometry by introducing suitable Clifford structures. A curved *momentum* space geometry was studied by [50]. Toller [73] has explored the different possible geometries associated with the maximal acceleration principle and the physical implications of the meaning of an "observer", "measuring device" in the cotangent space.

The $U(1,3) = SU(1,3) \otimes U(1)$ Group transformations, which leave invariant the phase-space intervals under rotations, velocity and acceleration boosts, were found by Low [74] and can be simplified drastically when the velocity/acceleration boosts are taken to lie in the z-direction, leaving the transverse directions x, y, p_x, p_y intact; i. e., the $U(1,1) = SU(1,1) \otimes U(1)$ subgroup transformations that leave invariant the phase-space interval are given by (in units of $\hbar = c = 1$)

$$\begin{aligned}(d\omega)^2 &= (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} = \\ &= (d\tau)^2 \left[1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}\right] = \\ &= (d\tau)^2 \left[1 - \frac{m^2 g^2(\tau)}{m_P^2 A_{max}^2}\right],\end{aligned} \tag{2.1}$$

where we have factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in eq-(2.1) and the maximal proper-force is set to be $b \equiv m_P A_{max}$. Here m_P is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, may also be interpreted as the maximal string tension when L_P is the Planck scale.

The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass m in the z-direction which we take to be defined by the X coordinate. The interval $(d\omega)^2$ described by Low [74] is $U(1,3)$-invariant for the most general transformations in the $8D$ phase-space. These transformations are rather elaborate, so we refer to the references [74] for details. The appearance of the $U(1,3)$ group in $8D$ Phase Space is not too surprising since it could be seen as the "complex doubling" version of the Lorentz group $SO(1,3)$. Low discussed the irreducible unitary representations of such $U(1,3)$ group and the relevance for the strong interactions of quarks and hadrons since $U(1,3)$, with 16 generators, contains the $SU(3)$ group.

The analog of the Lorentz relativistic factor in eq-(2.1) involves the ratios of two proper *forces*. One variable force is given by $mg(\tau)$ and the maximal proper force sustained by an *elementary* particle of mass m_P (a *Planckton*) is assumed to be $F_{max} = m_{Planck} c^2/L_P$. When $m = m_P$, the ratio-squared of the forces appearing in the relativistic factor of eq-(2.1) becomes then g^2/A^2_{max}, and the phase space interval coincides with the geometric interval discussed by [61], [54], [67], [22].

The transformations laws of the coordinates in that leave invariant the interval (2.1) were given by [74]:

$$T' = T\cosh\xi + \left(\frac{\xi_v X}{c^2} + \frac{\xi_a P}{b^2}\right)\frac{\sinh\xi}{\xi}\,, \tag{2.2a}$$

$$E' = E\cosh\xi + (-\xi_a X + \xi_v P)\,\frac{\sinh\xi}{\xi}\,, \tag{2.2b}$$

$$X' = X\cosh\xi + \left(\xi_v T - \frac{\xi_a E}{b^2}\right)\frac{\sinh\xi}{\xi}\,, \tag{2.2c}$$

$$P' = P\cosh\xi + \left(\frac{\xi_v E}{c^2} + \xi_a T\right)\frac{\sinh\xi}{\xi}\,. \tag{2.2d}$$

The ξ_v is velocity-boost rapidity parameter and the ξ_a is the force/acceleration-boost rapidity parameter of the primed-reference frame. They are defined respectively:

$$\tanh\left(\frac{\xi_v}{c}\right) = \frac{v}{c}, \quad \tanh\left(\frac{\xi_a}{b}\right) = \frac{ma}{m_P A_{max}}\,. \tag{2.3}$$

The *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs-(2.2a, 2.2d) is defined in terms of the velocity and acceleration boosts parameters ξ_v, ξ_a respectively as:

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}}\,. \tag{2.4}$$

Our definition of the rapidity parameters are *different* than those in [74].

Straightforward algebra allows us to verify that these transformations leave the interval of eq-(2.1) in classical phase space invariant. They are are fully consistent with Born's duality Relativity symmetry principle [49] $(Q,P) \rightarrow \rightarrow (P,-Q)$. By inspection we can see that under Born duality, the transformations in eqs-(2.2a, 2.2d) are *rotated* into each other, up to numerical b factors in order to match units. When on sets $\xi_a = 0$ in (2.2a, 2.2d) one recovers automatically the standard Lorentz transformations for the X,T and E,P variables *separately*, leaving invariant the intervals $dT^2 - dX^2 = (d\tau)^2$ and $(dE^2 - dP^2)/b^2$ separately.

When one sets $\xi_v = 0$ we obtain the transformations rules of the events in Phase space, from one reference-frame into another *uniformly*-accelerated frame of reference, $a = \text{const}$, whose acceleration-rapidity parameter is in this particular case:

$$\xi \equiv \frac{\xi_a}{b}, \quad \tanh(\xi) = \frac{ma}{m_P A_{max}}\,. \tag{2.5}$$

The transformations for pure acceleration-boosts in Phase Space are:

$$T' = T\cosh\xi + \frac{P}{b}\sinh\xi\,, \tag{2.6a}$$

$$E' = E\cosh\xi - bX\sinh\xi\,, \tag{2.6b}$$

$$X' = X\cosh\xi - \frac{E}{b}\sinh\xi\,, \tag{2.6c}$$

$$P' = P\cosh\xi + bT\sinh\xi\,. \tag{2.6d}$$

It is straightforward to verify that the transformations (2.6a, 2.6c) leave invariant the fully phase space interval (2.1) but *does not* leave invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$. Only the *combination*:

$$(d\omega)^2 = (d\tau)^2\left(1 - \frac{m^2 g^2}{m_P^2 A^2_{max}}\right) \tag{2.7a}$$

is truly left invariant under pure acceleration-boosts in Phase Space. Once again, can verify as well that these transformations satisfy Born's duality symmetry principle:

$$(T,X) \rightarrow (E,P), \ (E,P) \rightarrow (-T,-X) \tag{2.7b}$$

and $b \rightarrow \frac{1}{b}$. The latter Born duality transformation is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i. e. namely, a small/large radius duality, a winding modes/Kaluza-Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in Noncommutative Field Theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory.

The composition of two successive pure acceleration-boosts is another pure acceleration-boost with acceleration rapidity given by $\xi'' = \xi + \xi'$. The addition of *proper* forces (accelerations) follows the usual relativistic composition rule:

$$\tanh\xi'' = \tanh(\xi+\xi') = \\ = \frac{\tanh\xi+\tanh\xi'}{1+\tanh\xi\tanh\xi'} \Rightarrow \frac{ma''}{m_P A} = \frac{\frac{ma}{m_P A}+\frac{ma'}{m_P A}}{1+\frac{m^2 aa'}{m_P^2 A^2}} \tag{2.8}$$

and in this fashion the upper limiting *proper* acceleration is never *surpassed* like it happens with the ordinary Special Relativistic addition of velocities.

The group properties of the full combination of velocity and acceleration boosts eqs-(2.2a, 2.2d) in Phase Space requires much more algebra [68]. A careful study reveals that the composition *rule* of two successive full transformations is given by $\xi''=\xi+\xi'$ and the transformation laws are *preserved* if, and only if, the $\xi;\xi';\xi''\ldots$ parameters obeyed the suitable relations:

$$\frac{\xi_a}{\xi} = \frac{\xi'_a}{\xi'} = \frac{\xi''_a}{\xi''} = \frac{\xi''_a}{\xi+\xi'}\,, \tag{2.9a}$$

$$\frac{\xi_v}{\xi} = \frac{\xi'_v}{\xi'} = \frac{\xi''_v}{\xi''} = \frac{\xi''_v}{\xi+\xi'}\,. \tag{2.9b}$$

Finally we arrive at the composition law for the effective, velocity and acceleration boosts parameters ξ''; ξ''_v; ξ''_a respectively:

$$\xi''_v = \xi_v + \xi'_v\,, \tag{2.10a}$$

$$\xi''_a = \xi_a + \xi'_a\,, \tag{2.10b}$$

$$\xi'' = \xi + \xi'. \tag{2.10c}$$

The above relations among the parameters are required in order to prove the $U(1,1)$ *group* composition law of the transformations in order to have a truly Maximal-Acceleration Phase Space Relativity theory resulting from a Phase-Space change of coordinates in the cotangent bundle of spacetime.

2.1 Planck-scale Areas are invariant under acceleration boosts

Having displayed explicitly the Group transformations rules of the coordinates in Phase space we will show why *infinite* acceleration-boosts (which is *not* the same as infinite proper acceleration) preserve Planck-scale *Areas* [68] as a result of the fact that $b=(1/L_P^2)$ equals the *maximal* invariant force, or string tension, if the units of $\hbar=c=1$ are used.

At Planck-scale L_P intervals/increments in one reference frame we have by definition (in units of $\hbar=c=1$): $\Delta X = = \Delta T = L_P$ and $\Delta E = \Delta P = \frac{1}{L_P}$ where $b\equiv\frac{1}{L_P^2}$ is the maximal tension. From eqs-(2.6a, 2.6d) we get for the transformation rules of the finite intervals ΔX, ΔT, ΔE, ΔP, from one reference frame into another frame, in the *infinite* acceleration-boost limit $\xi\to\infty$,

$$\Delta T' = L_P(\cosh\xi+\sinh\xi)\to\infty$$

$$\Delta E' = \frac{1}{L_P}(\cosh\xi-\sinh\xi)\to 0 \tag{2.11b}$$

by a simple use of L'Hôpital's rule or by noticing that both $\cosh\xi;\sinh\xi$ functions approach infinity at the same rate

$$\Delta X' = L_P(\cosh\xi-\sinh\xi)\to 0\,, \tag{2.11c}$$

$$\Delta P' = \frac{1}{L_P}(\cosh\xi+\sinh\xi)\to\infty\,, \tag{2.11d}$$

where the discrete displacements of two events in Phase Space are defined: $\Delta X = X_2 - X_1 = L_P$, $\Delta E = E_2 - E_1 = \frac{1}{L_P}$, $\Delta T = T_2 - T_1 = L_P$ and $\Delta P = P_2 - P_1 = \frac{1}{L_P}$.

Due to the identity:

$$(\cosh\xi+\sinh\xi)(\cosh\xi-\sinh\xi) = \\ = \cosh^2\xi - \sinh^2\xi = 1 \tag{2.12}$$

one can see from eqs-(2.11a, 2.11d) that the Planck-scale *Areas* are truly *invariant* under *infinite* acceleration-boosts $\xi=\infty$:

$$\Delta X'\Delta P' = 0\times\infty = \\ = \Delta X\Delta P(\cosh^2\xi-\sinh^2\xi) = \Delta X\Delta P = \frac{L_P}{L_P} = 1\,, \tag{2.13a}$$

$$\Delta T'\Delta E' = \infty\times 0 = \\ = \Delta T\Delta E\,(\cosh^2\xi-\sinh^2\xi) = \Delta T\Delta E = \frac{L_P}{L_P} = 1\,, \tag{2.13b}$$

$$\Delta X'\Delta T' = 0\times\infty = \\ = \Delta X\Delta T\,(\cosh^2\xi-\sinh^2\xi) = \Delta X\Delta T = (L_P)^2\,, \tag{2.13c}$$

$$\Delta P'\Delta E' = \infty\times 0 = \\ = \Delta P\Delta E\,(\cosh^2\xi-\sinh^2\xi) = \Delta P\Delta E = \frac{1}{L_P^2}\,. \tag{2.13d}$$

It is important to emphasize that the invariance property of the minimal Planck-scale *Areas* (maximal Tension) is *not* an exclusive property of *infinite* acceleration boosts $\xi=\infty$, but, as a result of the identity $\cosh^2\xi-\sinh^2\xi=1$, for all values of ξ, the minimal Planck-scale *Areas* are *always* invariant under *any* acceleration-boosts transformations. Meaning physically, in units of $\hbar=c=1$, that the Maximal Tension (or maximal Force) $b=\frac{1}{L_P^2}$ is a true physical *invariant* universal quantity. Also we notice that the Phase-space areas, or cells, in units of $\hbar$, are also invariant! The pure-acceleration boosts transformations are "symplectic". It can be shown also that areas greater (smaller) than the Planck-area remain greater (smaller) than the invariant Planck-area under acceleration-boosts transformations.

The infinite acceleration-boosts are closely related to the infinite red-shift effects when light signals barely escape Black hole Horizons reaching an asymptotic observer with an infinite red shift factor. The important fact is that the Planck-scale *Areas* are truly maintained invariant under acceleration-boosts. This could reveal very important information about Black-holes Entropy and Holography.

3 Modified Newtonian Dynamics from Phase Space Relativity

3.1 The Machian Principle and Eddington-Dirac Large Numbers Coincidence

A natural action associated with the invariant interval in Phase-Space given by eq-(2.1) is:

$$S = m \int d\tau \sqrt{1 + \frac{m^2}{m_P^2 a^2}(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)} \,. \quad (3.1)$$

The proper-acceleration is orthogonal to the proper-velocity and this can be easily verified by differentiating the time-like proper-velocity squared:

$$V^2 = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = V^\mu V_\mu = 1 > 0 \Rightarrow \\ \Rightarrow \frac{dV^\mu}{d\tau}V_\mu = \frac{d^2x^\mu}{d\tau^2}V_\mu = 0 \,, \quad (3.2)$$

which implies that the proper-acceleration is space-like:

$$-g^2(\tau) = \frac{d^2x^\mu}{d\tau^2}\frac{d^2x_\mu}{d\tau^2} < 0 \Rightarrow \\ \Rightarrow S = m\int d\tau \sqrt{1 - \frac{m^2 g^2}{m_P^2 a^2}} = m \int d\omega \,, \quad (3.3)$$

where the analog of the Lorentz time-dilation factor in Phase-space is now given by

$$d\omega = d\tau \sqrt{1 - \frac{m^2 g^2(\tau)}{m_P^2 a^2}} \,, \quad (3.4a)$$

namely,

$$(d\omega)^2 = \Omega^2 d\tau^2 = \left[1 - \frac{m^2 g^2(\tau)}{m_P^2 a^2}\right] g_{\mu\nu} dx^\mu dx^\nu . \quad (3.4b)$$

The invariant proper interval is no longer the standard proper-time τ but is given by the quantity $\omega(\tau)$. The deep connection between the physics of maximal acceleration and Finsler geometry has been analyzed by [56]. The action is real-valued if, and only if, $m^2 g^2 < m_P^2 a^2$ in the same fashion that the action in Minkowski spacetime is real-valued if, and only if, $v^2 < c^2$. This is the physical reason why there is an upper bound in the proper-four force acting on a fundamental particle given by $(mg)_{bound} = m_P(c^2/L_P) = m_P^2$ in natural units of $\hbar = c = 1$.

The Eddington-Dirac large numbers coincidence (and an ultraviolet/infrared entanglement) can be easily implemented if one equates the upper bound on the proper-four force sustained by a fundamental particle, $(mg)_{bound} = m_P(c^2/L_P)$, with the proper-four force associated with the mass of the (observed) universe M_U, and whose *minimal* acceleration c^2/R is given in terms of an infrared-cutoff R (the Hubble horizon radius). Equating these proper-four forces gives

$$\frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} \Rightarrow \frac{M_U}{m_P} = \frac{R}{L_P} \sim 10^{61} \,, \quad (3.5)$$

from this equality of proper-four forces associated with a maximal/minimal acceleration one infers $M_U \sim 10^{61}\, m_{Planck} \sim 10^{61} 10^{19} m_{proton} = 10^{80} m_{proton}$ which is indeed consistent with observations and agrees with the Eddington-Dirac number 10^{80}:

$$N = 10^{80} = (10^{40})^2 \sim \left(\frac{F_e}{F_G}\right)^2 \sim \left(\frac{R}{r_e}\right)^2 , \quad (3.6)$$

where $F_e = e^2/r^2$ is the electrostatic force between an electron and a proton; $F_G = G m_e m_{proton}/r^2$ is the corresponding gravitational force and $r_e = e^2/m_e \sim 10^{-13}$cm is the classical electron radius (in units $\hbar = c = 1$).

One may notice that the above equation (3.5) is also consistent with the Machian postulate that the rest mass of a particle is determined via the gravitational potential energy due to the other masses in the universe. In particular, by equating:

$$m_i c^2 = G m_i \sum_j \frac{m_j}{|r_i - r_j|} = \frac{G m_i M_U}{R} \Rightarrow \frac{c^2}{G} = \frac{M_U}{R} . \quad (3.7)$$

Due to the negative binding energy, the composite mass m_{12} of a system of two objects of mass m_1, m_2 is not equal to the sum $m_1 + m_2 > m_{12}$. We can now arrive at the conclusion that the *minimal* acceleration c^2/R is also the same acceleration induced on a test particle of mass m by a spherical mass distribution M_U inside a radius R. The acceleration felt by a test particle of mass m sitting at the edge of the observable Universe (at the Hubble horizon radius R) is:

$$\frac{G M_U}{R^2} = a \,. \quad (3.8)$$

From the last two equations (3.7, 3.8) one gets the same expression for the *minimal* acceleration:

$$a = a_{min} = \frac{c^2}{R} \,, \quad (3.9)$$

which is of the same order of magnitude as the anomalous acceleration of the Pioneer and Galileo spacecrafts $a \sim 10^{-8}$ cm/s^2. A very plausible physical reason behind the observed anomalous Pioneer acceleration could be due to the fact that the universe is in accelerated expansion and motion (a non-inertial frame of reference) w. r. t the vacuum. Our proposal shares some similarities with the previous work of [6]. To our knowledge, the first person who *predicted* the Pioneer anomaly in 1978 was P. LaViolette [5], from an entirely different approach based on the novel theory of sub-quantum kinetics to explain the vacuum fluctuations, two years *prior* to the Anderson et al observations [7]. Nottale has invoked the Machian principle of inertia [3] adopting a local and global inertial coordinate system at the scale of the solar system in order to explain the origins of this Pioneer-Galileo anomalous constant acceleration. The Dirac-Eddington large number coincidences from vacuum fluctuations was studied by [8].

Let us examine closer the equality between the proper-four forces

$$\frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} \Rightarrow \frac{m_P}{L_P} = \frac{M_U}{R} = \frac{c^2}{G} . \qquad (3.10)$$

The last term in eq-(3.10) is directly obtained after implementing the Machian principle in eq-(3.7). Thus, one concludes from eq-(3.10) that as the universe evolves in time one must have the conserved ratio of the quantities $M_U/R = c^2/G = m_P/L_P$. This interesting possibility, advocated by Dirac long ago, for the fundamental constants $\hbar$, c, G, ... to vary over cosmological time is a plausible idea with the provision that the above ratios satisfy the relations in eq-(3.10) at any given moment of cosmological time. If the fundamental constants do not vary over time then the ratio $M_U/R = c^2/G$ must refer then to the *asymptotic* values of the Hubble horizon radius $R = R_{asymptotic}$. A related approach to the idea of an impassible upper asymptotic length R has been advocated by Scale Relativity [2] and in Khare [94] where a Cosmology based on non-Archimedean geometry was proposed by recurring to p-adic numbers. For example, a Non-Archimedean number addition law of two masses m_1, m_2 does not follow the naive addition rule $m_1 + m_2$ but instead:

$$m_1 \bullet m_2 = \frac{m_1 + m_2}{1 + (m_1 m_2/M_U^2)} ,$$

which is similar to the composition law of velocities in ordinary Relativity in terms of the speed of light. When the masses m_1, m_2 are much smaller than the universe mass M_U one recovers the ordinary addition law. Similar considerations follow in the Non-Archimedean composition law of lengths such that the upper length R_{asym} is never surpassed. For further references on p-adic numbers and Physics were refer to [40]. A Mersenne prime, $M_p = 2^p - 1 = prime$, for $p = prime$, p-adic hierarchy of scales in Particle physics and Cosmology has been discussed by Pitkannen and Noyes where many of the the fundamental energy scales, masses and couplings in Physics has been obtained [41], [42]. For example, the Mersenne prime $M_{127} = 2^{127} - 1 \sim 10^{38} \sim (m_{Planck}/m_{proton})^2$. The derivation of the Standard Model parameters from first principle has obtained by Smith [43] and Beck [47].

In [68] we proposed a plausible explanation of the variable fine structure constant phenomenon based on the maximal-acceleration relativity principle in phase-space by modifying the Robertson-Friedmann-Walker metric by a similar (acceleration-dependent) conformal factor as in eqs-(3.4). It led us to the conclusion that the universe could have emerged from the vacuum as a quantum bubble (or "brane-world") of Planck mass and Planck radius that expanded (w. r. t to the vacuum) at the speed of light with a *maximal* acceleration $a = c^2/L_p$. Afterwards the acceleration began to slow down as matter was being created from the vacuum, via an Unruh-Rindler-Hawking effect, from this initial maximal value c^2/L_p to the value of $c^2/R \sim 10^{-8}$cm/s^2 (of the same order of magnitude as the Pioneer anomalous acceleration). Namely, as the universe expanded, matter was being created from the vacuum via the Unruh-Rindler-Hawking effect (which must not to be confused with Hoyle's Steady State Cosmolgy) such that the observable mass of the universe enclosed within the observed Hubble horizon radius obeys (at any time) the relation $M_U \sim R$. Such latter relationship is very similar (up to a factor of 2) to the Schwarzschild black-hole horizon-radius relation $r_s = 2M$ (in units of $\hbar = c = G = 1$). As matter is being created out of the vacuum, the Hubble horizon radius grows accordingly such that $M_U/R = c^2/G$. Note that the Hubble horizon radius is one-half the Schwarzchild horizon radius $(1/2)(2GM_U/c^2) = (1/2)R_S$.

Lemaître's idea of the Universe as a "primordial atom" (like a brane-world) of Planck size has been also analyzed by [30] from a very different perspective than Born's Dual Phase Space Relativity. These authors have argued that one can have a compatible picture of the expansion of the Universe with the Eddington-Dirac large number coincidences if one invokes a variation of the fundamental constants with the cosmological evolution time as Dirac advocated long ago.

One of the most salient features of this section is that it agrees with the findings of [4] where a *geometric mean* relationship was found from first principles among the cosmological constant ρ_{vacuum}, the Planck area λ^2 and the AdS_4 throat size squared R^2 given by $(\rho_v)^{-1} = (\lambda)^2(R^2)$. Since the throat size of de Sitter space is the same as that of Anti de Sitter space, by setting the infrared scale R equal to the Hubble radius horizon observed *today* R_H and λ equal to the Planck scale one reproduces precisely the *observed* value of the vacuum energy density! [25]: $\rho \sim L_{Planck}^{-2} R_H^{-2} = L_P^{-4}(L_{Planck}/R_H)^2 \sim 10^{-122} M_{Planck}^4$.

Nottale's proposal [2] for the resolution to the cosmological constant problem is based on taking the Hubble scale R as an upper impassible scale and implementing the Scale Relativity principle so that in order to compare the vacuum energies of the Universe at the Planck scale $\rho(L_P)$ with the vacuum energy measured at the Hubble scale $\rho(R)$ one needs to include the Scale Relativistic correction factors which account for such apparent huge discrepancy: $\rho(L_P)/\rho(R) = (R/L_P)^2 \sim 10^{122}$. In contrast, the results of this work are based on Born's Dual Phase-Space Relativity principle. In the next sections we will review the dynamical consequences of the Yang's Noncommutative spacetime algebra comprised of *two* scales, the minimal Planck scale L_p (related to a minimum distance) and an upper infrared scale R related to a minimum momentum $p = \hbar/R$. Another interesting approach to dark matter, dark energy and the cosmological constant based on a vacuum condensate has been undertaken by [25].

We finalize this subsection by pointing out that the maximal/minimal angular velocity correspond to c/L_P and c/R respectively. A maximum angular velocity has important consequences in future Thomas-precession experiments [61], [73] whereas a minimal angular velocity has important consequences in galactic rotation measurements. The role of the Machian principle in constructing quantum cosmologies, models of dark energy, etc... has been studied in [52] and its relationship to modified Newtonian dynamics and fractals by [54], [3].

3.2 Modified Newtonian Dynamics from Phase-Space Relativity

Having displayed the cosmological features behind the proper-four forces equality (3.10) that relates the maximal/minimal acceleration in terms of the minimal/large scales and which is compatible with Eddington-Dirac's large number coincidences we shall derive next the *modified* Newtonian dynamics of a test particle which emerges from the Born's Dual Phase Space Relativity principle.

The modified Schwarzschild metric is defined in terms of the non-covariant acceleration as:

$$(d\omega)^2 = \Omega^2 (d\tau)^2 = = \left[1 + \frac{m^2 g_{\mu\nu}(d^2x^\mu/d\tau^2)(d^2x^\nu/d\tau^2)}{m_P^2 a^2}\right] g_{\mu\nu} dx^\mu dx^\nu \,,$$

$$-g^2(\tau) \equiv g_{\mu\nu}(d^2x^\mu/d\tau^2)(d^2x^\nu/d\tau^2) < 0\,. \qquad (3.11a)$$

A covariant acceleration in curved space-times is given by:

$$\frac{Dv^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\,.$$

A particle in free fall follows a geodesic with *zero* covariant acceleration. Hence, we shall use the non-covariant acceleration in order to compute the effects of the maximal acceleration of a test particle in Schwarzschild spacetimes.

The components of the non-covariant four-acceleration $d^2x^\mu/d\tau^2$ of a test particle of mass m moving in a Schwarzschild spacetime background can be obtained in a straightforward fashion after using the on-shell condition $g_{\mu\nu}P^\mu P^\nu = m^2$ in spherical coordinates (by solving the relativistic Hamilton-Jacobi equations). The explict components of the (space-like) proper-four acceleration can be found in [22], [36] in terms of two integration constants, the energy E and angular momentum L. The latter components yields the final expression for the conformal factor Ω^2 in the case of pure radial motion [22]:

$$\begin{aligned}&\Omega^2(m,a,M,E,r) = \\ &= 1 - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\Big\{(1-2M/r)^{-1}\left(\frac{M}{r^2}\right)^2 - \\ &- \left[4M^2(E/m)^2 r^{-4}(1-2M/r)^{-3}\right]\times \\ &\times\left[(E/m)^2-(1-2M/r)\right]\Big\}\,.\end{aligned} \qquad (3.12)$$

In the Newtonian limit, to a first order approximation, we can set $1-2M/r \sim 1$ in eq-(3.12) since we shall be concentrating in distances larger than the Schwarzschild radius $r > r_s = 2M$, the conformal factor Ω^2 in eq-(3.12) simplifies:

$$\begin{aligned}\Omega^2 \sim 1 - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\Big\{\left(\frac{M}{r^2}\right)^2 - \\ - \left[4M^2(E/m)^2r^{-4}\right]\left[(E/m)^2-1\right]\Big\},\end{aligned} \qquad (3.13)$$

the modified Schwarzschild metric component $g'_{00} = \Omega^2 g_{00} = = \Omega^2(1-2M/r) = 1+2\mathcal{U}'$ yields the *modified* gravitational potential $\mathcal{U}'$ in the *weak* field approximation

$$\begin{aligned}&g'_{00} = 1+2\mathcal{U}' \sim \\ &\sim 1 - \frac{2M}{r} - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\left(\frac{2M}{r^2}\right)^2 F(E/m)\end{aligned} \qquad (3.14)$$

with

$$F(E/m) = \left(\frac{E}{m}\right)^2 - \left(\frac{E}{m}\right)^4 + \frac{1}{4}\,, \qquad (3.15)$$

where $F(E/m) > 0$ in the Newtonian limit $E < m$. The modified radial acceleration which encodes the modified Newtonian dynamics and which violates the equivalence principle (since the acceleration now depends on the mass of the test particle m) is

$$\begin{aligned}a' = -\frac{\partial\mathcal{U}'}{\partial r} = -\frac{M}{r^2}\Big[1 + 8F\left(\frac{E}{m}\right)\left(\frac{m}{m_P}\right)^2\times \\ \times\left(\frac{M}{m_P}\right)\frac{1}{m_P^3 r^3}\Big] + O(r^{-6})\,,\end{aligned} \qquad (3.16)$$

this result is valid for distances $r \gg r_s = 2M$. We have set the maximal acceleration $a = \frac{c^2}{L_P} = m_P$ in units of $\hbar = c = = G = 1$. This explains the presence of the m_P factors in the denominators. The first term in eq-(3.16) is the standard Newtonian gravitational acceleration $-M/r^2$ and the second terms are the leading corrections of order $1/r^5$. The higher order corrections $O(r^{-6})$ appear when we do not set $1-2M/r \sim 1$ in the expression for the conformal factor Ω^2 and when we include the extra term in the product of Ω^2 with $g_{00} = (1-2M/r)$.

The conformal factor Ω^2 when $L \neq 0$ (rotational degrees of freedom are switched on) such that the test particle moves in the radial and transverse (angular) directions has been given in [22]:

$$\Omega^2 = 1 - \frac{m^2}{m_P^2 a^2} \left\{ \frac{1}{1-2M/r} \times \right.$$
$$\left. \times \left[-\frac{3ML^2}{m^2 r^4} + \frac{L^2}{m^2 r^3} - \frac{M}{r^2} \right]^2 \right\} +$$
$$+ \frac{m^2}{m_P^2 a^2} \left[-\frac{4L^2}{m^2 r^4} + \frac{4E^2 M^2}{m^2 r^4 (1-2M/r)^3} \right] \times$$
$$\times \left[\frac{E^2}{m^2} - (1-2M/r) \left(1 + \frac{L^2}{m^2 r^2} \right) \right]. \tag{3.17}$$

Following the same weak field approximation procedure $g'_{00} = \Omega^2(E, L, m) g_{00} = 1 + 2\mathcal{U}'$ yields the modified gravitational potential $\mathcal{U}'$ and modified Newtonian dynamics $a' = -\partial_r \mathcal{U}'$ that leads once again to a violation of the equivalence principle due to the fact that the acceleration depends on the values of the masses of the test particle.

4 Modified Newtonian Dynamics resulting from Yang's Noncommutative Spacetime Algebra

We end this work with some relevant remarks about the impact of Yang's Noncommutative spacetime algebra on modified Newtonian dynamics. Such algebra involves *two* length scales, the minimal Planck scale $L_P = \lambda$ and an upper infrared cutoff scale $\mathcal{R}$.

Recently in [134] an isomorphism between Yang's Noncommutative space-time algebra (involving *two* length scales) [136] and the *holographic area coordinates* algebra of C-spaces (Clifford spaces) was constructed via an AdS_5 spacetime (embedded in $6D$) which is instrumental in explaining the origins of an extra (infrared) scale $\mathcal{R}$ in conjunction to the (ultraviolet) Planck scale λ characteristic of C-spaces. Yang's Noncommutative space-time algebra allowed Tanaka [137] to explain the origins behind the *discrete* nature of the spectrum for the *spatial* coordinates and *spatial* momenta which yields a *minimum* length-scale λ (ultraviolet cutoff) and a minimum momentum $p = \hbar/\mathcal{R}$ (maximal length $\mathcal{R}$, infrared cutoff).

Related to the issue of area-quantization, the norm-squared $\mathbf{A}^2$ of the holographic Area operator $X_{AB}X^{AB}$ in Clifford-spaces has a correspondence with the quadratic Casimir operator $\lambda^4 \Sigma_{AB} \Sigma^{AB}$ of the conformal algebra $SO(4,2)$ ($SO(5,1)$ in the Euclideanized AdS_5 case). This holographic area-Casimir relationship does not differ much from the area-spin relation in Loop Quantum Gravity $\mathbf{A}^2 \sim \lambda^4 \sum j_i(j_i+1)$ in terms of the $SU(2)$ Casimir J^2 with eigenvalues $j(j+1)$, where the sum is taken over the spin network sites [111] and the minimal Planck scale emerges from a regularization procedure.

The Yang's algebra can be written in terms of the $6D$ angular momentum operators and a $6D$ pseudo-Euclidean metric η^{MN}:

$$\hat{M}^{\mu\nu} = \hbar \Sigma^{\mu\nu}, \quad \hat{M}^{56} = \hbar \Sigma^{56}, \tag{4.1}$$

$$\lambda \Sigma^{\mu 5} = \hat{x}^\mu, \quad \frac{\hbar}{\mathcal{R}} \Sigma^{\mu 6} = \hat{p}^\mu, \tag{4.2}$$

$$\mathcal{N} = \frac{\lambda}{\mathcal{R}} \Sigma^{56}, \tag{4.3}$$

as follows:

$$[\hat{p}^\mu, \mathcal{N}] = -i\eta^{66} \frac{\hbar}{R^2}\, \hat{x}^\mu, \tag{4.4}$$

$$[\hat{x}^\mu, \mathcal{N}] = i\eta^{55} \frac{L_P^2}{\hbar}\, \hat{p}^\mu, \tag{4.5}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = -i\eta^{55} L_P^2\, \Sigma^{\mu\nu}, \tag{4.6}$$

$$[\hat{p}^\mu, \hat{p}^\nu] = -i\eta^{66} \frac{\hbar^2}{R^2}\, \Sigma^{\mu\nu}, \tag{4.7}$$

$$[\hat{x}^\mu, \hat{p}^\mu] = i\hbar\eta^{\mu\nu}\, \mathcal{N}, \tag{4.8}$$

$$[\hat{x}^\mu, \Sigma^{\nu\rho}] = \eta^{\mu\rho} x^\nu - \eta^{\mu\nu} x^\rho, \tag{4.9}$$

$$[\hat{p}^\mu, \Sigma^{\nu\rho}] = \eta^{\mu\rho} p^\nu - \eta^{\mu\nu} p^\rho, \tag{4.10}$$

The dynamical consequences of the Yang's Noncommutative spacetime algebra can be derived from the quantum/classical correspondence:

$$\frac{1}{i\hbar} [\hat{A}, \hat{B}] \leftrightarrow \{A, B\}_{PB}, \tag{4.11}$$

i. e. commutators correspond to Poisson brackets. More precisely, to Moyal brackets in Phase Space. In the classical limit $\hbar \to 0$ Moyal brackets reduce to Poisson brackets. Since the coordinates and momenta are no longer commuting variables the classical Newtonian dynamics is going to be modified since the symplectic two-form $\omega^{\mu\nu}$ in Phase Space will have additional non-vanishing elements stemming from these non-commuting coordinates and momenta.

In particular, the modified brackets read now:

$$\{\{A(x,p), B(x,p)\}\} = \partial_\mu A \omega^{\mu\nu} \partial_\nu B =$$
$$= \{A(x,p), B(x,p)\}_{PB} \{x^\mu, p^\nu\} +$$
$$+ \frac{\partial A}{\partial x^\mu} \frac{\partial B}{\partial x^\nu} \{x^\mu, x^\nu\} + \frac{\partial A}{\partial p^\mu} \frac{\partial B}{\partial p^\nu} \{p^\mu, p^\nu\}. \tag{4.12}$$

If the coordinates and momenta were commuting variables the modified bracket will reduce to the first term only:

$$\{\{A(x,p), B(x,p)\}\} =$$
$$= \{A(x,p), B(x,p)\}_{PB} \{x^\mu, p^\nu\} =$$
$$= \left[\frac{\partial A}{\partial x^\mu} \frac{\partial B}{\partial p^\nu} - \frac{\partial A}{\partial p^\mu} \frac{\partial B}{\partial x^\nu} \right] \eta^{\mu\nu} \mathcal{N}. \tag{4.13}$$

The ordinary Heisenberg (canonical) algebra is recovered when $\mathcal{N} \to 1$ in eq-(4.13).

In the nonrelativistic limit, the modified dynamical equations are:

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{\partial H}{\partial p^j}\{x^i, p^j\} + \frac{\partial H}{\partial x^j}\{x^i, x^j\}\,, \quad (4.14)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = -\frac{\partial H}{\partial x^j}\{x^i, p^j\} + \frac{\partial H}{\partial p^j}\{p^i, p^j\}. \quad (4.15)$$

The non-relativistic Hamiltonian for a central potential $V(r)$ is:

$$H = \frac{p_i p^i}{2m} + V(r)\,, \quad r = \Big[\sum_i x_i x^i\Big]^{1/2}. \quad (4.16)$$

Defining the magnitude of the central force by $F = -\frac{\partial V}{\partial r}$ and using $\frac{\partial r}{\partial x^i} = \frac{x_i}{r}$ one has the modified dynamical equations of motion (4.14, 4.15):

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{p_j}{m}\delta^{ij} - F\frac{x_j}{r}L_P^2\Sigma^{ij}\,, \quad (4.16a)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = F\frac{x_j}{r}\delta^{ij} + \frac{p_j}{m}\frac{\Sigma^{ij}}{R^2}. \quad (4.16b)$$

The angular momentum two-vector Σ^{ij} can be written as the *dual* of a vector $\vec{J}$ as follows $\Sigma^{ij} = \epsilon^{ijk}J_k$ so that:

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{p^i}{m} - L_P^2 F\frac{x_j}{r}\epsilon^{ijk}J_k\,, \quad (4.17a)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = F\frac{x^i}{r} + \frac{p_j}{m}\frac{\epsilon^{ijk}J_k}{R^2}. \quad (4.17b)$$

For planar motion (central forces) the cross-product of $\vec{J}$ with $\vec{p}$ and $\vec{x}$ is not zero since $\vec{J}$ points in the perpendicular direction to the plane. Thus, one will have nontrivial corrections to the ordinary Newtonian equations of motion induced from Yang's Noncommutative spacetime algebra in the nonrelativistic limit. When $\vec{J} = 0$, pure radial motion, there are no corrections. This is not the case when we studied the modified Newtonian dynamics in the previous section of the modified Schwarzschild field due to the maximal-acceleration relativistic effects. Therefore, the two routes to obtain modifications of Newtonian dynamics are very different.

Concluding, eqs-(4.16, 4.17) determine the *modified* Newtonian dynamics of a test particle under the influence of a central potential explicitly in terms of the two L_P, R minimal/maximal scales. When $L_P \to 0$ and $R \to \infty$ one recovers the ordinary Newtonian dynamics $v^i = (p^i/m)$ and $F(x^i/r) = m(dv^i/dt)$. The unit vector in the radial direction has for components $\hat{r} = (\vec{r}/r) = (x^1/r, x^2/r, x^3/r)$.

It is warranted to study the full relativistic dynamics as well, in particular the *modified* relativistic dynamics of the de-Sitter rigid top [135] due to the effects of Yang's Noncommutative spacetime algebra with a lower and an upper scale. The de Sitter rigid Top can be generalized further to Clifford spaces since a Clifford-polyparticle has more degrees of freedom than a relativistic top in ordinary spacetimes [46] and, naturally, to study the *modified* Nambu-Poisson dynamics of p-branes [49] as well. A different physical approach to the theory of large distance physics based on certain two-dim nonlinear sigma models has been advanced by Friedan [51].

An Extended Relativity theory with both an upper and lower scale can be formulated in the Clifford extension of Phase Spaces along similar lines as [1], [68] by adding the Clifford-valued polymomentum degrees of freedom to the Clifford-valued holographic coordinates. The Planck scale L_P and the minimum momentum $(\hbar/R)$ are introduced to match the dimensions in the Clifford-Phase Space interval in D-dimensions as follows:

$$\begin{aligned} d\Sigma^2 &= <dX^\dagger dX> + \frac{1}{\mathcal{F}^2}<dP^\dagger dP> = \\ &= \left(\frac{d\sigma}{L_P^{D-1}}\right)^2 + dx_\mu dx^\mu + \frac{dx_{\mu\nu}dx^{\mu\nu}}{L_P^2} + \\ &+ \frac{dx_{\mu\nu\rho}dx^{\mu\nu\rho}}{L_P^4} + \ldots + \frac{1}{\mathcal{F}^2}\Bigg[\left(\frac{d\tilde{\sigma}}{(\hbar/R)^{D-1}}\right)^2 + \\ &+ dp_\mu dp^\mu + \frac{dp_{\mu\nu}dp^{\mu\nu}}{(\hbar/R)^2} + \frac{dp_{\mu\nu\rho}dp^{\mu\nu\rho}}{(\hbar/R)^4} + \ldots\Bigg]. \end{aligned} \quad (4.18)$$

All the terms in eq-(4.18) have dimensions of length^2 and the maximal force is:

$$\mathcal{F} = \frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} = \frac{c^4}{G}. \quad (4.19)$$

The relevance of studying this extended Relativity in a Clifford-extended Phase Space is that it is the proper arena to construct a Quantum Cosmology compatible with Non-Archimedean Geometry, Yang's Noncommutative spacetime algebra [136] and Scale Relativity [2] with an upper and lower limiting scales, simultaneously. This clearly deserves further investigation.

Acknowledgments

We are indebted to M. Bowers for assistance.

References

1. Castro C., Pavšič M. The extended relativity theory in Clifford spaces. CDS–CERN: EXT–2004–127.
2. Nottale L. La relativite dans tous ses etats. Hachette Lit., Paris 1999. Nottale L. Fractal spacetime and microphysics, towards scale relativity. World Scientific, Singapore, 1992.
3. Nottale L. The pioneer anomalous acceleration: a measurement of the cosmological constant at the scale of the Solar system.

arXiv: gr-qc/0307042. Scale relativistic cosmology. *Chaos, Solitons and Fractals*, v. 16, 2003, 539.

4. Castro C. *Mod. Phys. Letters*, v. A17, 2002, 2095–2103.
5. LaViolette P. *Int. Jour. Gen. Sys.*, v. 11, 1985, 329. *Appl. J.*, v. 301, 1986, 544. Subquantum Kinetics. Second edition, Starlane Publishers, Alexandria, VA, 2003.
6. Rosales J. The Pioneer's anomalous Doppler drift as a berry phase. arXiv: gr-qc/0401014. The Pioneer's acceleration anomaly and Hubble's constant. arXiv: gr-qc/0212019. Rosales J., Sanchez Gomez J. The Pioneer effect as a manifestation of the cosmic expansion in the solar system. arXiv: gr-qc/9810085.
7. Anderson J. D. et al. *Phys. Rev. Lett.*, v. 81, 1998, 2858.
8. Sidharth B. G. *Chaos, Solitons and Fractals*, v. 12, 2001, 1101.
9. Ahluwalia D. V. and Burgard C. *General Relativity and Gravitation*, v. 28 (10), 1996, 1163. Ahluwalia D. V. *General Relativity and Gravitation*, v. 29 (12), 1997, 1491. Ahluwalia D. V. *Phys. Letters*, v. A275, 2000, 31. Adunas G., Rodriguez-Milla E. and Ahluwalia D. V. *Phys. Letters*, v. B485, 2000, 215.
10. Born M. *Proc. Royal Society*, v. A165, 1938, 291. *Rev. Mod. Physics*, v. 21, 1949, 463.
11. Caianiello E. Is there a maximal acceleration? *Lett. Nuovo Cimento*, v. 32, 1981, 65.
12. Nesterenko V. *Class. Quant. Grav.*, v. 9, 1992, 1101; *Phys. Lett.*, v. B327, 1994, 50.
13. Bozza V., Feoli A., Lambiase G., Papini G., and Scarpetta G. *Phys. Lett.*, v. A283, 2001, 53. Nesterenko V., Feoli A., Lambiase G. and Scarpetta G. *Phys. Rev.*, v. D60, 1999, 065001.
14. Capozziello S., Lambiase G., Scarpetta G., *Int. Jour. Theor. Physics*, v. 39, 2000, 15.
15. Rama K. Classical velocity in kappa-deformed Poincaré algebra and amaximal acceleration. arXiv: hep-th/0209129.
16. Brandt H. *Contemporary Mathematics*, v. 196, 1996, 273. *Chaos, Solitons and Fractals*, v. 10 (2–3), 1999, 267.
17. Schuller F. *Annals of Phys.*, v. 299, 2002, 174.
18. Lukierski J., Nowicki A., Ruegg H., Tolstoy V. *Phys. Letters*, v. 264, 1991, 331. Lukierski J., Ruegg H., Zakrzewski W. *Ann. Phys.*, v. 243, 1995, 90. Lukierski J., Nowicki A. Double Special Relativity verus kappa-deformed dynamics. arXiv: hep-th/0203065.
19. Webb J., Murphy M., Flambaum V., Dzuba V., Barrow J., Churchill C., Prochaska J., and Wolfe A. Further evidence for Cosmological Evolution of the Fine Structure Constant. *Monthly Notices of the Royal Astronomical Society*, v. 327, 2001, 1208.
20. Uzan J. P. The fundamental constants and their variations: observational status and theoretical motivations. arXiv: hep-ph/0205340.
21. Lambiase G., Papini G., Scarpetta G. Maximal acceleration corrections to the Lamb shift of one electron atoms. *Nuovo Cimento*, v. B112, 1997, 1003. arXiv: hep-th/9702130. Lambiase G., Papini G., Scarpetta G. *Phys. Letters*, v. A224, 1998, 349. Papini G. Shadows of a maximal acceleration. arXiv: gr-qc/0211011.
22. Feoli A., Lambiase G., Papini G., and Scarpetta G. Schwarzschild field with maximal acceleration corections. *Phys. Letters*, v. A263, 1999, 147. arXiv: gr-qc/9912089. Capozziello S., Feoli A., Lambiase G., Papini G., and Scarpetta G. Massive scalar particles in a Modified Schwarzschild Geometry. *Phys. Letters*, v. A268, 2000, 247. arXiv: gr-qc/0003087. Bozza V., Lambiase G., Papini G., and Scarpetta G. Quantum violations of the Equivalence Principle in a Modfied Schwarzschild Geometry: neutrino oscillations. Maximal acceleration corections. arXiv: hep-ph/0012270.
23. Khare A., Rvachev V. L. *Foundations of Physics*, v. 30, No. 1, 2000, 139.
24. Misner C., Thorne K., and Wheeler J. Gravitation. (See Chapter 25), W. H. Freeman and Co., San Francisco, 1973.
25. Sarfatti J. Developments in Quantum Physics. Ed. by Frank Columbus and Volodymyr Krasnoholovets, *Wheeler's World: It from Bit*, p. 41084, NOVA Scientific Publishers ISBN 1-59454-003-9, http://qedcorp.com/APS/Nova.pdf; Vacuum coherence cosmology model. Talk at the GR17 Conference, July 2004, Dublin.
26. Castro C., *Int. J. Mod. Phys.*, vol. A18, 2003, 5445. arXiv: hep-th/0210061.
27. Castellani L. *Phys. Letters*, v. B327, 1994, 22. *Comm. Math. Phys.*, v. 171, 1995, 383.
28. Amelino-Camelia G. *Phys. Letters*, v. B510, 2001, 255. *Int. J. Mod. Phys.*, v. D11, 2002, 35. *Int. J. Mod. Phys.*, v. D11, 2002, 1643.
29. Greisen K. *Phys. Rev. Lett.*, v. 16, 1966, 748. Zatsepin G. T., Kurmin V. *Soviet Phys. JETP Lett.*, v. 4, 1966, 78.
30. Kafatos M., Roy S., and Amoroso R. Scaling in cosmology and the arrow of time. *Studies in the Structure of Time*, Ed. by Buccheri et al., Kluwer Academic, Plenum Publishers, New York, 2000, 191–200. Glanz J. *Science*, v. 282, 1998, 2156.
31. Toller M. The geometry of maximal acceleration. arXiv: hep-th/0312016. *Int. Jour. Theor. Physics*, v. 29, 1990, 963. *Nuovo Cimento*, v. B40, 1977, 27.
32. Low S. *Jour. Phys. A Math. Gen.*, v. 35, 2002, 5711. *J. Math. Phys.*, v. 38, 1997, 2197.
33. Vacaru S. Non-holonomic Clifford-structures and noncommutative Riemann-Finsler geometry. arXiv: math.DG/0408121. Vacaru S. and Vicol N. Nonlinear connections and Clifford structures. arXiv: math.DG/0205190. Vacaru S. (Non) commutative Finsler geometry from String/M theory. arXiv: hep-th/0211068. Vacaru S., Nadejda A. *Int. J. Math. Sci.*, v. 23, 2004, 1189–1233. Vacaru S. Clifford structures and spinors on spaces with local anisotropy. *Buletinul Academiei de Stiinte a Republicii Moldova*, *Fizica si Tehnica*, (Izvestia Academii Nauk Respubliky Moldova, fizica i tehnika), v. 3, 1995, 53–62. Vacaru S. Superstrings in higher order extensions of Finsler superspaces. *Nucl. Phys.*, v. B434, 1997, 590–656.
34. Ashtekar A., Rovelli C., and Smolin L., *Phys. Rev. Lett.*, v. 69, 1992, 237. Rovelli C. A dialog on quantum gravity. arXiv: hep-th/0310077. Freidel L., Livine E., and Rovelli C. *Class. Quant. Grav.*, v. 20, 2003, 1463–1478. Smolin L. How far are we from the quantum theory of gravity? arXiv: hep-th/0303185.

35. Castro C. On noncommutative Yang's space-time algebra, holography, area quantization and C-space Relativity. Submitted to *Class. Quan. Grav.* See also CERN–EXT–2004–090.
36. Yang C. N. *Phys. Rev.*, v. 72, 1947, 874. *Proc. of the Intern. Conf. on Elementary Particles*, 1965, Kyoto, 322–323.
37. Tanaka S. *Nuovo Cimento*, v. 114B, 1999, 49. Tanaka S. Noncommutative field theory on Yang's space-time algebra, covariant Moyal star products and matrix model. arXiv: hep-th/0406166. Space-time quantization and nonlocal field theory. arXiv: hep-th/0002001. Yang's quantized space-time algebra and holographic hypothesis. arXiv: hep-th/0303105.
38. Smith F. *Int. Journal Theor. Phys.*, v. 24, 1985, 155. *Int. Jour. Theor. Phys.*, v. 25, 1985, 355. From sets to quarks. arXiv: hep-ph/9708379 and CERN–CDS: EXT–2003–087. Gonzalez-Martin G. Physical geometry. University of Simon Bolivar Publ., Caracas, 2000, 265 pages, ISBN: 9800767495. Gonzalex-Martin G. The proton/electron geometric mass ratio. arXiv: physics/0009052. Gonzalez-Martin G. The fine structure constant from relativistic groups. arXiv: physics/0009051.
39. Beck C. *Spatio-Temporal Vacuum Fluctuations of Quantized Fields Advances in Nonlinear Dynamics*, v. 21, World Scientific, Singapore 2000. Chaotic strings and standard model parameters. *Physica* v. 171D, 2002, 72. arXiv: hep-th/0105152.
40. Vladimorov V., Volovich I., Zelenov I. P-adic Numbers in Mathematical Physics. World Scientific, Singapore, 1994. Brekke L., Freund P. *Phys. Reports*, v. 231, 1993, 1.
41. Pitkannen M. Topological geometrodynamics I, II. *Chaos, Solitons and Fractals*, v. 13, No. 6, 2002, 1205 and 1217.
42. Noyes P. Bit-String Physics: A discrete and finite approach to Natural Philosophy. Ed. by J. C. van der Berg, *Series of Knots in Physics*, v. 27, World Scientific, Singapore, 2001.
43. Smolin L., Kowalski-Glikman J. Triply Special Relativity. arXiv: hep-th/0406276. McGaugh S. S. Modified Newtonian Dynamics as an alternative to Dark Matter. arXiv: astro-ph/0204521. Aguirre A. Alternatives to Dark Matter. arXiv: astro-ph/0305572.
44. Nam Chang L. Some consequences of the Hypothesis of Minimal Lengths. arXiv: hep-th/0405059.
45. Castro C. *Foundations of Physics*, v. 8, 2000, 1301.
46. Castro C. *Foundations of Physics*, v. 34, No. 7, 2004, 107.
47. Armenta J., Nieto J. A. The de Sitter Relativistic Top Theory. arXiv: 0405254.
48. Vilela Mendes R. *J. Phys.*, v. A27, 1994, 8091–8104. Vilela-Mendes R. Some consequences of a noncommutative space-time structure. arXiv: hep-th/0406013. Chryssomalakos C., Okon E. Linear form of 3-scale Special Relativity algebra and the relevance of stability. arXiv: 0407080.
49. Nambu Y. *Phys. Rev.*, v. D7, 1973, 2405.
50. Moffat J. Quantun Gravity momentum representation an maximum invariant energy. arXiv: gr-qc/0401117.
51. Friedan D. Two talks on a tentative theory of large distance physics. arXiv: hep-th/0212268. A tentative theory of large distance physics. arXiv: hep-th/0204131.
52. Cole D. Contrasting quantum cosmologies. arXiv: gr-qc/0312045. Vishwakarma R. G. A Machian model of Dark Energy. arXiv: gr-qc/0205075. Buchalter A. On the time variations of c, G and h and the dynamics of the Cosmic expansion. arXiv: astro-ph/0403202.
53. Roscoe D. *General Relativity and Gravitation*, v. 36, 2004, 3. *General Relativity and Gravitation*, v. 34, 2002, 577. *Astrophysics and Space Science*, v. 285, No. 2, 2003, 459.
54. Steer D., Parry M. *Int. Jour. of Theor. Physics*, v. 41, No. 11, 2002, 2255.
55. Amati D., Ciafaloni M., Veneziano G. *Phys. Letters*, v. B197, 1987, 81. Amati D., Ciafaloni M., Veneziano G. *Phys. Letters*, v. B216, 1989, 41. Gross D., Mende P. *Phys. Letters*, v. B197, 1987, 129. Gross D., Mende P. *Nucl. Phys.*, v. B303, 1988, 407.

The Extended Relativity Theory in Clifford Spaces

Carlos Castro* and Matej Pavšič†

*Center for Theor. Studies of Physical Systems, Clark Atlanta University, Atlanta, USA
E-mail: czarlosromanov@yahoo.com; castro@ctsps.cau.edu

†Jožef Stefan Institute, Jamova 39, SI-1000 Ljubljana, Slovenia
E-mail: Matej.Pavsic@ijs.si

An introduction to some of the most important features of the Extended Relativity theory in Clifford-spaces (C-spaces) is presented whose "point" coordinates are non-commuting Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes... degrees of freedom associated with the collective particle, string, membrane, p-brane... dynamics of p-loops (closed p-branes) in target D-dimensional spacetime backgrounds. C-space Relativity naturally incorporates the ideas of an invariant length (Planck scale), maximal acceleration, non-commuting coordinates, supersymmetry, holography, higher derivative gravity with torsion and variable dimensions/signatures. It permits to study the dynamics of all (closed) p-branes, for all values of p, on a unified footing. It resolves the ordering ambiguities in QFT, the problem of time in Cosmology and admits superluminal propagation (tachyons) without violations of causality. A discussion of the maximal-acceleration Relativity principle in phase-spaces follows and the study of the invariance group of symmetry transformations in phase-space allows to show why Planck areas are *invariant* under acceleration-boosts transformations. This invariance feature suggests that a maximal-string tension principle may be operating in Nature. We continue by pointing out how the relativity of signatures of the underlying n-dimensional spacetime results from taking different n-dimensional slices through C-space. The conformal group in spacetime emerges as a natural subgroup of the Clifford group and Relativity in C-spaces involves natural *scale* changes in the sizes of physical objects without the introduction of forces nor Weyl's gauge field of dilations. We finalize by constructing the generalization of Maxwell theory of Electrodynamics of point charges to a theory in C-spaces that involves extended charges coupled to antisymmetric tensor fields of arbitrary rank. In the concluding remarks we outline briefly the current promising research programs and their plausible connections with C-space Relativity.

Contents

1 Introduction

In recent years it was argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein's general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature [8]. A theory involving spacetime *resolutions* was developed long ago by Nottale [23] where the Planck scale was postulated as the minimum observer independent invariant resolution [23] in Nature. Since "points" cannot be observed physically with an ultimate resolution, it is reasonable to postulate that they are smeared out into fuzzy balls. In refs.[8] it was assumed that those balls have the Planck radius and arbitrary dimension. For this reason it was argued in refs. [8] that one should construct a theory which includes all dimensions (and signatures) on the equal footing. In [8] this Extended Scale Relativity principle was applied to the quantum mechanics of p-branes which led to the construction of Clifford-space (C-space) where all p-branes were taken to be on the same footing, in the sense that the transformations in C-space reshuffled a string history for a five-brane history, a membrane history for a string history, for example.

Clifford algebras contained the appropriate algebraic-geometric features to implement this principle of polydimensional transformations [14]–[17]. In [14]–[16] it was proposed that every physical quantity is in fact a *polyvector*, that is, a Clifford number or a Clifford aggregate. Also, spinors are the members of left or right minimal ideals of Clifford algebra, which may provide the framework for a deeper understanding of sypersymmetries, i. e., the transformations relating bosons and fermions. The Fock-Stueckelberg theory of a relativistic particle can be embedded in the Clifford algebra of spacetime [15, 16]. Many important aspects of Clifford algebra are described in [1], [6], [7], [3], [15, 16, 17], [5], [48]. It is our belief that this may lead to the proper formulation of string and M theory.

A geometric approach to the physics of the Standard Model in terms of Clifford algebras was advanced by [4]. It was realized in [43] that the $Cl(8)$ Clifford algebra contains the 4 fundamental nontrivial representations of *Spin*(8) that accommodate the chiral fermions and gauge bosons of the Standard Model and which also includes gravitons via the McDowell-Mansouri-Chamseddine-West formulation of gravity, which permits to construct locally, in $D=8$, a geometric Lagrangian for the Standard Model plus Gravity. Furthermore, discrete Clifford-algebraic methods based on hyperdiamond-lattices have been instrumental in constructing E_8 lattices and deriving the values of the force-strengths (coupling constants) and masses of the Standard Model with remarkable precision by [43]. These results have recently been corroborated by [46] for Electromagnetism, and by [47], where all the Standard Model parameters were obtained from first principles, despite the contrary orthodox belief that it is senseless to "derive" the values of the fundamental constants in Nature from first principles, from pure thought alone; i. e. one must invoke the Cosmological Anthropic Principle to explain why the constants of Nature have they values they have.

Using these methods the bosonic p-brane propagator, in the quenched mini superspace approximation, was constructed in [18, 19]; the logarithmic corrections to the black hole entropy based on the geometry of Clifford space (in short C-space) were obtained in [21]; the modified nonlinear de Broglie dispersion relations, the corresponding minimal-length stringy [11] and p-brane uncertainty relations also admitted a C-space interpretation [10], [19]. A generalization of Maxwell theory of electromagnetism in C-spaces comprised of extended charges coupled to antisymmetric tensor fields of arbitrary rank was attained recently in [75]. The resolution of the ordering ambiguities of QFT in curved spaces was resolved by using polyvectors, or Clifford-algebra valued objects [26]. One of the most remarkable features of the Extended Relativity in C-spaces is that a higher derivative Gravity with Torsion in ordinary spacetime follows naturally from the analog of the Einstein-Hlbert action in *curved* C-space [20].

In this new physical theory the arena for physics is no longer the ordinary spacetime, but a more general manifold of Clifford algebra valued objects, noncommuting polyvectors. Such a manifold has been called a pan-dimensional continuum [14] or C-space [8]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also closed lines, surfaces, volumes, . . . , called 0-loops (points), 1-loops (closed strings), 2-loops (closed membranes), 3-loops, etc. It is a sort of a *dimension* category, where the role of functorial maps is played by C-space transformations which reshuffles a p-brane history for a p'-brane history or a mixture of all of them, for example. The above geometric objects may be considered as to corresponding to the well-known physical objects, namely closed p-branes. Technically those transformations in C-space that reshuffle objects of different dimensions are generalizations of the ordinary Lorentz transformations to C-space.

C-space Relativity involves a generalization of Lorentz invariance (and not a deformation of such symmetry) involving superpositions of p-branes (p-loops) of all possible dimensions. The Planck scale is introduced as a natural parameter that allows us to bridge extended objects of different dimensionalities. Like the speed of light was need in Einstein Relativity to fuse space and time together in the Minkowski spacetime interval. Another important point is that the Conformal Group of four-dimensional spacetime is a consequence of the Clifford algebra in *four-dimensions* [25] and it emphasizes the fact why the natural dilations/contractions of objects in C-space is *not* the same physical phenomenon than what occurs in Weyl's geometry which requires introducing, by hand, a gauge field of dilations. Objects move dilationally, in the absence of forces, for a different physical reasoning than in Weyl's geometry: they move dilationally because of inertia. This was discussed long ago in refs. [27, 28].

This review is organized as follows: section 2 is dedicated to extending ordinary Special Relativity theory, from Minkowski spacetime to C-spaces, where the introduction of the invariant Planck scale is required to bridge objects, p-branes, of different dimensionality.

The generalized dynamics of particles, fields and branes in C-space is studied in section 3. This formalism allows us to construct for the first time, to our knowledge, a *unified* action which comprises the dynamics of *all* p-branes in C-spaces, for all values of p, in one single footing (see also [15]). In particular, the polyparticle dynamics in C-space, when reduced to 4-dimensional spacetime leads to the Stuckelberg formalism and the solution to the problem of time in Cosmology [15].

In section 4 we begin by discussing the geometric Clifford calculus that allows us to reproduce all the standard results in differential and projective geometry [41]. The resolution of the ordering ambiguities of QFT in curved spaces follows next when we review how it can be resolved by using polyvectors, or Clifford-algebra valued objects [26]. Afterwards we construct the Generalized Gravitational Theories in Curved C-spaces, in particular it is shown how Higher derivative Gravity with Torsion in ordinary spacetime follows naturaly from the Geometry of C-space [20].

In section 5 we discuss the Quantization program in C-spaces, and write the C-space Klein-Gordon and Dirac equations [15]. The coresponding bosonic/fermionic p-brane loop-wave equations were studied by [12], [13] without employing Clifford algebra and the concept of C-space.

In section 6 we review the Maximal-Acceleration Relativity in Phase-Spaces [127], starting with the construction of the submaximally-accelerated particle action of [53] using Clifford algebras in phase-spaces; the $U(1, 3)$ invariance transformations [74] associated with an 8-dimensional phase space, and show why the minimal Planck-Scale areas are invariant under pure acceleration boosts which suggests that there could be a principle of maximal-tension (maximal acceleration) operating in string theory [68].

In section 7 we discuss the important point that the notion of spacetime signature is relative to a chosen n-dimensional subspace of 2^n-dimensional Clifford space. Different subspaces V_n — different sections through C-space — have in general different signature [15] We show afterwards how the Conformal algebra of spacetime emerges from the Clifford algebra [25] and emphasize the physical *differences* between our model and the one based on Weyl geometry. At the end we show how Clifford algebraic methods permits one to generalize Maxwell theory of Electrodynamics (associated with ordinary point-charges) to a generalized Maxwell theory in Clifford spaces involving *extended* charges and p-forms of arbitrary rank [75].

In the concluding remarks, we briefly discuss the possible avenues of future research in the construction of QFT in C-spaces, Quantum Gravity, Noncommutative Geometry, and other lines of current promising research in the literature.

2 Extending Relativity from Minkowski spacetime to C-space

We embark into the construction of the extended relativity theory in C-spaces by a natural generalization of the notion of a spacetime interval in Minkowski space to C-space [8, 14, 16, 15, 17]:

$$dX^2 = d\sigma^2 + dx_\mu dx^\mu + dx_{\mu\nu} dx^{\mu\nu} + \ldots\,, \tag{1}$$

where $\mu_1 < \mu_2 < \ldots$. The Clifford valued polyvector:*

$$X = X^M E_M = \sigma \underline{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \ldots + \\ + x^{\mu_1 \mu_2 \ldots \mu_D} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \ldots \wedge \gamma_{\mu_D} \tag{2}$$

denotes the position of a point in a manifold, called Clifford space or C-space. The series of terms in (2) terminates at a *finite* grade depending on the dimension D. A Clifford algebra $Cl(r, q)$ with $r+q = D$ has 2^D basis elements. For simplicity, the gammas γ^μ correspond to a Clifford algebra associated with a flat spacetime:

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}\,, \tag{3}$$

but in general one could extend this formulation to curved spacetimes with metric $g^{\mu\nu}$ (see section 4).

The connection to strings and p-branes can be seen as follows. In the case of a closed string (a 1-loop) embedded in a target flat spacetime background of D-dimensions, one represents the projections of the closed string (1-loop) onto the embedding spacetime coordinate-planes by the variables $x^{\mu\nu}$. These variables represent the respective *areas* enclosed by the projections of the closed string (1-loop) onto the corresponding embedding spacetime planes. Similary, one can embed a closed membrane (a 2-loop) onto a D-dim flat spacetime, where the projections given by the antisymmetric variables $x^{\mu\nu\rho}$ represent the corresponding *volumes* enclosed by the projections of the 2-loop along the hyperplanes of the flat target spacetime background.

This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p = 0, 1, 2, 3, \ldots$. The $p = 0$ value represents the center of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}, \ldots$ have been *coined* in the string-brane literature [24]. as the *holographic* areas, volumes, ... projections of the nested family of p-loops (closed p-branes) onto the embedding spacetime coordinate planes/hyperplanes. In ref. [17]

*If we do not restrict indices according to $\mu_1 < \mu_2 < \mu_3 < \ldots$, then the factors $1/2!$, $1/3!$, respectively, have to be included in front of every term in the expansion (1).

they were interpreted as the generalized centre of mass coordinates of an extended object. Extended objects were thus modeled in C-space.

The scalar coordinate σ entering a polyvector X is a measure associated with the p-brane's world manifold V_{p+1} (e. g., the string's 2-dimensional worldsheet V_2): it is proportional to the $(p+1)$-dimensional area/volume of V_{p+1}. In other words, σ is proportional to the areal-time parameter of the Eguchi-Schild formulation of string dynamics [126, 37, 24].

We see in this generalized scheme the objects as observed in spacetime (which is a section through C-space) need not be infinitely extended along time-like directions. They need not be infinitely long world lines, world tubes. They can be finite world lines, world tubes. The σ coordinate measures how long are world lines, world tubes. During evolution they can becomes longer and longer or shorter and shorter.

If we take the differential dX of X and compute the scalar product among two polyvectors $\langle dX^\dagger dX\rangle_0 \equiv \mathrm{d}X^\dagger * \mathrm{d}X \equiv |\mathrm{d}X|^2$ we obtain the C-space extension of the particles proper time in Minkowski space. The symbol $X^\dagger$ denotes the *reversion* operation and involves reversing the order of all the basis γ^μ elements in the expansion of X. It is the analog of the transpose (Hermitian) conjugation. The C-space proper time associated with a polyparticle motion is then the expression (1) which can be written more explicitly as:

$$\begin{aligned} |\mathrm{d}X|^2 &= G_{MN}\,\mathrm{d}X^M \mathrm{d}X^N = \mathrm{d}S^2 = \\ &= \mathrm{d}\sigma^2 + L^{-2} dx_\mu dx^\mu + L^{-4} dx_{\mu\nu} dx^{\mu\nu} + \ldots + \\ &+ L^{-2D} \mathrm{d}x_{\mu_1\ldots\mu_D}\,\mathrm{d}x^{\mu_1\cdots\mu_D}, \end{aligned} \tag{4}$$

where $G_{MN} = E_M^\dagger * E_N$ is the C-space metric.

Here we have introduced the Planck scale L since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops, ..., p-loops. Einstein introduced the speed of light as a universal absolute invariant in order to "unite" space with time (to match units) in the Minkowski space interval:

$$ds^2 = c^2 dt^2 + dx_i dx^i.$$

A similar unification is needed here to "unite" objects of different dimensions, such as x^μ, $x^{\mu\nu}$, etc... . The Planck scale then emerges as another universal invariant in constructing an extended relativity theory in C-spaces [8].

Since the D-dimensional Planck scale is given explicitly in terms of the Newton constant: $L_D = (G_N)^{1/(D-2)}$, in natural units of $\hbar = c = 1$, one can see that when $D = \infty$ the value of L_D is then $L_\infty = G^0 = 1$ (assuming a finite value of G). Hence in $D = \infty$ the Planck scale has the natural value of unity. However, if one wishes to avoid any serious algebraic divergence problems in the series of terms appearing in the expansion of the analog of proper time in C-spaces, in the extreme case when $D = \infty$, from now on we shall focus solely on a *finite* value of D. In this fashion we avoid any serious algebraic convergence problems. We shall not be concerned in this work with the representations of Clifford algebras in different dimensions and with different signatures.

The line element $\mathrm{d}S$ as defined in (4) is *dimensionless*. Alternatively, one can define [8, 9] the line element whose dimension is that of the D-volume so that:

$$\begin{aligned} \mathrm{d}\Sigma^2 &= L^{2D}\mathrm{d}\sigma^2 + L^{2D-2}\mathrm{d}x_\mu \mathrm{d}^\mu + \\ &+ L^{2D-4}\mathrm{d}x_{\mu\nu}\mathrm{d}x^{\mu\nu} + \ldots + \mathrm{d}x_{\mu_1\ldots\mu_D}\mathrm{d}x^{\mu_1\cdots\mu_D}. \end{aligned} \tag{5}$$

Let us use the relation

$$\gamma_{\mu_1} \wedge \ldots \wedge \gamma_{\mu_D} = \gamma \epsilon_{\mu_1\ldots\mu_D} \tag{6}$$

and write the *volume element* as

$$\mathrm{d}x^{\mu_1\cdots\mu_D}\gamma_{\mu_1} \wedge \ldots \wedge \gamma_{\mu_D} \equiv \gamma \mathrm{d}\tilde\sigma\,, \tag{7}$$

where

$$\mathrm{d}\tilde\sigma \equiv \mathrm{d}x^{\mu_1\cdots\mu_D}\epsilon_{\mu_1\ldots\mu_D}\,. \tag{8}$$

In all expressions we assume the ordering prescription $\mu_1 < \mu_2 < \ldots < \mu_r$, $r = 1, 2, \ldots, D$. The line element can then be written in the form

$$\begin{aligned} \mathrm{d}\Sigma^2 &= L^{2D}\mathrm{d}\sigma^2 + L^{2D-2}\mathrm{d}x_\mu \mathrm{d}x^\mu + \\ &+ L^{2D-4}\mathrm{d}x_{\mu\nu}\mathrm{d}x^{\mu\nu} + \ldots + |\gamma|^2\,\mathrm{d}\tilde\sigma^2, \end{aligned} \tag{9}$$

where

$$|\gamma|^2 \equiv \gamma^\dagger * \gamma\,. \tag{10}$$

Here γ is the pseudoscalar basis element and can be written as $\gamma_0 \wedge \gamma_1 \wedge \ldots \gamma_{D-1}$. In flat spacetime M_D we have that $|\gamma|^2 = +1$ or -1, depending on dimension and signature. In M_4 with signature $(+---)$ we have $\gamma^\dagger * \gamma = \gamma^\dagger\gamma = \gamma^2 = -1$ $(\gamma \equiv \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3)$, whilst in M_5 with signature $(+----)$ it is $\gamma^\dagger\gamma = 1$.

The analog of Lorentz transformations in C-spaces which transform a polyvector X into another poly-vector X' is given by

$$X' = RXR^{-1} \tag{11}$$

with

$$R = e^{\theta^A E_A} = \exp\left[(\theta I + \theta^\mu\gamma_\mu + \theta^{\mu_1\mu_2}\gamma_{\mu_1}\wedge\gamma_{\mu_2}\ldots)\right] \tag{12}$$

and also

$$R^{-1} = e^{-\theta^A E_A} = \exp[-(\theta I + \theta^\nu\gamma_\nu + \theta^{\nu_1\nu_2}\gamma_{\nu_1}\wedge\gamma_{\nu_2}\ldots)] \tag{13}$$

where the theta parameters in (12), (13) are the components of the Clifford-value parameter $\Theta = \theta^M E_M$:

$$\theta; \theta^\mu; \theta^{\mu\nu}; \ldots \tag{14}$$

they are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation, one can write the norm of a poly-vectors in terms of the trace operation as: $||X||^2 = Trace X^2$. Hence under C-space Lorentz transformation the norms of poly-vectors behave like follows:

$$Trace\, X'^2 = Trace\,[RX^2R^{-1}] = \\ = Trace\,[RR^{-1}X^2] = Trace\, X^2. \tag{15}$$

These norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and $RR^{-1} = 1$. If one writes the invariant norm in terms of the *reversal* operation $<X^\dagger X>_s$ this will constrain the explicit form of the terms in the exponential which define the rotor R so the rotor R obeys the analog condition of an orthogonal rotation matrix $R^\dagger = R^{-1}$. Hence the appropriate poly-rotations of poly-vectors which preserve the norm must be:

$$||(X')^2|| = < X'^\dagger X' >_s = \\ = < (R^{-1})^\dagger X^\dagger R^\dagger R X R^{-1} >_s = \\ = < R X^\dagger X R^{-1} >_s = < X^\dagger X >_s = ||X^2||\,, \tag{16}$$

where once again, we made use of the analog of the cyclic property of the trace, $<RX^\dagger X R^{-1}>_s = <X^\dagger X>_s$.

This way of rewriting the inner product of poly-vectors by means of the reversal operation that reverses the order of the Clifford basis generators: $(\gamma^\mu \wedge \gamma^\nu)^\dagger = \gamma^\nu \wedge \gamma^\mu$, etc... has some subtleties. The analog of an orthogonal matrix in Clifford spaces is $R^\dagger = R^{-1}$ such that

$$< X'^\dagger X' >_s = < (R^{-1})^\dagger X^\dagger R^\dagger R X R^{-1} >_s = \\ = < R X^\dagger X R^{-1} >_s = < X^\dagger X >_s = \text{invariant}.$$

This condition $R^\dagger = R^{-1}$, of course, will *restrict* the type of terms allowed inside the exponential defining the rotor R because the *reversal* of a p-vector obeys

$$(\gamma_{\mu_1} \wedge \gamma_{\mu_2} \ldots \wedge \gamma_{\mu_p})^\dagger = \gamma_{\mu_p} \wedge \gamma_{\mu_{p-1}} \ldots \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1} = \\ = (-1)^{p(p-1)/2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \ldots \wedge \gamma_{\mu_p}\,.$$

Hence only those terms that change sign (under the reversal operation) are permitted in the exponential defining $R = \exp[\theta^A E_A]$.

Another possibility is to *complexify* the C-space polyvector valued coordinates $Z = Z^A E_A = X^A E_A + i Y^A E_A$ and the boosts/rotation parameters θ allowing the unitary condition $\bar{U}^\dagger = U^{-1}$ to hold in the generalized Clifford unitary transformations $Z' = U Z U^\dagger$ associated with the complexified polyvector $Z = Z^A E_A$ such that the interval

$$<d\bar{Z}^\dagger dZ>_s = d\bar{\Omega} d\Omega + d\bar{z}^\mu dz_\mu + d\bar{z}^{\mu\nu} dz_{\mu\nu} + d\bar{z}^{\mu\nu\rho} dz_{\mu\nu\rho} + \ldots$$

remains invariant (upon setting the Planck scale $\Lambda = 1$).

The unitary condition $\bar{U}^\dagger = U^{-1}$ under the *combined* reversal and complex-conjugate operation will constrain the form of the complexified boosts/rotation parameters θ^A appearing in the rotor: $U = \exp\left[\theta^A E_A\right]$. The theta parameters θ^A are either purely real or purely imaginary depending if the reversal ${E_A}^\dagger = \pm E_A$, to ensure that an overall *change* of sign occurs in the terms $\theta^A E_A$ inside the exponential defining U so that $\bar{U}^\dagger = U^{-1}$ holds and the norm $<\bar{Z}^\dagger Z>_s$ remains invariant under the analog of unitary transformations in *complexified* C-spaces. These techniques are not very different from Penrose Twistor spaces. As far as we know a Clifford-Twistor space construction of C-spaces has not been performed so far.

Another alternative is to define the polyrotations by $R = \exp(\Theta^{AB}[E_A, E_B])$ where the commutator $[E_A, E_B] = F_{ABC} E_C$ is the C-space analog of the $i\,[\gamma_\mu, \gamma_\nu]$ commutator which is the generator of the Lorentz algebra, and the theta parameters Θ^{AB} are the C-space analogs of the rotation/boots parameters $\theta^{\mu\nu}$. The diverse parameters Θ^{AB} are purely real or purely imaginary depending whether the reversal $[E_A, E_B]^\dagger = \pm\,[E_A, E_B]$ to ensure that $R^\dagger = R^{-1}$ so that the scalar part $<X^\dagger X>_s$ remains invariant under the transformations $X' = R X R^{-1}$. This last alternative seems to be more physical because a poly-rotation should map the E_A direction into the E_B direction in C-spaces, hence the meaning of the generator $[E_A, E_B]$ which extends the notion of the $[\gamma_\mu, \gamma_\nu]$ Lorentz generator.

The above transformations are *active transformations* since the transformed Clifford number X' (polyvector) is different from the "original" Clifford number X. Considering the transformations of components we have

$$X' = X'^M E_M = L^M{}_N\, X^N E_M\,. \tag{17}$$

If we compare (17) with (11) we find

$$L^M{}_N E_N = R E_N R^{-1} \tag{18}$$

from which it follows that

$$L^M{}_N = \langle E^M R E_N R^{-1} \rangle_0 \equiv E^M * (R E_N R^{-1}) = E^M * E'_N, \tag{19}$$

where we have labelled E'_N as new basis element since in the active interpretation one may perform either a change of the polyvector components or a change of the basis elements. The $\langle\ \ \rangle_0$ means the scalar part of the expression and "$*$" the scalar product. Eq-(19) has been obtained after multiplying (18) from the left by E^J, taking into account that $\langle E^J E_N \rangle_0 \equiv E^J * E_N = \delta^J{}_N$, and renamiming the index J into M.

3 Generalized dynamics of particles, fields and branes in C-space

An immediate application of this theory is that one may consider "strings" and "branes" in C-spaces as a unifying description of *all* branes of different dimensionality. As we have already indicated, since spinors are in left/right ideals of a Clifford algebra, a supersymmetry is then naturally incorporated into this approach as well. In particular, one can have world manifold and target space supersymmetry *simultaneously* [15]. We hope that the C-space "strings" and "branes" may lead us towards discovering the physical foundations of string and M-theory. For other alternatives to supersymmetry see the work by [50]. In particular, Z_3 generalizations of supersymmetry based on ternary algebras and Clifford algebras have been proposed by Kerner [128] in what has been called Hypersymmetry.

3.1 The Polyparticle Dynamics in C-space

We will now review the theory [15, 17] in which an extended object is modeled by the components $\sigma,\ x^\mu, x^{\mu\nu}, \ldots$ of the Clifford valued polyvector (2). By assumption the extended objects, as observed from Minkowski spacetime, can in general be localized not only along space-like, but also along time-like directions [15, 17]. In particular, they can be "instantonic" p-loops with either space-like or time-like orientation. Or they may be long, but finite, tube-like objects. The theory that we consider here goes beyond the ordinary relativity in Minkowski spacetime, therefore such localized objects in Minkowski spacetime pose no problems. They are postulated to satisfy the dynamical principle which is formulated in C-space. All conservation laws hold in C-space where we have infinitely long world "lines" or Clifford lines. In Minkowski spacetime M_4 — which is a subspace of C-space — we observe the intersections of Clifford lines with M_4. And those intersections appear as localized extended objects, p-loops, described above.

Let the motion of such an extended object be determined by the action principle

$$I = \kappa \int \mathrm{d}\tau\, (\dot{X}^\dagger * \dot{X})^{1/2} = \kappa \int \mathrm{d}\tau\, (\dot{X}^A \dot{X}_A)^{1/2}\,, \qquad (20)$$

where κ is a constant, playing the role of "mass" in C-space, and τ is an arbitrary parameter. The C-space velocities $\dot{X}^A = \mathrm{d}X^A/\mathrm{d}\tau = (\dot{\sigma}, \dot{x}^\mu, \dot{x}^{\mu\ nu}, \ldots)$ are also called "holographic" velocities.

The equation of motion resulting from (20) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{X}^A}{\sqrt{\dot{X}^B \dot{X}_B}}\right) = 0\,. \qquad (21)$$

Taking $\dot{X}^B \dot{X}_B = \text{constant} \neq 0$ we have that $\ddot{X}^A = 0$, so that $x^A(\tau)$ is a straight worldline in C-space. The components x^A then change linearly with the parameter τ. This means that the extended object position x^μ, effective area $x^{\mu\nu}$, 3-volume $x^{\mu\nu\alpha}$, 4-volume $x^{\mu\nu\alpha\beta}$, etc., they all change with time. That is, such object experiences a sort of generalized dilational motion [17].

We shall now review the procedure exposed in ref. [17] according to which in such a generalized dynamics an object may be accelerated to faster than light speeds as viewed from a 4-dimensional Minkowski space, which is a subspace of C-space. For a different explanation of superluminal propagation based on the modified nonlinear de Broglie dispersion relations see [68].

The canonical momentum belonging to the action (20) is

$$P_A = \frac{\kappa \dot{X}_A}{(\dot{X}^B \dot{X}_B)^{1/2}}\,. \qquad (22)$$

When the denominator in eq.-(22) is zero the momentum becomes infinite. We shall now calculate the speed at which this happens. This will be the *maximum speed* that an object accelerating in C-space can reach. Although an initially slow object cannot accelerate beyond that speed limit, this does not automatically exclude the possibility that fast objects traveling at a speed above that limit may exist. Such objects are C-space analog of tachyons [31, 32]. All the well known objections against tachyons should be reconsidered for the case of C-space before we could say for sure that C-space tachyons do not exist as freely propagating objects. We will leave aside this interesting possibility, and assume as a working hypothesis that there is no tachyons in C-space.

Vanishing of $\dot{X}^B \dot{X}_B$ is equivalent to vanishing of the C-space line element

$$\begin{aligned} \mathrm{d}X^A \mathrm{d}X_A = \mathrm{d}\sigma^2 + \left(\frac{\mathrm{d}x^0}{L}\right)^2 - \left(\frac{\mathrm{d}x^1}{L}\right)^2 - \left(\frac{\mathrm{d}x^{01}}{L^2}\right)^2 \ldots \\ \ldots + \left(\frac{\mathrm{d}x^{12}}{L^2}\right)^2 - \left(\frac{\mathrm{d}x^{123}}{L^3}\right)^2 - \left(\frac{\mathrm{d}x^{0123}}{L^4}\right)^2 + \ldots = 0\,, \end{aligned} \qquad (23)$$

where by "..." we mean the terms with the remaining components such as x^2, x^{01}, x^{23}, ..., x^{012}, etc. The C-space line element is associated with a particular *choice* of C-space metric, namely $G_{MN} = E^\dagger_M * E_N$. If the basis E_M, $M = 1, 2, \ldots, 2^D$ is generated by the flat space γ^μ satisfying (3), then the C-space has the diagonal metric of eq.-(23) with $+, -$ signa. In general this is not necessarily so and the C-space metric is a more complicated expression. We take now dimension of spacetime being 4, so that x^{0123} is the highest grade coordinate. In eq.-(23) we introduce a length parameter L. This is necessary, since $x^0 = ct$ has dimension of length, x^{12} of length square, x^{123} of length to the third power, and x^{0123} of length to the forth power. It is natural to assume that L is the *Planck length*, that is $L = 1.6{\times}10^{-35}$m.

Let us assume that the coordinate time $t = x^0/c$ is the parameter with respect to which we define the speed V in C-space.

So we have

$$\begin{aligned} V^2 = -\left(L\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^{01}}{L^2}\right)^2 \ldots \\ \ldots - \left(\frac{1}{L}\frac{\mathrm{d}x^{12}}{\mathrm{d}t}\right)^2 + \left(\frac{1}{L^2}\frac{\mathrm{d}x^{123}}{\mathrm{d}t}\right)^2 + \left(\frac{1}{L^3}\frac{\mathrm{d}x^{0123}}{\mathrm{d}t}\right)^2 - \ldots \end{aligned} \qquad (24)$$

From eqs.-(23), (24) we find that the maximum speed is the maximum speed is given by

$$V^2 = c^2 \,. \tag{25}$$

First, we see, the maximum speed squared V^2 contains not only the components of the 1-vector velocity $\mathrm{d}x^1/\mathrm{d}t$, as it is the case in the ordinary relativity, but also the multivector components such as $\mathrm{d}x^{12}/\mathrm{d}t$, $\mathrm{d}x^{123}/\mathrm{d}t$, etc.

The following special cases when only certain components of the velocity in C-space are different from zero, are of particular interest:

(i) Maximum 1-vector speed

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = c = 3.0{\times}10^8 \mathrm{m/s}\,;$$

(ii) Maximum 3-vector speed

$$\frac{\mathrm{d}x^{123}}{\mathrm{d}t} = L^2 c = 7.7{\times}10^{-62} \mathrm{m}^3/\mathrm{s}\,;$$

$$\frac{\mathrm{d}\sqrt[3]{x^{123}}}{\mathrm{d}t} = 4.3{\times}10^{-21} \mathrm{m/s} \quad \text{(diameter speed)}\,;$$

(iii) Maximum 4-vector speed

$$\frac{\mathrm{d}x^{0123}}{\mathrm{d}t} = L^3 c = 1.2{\times}10^{-96} \mathrm{m}^4/\mathrm{s}$$

$$\frac{\mathrm{d}\sqrt[4]{x^{0123}}}{\mathrm{d}t} = 1.05{\times}10^{-24} \mathrm{m/s} \quad \text{(diameter speed)}\,.$$

Above we have also calculated the corresponding diameter speeds for the illustration of how fast the object expands or contracts.

We see that the maximum multivector speeds are very small. The diameters of objects change very slowly. Therefore we normally do not observe the dilatational motion.

Because of the positive sign in front of the σ and x^{12}, x^{012}, etc., terms in the quadratic form (23) there are no limits to corresponding 0-vector, 2-vector and 3-vector speeds. But if we calculate, for instance, the energy necessary to excite 2-vector motion we find that it is very high. Or equivalently, to the relatively modest energies (available at the surface of the Earth), the corresponding 2-vector speed is very small. This can be seen by calculating the energy

$$p^0 = \frac{\kappa c^2}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{26}$$

(a) for the case of pure 1-vector motion by taking $V = \mathrm{d}x^1/\mathrm{d}t$, and

(b) for the case of pure 2-vector motion by taking $V = \mathrm{d}x^{12}/(L\mathrm{d}t)$.

By equating the energies belonging to the cases (a) and (b) we have

$$p^0 = \frac{\kappa c^2}{\sqrt{1 - \left(\frac{1}{c}\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2}} = \frac{\kappa c^2}{\sqrt{1 - \left(\frac{1}{Lc}\frac{\mathrm{d}x^{12}}{\mathrm{d}t}\right)^2}}\,, \tag{27}$$

which gives

$$\frac{1}{c}\frac{\mathrm{d}x^1}{\mathrm{d}t} = \frac{1}{Lc}\frac{\mathrm{d}x^{12}}{\mathrm{d}t} = \sqrt{1 - \left(\frac{\kappa c^2}{p_0}\right)^2}\,. \tag{28}$$

Thus to the energy of an object moving translationally at $\mathrm{d}x^1/\mathrm{d}t = 1\,\mathrm{m/s}$, there corresponds the 2-vector speed $\mathrm{d}x^{12}/\mathrm{d}t = L\,\mathrm{d}x^1/\mathrm{d}t = 1.6{\times}10^{-35}$ m^2/s (diameter speed 4×10^{-18} m/s). This would be a typical 2-vector speed of a macroscopic object. For a microscopic object, such as the electron, which can be accelerated close to the speed of light, the corresponding 2-vector speed could be of the order of 10^{-26} m^2/s (diameter speed 10^{-13} m/s). In the examples above we have provided rough estimations of possible 2-vector speeds. Exact calculations should treat concrete situations of collisions of two or more objects, assume that not only 1-vector, but also 2-vector, 3-vector and 4-vector motions are possible, and take into account the conservation of the polyvector momentum P_A.

Maximum 1-vector speed, i. e., the usual speed, can exceed the speed of light when the holographic components such as $\mathrm{d}\sigma/\mathrm{d}t$, $\mathrm{d}x^{12}/\mathrm{d}t$, $\mathrm{d}x^{012}/\mathrm{d}t$, *etc., are different from zero* [17]. This can be immediately verified from eqs.-(23), (24). The speed of light is no longer such a strict barrier as it appears in the ordinary theory of relativity in M_4. In C-space a particle has extra degrees of freedom, besides the translational degrees of freedom. The scalar, σ, the bivector, x^{12} (in general, x^{rs}, $r, s = 1, 2, 3$) and the three vector, x^{012} (in general, x^{0rs}, $r, s = 1, 2, 3$), contributions to the C-space quadratic form (23) have positive sign, which is just opposite to the contributions of other components, such as x^r, x^{0r}, x^{rst}, $x^{\mu\nu\rho\sigma}$. Because some terms in the quadratic form have $+$ and some $-$ sign, the absolute value of the 3-velocity $\mathrm{d}x^r/\mathrm{d}x^0$ can be greater than c.

It is known that when tachyons can induce a breakdown of causality. The simplest way to see why causality is violated when tachyons are used to exchange signals is by writing the temporal displacements $\delta t = t^B - t^A$ between two events (in Minkowski space-time) in two different frames of reference:

$$\begin{aligned}(\delta t)' &= (\delta t)\cosh(\xi) + \frac{\delta x}{c}\sinh(\xi) = (\delta t)\bigg[\cosh(\xi) + \\ &+ \left(\frac{1}{c}\frac{\delta x}{\delta t}\right)\sinh(\xi)\bigg] = (\delta t)[\cosh(\xi) + (\beta_{tach.})\sinh(\xi)]\end{aligned} \tag{29}$$

the boost parameter ξ is defined in terms of the velocity as $\beta_{frame} = v_{frame}/c = \tanh(\xi)$, where v_{frame} is is the relative velocity (in the x-direction) of the two reference frames and can be written in terms of the Lorentz-boost rapidity parameter ξ by using hyperbolic functions. The Lorentz dilation factor is $\cosh(\xi) = (1-\beta^2_{frame})^{-1/2}$; whereas

$\beta_{tachyon} = v_{tachyon}/c$ is the beta parameter associated with the tachyon velocity $\delta x/\delta t$. By emitting a tachyon along the *negative* x -direction one has $\beta_{tachyon} < 0$ and such that its velocity exceeds the speed of light $|\beta_{tachyon}| > 1$.

A reversal in the sign of $(\delta t)' < 0$ in the above boost transformations occurs when the tachyon velocity $|\beta_{tachyon}| > 1$ and the relative velocity of the reference frames $|\beta_{frame}| < 1$ obey the inequality condition:

$$(\delta t)' = (\delta t)[\cosh(\xi) - |\beta_{tachyon}|\sinh(\xi)] < 0 \Rightarrow \Rightarrow 1 < \frac{1}{\tanh(\xi)} = \frac{1}{\beta_{frame}} < |\beta_{tachyon}| \tag{30}$$

thereby resulting in a causality violation in the primed reference frame since the effect (event B) occurs *before* the cause (event A) in the *primed* reference frame.

In the case of subluminal propagation $|\beta_{particle}| < 1$ there is no causality violation since one would have:

$$(\delta t)' = (\delta t)[\cosh(\xi) - |\beta_{particle}|\sinh(\xi)] > 0 \tag{31}$$

due to the hyperbolic trigonometric relation:

$$\cosh^2(\xi) - \sinh^2(\xi) = 1 \Rightarrow \cosh(\xi) - \sinh(\xi) \geqslant 0\,. \tag{32}$$

In the theory considered here, there are no tachyons in C-space, because physical signals in C-space are constrained to live *inside* the C-space-light cone, defined by eq.-(23). However, certain worldlines in C-space, when projected onto the subspace M_4, can appear as worldlines of ordinary tachyons outside the light-cone in M_4. The physical analog of photons in C-space corresponds to tensionless p-loops, i. e., *tensionless* closed branes, since the analog of mass m in C-space is the maximal p-loop tension. By "maximal p-loop" we mean the loop with the maximum value of p associated with the hierarchy of p-loops (closed p-branes): $p = 0, 1, 2, \dots$ living in the embedding target spacetime. One must not confuse the Stueckelberg parameter σ with the C-space Proper-time Σ eq.-(5); so one could have a world line in C-space such that

$$\mathrm{d}\Sigma = 0 \leftrightarrow C\text{-space photon} \leftrightarrow \begin{array}{l}\text{Tensionless branes with}\\ \text{a monotonically increasing}\\ \text{Stueckelberg parameter } \sigma.\end{array}$$

In C-space the dynamics refers to a larger space. Minkowski space is just a subspace of C-space. "Wordlines" now live in C-space that can be projected onto the Minkowski subspace M_4. Concerning tachyons and causality within the framework of the C-space relativity, the authors of this review propose two different explanations, described below.

According to one author (C. C.) one has to take into account the fact that one is enlarging the ordinary Lorentz group to a larger group of C-space Lorentz transformations which involve poly-rotations and generalizations of boosts transformations. In particular, the C-space generalization of the ordinary boost transformations associated with the boost rapidity parameter ξ such that $\tanh(\xi) = \beta_{frame}$ will involve now the family of C-space boost rapidity parameters θ^{t1}, θ^{t12}, θ^{t123}, $\dots$ $\theta^{t123\dots}$, $\dots$ since boosts are just (poly) rotations along directions involving the *time* coordinate. Thus, one is replacing the ordinary boost transformations in Minkowski spacetime for the more general C-space boost transformations as we go from one frame of reference to another frame of reference.

Due to the linkage among the C-space coordinates (polydimensional covariance) when we envision an ordinary boost along the x^1-direction, we must not forget that it is also? interconnected to the area-boosts in the x^{12}-direction as well, and, which in turn, is also linked to the x^2 direction. Because the latter direction is *transverse* to the original tachyonic? x^1-motion? the latter x^2-boosts? won't affect things and we may concentrate? on the area-boosts along the x^{12} direction involving the θ^{t12} parameter that will appear in the C-space boosts and which contribute to a crucial extra term in the transformations such that no sign-change in $\delta t'$? will occur.

More precisely, let us set *all* the values of the theta parameters to zero *except* the parameters θ^{t1} and θ^{t12} related to the ordinary boosts in the x^1 direction and area-boosts in the x^{12} directions of C-space. This requires, for example, that one has at least one spatial-area component, and one temporal coordinate, which implies that the dimensions must be at least $D = 2 + 1 = 3$. Thus, we have in this case:

$$X' = RXR^{-1} = e^{\theta^{t1}\gamma_t\wedge\gamma_1 + \theta^{t12}\gamma_t\wedge\gamma_1\wedge\gamma_2} \times \times X^M E_M e^{-\theta^{t1}\gamma_t\wedge\gamma_1 - \theta^{t12}\gamma_t\wedge\gamma_1\wedge\gamma_2} \Rightarrow X'^N = L^N_M X^M, \tag{33}$$

where as we shown previously $L^N_M = <E^N R E_M R^{-1}>_0$. When one concentrates on the transformations of the time coordinate, we have now that the C-space boosts do *not* coincide with ordinary boosts in the x^1 direction:

$$t' = L^t_M X^M = <E^t R E_M R^{-1}>_0 X^M \neq (L^t_t)t + (L^t_1)x^1, \tag{34}$$

because of the extra non-vanishing θ parameter θ^{t12}.

This is because the rotor R includes the extra generator $\theta^{t12}\gamma_t \wedge \gamma_1 \wedge \gamma_2$ which will bring extra terms into the transformations; i. e. it will rotate the $E_{[12]}$ bivector-basis, that couples to the holographic coordinates x^{12}, into the E_t direction which is being contracted with the E^t element in the definition of L^t_M. There are extra terms in the C-space boosts because the poly-particle dynamics is taking place in C-space and all coordinates X^M which contain the t, x^1, x^{12} directions will contribute to the C-space boosts in $D = 3$, since one is projecting down the dynamics from C-space onto the (t, x^1) plane when one studies the motion of the tachyon in M_4.

Concluding, in the case when one sets all the θ parameters to zero, except the θ^{t1} and θ^{t12}, the $X' = RX^M E_M R^{-1}$

transformations will be:

$$(\delta t)' = L^t_M(\theta^{t1}; \theta^{t12})(\delta X^M) \neq L^t_t(\delta t) + L^t_1(\delta x^1)\,, \quad (35)$$

due to the presence of the *extra* term $L^t_{12}(\delta X^{12})$ in the transformations. In the more general case, when there are more non-vanishing θ parameters, the indices M of the X^M coordinates must be *restricted* to those directions in C-space which involve the t, x^1, x^{12}, $x^{123}\ldots$ directions as required by the C-space poly-particle dynamics. The generalized C-space boosts involve now the ordinary tachyon velocity component of the poly-particle as well as the generalized holographic areas, volumes, hyper-volumes... velocities $V^M = (\delta X^M/\delta t)$ associated with the poly-vector components of the Clifford-valued C-space velocity.

Hence, at the expense of *enlarging* the ordinary Lorentz boosts to the C-space Lorentz boosts, and the degrees of freedom of a point particle into an extended poly-particle by including the holographic coordinates, in C-space one can still have ordinary point-particle tachyons without changing the sign of δt, and without violating causality, due to the presence of the *extra* terms in the C-space boosts transformations which ensure us that the sign of $\delta t > 0$ is maintained as we go from one frame of reference to another one. Naturally, if one were to *freeze* all the θ parameters to zero except one θ^{t1} one would end up with the standard Lorentz boosts along the x^1-direction and a violation of causality would occur for tachyons as a result of the sign-change in $\delta t'$.

In future work we shall analyze in more detail if the condition $\delta t' = L^t_M(\delta X^M) > 0$ is satisfied for *any* physical values of the θ C-space boosts parameters and for *any* physical values of the holographic velocities consistent with the condition that the C-space velocity $V_M V^M \geqslant 0$. What one cannot have is a C-space tachyon; i. e. the physical signals in C-space must be constrained to live *inside* the C-space light-cone. The analog of "photons" in C-space are *tensionless* branes. The corresponding analog of C-space tachyons involve branes with imaginary tensions, not unlike ordinary tachyons $m^2 < 0$ of imaginary mass.

To sum up: Relativity in C-space demands *enlarging* the ordinary Lorentz group (boosts) to a larger symmetry group of C-space Lorentz group and enlarging the degrees of freedom by including Clifford-valued coordinates $X = X^M E_M$. This is the only way one can have a point-particle tachyonic speed in a Minkowski subspace without violating causality in C-space. Ordinary Lorentz boosts are incompatible with tachyons if one wishes to preserve causality. In C-space one requires to have, at least, two theta parameters θ^{t1} and θ^{t12} with the inclusion, at least, of the t, x^1, x^{12} coordinates in a C-space boost, to be able to enforce the condition $\delta t' > 0$ under (combined) boosts along the x^1 direction accompanied by an *area*-boost along the x^{12} direction of C-space. It is beyond the scope of this review to analyze all the further details of the full-fledged C-boosts transformations in order to check that the condition $\delta t' > 0$ is obeyed for *any* physical values of the θ parameters and holographic velocities.

According to the other author (M. P.), the problem of causality could be explained as follows. In the usual theory of relativity the existence of tachyons is problematic because one can arrange for situations such that tachyons are sent into the past. A tachyon T_1 is emitted from an apparatus worldline $\mathcal{C}$ at x^0_1 and a second tachyon T_2 can arrive to the same worldline $\mathcal{C}$ at an earlier time $x'^0 < x^0_1$ and trigger destruction of the apparatus. The spacetime event E' at which the apparatus is destroyed coincides with the event E at which the apparatus by initial assumption kept on functioning normally and later emitted T_1. So there is a paradox from the ordinary (constrained) relativistic particle dynamics.

There is no paradox if one invokes the unconstrained Stueckelberg description of superluminal propagation in M_4. It can be described as follows. A C-space worldline can be described in terms of five functions $x^\mu(\tau)$, $\sigma(\tau)$ (all other C-space coordinates being kept constant). In C-space we have the *constrained action* (20), whilst in Minkowski space we have a reduced, *unconstrained* action. A reduction of variables can be done by choosing a gauge in which $\sigma(\tau) = \tau$. It was shown in ref. [16, 15, 17] that the latter unconstrained action is equivalent to the well known Stueckelberg action [33, 34]. In other words, the Stueckelberg relativistic dynamics is embedded in C-space. In Stueckelberg theory all four spacetime coordinates x^μ are independent dynamical degrees of freedom that evolve in terms of an extra parameter σ which is invariant under Lorentz transformations in M_4.

From the C-space point of view, the evolution parameter σ is just one of the C-space coordintes X^M. By assumption, σ is monotonically increasing along particles' worldlines. Certain C-space worldlines may appear tachyonic from the point of view of M_4. If we now repeat the above experiment with the emission of the first and absorption of the second tachyon we find out that the second tachyon T_2 cannot reach the aparatus worldline earlier than it was emmitted from. Namely, T_2 can arrive at a C-space event E' with $x'^0 < x^0_1$, but the latter event does not coincide with the event E on the aparatus worldline, since although having the same coordinates $x'^\mu = x^\mu$, the events E and E' have different extra coordinates $\sigma' \neq \sigma$. In other words, E and E' are different points in C-space. Therefore T_2 cannot destroy the apparatus and there is no paradox.

If nature indeed obeys the dynamics in Clifford space, then a particle, as observed from the 4-dimensional Minkowski space, can be accelerated beyond the speed of light [17], provided that its extra degrees of freedom $x^{\mu\nu}$, $x^{\mu\nu\alpha}, \ldots$, are changing simultaneously with the ordinary position x^μ. But such a particle, although moving faster than light in the subspace M_4, is moving slower than light in C-space, since its speed V, defined in eq.-(24), is smaller than *c*. *In* *this respect, our particle is not tachyon at all!* In C-space we thus retain all the nice features of relativity, but in the subspace M_4 we have, as a particular case, the unconstrained Stueckelberg theory in which faster-than-light propagation is not paradoxical and is consistent with the quantum field theory as well [15]. This is so, because the unconstrained Stueckelberg theory is quite different from the ordinary (constrained) theory of relativity in M_4, and faster than light motion in the former theory is of totally different nature from the faster that light motion in the latter theory. The tachyonic "world lines" in M_4 are just projections of trajectories in C-space onto Minkowski space, however, the true world lines of M_4 must be interpreted always as being embedded onto a larger C-space, such that they cannot take part in the paradoxical arrangement in which future could influence the past. The well known objections against tachyons are not valid for our particle which moves according to the relativity in C-space.

We have described how one can obtain faster than light motion in M_4 from the theory of relativity in C-space. There are other possible ways to achieve superluminal propagation. One such approach is described in refs. [84]

An alternative procedure In ref. [9] an alternative factorization of the C-space line element has been undertaken. Starting from the line element $\mathrm{d}\Sigma$ of eq.-(5), instead of factoring out the $(dx^0)^2$ element, one may factor out the $(\mathrm{d}\Omega)^2 \equiv \equiv L^{2D}\mathrm{d}\sigma^2$ element, giving rise to the generalized "holographic" velocities measured w. r. t the Ω parameter, for example the areal-time parameter in the Eguchi-Schild formulation of string dynamics [126], [37], [24], instead of the x^0 parameter (coordinate clock). One then obtains

$$\mathrm{d}\Sigma^2 = \mathrm{d}\Omega^2\left[1 + L^{2D-2}\frac{\mathrm{d}x_\mu}{\mathrm{d}\Omega}\frac{\mathrm{d}x^\mu}{\mathrm{d}\Omega} + \right. \\ \left. + L^{2D-4}\frac{\mathrm{d}x_{\mu\nu}}{\mathrm{d}\Omega}\frac{\mathrm{d}x^{\mu\nu}}{\mathrm{d}\Omega} + \ldots + |\gamma|^2\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\Omega}\right)^2\right]. \tag{36}$$

The idea of ref. [9] was to restrict the line element (36) to the non tachyonic values which imposes un upper limit on the holographic velocities. The motivation was to find a lower bound of length scale. This upper holographic-velocity bound does not necessarily translate into a lower bound on the values of lengths, areas, volumes... without the introduction of quantum mechanical considerations. One possibility could be that the upper limiting speed of light and the upper bound of the momentum $m_p c$ of a Planck-mass elementary particle (the so-called *Planckton* in the literature) generalizes now to an upper-bound in the p-loop holographic velocities and the p-loop holographic momenta associated with elementary closed p-branes whose tensions are given by powers of the Planck mass. And the latter upper bounds on the holographic p-loop momenta implies a lower-bound on the holographic areas, volumes, ..., resulting from the string/brane uncertainty relations [11], [10], [19]. Thus, Quantum Mechanics is required to implement the postulated principle of minimal lengths, areas, volumes... and which cannot be derived from the classical geometry alone. The emergence of minimal Planck areas occurs also in the Loop Quantum Gravity program [111] where the expectation values of the Area operator are given by multiples of Planck area.

Recently in [134] an isomorphism between Yang's Noncommutative space-time algebra (involving *two* length scales) [136] and the *holographic area coordinates* algebra of C-spaces (Clifford spaces) was constructed via an AdS_5 space-time which is instrumental in explaining the origins of an extra (infrared) scale R in conjunction to the (ultraviolet) Planck scale λ characteristic of C-spaces. Yang's Noncommutative space-time algebra allowed Tanaka [137] to explain the origins behind the *discrete* nature of the spectrum for the *spatial* coordinates and *spatial* momenta which yields a *minimum* length-scale λ (ultraviolet cutoff) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff). In particular, the norm-squared $\mathbf{A}^2$ of the holographic Area operator $X_{AB}X^{AB}$ has a correspondence with the quadratic Casimir operator $\Sigma_{AB}\Sigma^{AB}$ of the conformal algebra $SO(4,2)$ ($SO(5,1)$ in the Euclideanized AdS_5 case). This holographic area-Casimir relationship does not differ much from the area-spin relation in Loop Quantum Gravity $\mathbf{A}^2 \sim \sim \lambda^4 \sum j_i\,(j_i + 1)$ in terms of the $SU(2)$ Casimir J^2 with eigenvalues $j\,(j+1)$ and where the sum is taken over the spin network sites.

3.2 A unified theory of all p-Branes in C-spaces

The generalization to C-spaces of string and p-brane actions as embeddings of world-manifolds onto target spacetime backgrounds involves the embeddings of polyvector-valued world-manifolds (of dimensions 2^d) onto polyvector-valued target spaces (of dimensions 2^D), given by the Clifford-valued maps $X = X(\Sigma)$ (see [15]). These are maps from the Clifford-valued world-manifold, parametrized by the polyvector-valued variables Σ, onto the Clifford-valued target space parametrized by the polyvector-valued coordinates X. Physically one envisions these maps as taking an n-dimensional simplicial cell (n-loop) of the world-manifold onto an m-dimensional simplicial cell (m-loop) of the target C-space manifold; i. e. maps from n-dim objects onto m-dim objects generalizing the old maps of taking points onto points. One is basically dealing with a dimension-category of objects. The size of the simplicial cells (p-loops), upon quantization of a generalized harmonic oscillator, for example, are given by multiples of the Planck scale, in area, volume, hypervolume units or Clifford-bits.

In compact multi-index notation $X = X^M\Gamma_M$ one denotes for each one of the components of the target space polyvector X:

$$X^M \equiv X^{\mu_1\mu_2\ldots\mu_r},\ \mu_1 < \mu_2 < \ldots < \mu_r \tag{37}$$

and for the world-manifold polyvector $\Sigma = \Sigma^A E_A$:

$$\Sigma^A \equiv \xi^{a_1 a_2 \dots a_s}, a_1 < a_2 < \dots < a_s \,, \tag{38}$$

where $\Gamma_M = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu}, \dots)$ and $E_A = (\underline{1}, e_a, e_{ab}, \dots)$ form the basis of the target manifold and world manifold Clifford algebra, respectively. It is very important to order the indices within each multi-index M and A as shown above. The above Clifford-valued coordinates X^M, Σ^A correspond to antisymmetric tensors of ranks r, s in the target spacetime background and in the world-manifold, respectively.

There are many different ways to construct C-space brane actions which are on-shell equivalent to the analogs of the Dirac-Nambu-Goto action for extended objects and that are given by the world-volume spanned by the branes in their motion through the target spacetime background.

One of these actions is the Polyakov-Howe-Tucker one:

$$I = \frac{T}{2}\int [D\Sigma] \sqrt{|H|} \left[H^{AB} \partial_A X^M \partial_B X^N G_{MN} + (2 - 2^d) \right] \tag{39}$$

with the 2^d-dim world-manifold measure:

$$[D\Sigma] = (d\xi)(d\xi^a)(d\xi^{a_1 a_2})(d\xi^{a_1 a_2 a_3}) \dots \tag{40}$$

Upon the algebraic elimination of the auxiliary world-manifold metric H^{AB} from the action (39), via the equations of motion, yields for its on-shell solution the pullback of the target C-space metric onto the C-space world-manifold:

$$H_{AB}(on - shell) = G_{AB} = \partial_A X^M \partial_B X^N G_{MN} \tag{41}$$

upon inserting back the on-shell solutions (41) into (39) gives the Dirac-Nambu-Goto action for the C-space branes directly in terms of the C-space determinant, or measure, of the induced C-space world-manifold metric G_{AB}, as a result of the embedding:

$$I = T \int [\mathrm{D}\Sigma] \sqrt{\mathrm{Det}(\partial_A X^M \partial_B X^N G_{MN})} \,. \tag{42}$$

However in C-space, the Polyakov-Howe-Tucker action admits an even further generalization that is comprised of two terms $S_1 + S_2$. The first term is [15]:

$$S_1 = \int [\mathrm{D}\Sigma] |E| E^A E^B \partial_A X^M \partial_B X^N \Gamma_M \Gamma_N \,. \tag{43}$$

Notice that this is a generalized action which is written in terms of the C-space coordinates $X^M(\Sigma)$ and the C-space analog of the target-spacetime vielbein/frame one-forms $e^m = e^m{}_\mu dx^\mu$ given by the Γ^M variables. The auxiliary world-manifold vielbein variables e^a, are given now by the Clifford-valued frame E^A variables.

In the conventional Polyakov-Howe-Tucker action, the auxiliary world-manifold metric h^{ab} associated with the standard p-brane actions is given by the usual scalar product of the frame vectors $e^a, e^b = e^a_\mu e^b_\nu g^{\mu\nu} = h^{ab}$. Hence, the C-space world-manifold metric H^{AB} appearing in (41) is given by scalar product $<(E^A)^\dagger E^B>_0 = H^{AB}$, where $(E^A)^\dagger$ denotes the reversal operation of E^A which requires reversing the ordering of the vectors present in the Clifford aggregate E^A.

Notice, however, that the form of the action (43) is far more general than the action in (39). In particular, the S_1 itself can be decomposed further into two additional pieces by rewriting the Clifford product of two basis elements into a symmetric plus an antisymmetric piece, respectively:

$$E^A E^B = \frac{1}{2}\{E^A, E^B\} + \frac{1}{2}[E^A, E^B] \,, \tag{44}$$

$$\Gamma_M \Gamma_N = \frac{1}{2}\{\Gamma_M, \Gamma_N\} + \frac{1}{2}[\Gamma_M, \Gamma_N]. \tag{45}$$

In this fashion, the S_1 component has *two* kinds of terms. The first term containing the symmetric combination is just the analog of the standard non-linear sigma model action, and the second term is a Wess-Zumino-like term, containing the antisymmetric combination. To extract the non-linear sigma model part of the generalized action above, we may simply take the scalar product of the vielbein-variables as follows:

$$(S_1)_{sigma} = \\ = \frac{T}{2}\int [\mathrm{D}\Sigma] |E| <(E^A \partial_A X^M \Gamma_M)^\dagger (E^B \partial_B X^N \Gamma_N)>_0 \tag{46}$$

where once again we have made use of the reversal operation (the analog of the hermitian adjoint) before contracting multi-indices. In this fashion we recover again the Clifford-scalar valued action given by [15].

Actions like the ones presented here in terms of derivatives with respect to quantities with multi-indices can be mapped to actions involving *higher* derivatives, in the same fashion that the C-space scalar curvature, the analog of the Einstein-Hilbert action, could be recast as a higher derivative gravity with torsion (reviewed in sec. 4). Higher derivatives actions are also related to theories of Higher spin fields [117] and W-geometry, W-algebras [116], [122]. For the role of Clifford algerbras to higher spin theories see [51].

The S_2 (scalar) component of the C-space brane action is the usual cosmological constant term given by the C-space determinant $|E| = \det(H^{AB})$ based on the scalar part of the geometric product $<(E^A)^\dagger E^B>_0 = H^{AB}$

$$S_2 = \frac{T}{2}\int [\mathrm{D}\Sigma] |E| \,, (2 - 2^d) \,, \tag{47}$$

where the C-space determinant $|E| = \sqrt{|\det(H^{AB})|}$ of the $2^d \times 2^d$ generalized world-manifold metric H^{AB} is given by:

$$\det(H^{AB}) = \frac{1}{(2^d)!} \epsilon_{A_1 A_2 \dots A_{2^d}} \epsilon_{B_1 B_2 \dots B_{2^d}} \times \\ \times H^{A_1 B_1} H^{A_2 B_2} \dots H^{A_{2^d} B_{2^d}} \,. \tag{48}$$

The $\epsilon_{A_1 A_2 \ldots A_{2d}}$ is the totally antisymmetric tensor density in C-space.

There are many different forms of p-brane actions, with and without a cosmological constant [123], and based on a new integration measure by recurring to auxiliary scalar fields [115], that one could have used to construct their C-space generalizations. Since all of them are on-shell equivalent to the Dirac-Nambu-Goto p-brane actions, we decided to focus solely on those actions having the Polyakov-Howe-Tucker form.

4 Generalized gravitational theories in curved C-spaces: higher derivative gravity and torsion from the geometry of C-space

4.1 Ordinary space

4.1.1 Clifford algebra based geometric calculus in curved space(time)

Clifford algebra is a very useful tool for description of geometry, especially of curved space V_n. Let us first review how it works in curved space(time). Later we will discuss a generalization to curved Clifford space [20].

We would like to make those techniques accessible to a wide audience of physicists who are not so familiar with the rigorous underlying mathematics, and demonstrate how Clifford algebra can be straightforwardly employed in the theory of gravity and its generalization. So we will leave aside the sophisticated mathematical approach, and rather follow as simple line of thought as possible, a praxis that is normally pursued by physicists. For instance, physicists in their works on general relativity employ a mathematical formulation and notation which is much simpler from that of purely mathematical or mathematically oriented works. For rigorous mathematical treatment the reader is advised to study, refs. [1, 76, 77, 78, 79].

Let the vector fields γ_μ, $\mu = 1, 2, \ldots, n$ be a coordinate basis in V_n satisfying the Clifford algebra relation

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}\left(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu\right) = g_{\mu\nu}\,, \tag{49}$$

where $g_{\mu\nu}$ is the metric of V_n. In curved space γ_μ and $g_{\mu\nu}$ cannot be constant but necessarily depend on position x^μ. An arbitrary vector is a linear superposition [1]

$$a = a^\mu \gamma_\mu\,, \tag{50}$$

where the components a^μ are *scalars* from the geometric point of view, whilst γ_μ are *vectors*.

Besides the basis $\{\gamma_\mu\}$ we can introduce the reciprocal basis* $\{\gamma^\mu\}$ satisfying

$$\gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) = g^{\mu\nu}\,, \tag{51}$$

*In Appendix A of the Hesteness book [1] the frame $\{\gamma^\mu\}$ is called *dual* frame because the duality operation is used in constructing it.

where $g^{\mu\nu}$ is the covariant metric tensor such that $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$, $\gamma^\mu \gamma_\nu + \gamma_\nu \gamma^\mu = 2\delta^\mu{}_\nu$ and $\gamma^\mu = g^{\mu\nu}\gamma_\nu$.

Following ref. [1] (see also [15]) we consider the *vector derivative* or *gradient* defined according to

$$\partial \equiv \gamma^\mu \partial_\mu\,, \tag{52}$$

where ∂_μ is an operator whose action depends on the quantity it acts on [26].

Applying the vector derivative ∂ on a *scalar* field ϕ we have

$$\partial\phi = \gamma^\mu \partial_\mu \phi\,, \tag{53}$$

where $\partial_\mu \phi \equiv (\partial/\partial x^\mu)\phi$ coincides with the partial derivative of ϕ.

But if we apply it on a *vector* field a we have

$$\partial a = \gamma^\mu \partial_\mu (a^\nu \gamma_\nu) = \gamma^\mu (\partial_\mu a^\nu \gamma_\nu + a^\nu \partial_\mu \gamma_\nu)\,. \tag{54}$$

In general γ_ν is not constant; it satisfies the relation to works [1, 15]

$$\partial_\mu \gamma_\nu = \Gamma^\alpha_{\mu\nu} \gamma_\alpha\,, \tag{55}$$

where $\Gamma^\alpha_{\mu\nu}$ is the *connection*. Similarly, for $\gamma^\nu = g^{\nu\alpha}\gamma_\alpha$ we have

$$\partial_\mu \gamma^\nu = -\Gamma^\nu_{\mu\alpha} \gamma^\alpha\,. \tag{56}$$

The *non commuting* operator ∂_μ so defined determines *the parallel transport* of a basis vector γ^ν. Instead of the symbol ∂_μ Hestenes uses $\Box_\mu$, whilst Wheeler et. al. [36] use ∇_μ and call it "covariant derivative". In modern, mathematically oriented literature more explicit notation such as D_{γ_μ} or ∇_{γ_μ} is used. However, such a notation, although mathematically very relevant, would not be very practical in long computations. We find it very convenient to keep the symbol ∂_μ for components of the geometric operator $\partial = \gamma^\mu \partial_\mu$. When acting on a scalar field the derivative ∂_μ happens to be commuting and thus behaves as the ordinary partial derivative. When acting on a vector field, ∂_μ is a *non commuting operator*. In this respect, there can be no confusion with partial derivative, because the latter normally acts on *scalar fields*, and in such a case partial derivative and ∂_μ are one and the same thing. However, when acting on a vector field, the derivative ∂_μ is non commuting. Our operator ∂_μ when acting on γ_μ or γ^μ should be distinguished from the ordinary — *commuting* — partial derivative, let be denoted $\gamma^\nu{}_{,\mu}$, usually used in the literature on the Dirac equation in curved spacetime. The latter derivative is not used in the present paper, so there should be no confusion.

Using (55), eq.-(54) becomes

$$\partial a = \gamma^\mu \gamma_\nu (\partial_\mu a^\nu + \Gamma^\nu_{\mu\alpha} a^\alpha) \equiv \gamma^\mu \gamma_\nu \mathrm{D}_\mu a^\nu = \gamma^\mu \gamma^\nu \mathrm{D}_\mu a_\nu \tag{57}$$

where D_μ is the covariant derivative of tensor analysis.

Decomposing the Clifford product $\gamma^\mu \gamma^\nu$ into its symmetric and antisymmetric part [1]

$$\gamma^\mu \gamma^\nu = \gamma^\mu \cdot \gamma^\nu + \gamma^\mu \wedge \gamma^\nu\,, \tag{58}$$

where

$$\gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu} \tag{59}$$

is the *inner product* and

$$\gamma^\mu \wedge \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{60}$$

the *outer product*, we can write eq.-(57) as

$$\begin{aligned} \partial a = g^{\mu\nu}\,\mathrm{D}_\mu a_\nu + \gamma^\mu \wedge \gamma^\nu \mathrm{D}_\mu a_\nu = \\ = \mathrm{D}_\mu a^\mu + \frac{1}{2}\gamma^\mu \wedge \gamma^\nu(\mathrm{D}_\mu a_\nu - \mathrm{D}_\nu a_\mu)\,. \end{aligned} \tag{61}$$

Without employing the expansion in terms of γ_μ we have simply

$$\partial a = \partial \cdot a + \partial \wedge a\,. \tag{62}$$

Acting twice on a vector by the operator ∂ we have*

$$\begin{aligned} \partial\partial a &= \gamma^\mu\partial_\mu(\gamma^\nu\partial_\nu)(a^\alpha\gamma_\alpha) = \gamma^\mu\gamma^\nu\gamma_\alpha \mathrm{D}_\mu \mathrm{D}_\nu a^\alpha = \\ &= \gamma_\alpha \mathrm{D}_\mu \mathrm{D}^\mu a^\alpha + \frac{1}{2}(\gamma^\mu\wedge\gamma^\nu)\gamma_\alpha[\mathrm{D}_\mu,\mathrm{D}_\nu]a^\alpha = \\ &= \gamma_\alpha \mathrm{D}_\mu \mathrm{D}^\mu a^\alpha + \gamma^\mu(R_{\mu\rho}a^\rho + K_{\mu\alpha}{}^\rho \mathrm{D}_\rho a^\alpha) + \\ &+ \frac{1}{2}(\gamma^\mu\wedge\gamma^\nu\wedge\gamma_\alpha)(R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha)\,. \end{aligned} \tag{63}$$

We have used

$$[\mathrm{D}_\mu, \mathrm{D}_\nu]a^\alpha = R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha\,, \tag{64}$$

where

$$K_{\mu\nu}{}^\rho = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \tag{65}$$

is *torsion* and $R_{\mu\nu\rho}{}^\alpha$ the *curvature tensor*. Using eq.-(55) we find

$$[\partial_\alpha, \partial_\beta]\gamma_\mu = R_{\alpha\beta\mu}{}^\nu\gamma_\nu\,, \tag{66}$$

from which we have

$$R_{\alpha\beta\mu}{}^\nu = ([[\partial_\alpha,\partial_\beta]\gamma_\mu)\cdot\gamma^\nu\,. \tag{67}$$

Thus in general the commutator of derivatives ∂_μ acting on a vector does not give zero, but is given by the curvature tensor.

In general, for an r-vector $A = a^{\alpha_1\ldots\alpha_r}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}$ we have

$$\begin{aligned} \partial\partial\ldots\partial A &= (\gamma^{\mu_1}\partial_{\mu_1})(\gamma^{\mu_2}\partial_{\mu_2})\ldots(\gamma^{\mu_k}\partial_{\mu_k})\times \\ &\times(a^{\alpha_1\ldots\alpha_r}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}) = \gamma^{\mu_1}\gamma^{\mu_2}\ldots \\ &\ldots\gamma^{\mu_k}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}\mathrm{D}_{\mu_1}\mathrm{D}_{\mu_2}\ldots\mathrm{D}_{\mu_k}a^{\alpha_1\ldots\alpha_r}\,. \end{aligned} \tag{68}$$

*We use $(a\wedge b)\,c = (a\wedge b)\cdot c + a\wedge b\wedge c$ [1] and also $(a\wedge b)\cdot c = (b\cdot c)\,a - (a\cdot c)\,b$.

4.1.2 Clifford algebra based geometric calculus and resolution of the ordering ambiguity for the product of momentum operators

Clifford algebra is a very useful tool for description of geometry of curved space. Moreover, as shown in ref. [26] it provides a resolution of the long standing problem of the ordering ambiguity of quantum mechanics in curved space. Namely, eq.-(52) for the vector derivative suggests that the momentum operator is given by

$$p = -i\,\partial = -i\,\gamma^\mu\partial_\mu\,. \tag{69}$$

One can consider three distinct models:

(i) *The non relativistic particle* moving in $n$dimensional curved space. Then, $\mu = 1, 2, \ldots, n$, and signature is $(++++\ldots)$;

(ii) *The relativistic particle* in curved spacetime, described by the *Schild action* [37]. Then, $\mu = 0, 1, 2, \ldots, n-1$ and signature is $(+---\ldots)$;

(iii) *The Stueckelberg unconstrained particle* [33, 34, 35, 29].

In all three cases the classical action has the form

$$I[X^\mu] = \frac{1}{2\Lambda}\int \mathrm{d}\tau\, g_{\mu\nu}(x)\dot{X}^\mu\dot{X}^\nu \tag{70}$$

and the corresponding Hamiltonian is

$$H = \frac{\Lambda}{2}\,g^{\mu\nu}(x)p_\mu p_\nu = \frac{\Lambda}{2}\,p^2\,. \tag{71}$$

If, upon quantization we take for the momentum operator $p_\mu = -i\,\partial_\mu$, then the ambiguity arises of how to write the quantum Hamilton operator. The problem occurs because the expressions $g^{\mu\nu}p_\mu p_\nu$, $p_\mu g^{\mu\nu}p_\nu$ and $p_\mu p_\nu g^{\mu\nu}$ are not equivalent.

But, if we rewrite H as

$$H = \frac{\Lambda}{2}\,p^2\,, \tag{72}$$

where $p = \gamma^\mu p_\mu$ is the *momentum vector* which upon quantization becomes the momentum vector operator (69), we find that there is no ambiguity in writing the square p^2. When acting with H on a *scalar* wave function ϕ we obtain the unambiguous expression

$$H\phi = \frac{\Lambda}{2}p^2\phi = \frac{\Lambda}{2}(-i)^2(\gamma^\mu\partial_\mu)(\gamma^\nu\partial_\nu)\phi = -\frac{\Lambda}{2}\mathrm{D}_\mu\mathrm{D}^\mu\phi \tag{73}$$

in which there is no curvature term R. We expect that a term with R will arise upon acting with H on a *spinor* field ψ.

4.2 C-space

Let us now consider C-space and review the procedure of ref. [20]. A basis in C-space is given by

$$E_A = \{\gamma, \gamma_\mu, \gamma_\mu\wedge\gamma_\nu, \gamma_\mu\wedge\gamma_\nu\wedge\gamma_\rho, \ldots\}\,, \tag{74}$$

where in an r-vector $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r}$ we take the indices so that $\mu_1 < \mu_2 < \ldots < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X = X^A E_A = s\,\gamma + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \ldots \,. \tag{75}$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, etc., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops, etc., onto the coordinate planes. By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string.

In order to avoid using the powers of the Planck scale length parameter L in the expansion of the polyvector X we use the dilatationally invariant units [15] in which L is set to 1. The dilation invariant physics was discussed from a different perspective also in refs. [23, 21].

In a flat C-space the basis vectors E^A are constants. In a curved C-space this is no longer true. Each E_A is a function of the C-space coordinates

$$X^A = \{s, x^\mu, x^{\mu\nu}, \ldots\} \tag{76}$$

which include scalar, vector, bivector, . . . , r-vector, . . . , coordinates.

Now we define the connection $\tilde{\Gamma}^C_{AB}$ in C-space according to

$$\partial_A E_B = \tilde{\Gamma}^C_{AB} E_C \,, \tag{77}$$

where $\partial_A \equiv \partial/\partial X^A$ is the derivative in C-space. This definition is analogous to the one in ordinary space. Let us therefore define the C-space curvature as

$$\mathcal{R}_{ABC}{}^D = ([\partial_A, \partial_B] E_C) * E^D \,, \tag{78}$$

which is a straightforward generalization of the relation (67). The "star" means the *scalar product* between two polyvectors A and B, defined as

$$A * B = \langle A\, B \rangle_S \,, \tag{79}$$

where "S" means "the scalar part" of the geometric product AB.

In the following we shall explore the above relation for curvature and see how it is related to the curvature of the ordinary space. Before doing that we shall demonstrate that the derivative with respect to the bivector coordinate $x^{\mu\nu}$ is equal to the commutator of the derivatives with respect to the vector coordinates x^μ.

Returning now to eq.-(77), the differential of a C-space basis vector is given by

$$\mathrm{d}E_A = \frac{\partial E_A}{\partial X^B}\,\mathrm{d}X^B = \Gamma^C_{AB}\, E_C\, \mathrm{d}X^B \,. \tag{80}$$

In particular, for $A = \mu$ and $E_A = \gamma_\mu$ we have

$$\begin{aligned} \mathrm{d}\gamma_\mu &= \frac{\partial \gamma_\mu}{\partial X^\nu}\mathrm{d}x^\nu + \frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}}\mathrm{d}x^{\alpha\beta} + \ldots = \\ &= \tilde{\Gamma}^A_{\nu\mu} E_A \mathrm{d}x^\nu + \tilde{\Gamma}^A_{[\alpha\beta]\mu} E_A \mathrm{d}x^{\alpha\beta} + \ldots = \\ &= (\tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu}\gamma_\rho \wedge \gamma_\sigma + \ldots)\,\mathrm{d}x^\nu + \\ &+ (\tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu}\gamma_\rho \wedge \gamma_\sigma + \ldots)\,\mathrm{d}x^{\alpha\beta} + \ldots \,. \end{aligned} \tag{81}$$

We see that the differential $\mathrm{d}\gamma_\mu$ is in general a polyvector, i. e., a Clifford aggregate. In eq.-(81) we have used

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu}\gamma_\rho \wedge \gamma_\sigma + \ldots \,, \tag{82}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu}\gamma_\rho \wedge \gamma_\sigma + \ldots \,. \tag{83}$$

Let us now consider a *restricted* space in which the derivatives of γ_μ with respect to x^ν and $x^{\alpha\beta}$ do not contain higher rank multivectors. Then eqs.-(82), (83) become

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha \,, \tag{84}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho \,. \tag{85}$$

Further we assume that:

(i) The components $\tilde{\Gamma}^\alpha_{\nu\mu}$ of the C-space connection $\tilde{\Gamma}^C_{AB}$ coincide with the connection $\Gamma^\alpha_{\nu\mu}$ of an ordinary space;

(ii) The components $\tilde{\Gamma}^\rho_{[\alpha\beta]\mu}$ of the C-space connection coincide with the curvature tensor $R_{\alpha\beta\mu}{}^\rho$ of an ordinary space.

Hence, eqs.-(84), (85) read

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \Gamma^\alpha_{\nu\mu}\gamma_\alpha \,, \tag{86}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = R_{\alpha\beta\mu}{}^\rho \gamma_\rho \,, \tag{87}$$

and the differential (81) becomes

$$\mathrm{d}\gamma_\mu = \Big(\Gamma^\rho_{\alpha\mu}\mathrm{d}x^\alpha + \frac{1}{2} R_{\alpha\beta\mu}{}^\rho \mathrm{d}x^{\alpha\beta}\Big)\gamma_\rho \,. \tag{88}$$

The same relation was obtained by Pezzaglia [14] by using a different method, namely by considering how polyvectors change with position. The above relation demonstrates that a geodesic in C-space is not a geodesic in ordinary spacetime. Namely, in ordinary spacetime we obtain Papapetrou's equation. This was previously pointed out by Pezzaglia [14].

Although a C-space connection does not transform like a C-space tensor, some of its components, i. e., those of eq.-(85), may have the transformation properties of a tensor in an ordinary space.

Under a general coordinate transformation in C-space

$$X^A \to X'^A = X'^A(X^B) \tag{89}$$

the connection transforms according to*

$$\tilde{\Gamma}'^C_{AB} = \frac{\partial X'^C}{\partial X^E}\frac{\partial X^J}{\partial X'^A}\frac{\partial X^K}{\partial X'^B}\tilde{\Gamma}^E_{JK} + \frac{\partial X'^C}{\partial X^J}\frac{\partial^2 X^J}{\partial X'^A \partial X'^B}. \tag{90}$$

In particular, the components which contain the bivector index $A = [\alpha\beta]$ transform as

$$\tilde{\Gamma}'^\rho_{[\alpha\beta]\mu} = \frac{\partial X'^\rho}{\partial X^E}\frac{\partial X^J}{\partial \sigma'^{\alpha\beta}}\frac{\partial X^K}{\partial x'^\mu}\tilde{\Gamma}^E_{JK} + \frac{\partial x'^\rho}{\partial X^J}\frac{\partial^2 X^J}{\partial \sigma'^{\alpha\beta}\partial x'^\mu}. \tag{91}$$

Let us now consider a particular class of coordinate transformations in C-space such that

$$\frac{\partial x'^\rho}{\partial x^{\mu\nu}} = 0\,, \qquad \frac{\partial x^{\mu\nu}}{\partial x'^\alpha} = 0\,. \tag{92}$$

Then the second term in eq.-(91) vanishes and the transformation becomes

$$\tilde{\Gamma}'^\rho_{[\alpha\beta]\mu} = \frac{\partial X'^\rho}{\partial x^\epsilon}\frac{\partial x^{\rho\sigma}}{\partial \sigma'^{\alpha\beta}}\frac{\partial x^\gamma}{\partial x'^\mu}\tilde{\Gamma}^\epsilon_{[\rho\sigma]\gamma}\,. \tag{93}$$

Now, for the bivector whose components are $\mathrm{d}x^{\alpha\beta}$ we have

$$\mathrm{d}\sigma'^{\alpha\beta}\gamma'_\alpha \wedge \gamma'_\beta = \mathrm{d}x^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta\,. \tag{94}$$

Taking into account that in our particular case (92) γ_α transforms as a basis vector in an ordinary space

$$\gamma'_\alpha = \frac{\partial x^\mu}{\partial x'^\alpha}\gamma_\mu\,, \tag{95}$$

we find that (94) and (95) imply

$$\mathrm{d}\sigma'^{\alpha\beta}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} = \mathrm{d}x^{\mu\nu}\,, \tag{96}$$

which means that

$$\frac{\partial x^{\mu\nu}}{\partial \sigma'^{\alpha\beta}} = \frac{1}{2}\left(\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} - \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x^\mu}{\partial x'^\beta}\right) \equiv \frac{\partial x^{[\mu}}{\partial x'^\alpha}\frac{\partial x^{\nu]}}{\partial x'^\beta}. \tag{97}$$

The transformation of the bivector coordinate $x^{\mu\nu}$ is thus determined by the transformation of the vector coordinates x^μ. This is so because the basis bivectors are the wedge products of basis vectors γ_μ.

From (93) and (97) we see that $\tilde{\Gamma}^\epsilon_{[\rho\sigma]\gamma}$ transforms like a 4th-rank tensor in an ordinary space.

Comparing eq.-(87) with the relation (66) we find

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = [\partial_\alpha, \partial_\beta]\gamma_\mu\,. \tag{98}$$

*This can be derived from the relation $\mathrm{d}E'_A = \frac{\partial E'_A}{\partial X'^B}\mathrm{d}X'^B$, where $E'_A = \frac{\partial X^D}{\partial X'^A}E_D$ and $\mathrm{d}X'^B = \frac{\partial X'^B}{\partial X^C}\mathrm{d}X^C$.

The derivative of a basis vector with respect to the bivector coordinates $x^{\alpha\beta}$ is equal to the commutator of the derivatives with respect to the vector coordinates x^α.

The above relation (98) holds for the basis vectors γ_μ. For an arbitrary polyvector

$$A = A^A E_A = s\gamma + a^\alpha\gamma_\alpha + a^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta + \ldots \tag{99}$$

we will assume the validity of the following relation

$$\frac{\mathrm{D}A^A}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]A^A\,, \tag{100}$$

where $\mathrm{D}/\mathrm{D}x^{\mu\nu}$ is the covariant derivative, defined in analogous way as in eqs. (57):

$$\frac{\mathrm{D}A^A}{\mathrm{D}X^B} = \frac{\partial A^A}{\partial X^B} + \tilde{\Gamma}^A_{BC}A^C\,. \tag{101}$$

From eq.-(100) we obtain

$$\frac{\mathrm{D}s}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]s = K_{\mu\nu}{}^\rho\partial_\rho s\,, \tag{102}$$

$$\frac{\mathrm{D}a^\alpha}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]a^\alpha = R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho\mathrm{D}_\rho a^\alpha\,. \tag{103}$$

Using (101) we have that

$$\frac{\mathrm{D}s}{\mathrm{D}x^{\mu\nu}} = \frac{\partial s}{\partial x^{\mu\nu}} \tag{104}$$

and also follows

$$\frac{\mathrm{D}a^\alpha}{\mathrm{D}x^{\mu\nu}} = \frac{\partial a^\alpha}{\partial x^{\mu\nu}} + \tilde{\Gamma}^\alpha_{[\mu\nu]\rho}a^\rho = \frac{\partial a^\alpha}{\partial x^{\mu\nu}} + R_{\mu\nu\rho}{}^\alpha a^\rho\,, \tag{105}$$

where, according to (ii), $\tilde{\Gamma}^\alpha_{[\mu\nu]\rho}$ has been identified with curvature. So we obtain, after inserting (104), (105) into (102), (103) that:

(a) The partial derivatives of the coefficients s and a^α, which are Clifford scalars†, with respect to $x^{\mu\nu}$ are related to *torsion*:

$$\frac{\partial s}{\partial x^{\mu\nu}} = K_{\mu\nu}{}^\rho\partial_\rho s\,, \tag{106}$$

$$\frac{\partial a^\alpha}{\partial x^{\mu\nu}} = K_{\mu\nu}{}^\rho\mathrm{D}_\rho a^\alpha\,; \tag{107}$$

(b) Whilst the derivative of the basis vectors with respect to $x^{\mu\nu}$ are related to *curvature*:

$$\frac{\partial \gamma_\alpha}{\partial x^{\mu\nu}} = R_{\mu\nu\alpha}{}^\beta\gamma_\beta\,. \tag{108}$$

In other words, the dependence of coefficients s and a^α on $x^{\mu\nu}$ indicates the presence of torsion. On the contrary, when basis vectors γ_α depend on $x^{\mu\nu}$ this indicates that the corresponding vector space has non vanishing curvature.

†In the geometric calculus based on Clifford algebra, the coefficients such as s, a^α, $a^{\alpha\beta}$,..., are called *scalars* (although in tensor calculus they are called scalars, vectors and tensors, respectively), whilst the objects γ_α, $\gamma_\alpha \wedge \gamma_\beta, \ldots$, are called *vectors, bivectors*, etc.

4.3 On the relation between the curvature of C-space and the curvature of an ordinary space

Let us now consider the C-space curvature defined in eq.-(78). The indices A,B, can be of vector, bivector, etc., type. It is instructive to consider a particular example.

$A=[\mu\nu]$, $B=[\alpha\beta]$, $C=\gamma$, $D=\delta$

$$\left(\left[\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\alpha\beta}}\right]\gamma_\gamma\right)\cdot\gamma^\delta = \mathcal{R}_{[\mu\nu][\alpha\beta]\gamma}{}^\delta . \qquad (109)$$

Using (87) we have

$$\frac{\partial}{\partial x^{\mu\nu}}\frac{\partial}{\partial x^{\alpha\beta}}\gamma_\gamma = \frac{\partial}{\partial x^{\mu\nu}}(R_{\alpha\beta\gamma}{}^\rho\gamma_\rho) = R_{\alpha\beta\gamma}{}^\rho R_{\mu\nu\rho}{}^\sigma\gamma_\sigma \quad (110)$$

where we have taken

$$\frac{\partial}{\partial x^{\mu\nu}}R_{\alpha\beta\gamma}{}^\rho = 0 \,, \qquad (111)$$

which is true in the case of vanishing torsion (see also an explanation that follows after the next paragraph). Inserting (110) into (109) we find

$$\mathcal{R}_{[\mu\nu][\alpha\beta]\gamma}{}^\delta = R_{\mu\nu\gamma}{}^\rho R_{\alpha\beta\rho}{}^\delta - R_{\alpha\beta\gamma}{}^\rho R_{\mu\nu\rho}{}^\delta \,, \qquad (112)$$

which is the product of two usual curvature tensors. We can proceed in analogous way to calculate the other components of $\mathcal{R}_{ABC}{}^D$ such as $\mathcal{R}_{[\alpha\beta\gamma\delta][\rho\sigma]\epsilon}{}^\mu$, $\mathcal{R}_{[\alpha\beta\gamma\delta][\rho\sigma\tau\kappa]\epsilon}{}^{[\mu\nu]}$, etc. These contain higher powers of the curvature in an ordinary space. All this is true in our restricted C-space given by eqs.-(84), (85) and the assumptions (i), (ii) bellow those equations. By releasing those restrictions we would have arrived at an even more involved situation which is beyond the scope of the present paper.

After performing the contractions of (112) and the corresponding higher order relations we obtain the expansion of the form

$$\mathcal{R} = R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \dots \,. \qquad (113)$$

So we have shown that the C-space curvature can be expressed as the sum of the products of the ordinary spacetime curvature. This bears a resemblance to the string effective action in curved spacetimes given by sums of powers of the curvature tensors based on the quantization of non-linear sigma models [118].

If one sets aside the algebraic convergence problems when working with Clifford algebras in infinite dimensions, one can consider the possibility of studying Quantum Gravity in a very large number of dimensions which has been revisited recently [83] in connection to a perturbative renormalizable quantum theory of gravity in infinite dimensions. Another interesting possibility is that an infinite series expansion of the powers of the scalar curvature could yield the recently proposed modified Lagrangians $R+1/R$ of gravity to accommodate the cosmological accelerated expansion of the Universe [131], after a judicious choice of the algebraic coefficients is taken. One may notice also that having a vanishing cosmological constant in C-space, $\mathcal{R}=\Lambda=0$ does not necessarily imply that one has a vanishing cosmological constant in ordinary spacetime. For example, in the very special case of homogeneous symmetric spacetimes, like spheres and hyperboloids, where all the curvature tensors are proportional to suitable combinations of the metric tensor times the scalar curvature, it is possible to envision that the net combination of the sum of all the powers of the curvature tensors may cancel-out giving an overall zero value $\mathcal{R}=0$. This possibility deserves investigation.

Let us now show that for vanishing torsion the curvature is independent of the bivector coordinates $x^{\mu\nu}$, as it was taken in eq.-(111). Consider the basic relation

$$\gamma_\mu\cdot\gamma_\nu = g_{\mu\nu} \,. \qquad (114)$$

Differentiating with respect to $x^{\alpha\beta}$ we have

$$\begin{aligned}\frac{\partial}{\partial x^{\alpha\beta}}(\gamma_\mu\cdot\gamma_\nu) = \frac{\partial\gamma_\mu}{\partial x^{\alpha\beta}}\cdot\gamma_\nu + \gamma_\mu\cdot\frac{\partial\gamma_\nu}{\partial x^{\alpha\beta}} = \\ = R_{\alpha\beta\mu\nu} + R_{\alpha\beta\nu\mu} = 0 \,.\end{aligned} \qquad (115)$$

This implies that

$$\frac{\partial g_{\mu\nu}}{\partial\sigma_{\alpha\beta}} = [\partial_\alpha, \partial_\beta]g_{\mu\nu} = 0 \,. \qquad (116)$$

Hence the metric, in this particular case, is independent of the holographic (bivector) coordinates. Since the curvature tensor — when torsion is zero — can be written in terms of the metric tensor and its derivatives, we conclude that not only the metric, but also the curvature is independent of $x^{\mu\nu}$. In general, when the metric has a dependence on the holographic coordinates one expects further corrections to eq.-(112) that would include torsion.

5 On the quantization in C-spaces

5.1 The momentum constraint in C-space

A detailed discussion of the physical properties of all the components of the polymomentum P in four dimensions and the emergence of the physical mass in Minkowski spacetime has been provided in the book [15]. The polymomentum in $D=4$, canonically conjugate to the position polyvector

$$X = \sigma + x^\mu\gamma_\mu + \gamma^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + \xi^\mu\gamma_5\gamma_\mu + s\gamma_5 \qquad (117)$$

can be written as:

$$P = \mu + p^\mu\gamma_\mu + S^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + \pi^\mu\gamma_5\gamma_\mu + m\gamma_5 \,, \qquad (118)$$

where besides the vector components p^μ we have the scalar component μ, the 2-vector components $S^{\mu\nu}$, that are connected to the spin as shown by [14]; the pseudovector components π^μ and the pseudoscalar component m.

The most salient feature of the polyparticle dynamics in C-spaces [15] is that one can start with a *constrained* action in C-space and arrive, nevertheless, at an *unconstrained* Stuckelberg action in Minkowski space (a subspace of C-space) in which $p_\mu p^\mu$ is a constant of motion. The true constraint in C-space is:

$$P_A P^A = \mu^2 + p_\mu p^\mu - 2S^{\mu\nu} S_{\mu\nu} + \pi_\mu \pi^\mu - m^2 = M^2\,, \quad (119)$$

where M is a *fixed* constant, the mass in C-space. The pseudoscalar component m is a variable, like μ, p_μ, $S^{\mu\nu}$, and π^μ, which altogether are constrained according to eq.-(119). It becomes the physical mass in Minkowski spacetime in the special case when other extra components vanish, i. e., when $\mu = 0$, $S^{\mu\nu} = 0$ and $\pi^\mu = 0$. This justifies using the notation m for mass. This is basically the distinction between the mass in Minkowski space which is a constant of motion $p_\mu p^\mu$ and the fixed mass M in C-space. The variable m is canonically conjugate to s which acquires the role of the Stuckelberg evolution parameter s that allowed ref. [29, 15] to propose a natural solution of the problem of time in quantum gravity. The polyparticle dynamics in C-space is a generalization of the relativistic Regge top construction which has recently been studied in de Sitter spaces by [135].

A derivation of a charge, mass, and spin relationship of a polyparticle can be obtained from the above polymomentum constraint in C-space if one relates the norm of the axial-momentum component π^μ of the polymomentum P to the charge [80]. It agrees exactly with the recent charge-mass-spin relationship obtained by [44] based on the Kerr-Newman black hole metric solutions of the Einstein-Maxwell equations. The naked singularity Kerr-Newman solutions have been interpreted by [45] as Dirac particles. Further investigation is needed to understand better these relationships, in particular, the deep reasons behind the *charge* assignment to the norm of the axial-vector π^μ component of the polymomentum which suggests that mass has a gravitational, electromagnetic and rotational aspects to it. In a Kaluza-Klein reduction from $D = 5$ to $D = 4$ it is well known that the electric charge is related to the p_5 component of the momentum. Hence, charge bears a connection to an internal momentum.

5.2 C-space Klein-Gordon and Dirac wave equations

The ordinary Klein-Gordon equation can be easily obtained by implementing the on-shell constraint $p^2 - m^2 = 0$ as an operator constraint on the physical states after replacing p_μ for $-i\partial/\partial x^\mu$ (we use units in which $\hbar = 1, c = 1$):

$$\left(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + m^2\right)\phi = 0. \quad (120)$$

The C-space generalization follows from the $P^2 - M^2 = 0$ condition by replacing

$$P_A \to -i\frac{\partial}{\partial X^A} = -i\left(\frac{\partial}{\partial \sigma}, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^{\mu\nu}}, \ldots\right), \quad (121)$$

$$\left(\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial x^\mu \partial x_\mu} + \frac{\partial^2}{\partial x^{\mu\nu} \partial x_{\mu\nu}} + \ldots + M^2\right)\Phi = 0, \quad (122)$$

where we have set $L = \hbar = c = 1$ for convenience purposes and the C-space scalar field $\Phi\,(\sigma, x^\mu, x^{\mu\nu}, \ldots)$ is a polyvector-valued *scalar* function of *all* the C-space variables. This is the Klein-Gordon equation associated with a free scalar polyparticle in C-space.

A wave equation for a generalized C-space harmonic oscillator requires to introduce the potential of the form $V = \kappa X^2$ that admits straightforward solutions in terms of Gaussians and Hermite polynomials similar to the ordinary point-particle oscillator. There are now collective excitations of the Clifford-oscillator in terms of the number of Clifford-bits and which represent the quanta of areas, volumes, hyper-volumes, ..., associated with the p-loops oscillations in Planck scale units. The logarithm of the degeneracy of the first collective state of the C-space oscillator, as a function of the number of bits, bears the same functional form as the Bekenstein-Hawking black hole entropy, with the upshot that one recovers, in a natural way, the logarithmic corrections to the black-hole entropy as well, if one identifies the number of Clifford-bits with the number of area-quanta of the black hole horizon. For further details about this derivation and the emergence of the Schwarzschild horizon radius relation, the Hawking temperature, the maximal Planck temperature condition, etc., we refer to [21]. Perhaps the most important consequence of this latter view of black hole entropy is the possibility that there is a ground state of quantum spacetime, resulting from of a Bose-Einstein condensate of the C-space harmonic oscillator.

A C-space version of the Dirac Equation, representing the dynamics of spinning-polyparticles (theories of extended-spin, extended charges) is obtained via the square-root procedure of the Klein-Gordon equation:

$$-i\left(\frac{\partial}{\partial \sigma} + \gamma^\mu \frac{\partial}{\partial x_\mu} + \gamma^\mu \wedge \gamma^\nu \frac{\partial}{\partial x_{\mu\nu}} + \ldots\right)\Psi = M\Psi\,, \quad (123)$$

where $\Psi(\sigma, x^\mu, x^{\mu\nu}, \ldots)$ is a polyvector-valued function, a Clifford-number, $\Psi = \Psi^A E_A$ of *all* the C-space variables. For simplicity we consider here a *flat* C-space in which the metric $G_{AB} = E_A^\dagger * E_B = \eta_{AB}$ is diagonal, η_{AB} being the C-space analog of Minkowski tensor. In curved C-space the equation (123) should be properly generalized. This goes beyond the scope of the present paper.

Ordinary spinors are nothing but elements of the left/right ideals of a Clifford algebra. So they are automatically contained in the polyvector valued wave function Ψ. The ordinary Dirac equation can be obtained when Ψ is independent of the extra variables associated with a polyvector-valued coordinates X (i. e., of $x^{\mu\nu}$, $x^{\mu\nu\rho}, \dots$). For details see [15].

Thus far we have written ordinary wave equations in C-space, that is, we considered the wave equations for a "point particle" in C-space. From the perspective of the 4-dimensional Minkowski spacetime the latter "point particle" has, of course, a much richer structure then a mere point: it is an extended object, modeled by coordinates x^{μ}, $x^{\mu\nu}, \dots$ But such modeling does not embrace all the details of an extended object. In order to provide a description with more details, one can considere not the "point particles" in C-space, but *branes* in C-space. They are described by the embeddings $X = X(\Sigma)$, that is $X^M = X^M(\Sigma^A)$, considered in sec. 3.2. Quantization of such branes can employ wave functional equation, or other methods, including the second quantization formalism. For a more detailed study detailed study of the second quantization of extended objects using the tools of Clifford algebra see [15].

Without emplying Clifford algebra a lot of illuminating work has been done in relation to description of branes in terms of p-loop coordinates [132]. A bosonic/fermionic p-brane wave-functional equation was presented in [12], generalizing the closed-string (loop) results in [13] and the the quantum bosonic p-brane propagator, in the quenched-reduced minisuperspace approximation, was attained by [18]. In the latter work branes are described in terms of the collective coordinates which are just the highest grade components in the expansion of a poplyvector X given in eq.-(2). This work thus paved the way for the next logical step, that is, to consider other multivector components of X in a unified description of all branes.

Notice that the approach based on eqs.-(122), (123) is different from that by Hestenes [1] who proposed an equation which is known as the Dirac-Hestenes equation. Dirac's equation using quaternions (related to Clifford algebras) was first derived by Lanczos [91]. Later on the Dirac-Lanczos equation was rediscovered by many people, in particular by Hestenes and Gursey [92] in what became known as the Dirac-Hestenes equation. The former Dirac-Lanczos equation is Lorentz *covariant* despite the fact that it singles out an arbitrary but unique direction in ordinary space: the *spin* quantization axis. Lanczos, without knowing, had anticipated the existence of isospin as well. The Dirac-Hestenes equation $\partial \Psi e_{21} = m \Psi e_0$ is *covariant* under a change of frame [133], [93]. $e'_{\mu} = U e_{\mu} U^{-1}$ and $\Psi' = \Psi U^{-1}$ with U an element of the $Spin_{+}(1,3)$ yielding $\partial \Psi' e'_{21} = m \Psi' e'_0$. As Lanczos had anticipated, in a new frame of reference, the spin quantization axis is also rotated appropriately, thus there is no breakdown of covariance by introducing bivectors in the Dirac-Hestenes equation.

However, subtleties still remain. In the Dirac-Hestenes equation instead of the imaginary unit i there occurs the bivector $\gamma_1 \gamma - 2$. Its square is -1 and commutes with all the elements of the Dirac algebra which is just a desired property. But on the other hand, the introduction of a bivector into an equation implies a selection of a preferred orientation in spacetime; i. e. the choice of the spin quantization axis in the original Dirac-Lanczos quaternionic equation. How is such preferred orientation (spin quantization axis) determined? Is there some dynamical symmetry which determines the preferred orientation (spin quantization axis)? is there an action which encodes a hidden dynamical principle that selects *dynamically* a preferred spacetime orientation (spin quantization axis)?

Many subtleties of the Dirac-Hesteness equation and its relation to the ordinary Dirac equation and the Seiberg-Witten equation are investigated from the rigorous mathematical point of view in refs. [93]. The approach in refs. [16, 15, 17, 8], reviewed here, is different. We start from the usual formulation of quantum theory and extend it to C-space. We retain the imaginary unit i. Next step is to give a geometric interpretation to i. Instead of trying to find a geometric origin of i in *spacetime* we adopt the interpretation proposed in [15] according to which the i is the bivector of the 2-dimensional *phase space* (whose direct product with the n-dimensional configuration space gives the $2n$-dimensional phase space)*. This appears to be a natural assumption due to the fact that complex valued quantum mechanical wave functions involve momenta p_{μ} and coordinates x^{μ} (e. g., a plane wave is given by $\exp[i p_{\mu} x^{\mu}]$, and arbitrary wave packet is a superposition of plane waves).

6 Maximal-acceleration Relativity in phase-spaces

In this section we shall discuss the maximal acceleration Relativity principle [68] based on Finsler geometry which does not destroy, nor deform, Lorentz invariance. Our discussion differs from the pseudo-complex Lorentz group description by Schuller [61] related to the effects of maximal acceleration in Born-Infeld models that also maintains Lorentz invariance, in contrast to the approaches of Double Special Relativity (DSR). In addition one does not need to modify the energy-momentum addition (conservation) laws in the scattering of particles which break translational invariance. For a discussions on the open problems of Double Special Relativity theories based on kappa-deformed Poincaré symmetries [63] and motivated by the anomalous Lorentz-violating dispersion relations in the ultra high energy cosmic rays [71, 72, 73], we refer to [70].

Related to the minimal Planck scale, an upper limit on the maximal acceleration principle in Nature was proposed by long ago Cainello [52]. This idea is a direct consequence of a suggestion made years earlier by Max Born on a Dual Relativity principle operating in phase spaces [49], [74] wherethere

*Yet another interpretation of the imaginary unit i present in the Heisenberg uncertainty relations has been undertaken by Finkelstein and collaborators [96].

is an upper bound on the four-force (maximal string tension or tidal forces in the string case) acting on a particle as well as an upper bound in the particle velocity. One can combine the maximum speed of light with a minimum Planck scale into a maximal proper-acceleration $a = c^2/L$ within the framework of Finsler geometry [56]. For a recent status of the geometries behind maximal-acceleration see [73]; its relation to the Double Special Relativity programs was studied by [55] and the possibility that Moyal deformations of Poincaré algebras could be related to the kappa-deformed Poincaré algebras was raised in [68]. A thorough study of Finsler geometry and Clifford algebras has been undertaken by Vacaru [81] where Clifford/spinor structures were defined with respect to Nonlinear connections associated with certain nonholonomic modifications of Riemann-Cartan gravity.

Other several new physical implications of the maximal acceleration principle in Nature, like neutrino oscillations and other phenomena, have been studied by [54], [67], [42]. Recently, the variations of the fine structure constant α [64], with the cosmological accelerated expansion of the Universe, was recast as a renormalization group-like equation governing the cosmological red shift (Universe scale) variations of α based on this maximal acceleration principle in Nature [68]. The fine structure constant was smaller in the past. Pushing the cutoff scale to the minimum Planck scale led to the intriguing result that the fine structure constant could have been extremely small (zero) in the early Universe and that all matter in the Universe could have emerged via the Unruh-Rindler-Hawking effect (creation of radiation/matter) due to the acceleration w. r. t the vacuum frame of reference. For reviews on the alleged variations of the fundamental constants in Nature see [65] and for more astonishing variations of αdriven by quintessence see [66].

6.1 Clifford algebras in phase space

We shall employ the procedure described in [15] to construct the Phase Space Clifford algebra that allowed [127] to reproduce the sub-maximally accelerated particle action of [53].

For simplicity we will focus on a two-dim phase space. Let e_p, e_q be the Clifford-algebra basis elements in a two-dim phase space obeying the following relations [15]:

$$e_p \cdot e_q \equiv \frac{1}{2}(e_q e_p + e_p e_q) = 0 \tag{124}$$

and $e_p e_p = e_q e_q = 1$.

The Clifford product of e_p, e_q is by definition the sum of the scalar and the wedge product:

$$e_p e_q = e_p \cdot e_q + e_p \wedge e_q = 0 + e_p \wedge e_q = i\,, \tag{125}$$

such that $i^2 = e_p e_q e_p e_q = -1$. Hence, the imaginary unit i, $i^2 = -1$ admits a very natural interpretation in terms of Clifford algebras, i. e., it is represented by the wedge product $i = e_p \wedge e_q$, a phase-space area element. Such imaginary unit allows us to express vectors in a C-phase space in the form:

$$\begin{aligned} Q &= q e_q + p e_q\,, \\ Q \cdot e_q &= q + p e_p \cdot e_q = q + ip = z\,, \\ e_q \cdot Q &= q + p e_q \cdot e_p = q - ip = z^*\,, \end{aligned} \tag{126}$$

which reminds us of the creation/annihilation operators used in the harmonic oscillator.

We shall now review the steps in [127] to reproduce the sub-maximally accelerated particle action [53]. The phase-space analog of the spacetime action is:

$$dQdQ = (dq)^2 + (dp)^2 \Rightarrow S = m\int\sqrt{(dq)^2 + (dp)^2}\,. \tag{127}$$

Introducing the appropriate length/mass scale parameters in order to have consistent units yields:

$$S = m\int\sqrt{(dq)^2 + \left(\frac{L}{m}\right)^2 (dp)^2}\,, \tag{128}$$

where we have introduced the Planck scale L and have chosen the natural units $\hbar = c = 1$. A detailed physical discussion of the dilational invariant system of units $\hbar = c = G = 4\pi\epsilon_0 = 1$ was presented in ref. [15]. G is the Newton constant and ϵ_0 is the permittivity of the vacuum.

Extending this two-dim result to a $2n$-dim phase space result requires to have for Clifford basis the elements e_{p_μ}, e_{q_μ}, where $\mu = 1, 2, 3, \ldots n$. The action in the $2n$-dim phase space is:

$$\begin{aligned} S &= m\int\sqrt{(dq^\mu dq_\mu) + \left(\frac{L}{m}\right)^2 (dp^\mu dp_\mu)} = \\ &= m\int d\tau\sqrt{1 + \left(\frac{L}{m}\right)^2 (dp^\mu/d\tau)(dp_\mu/d\tau)}\,, \end{aligned} \tag{129}$$

where we have factored-out of the square-root the infinitesimal proper-time displacement $(d\tau)^2 = dq^\mu dq_\mu$.

One can recognize the action (129), up to a numerical factor of m/a, where a is the proper acceleration, as the same action for a sub-maximally accelerated particle given by Nesterenko [53] by rewriting $(dp^\mu/d\tau) = m(d^2x^\mu/d\tau^2)$:

$$S = m\int d\tau\sqrt{1 + L^2 (d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}\,. \tag{130}$$

Postulating that the maximal proper-acceleration is given in terms of the speed of light and the minimal Planck scale by $a = c^2/L = 1/L$, the action above gives the Nesterenko action, up to a numerical m/a factor:

$$S = m\int d\tau\sqrt{1 + a^{-2} (d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}\,. \tag{131}$$

The proper-acceleration is orthogonal to the proper-velocity and this can be easily verified by differentiating the time-like proper-velocity squared:

$$V^2 = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = V^\mu V_\mu = 1 > 0 \Rightarrow \\ \Rightarrow \frac{dV^\mu}{d\tau} V_\mu = \frac{d^2 x^\mu}{d\tau^2} V_\mu = 0\,, \qquad (132)$$

which implies that the proper-acceleration is space-like:

$$g^2(\tau) = -\frac{d^2 x^\mu}{d\tau^2}\frac{d^2 x_\mu}{d\tau^2} > 0 \Rightarrow \\ \Rightarrow S = m\int d\tau \sqrt{1-\frac{g^2}{a^2}} = m\int d\omega\,, \qquad (133)$$

where the analog of the Lorentz time-dilation factor for a sub-maximally accelerated particle is given by

$$d\omega = d\tau\sqrt{1-\frac{g^2(\tau)}{a^2}}\,. \qquad (134)$$

Therefore the dynamics of a sub-maximally accelerated particle can be reinterpreted as that of a particle moving in the spacetime tangent bundle whose Finsler-like metric is

$$(d\omega)^2 = g_{\mu\nu}(x^\mu, dx^\mu)dx^\mu dx^\nu = (d\tau)^2\left(1-\frac{g^2(\tau)}{a^2}\right). \qquad (135)$$

The invariant time now is no longer the standard proper-time τ but is given by the quantity $\omega(\tau)$. The deep connection between the physics of maximal acceleration and Finsler geometry has been analyzed by [56]. This sort of actions involving second derivatives have also been studied in the construction of actions associated with rigid particles (strings) [57], [58], [59], [60] among others.

The action is real-valued if, and only if, $g^2 < a^2$ in the same fashion that the action in Minkowski spacetime is real-valued if, and only if, $v^2 < c^2$. This is the physical reason why there is an upper bound in the proper-acceleration. In the special case of uniformly-accelerated motion $g(\tau) = g_0 =$ = constant, the trajectory of the particle in Minkowski spacetime is a hyperbola.

Most recently, an Extended Relativity Theory in Born-Clifford-Phase spaces with an *upper* and *lower* length scales (infrared/ultraviolet cutoff) has been constructed [138]. The invariance symmetry associated with an $8D$ Phase Space leads naturally to the real Clifford algebra $Cl(2,6,R)$ and complexified Clifford $Cl_C(4)$ algebra related to Twistors. The consequences of Mach's principle of inertia within the context of Born's Dual Phase Space Relativity Principle were also studied in [138] and they were compatible with the Eddington-Dirac large numbers coincidence and with the observed values of the anomalous Galileo-Pioneer acceleration. The modified Newtonian dynamics due to the upper/lower scales and modified Schwarzschild dynamics due the maximal acceleration were also provided.

6.2 Invariance under the $U(1,3)$ Group

In this section we will review in detail the principle of Maximal-acceleration Relativity [68] from the perspective of $8D$ Phase Spaces and the $U(1,3)$ Group. The $U(1,3) =$ $= SU(1,3)\otimes U(1)$ Group transformations, which leave invariant the phase-space intervals under rotations, velocity and acceleration boosts, were found by Low [74] and can be simplified drastically when the velocity/acceleration boosts are taken to lie in the z-direction, leaving the transverse directions x, y, p_x, p_y intact; i. e., the $U(1,1) = SU(1,1)\otimes U(1)$ subgroup transformations that leave invariant the phase-space interval are given by (in units of $\hbar = c = 1$)

$$(d\sigma)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} = \\ = (d\tau)^2\left[1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}\right] = \\ = (d\tau)^2\left[1 - \frac{m^2 g^2(\tau)}{m_P^2 A_{max}^2}\right], \qquad (136)$$

where we have factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in eq.-(136) and the maximal proper-force is set to be $b \equiv m_P A_{max}$. m_P is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, may also be interpreted as the maximal string tension when L_P is the Planck scale.

The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass m in the z-direction which we take to be X. Notice that the invariant interval $(d\sigma)^2$ in eq.-(136) is not strictly the *same* as the interval $(d\omega)^2$ of the Nesterenko action eq.-(131), which was invariant under a pseudo-complexification of the Lorentz group [61]. Only when $m = m_P$, the two intervals agree. The interval $(d\sigma)^2$ described by Low [74] is $U(1,3)$-invariant for the most general transformations in the $8D$ phase-space. These transformations are rather elaborate, so we refer to the references [74] for details. The analog of the Lorentz relativistic factor in eq.-(136) involves the ratios of two proper *forces*. One variable force is given by ma and the maximal proper force sustained by an *elementary* particle of mass m_P (a *Planckton*) is assumed to be $F_{max} = m_{Planck}c^2/L_P$. When $m = m_P$, the ratio-squared of the forces appearing in the relativistic factor of eq.-(136) becomes then g^2/A_{max}^2, and the phase space interval (136) coincides with the geometric interval of (131).

The transformations laws of the coordinates in that leave invariant the interval (136) are [74]:

$$T' = T\cosh\xi + \left(\frac{\xi_v X}{c^2} + \frac{\xi_a P}{b^2}\right)\frac{\sinh\xi}{\xi}\,, \qquad (137)$$

$$E' = E\cosh\xi + (-\xi_a X + \xi_v P)\frac{\sinh\xi}{\xi}\,, \qquad (138)$$

$$X' = X\cosh\xi + \left(\xi_v T - \frac{\xi_a E}{b^2}\right)\frac{\sinh\xi}{\xi}\,, \qquad (139)$$

$$P' = P\cosh\xi + \left(\frac{\xi_v E}{c^2} + \xi_a T\right)\frac{\sinh\xi}{\xi}\,. \tag{140}$$

The ξ_v is velocity-boost rapidity parameter and the ξ_a is the force/acceleration-boost rapidity parameter of the primed-reference frame. They are defined respectively (in the special case when $m = m_P$):

$$\tanh\left(\frac{\xi_v}{c}\right) = \frac{v}{c}\,,$$

$$\tanh\frac{\xi_a}{b} = \frac{ma}{m_P A_{max}}\,. \tag{141}$$

The *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs.-(137)–(140) is defined in terms of the velocity and acceleration boosts parameters ξ_v, ξ_a respectively as:

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}}\,. \tag{142}$$

Our definition of the rapidity parameters are *different* than those in [74].

Straightforward algebra allows us to verify that these transformations leave the interval of eq.-(136) in classical phase space invariant. They are are fully consistent with Born's duality Relativity symmetry principle [49] $(Q,P)\to \to (P,-Q)$. By inspection we can see that under Born duality, the transformations in eqs.-(137)–(140) are *rotated* into each other, up to numerical b factors in order to match units. When on sets $\xi_a = 0$ in (137)–(140) one recovers automatically the standard Lorentz transformations for the X, T and E, P variables *separately*, leaving invariant the intervals $dT^2 - dX^2 = (d\tau)^2$ and $(dE^2 - dP^2)/b^2$ separately.

When one sets $\xi_v = 0$ we obtain the transformations rules of the events in Phase space, from one reference-frame into another *uniformly*-accelerated frame of reference, $a = \text{const}$, whose acceleration-rapidity parameter is in this particular case:

$$\xi \equiv \frac{\xi_a}{b}\,, \qquad \tanh\xi = \frac{ma}{m_P A_{max}}\,. \tag{143}$$

The transformations for pure acceleration-boosts in are:

$$T' = T\cosh\xi + \frac{P}{b}\sinh\xi\,, \tag{144}$$

$$E' = E\cosh\xi - bX\sinh\xi\,, \tag{145}$$

$$X' = X\cosh\xi - \frac{E}{b}\sinh\xi\,, \tag{146}$$

$$P' = P\cosh\xi + bT\sinh\xi\,. \tag{147}$$

It is straightforward to verify that the transformations (144)–(146) leave invariant the fully phase space interval (136) but *does not* leave invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$. Only the *combination*:

$$(d\sigma)^2 = (d\tau)^2\left(1 - \frac{m^2 g^2}{m_P^2 A_{max}^2}\right) \tag{148}$$

is truly left invariant under pure acceleration-boosts (144)–(146). One can verify as well that these transformations satisfy Born's duality symmetry principle:

$$(T,X)\to(E,P)\,, \qquad (E,P)\to(-T,-X) \tag{149}$$

and $b \to \frac{1}{b}$. The latter Born duality transformation is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i. e. namely, a small/large radius duality, a winding modes/ Kaluza-Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in Noncommutative Field Theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory.

The composition of two successive pure acceleration-boosts is another pure acceleration-boost with acceleration rapidity given by $\xi'' = \xi + \xi'$. The addition of *proper* four-forces (accelerations) follows the usual relativistic composition rule:

$$\tanh\xi'' = \tanh(\xi+\xi') = \frac{\tanh\xi + \tanh\xi'}{1+\tanh\xi\tanh\xi'} \Rightarrow$$
$$\Rightarrow \frac{ma''}{m_P A} = \frac{\frac{ma}{m_P A} + \frac{ma'}{m_P A}}{1 + \frac{m^2 a a'}{m_P^2 A^2}}\,, \tag{150}$$

and in this fashion the upper limiting *proper* acceleration is never *surpassed* like it happens with the ordinary Special Relativistic addition of velocities.

The group properties of the full combination of velocity and acceleration boosts (137)–(140) requires much more algebra [68]. A careful study reveals that the composition *rule* of two succesive full transformations is given by $\xi'' = = \xi + \xi'$ and the transformation laws are *preserved* if, and only if, the ξ ; ξ' ; ξ'' … parameters obeyed the suitable relations:

$$\frac{\xi_a}{\xi} = \frac{\xi'_a}{\xi'} = \frac{\xi''_a}{\xi''} = \frac{\xi''_a}{\xi+\xi'}\,, \tag{151}$$

$$\frac{\xi_v}{\xi} = \frac{\xi'_v}{\xi'} = \frac{\xi''_v}{\xi''} = \frac{\xi''_v}{\xi+\xi'}\,. \tag{152}$$

Finally we arrive at the composition law for the effective, velocity and acceleration boosts parameters ξ''; ξ''_v; ξ''_a respectively:

$$\xi''_v = \xi_v + \xi'_v\,, \tag{153}$$

$$\xi''_a = \xi_a + \xi'_a\,, \tag{154}$$

$$\xi'' = \xi + \xi'\,. \tag{155}$$

The relations (151, 152, 153, 154, 155) are required in order to prove the *group* composition law of the transformations of (137)–(140) and, consequently, in order to have a truly Maximal-Acceleration Phase Space Relativity theory resulting from a phase-space change of coordinates in the cotangent bundle of spacetime.

6.3 Planck-Scale Areas are invariant under acceleration boosts

Having displayed explicitly the Group transformations rules of the coordinates in Phase space we will show why *infinite* acceleration-boosts (which is *not* the same as infinite proper acceleration) preserve Planck-Scale *Areas* [68] as a result of the fact that $b = (1/L_P^2)$ equals the *maximal* invariant force, or string tension, if the units of $\hbar = c = 1$ are used.

At Planck-scale L_P intervals/increments in one reference frame we have by definition (in units of $\hbar = c = 1$): $\Delta X = \Delta T = L_P$ and $\Delta E = \Delta P = \frac{1}{L_P}$ where $b \equiv \frac{1}{L_P^2}$ is the maximal tension. From eqs.-(137)–(140) we get for the transformation rules of the finite intervals ΔX, ΔT, ΔE, ΔP, from one reference frame into another frame, in the *infinite* acceleration-boost limit $\xi \to \infty$,

$$\Delta T' = L_P(\cosh\xi + \sinh\xi) \to \infty\,, \tag{156}$$

$$\Delta E' = \frac{1}{L_P}(\cosh\xi - \sinh\xi) \to 0 \tag{157}$$

by a simple use of L'Hôpital's rule or by noticing that both $\cosh\xi$; $\sinh\xi$ functions approach infinity at the same rate

$$\Delta X' = L_P(\cosh\xi - \sinh\xi) \to 0\,, \tag{158}$$

$$\Delta P' = \frac{1}{L_P}(\cosh\xi + \sinh\xi) \to \infty\,, \tag{159}$$

where the discrete displacements of two events in Phase Space are defined: $\Delta X = X_2 - X_1 = L_P$, $\Delta E = E_2 - E_1 = \frac{1}{L_P}$, $\Delta T = T_2 - T_1 = L_P$ and $\Delta P = P_2 - P_1 = \frac{1}{L_P}$.

Due to the identity:

$$(\cosh\xi + \sinh\xi)(\cosh\xi - \sinh\xi) = \cosh^2\xi - \sinh^2\xi = 1 \tag{160}$$

one can see from eqs.-(156)–(159) that the Planck-scale *Areas* are truly *invariant* under *infinite* acceleration-boosts $\xi = \infty$:

$$\Delta X'\Delta P' = 0\times\infty = \Delta X\Delta P(\cosh^2\xi - \sinh^2\xi) = \\ = \Delta X\Delta P = \frac{L_P}{L_P} = 1\,, \tag{161}$$

$$\Delta T'\Delta E' = \infty\times 0 = \Delta T\Delta E(\cosh^2\xi - \sinh^2\xi) = \\ = \Delta T\Delta E = \frac{L_P}{L_P} = 1\,, \tag{162}$$

$$\Delta X'\Delta T' = 0\times\infty = \Delta X\Delta T(\cosh^2\xi - \sinh^2\xi) = \\ = \Delta X\Delta T = (L_P)^2\,, \tag{163}$$

$$\Delta P'\Delta E' = \infty\times 0 = \Delta P\Delta E(\cosh^2\xi - \sinh^2\xi) = \\ = \Delta P\Delta E = \frac{1}{L_P^2}\,. \tag{164}$$

It is important to emphasize that the invariance property of the minimal Planck-scale *Areas* (maximal Tension) is *not* an exclusive property of *infinite* acceleration boosts $\xi = \infty$, but, as a result of the identity $\cosh^2\xi - \sinh^2\xi = 1$, for all values of ξ, the minimal Planck-scale *Areas* are *always* invariant under *any* acceleration-boosts transformations. Meaning physically, in units of $\hbar = c = 1$, that the Maximal Tension (or maximal Force) $b = \frac{1}{L_P^2}$ is a true physical *invariant* universal quantity. Also we notice that the Phase-space areas, or cells, in units of $\hbar$, are also invariant! The pure-acceleration boosts transformations are "symplectic". It can be shown also that areas greater (smaller) than the Planck-area remain greater (smaller) than the invariant Planck-area under acceleration-boosts transformations.

The infinite acceleration-boosts are closely related to the infinite red-shift effects when light signals barely escape Black hole Horizons reaching an asymptotic observer with an infinite red shift factor. The important fact is that the Planck-scale *Areas* are truly maintained invariant under acceleration-boosts. This could reveal very important information about Black-holes Entropy and Holography. The logarithmic corrections to the Black-Hole Area-Entropy relation were obtained directly from Clifford-algebraic methods in C-spaces [21], in addition to the derivation of the maximal Planck temperature condition and the Schwarzschild radius in terms of the Thermodynamics of a gas of p-loop-oscillators quanta represented by area-bits, volume-bits, ... hyper-volume-bits in Planck scale units. Minimal loop-areas, in Planck units, is also one of the most important consequences found in Loop Quantum Gravity long ago [111].

7 Some further important physical applications related to the C-space physics

7.1 Relativity of signature

In previous sections we have seen how Clifford algebra can be used in the formulation of the point particle classical and quantum theory. The metric of spacetime was assumed, as usually, to have the Minkowski signature, and we have used the choice $(+\,-\,-\,-)$. There were arguments in the literature of why the spacetime signature is of the Minkowski type [113, 43]. But there are also studies in which signature changes are admitted [112]. It has been found out [16, 15, 30] that within Clifford algebra the signature of the underlying space is a matter of choice of basis vectors amongst available Clifford numbers. We are now going to review those important topics.

Suppose we have a 4-dimensional space V_4 with signature

$(++++)$. Let e_μ, $\mu = 0, 1, 2, 3$, be basis vectors satisfying

$$e_\mu \cdot e_\nu \equiv \frac{1}{2}\,(e_\mu e_\nu + e_\nu e_\mu) = \delta_{\mu\nu}\ , \tag{165}$$

where $\delta_{\mu\nu}$ is the *Euclidean signature* of V_4. The vectors e_μ can be used as generators of Clifford algebra $\mathcal{C}_4$ over V_4 with a generic Clifford number (also called polyvector or Clifford aggregate) expanded in term of $e_J{=}(1{,}e_\mu{,}e_{\mu\nu}{,}e_{\mu\nu\alpha}{,}e_{\mu\nu\alpha\beta})$, $\mu < \nu < \alpha < \beta$,

$$\begin{aligned} A = a^J e_J = a + a^\mu e_\mu + a^{\mu\nu} e_\mu e_\nu \,+ \\ +\, a^{\mu\nu\alpha} e_\mu e_\nu e_\alpha + a^{\mu\nu\alpha\beta} e_\mu e_\nu e_\alpha e_\beta\,. \end{aligned} \tag{166}$$

Let us consider the set of four Clifford numbers $(e_0, e_i e_0)$, $i = 1, 2, 3$, and denote them as

$$\begin{aligned} e_0 &\equiv \gamma_0\,, \\ e_i e_0 &\equiv \gamma_i\,. \end{aligned} \tag{167}$$

The Clifford numbers γ_μ, $\mu = 0, 1, 2, 3$, satisfy

$$\frac{1}{2}\,(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu}\ , \tag{168}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the *Minkowski tensor*. We see that the γ_μ behave as basis vectors in a 4-dimensional space $V_{1,3}$ with signature $(+\,-\,-\,-)$. We can form a Clifford aggregate

$$\alpha = \alpha^\mu \gamma_\mu\,, \tag{169}$$

which has the properties of a *vector* in $V_{1,3}$. From the point of view of the space V_4 the same object α is a linear combination of a vector and bivector:

$$\alpha = \alpha^0 e_0 + \alpha^i e_i e_0\,. \tag{170}$$

We may use γ_μ as generators of the Clifford algebra $\mathcal{C}_{1,3}$ defined over the pseudo-Euclidean space $V_{1,3}$. The basis elements of $\mathcal{C}_{1,3}$ are $\gamma_J = (1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\alpha}, \gamma_{\mu\nu\alpha\beta})$, with $\mu < \nu < \alpha < \beta$. A generic Clifford aggregate in $\mathcal{C}_{1,3}$ is given by

$$\begin{aligned} B = b^J \gamma_J = b + b^\mu \gamma_\mu + b^{\mu\nu} \gamma_\mu \gamma_\nu \,+ \\ +\, b^{\mu\nu\alpha} \gamma_\mu \gamma_\nu \gamma_\alpha + b^{\mu\nu\alpha\beta} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta\,. \end{aligned} \tag{171}$$

With suitable choice of the coefficients $b^J = (b, b^\mu, b^{\mu\nu}, b^{\mu\nu\alpha}, b^{\mu\nu\alpha\beta})$ we have that B of eq.-(171) is equal to A of eq.-(166). Thus the same number A can be described either with e_μ which generate $\mathcal{C}_4$, or with γ_μ which generate $\mathcal{C}_{1,3}$. The expansions (171) and (166) exhaust all possible numbers of the Clifford algebras $\mathcal{C}_{1,3}$ and $\mathcal{C}_4$. Those expansions are just two different representations of the same set of Clifford numbers (also being called polyvectors or Clifford aggregates).

As an alternative to (167) we can choose

$$\begin{aligned} e_0 e_3 &\equiv \tilde\gamma_0\,, \\ e_i &\equiv \tilde\gamma_i\,, \end{aligned} \tag{172}$$

from which we have

$$\frac{1}{2}\,(\tilde\gamma_\mu\tilde\gamma_\nu + \tilde\gamma_\nu\tilde\gamma_\mu) = \tilde\eta_{\mu\nu} \tag{173}$$

with $\tilde\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Obviously $\tilde\gamma_\mu$ are basis vectors of a pseudo-Euclidean space $\widetilde{V}_{1,3}$ and they generate the Clifford algebra over $\widetilde{V}_{1,3}$ which is yet another representation of the same set of objects (i. e., polyvectors). The spaces V_4, $V_{1,3}$ and $\widetilde{V}_{1,3}$ are different slices through C-space, and they span different subsets of polyvectors. In a similar way we can obtain spaces with signatures $(+-++)$, $(++-+)$, $(+++-)$, $(-+--)$, $(--+-)$, $(---+)$ and corresponding higher dimensional analogs. But we cannot obtain signatures of the type $(++--)$, $(+-+-)$, etc. In order to obtain such signatures we proceed as follows.

4-space. First we observe that the bivector $\bar I$*4-space.* $e_3 e_4$ satisfies $\bar I^2 = -1$, commutes with e_1, e_2 and anticommutes with e_3, e_4. So we obtain that the set of Clifford numbers $\gamma_\mu = (e_1 \bar I, e_2 \bar I, e_3, e_3)$ satisfies

$$\gamma_\mu \cdot \gamma_\nu = \bar\eta_{\mu\nu}\ , \tag{174}$$

where $\bar\eta = \mathrm{diag}(-1, -1, 1, 1)$.

8-space. Let e_A be basis vectors of 8-dimensional vector space with signature $(+ + + + + + + +)$. Let us decompose

$$\begin{aligned} e_A = (e_\mu, e_{\bar\mu})\,, \quad \mu &= 0, 1, 2, 3\,, \\ \bar\mu &= \bar 0, \bar 1, \bar 2, \bar 3\,. \end{aligned} \tag{175}$$

The inner product of two basis vectors

$$e_A \cdot e_B = \delta_{AB}\,, \tag{176}$$

then splits into the following set of equations:

$$\begin{aligned} e_\mu \cdot e_\nu &= \delta_{\mu\nu}\,, \\ e_{\bar\mu} \cdot e_{\bar\nu} &= \delta_{\bar\mu\bar\nu}\,, \\ e_\mu \cdot e_{\bar\nu} &= 0\,. \end{aligned} \tag{177}$$

The number $\bar I = e_{\bar 0} e_{\bar 1} e_{\bar 2} e_{\bar 3}$ has the properties

$$\begin{aligned} \bar I^2 &= 1\,, \\ \bar I e_\mu &= e_\mu \bar I\,, \\ \bar I e_{\bar\mu} &= -e_{\bar\mu} \bar I\,. \end{aligned} \tag{178}$$

The set of numbers

$$\begin{aligned} \gamma_\mu &= e_\mu\,, \\ \gamma_{\bar\mu} &= e_{\bar\mu} \bar I \end{aligned} \tag{179}$$

satisfies

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &= \delta_{\mu\nu} \,, \\ \gamma_{\bar\mu} \cdot \gamma_{\bar\nu} &= -\delta_{\mu\nu} \,, \\ \gamma_\mu \cdot \gamma_{\bar\mu} &= 0 \,. \end{aligned} \tag{180}$$

The numbers $(\gamma_\mu, \gamma_{\bar\mu})$ thus form a set of basis vectors of a vector space $V_{4,4}$ with signature $(++++----)$.

10-space. Let $e_A = (e_\mu, e_{\bar\mu})$, $\mu = 1,2,3,4,5$; $\bar\mu = \bar1,\bar2,\bar3,\bar4,\bar5$ be basis vectors of a 10-dimensional Euclidean space V_{10} with signature $(+++\ldots)$. We introduce $\bar I = e_{\bar1}e_{\bar2}e_{\bar3}e_{\bar4}e_{\bar5}$ which satisfies

$$\begin{aligned} \bar I^2 &= 1 \,, \\ e_\mu \bar I &= -\bar I e_\mu \,, \\ e_{\bar\mu} \bar I &= \bar I e_{\bar\mu} \,. \end{aligned} \tag{181}$$

Then the Clifford numbers

$$\begin{aligned} \gamma_\mu &= e_\mu \bar I \,, \\ \gamma_{\bar\mu} &= e_\mu \end{aligned} \tag{182}$$

satisfy

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &= -\delta_{\mu\nu} \,, \\ \gamma_{\bar\mu} \cdot \gamma_{\bar\nu} &= \delta_{\bar\mu\bar\nu} \,, \\ \gamma_\mu \cdot \gamma_{\bar\mu} &= 0 \,. \end{aligned} \tag{183}$$

The set $\gamma_A = (\gamma_\mu, \gamma_{\bar\mu})$ therefore spans the vector space of signature $(-----+++++)$.

The examples above demonstrate how vector spaces of various signatures are obtained within a given set of polyvectors. Namely, vector spaces of different signature are different subsets of polyvectors within the same Clifford algebra. In other words, vector spaces of different signature are different subspaces of C-space, i. e., different sections through C-space*.

This has important physical implications. We have argued that physical quantities are polyvectors (Clifford numbers or Clifford aggregates). Physical space is then not simply a vector space (e.g., Minkowski space), but a space of polyvectors, called C-space, a pandimensional continuum of points, lines, planes, volumes, etc., altogether. Minkowski space is then just a subspace with pseudo-Euclidean signature. Other subspaces with other signatures also exist within the pandimensional continuum C and they all have physical significance. If we describe a particle as moving in Minkowski spacetime $V_{1,3}$ we consider only certain physical aspects of the object considered. We have omitted its other physical properties like spin, charge, magnetic moment, etc. We can as well describe the same object as moving in an Euclidean space V_4. Again such a description would reflect only a part of the underlying physical situation described by Clifford algebra.

*What we consider here should not be confused with the well known fact that Clifford algebras associated with vector spaces of different signatures (p, q), with $p + q = n$, are not all isomorphic.

7.2 Clifford space and the conformal group

7.2.1 Line element in C-space of Minkowski spacetime

In 4-dimensional spacetime a polyvector and its square (1) can be written as

$$\mathrm{d}X = \mathrm{d}\sigma + \mathrm{d}x^\mu\gamma_\mu + \frac{1}{2}\,\mathrm{d}x^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + \mathrm{d}\tilde x^\mu\, I\gamma_\mu + \mathrm{d}\tilde\sigma I \,, \tag{184}$$

$$|\mathrm{d}X|^2 = \mathrm{d}\sigma^2 + \mathrm{d}x^\mu\mathrm{d}x_\mu + \frac{1}{2}\,\mathrm{d}x^{\mu\nu}\mathrm{d}x_{\mu\nu} - \mathrm{d}\tilde x^\mu\mathrm{d}\tilde x_\mu - \mathrm{d}\tilde\sigma^2 . \tag{185}$$

The minus sign in the last two terms of the above quadratic form occurs because in 4-dimensional spacetime with signature $(+---)$ we have $I^2 = (\gamma_0\gamma_1\gamma_2\gamma_3)(\gamma_0\gamma_1\gamma_2\gamma_3) = -1$, and $I^\dagger I = (\gamma_3\gamma_2\gamma_1\gamma_0)(\gamma_0\gamma_1\gamma_2\gamma_3) = -1$.

In eq.-(185) the line element $\mathrm{d}x^\mu\mathrm{d}x_\mu$ of the ordinary special or general relativity is replaced by the line element in Clifford space. A "square root" of such a generalized line element is $\mathrm{d}X$ of eq.-(184). The latter object is a *polyvector*, a differential of the coordinate polyvector field

$$X = \sigma + x^\mu\gamma_\mu + \frac{1}{2}\,x^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + \tilde x^\mu I\gamma_\mu + \tilde\sigma I \,, \tag{186}$$

whose square is

$$|X|^2 = \sigma^2 + x^\mu x_\mu + \frac{1}{2}\,x^{\mu\nu}x_{\mu\nu} - \tilde x^\mu\tilde x_\mu - \tilde\sigma^2 \,. \tag{187}$$

The polyvector X contains not only the vector part $x^\mu\gamma_\mu$, but also a *scalar part* σ, *tensor part* $x^{\mu\nu}\gamma_\mu\wedge\gamma_\nu$, *pseudovector part* $\tilde x^\mu\, I\gamma_\mu$ and pseudoscalar part $\tilde\sigma I$. Similarly for the differential $\mathrm{d}X$.

When calculating the quadratic forms $|X|^2$ and $|\mathrm{d}X|^2$ one obtains in 4-dimensional spacetime with pseudo euclidean signature $(+---)$ the minus sign in front of the squares of the pseudovector and pseudoscalar terms. This is so, because in such a case the pseudoscalar unit square in flat spacetime is $I^2 = I^\dagger I = -1$. In 4-dimensions $I^\dagger = I$ regardless of the signature.

Instead of Lorentz transformations — pseudo rotations in spacetime — which preserve $x^\mu x_\mu$ and $\mathrm{d}x^\mu\mathrm{d}x_\mu$ we have now more general rotations — rotations in C-space — which preserve $|X|^2$ and $|\mathrm{d}X|^2$.

7.2.2 C-space and conformal transformations

From (185) and (187) we see [25] that a subgroup of the Clifford Group, or rotations in C-space is the group $SO(4,2)$. The transformations of the latter group rotate x^μ, σ, $\tilde\sigma$, but leave $x^{\mu\nu}$ and $\tilde x^\mu$ unchanged. Although according to our assumption physics takes place in full C-space, it is very instructive to consider a subspace of C-space, that we shall call *conformal space* whose isometry group is $SO(4,2)$.

Coordinates can be given arbitrary symbols. Let us now use the symbol η^μ instead of x^μ, and η^5,η^6 instead of $\tilde\sigma$, σ. In other words, instead of $(x^\mu, \tilde{\sigma}, \sigma)$ we write $(\eta^\mu, \eta^5, \eta^6) \equiv \eta^a$, $\mu = 0, 1, 2, 3$, $a = 0, 1, 2, 3, 5, 6$. The quadratic form reads

$$\eta^a \eta_a = g_{ab} \eta^a \eta^b \tag{188}$$

with

$$g_{ab} = \mathrm{diag}(1, -1, -1, -1, -1, 1) \tag{189}$$

being the diagonal metric of the flat 6-dimensional space, a subspace of C-space, parametrized by coordinates η^a. The transformations which preserve the quadratic form (188) belong to the group $SO(4, 2)$. It is well known [38, 39] that the latter group, when taken on the cone

$$\eta^a \eta_a = 0 \tag{190}$$

is isomorphic to the 15-parameter group of conformal transformations in 4-dimensional spacetime [40].

Let us consider first the rotations of η^5 and η^6 which leave coordinates η^μ unchanged. The transformations that leave $-(\eta^5)^2 + (\eta^6)^2$ invariant are

$$\begin{aligned} \eta'^5 &= \eta^5 \cosh\alpha + \eta^6 \sinh\alpha \\ \eta'^6 &= \eta^5 \sinh\alpha + \eta^6 \cosh\alpha \,, \end{aligned} \tag{191}$$

where α is a parameter of such pseudo rotations.

Instead of the coordinates η^5, η^6 we can introduce [38, 39] new coordinates κ, λ according to

$$\begin{aligned} \kappa &= \eta^5 - \eta^6 \,, \\ \lambda &= \eta^5 + \eta^6 \,. \end{aligned} \tag{192}$$

In the new coordinates the quadratic form (188) reads

$$\eta^a \eta_a = \eta^\mu \eta_\mu - (\eta^5)^2 - (\eta^6)^2 = \eta^\mu \eta_\mu - \kappa\lambda \,. \tag{193}$$

The transformation (191) becomes

$$\kappa' = \rho^{-1} \kappa \,, \tag{194}$$

$$\lambda' = \rho\lambda \,, \tag{195}$$

where $\rho = e^\alpha$. This is just a dilation of κ and the inverse dilation of λ.

Let us now introduce new coordinates x^μ*

$$\eta^\mu = \kappa x^\mu \,. \tag{196}$$

Under the transformation (196) we have

$$\eta'^\mu = \eta^\mu \,, \tag{197}$$

but

$$x'^\mu = \rho x^\mu \,, \tag{198}$$

the latter transformation is *dilatation* of coordinates x^μ.

*These new coordinates x^μ should not be confused with coordinate x^μ used in section 2.

Considering now a line element

$$\mathrm{d}\eta^a \mathrm{d}\eta_a = \mathrm{d}\eta^\mu \mathrm{d}\eta_\mu - \mathrm{d}\kappa \mathrm{d}\,, \lambda \tag{199}$$

we find that *on the cone* $\eta^a \eta_a = 0$ it is

$$\mathrm{d}\eta^a \mathrm{d}\eta_a = \kappa^2\, \mathrm{d}x^\mu \mathrm{d}x_\mu \tag{200}$$

even if κ is not constant. Under the transformation (194) we have

$$\mathrm{d}\eta'^a \mathrm{d}\eta'_a = \mathrm{d}\eta^a \mathrm{d}\eta_a \,, \tag{201}$$

$$\mathrm{d}x'^\mu \mathrm{d}x'_\mu = \rho^2\, \mathrm{d}x^\mu \mathrm{d}x_\mu \,. \tag{202}$$

The last relation is a *dilatation* of the 4-dimensional line element related to coordinates x^μ. In a similar way also other transformations of the group $SO(4, 2)$ that preserve (190) and (201) we can rewrite in terms of of the coordinates x^μ. So we obtain — besides dilations — translations, Lorentz transformations, and special conformal transformations; altogether they are called *conformal transformations*. This is a well known old observation [38, 39] and we shall not discuss it further. What we wanted to point out here is that conformal group $SO(4, 2)$ is a subgroup of the Clifford group.

7.2.3 On the physical interpretation of the conformal group $SO(4, 2)$

In order to understand the physical meaning of the transformations (196) from the coordinates η^μ to the coordinates x^μ let us consider the following transformation in 6-dimensional space V_6:

$$\begin{aligned} x^\mu &= \kappa^{-1} \eta^\mu \,, \\ \alpha &= -\kappa^{-1} \,, \\ \Lambda &= \lambda - \kappa^{-1} \eta^\mu \eta_\mu \,. \end{aligned} \tag{203}$$

This is a transformation from the coordinates $\eta^a = (\eta^\mu, \kappa, \lambda)$ to the new coordinates $x^a = (x^\mu, \alpha, \Lambda)$. No extra condition on coordinates, such as (190), is assumed now. If we calculate the line element in the coordinates η^a and x^a, respectively, we find the the following relation [27]

$$\mathrm{d}\eta^\mu \mathrm{d}\eta^\nu\, g_{\mu\nu} - \mathrm{d}\kappa\, \mathrm{d}\lambda = \alpha^{-2} (\mathrm{d}x^\mu \mathrm{d}x^\nu\, g_{\mu\nu} - \mathrm{d}\alpha \mathrm{d}\Lambda) \,. \tag{204}$$

We can interpret a transformation of coordinates *passively* or *actively*. Geometric calculus clarifies significantly the meaning of passive and active transformations. Under a *passive transformation* a vector remains the same, but its components and basis vector change. For a vector $\mathrm{d}\eta = \mathrm{d}\eta^a \gamma_a$ we have

$$\mathrm{d}\eta' = \mathrm{d}\eta'^a \gamma'_a = \mathrm{d}\eta^a \gamma_a = \mathrm{d}\eta \tag{205}$$

with

$$\mathrm{d}\eta'^a = \frac{\partial \eta'^a}{\partial \eta^b}\, \mathrm{d}\eta^b \tag{206}$$

and

$$\gamma'_a = \frac{\partial \eta^b}{\partial \eta'^a}\, \gamma_b\,. \tag{207}$$

Since the vector is invariant, so it is its square:

$$\mathrm{d}\eta'^2 = \mathrm{d}\eta'^a \gamma'_a\, \mathrm{d}\eta'^b \gamma'_b = \mathrm{d}\eta'^a \mathrm{d}\eta'^b g'_{ab} = \mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab}\,. \tag{208}$$

From (207) we read that the well known relation between new and old coordinates:

$$g'_{ab} = \frac{\partial \eta^c}{\partial \eta'^a} \frac{\partial \eta^d}{\partial \eta'^b}\, g_{cd}\,. \tag{209}$$

Under an *active transformation* a vector changes. This means that in a fixed basis the components of a vector change:

$$\mathrm{d}\eta' = \mathrm{d}\eta'^a \gamma_a \tag{210}$$

with

$$\mathrm{d}\eta'^a = \frac{\partial \eta'^a}{\partial \eta^b}\, \mathrm{d}\eta^b\,. \tag{211}$$

The transformed vector $\mathrm{d}\eta'$ is different from the original vector $\mathrm{d}\eta = \mathrm{d}\eta^a \gamma_a$. For the square we find

$$\mathrm{d}\eta'^2 = \mathrm{d}\eta'^a \mathrm{d}\eta'^b g_{ab} = \frac{\partial \eta'^a}{\partial \eta^c} \frac{\partial \eta'^b}{\partial \eta^d}\, \mathrm{d}\eta^c \mathrm{d}\eta^d g_{ab}\,, \tag{212}$$

i. e., the transformed line element $\mathrm{d}\eta'^2$ is different from the original line element.

Returning now to the coordinate transformation (203) with the identification $\eta'^a = x^a$, we can interpret eq.-(204) passively or actively.

In the *passive interpretation* the metric tensor and the components $\mathrm{d}\eta^a$ change under a transformation, so that in our particular case the relation (208) becomes

$$\begin{aligned} \mathrm{d}x^a\, \mathrm{d}x^b\, g'_{ab} &= \alpha^{-2}(\mathrm{d}x^\mu \mathrm{d}x^\nu\, g_{\mu\nu} - \mathrm{d}\alpha\, \mathrm{d}\Lambda) = \\ &= \mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab} = \mathrm{d}\eta^\mu \mathrm{d}\eta^\nu g_{\mu\nu} - \mathrm{d}\kappa\, \mathrm{d}\lambda \end{aligned} \tag{213}$$

with

$$\begin{aligned} g'_{ab} &= \alpha^{-2} \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \\ g_{ab} &= \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}. \end{aligned} \tag{214}$$

In the above equation the same infinitesimal distance squared is expressed in two different coordinates η^a or x^a.

In *active interpretation*, only $\mathrm{d}\eta^a$ change, whilst the metric remains the same, so that the transformed element is

$$\begin{aligned} \mathrm{d}x^a\, \mathrm{d}x^b\, g_{ab} &= \mathrm{d}x^\mu \mathrm{d}x^\nu\, g_{\mu\nu} - \mathrm{d}\alpha\, \mathrm{d}\Lambda = \\ &= \kappa^{-2}\, \mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab} = \kappa^{-2}(\mathrm{d}\eta^\mu \mathrm{d}\eta^\nu g_{\mu\nu} - \mathrm{d}\kappa\, \mathrm{d}\lambda)\,. \end{aligned} \tag{215}$$

The transformed line element $\mathrm{d}x^a \mathrm{d}x_a$ is physically different from the original line element $\mathrm{d}\eta^a \mathrm{d}\eta_a$ by a factor $\alpha^2 = \kappa^{-2}$.

A rotation (191) in the plane (η^5, η^6) i. e. the transformation (194), (195) of (κ, λ) manifests in the new coordinates x^a as a *dilatation* of the line element $\mathrm{d}x^a \mathrm{d}x_a = \kappa^{-2}\, \mathrm{d}\eta^a \eta_a$:

$$\mathrm{d}x'^a \mathrm{d}x'_a = \rho^2 \mathrm{d}x^a \mathrm{d}x_a\,. \tag{216}$$

All this is true in the full space V_6. On the cone $\eta^a \eta_a = 0$ we have $\Lambda = \lambda - \kappa \eta^\mu \eta_\mu = 0$, $\mathrm{d}\Lambda = 0$ so that $\mathrm{d}x^a \mathrm{d}x_a = \mathrm{d}x^\mu \mathrm{d}x_\mu$ and we reproduce the relations (202) which is a dilatation of the 4-dimensional line element. It can be interpreted either passively or actively. In general, the pseudo rotations in V_6, that is, the transformations of the 15-parameter group $SO(4,2)$ when expressed in terms of coordinates x^a, assume on the cone $\eta^a \eta_a = 0$ the form of the ordinary conformal transformations. They all can be given the active interpretation [27, 28].

We started from the new paradigm that physical phenomena actually occur not in spacetime, but in a larger space, the so called *Clifford space* or C-space which is a manifold associated with the Clifford algebra generated by the basis vectors γ_μ of spacetime. An arbitrary element of Clifford algebra can be expanded in terms of the objects E_A, $A = 1, 2, \ldots, 2^D$, which include, when $D = 4$, the scalar unit $\mathbf{1}$, vectors γ_μ, bivectors $\gamma_\mu \wedge \gamma_\nu$, pseudovectors $I\gamma_\mu$ and the pseudoscalar unit $I \equiv \gamma_5$. C-space contains 6-dimensional subspace V_6 spanned* by $\mathbf{1}$, γ_μ, and γ_5. The metric of V_6 has the signature $(+ - - - - +)$. It is well known that the rotations in V_6, when taken on the conformal cone $\eta^a \eta_a = 0$, are isomorphic to the non linear transformations of the conformal group in spacetime. Thus we have found out that C-space contains — as a subspace — the 6-dimensional space V_6 in which the conformal group acts linearly. From the physical point of view this is an important and, as far as we know, a novel finding, although it might look mathematically trivial. *So far it has not been clear what could be a physical interpretation of the 6 dimensional conformal space*. Now we see that it is just a subspace of Clifford space. The two extra dimensions, parameterized by κ and λ, are not the ordinary extra dimensions; they are coordinates of Clifford space C_4 of the 4-dimensional Minkowski spacetime V_4.

We take C-space seriously as an arena in which physics takes place. The theory is a very natural, although not trivial, extension of the special relativity in spacetime. In special relativity the transformations that preserve the quadratic form

*It is a well known observation that the generators L_{ab} of $SO(4,2)$ can be realized in terms of $\mathbf{1}$, γ_μ, and γ_5. Lorentz generators are $M_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]$, dilatations are generated by $D = L_{65} = -\frac{1}{2}\gamma_5$, translations by $P_\mu = L_{5\mu} + L_{6\mu} = \frac{1}{2}\gamma_\mu(1 - i\gamma_5)$ and the special conformal transformations by $L_{5\mu} - L_{6\mu} = \frac{1}{2}\gamma_\mu(1 + i\gamma_5)$. This essentially means that the generators are $L_{ab} = -\frac{i}{4}[e_a, e_b]$ with $e_a = (\gamma_\mu, \gamma_5, \mathbf{1})$, where care must be taken to replace commutators $[\mathbf{1}, \gamma_5]$ and $[\mathbf{1}, \gamma_\mu]$ with $2\gamma_5$ and $2\gamma_\mu$.

are given an *active interpretation*: they relate the objects or the systems of reference in *relative translational motion*. Analogously also the transformations that preserve the quadratic form (185) or (187) in C-space should be given an active interpretation. We have found that among such transformations (rotations in C-space) there exist the transformations of the group $SO(4,2)$. Those transformations also should be given an active interpretation as the transformations that relate different physical objects or reference frames. Since in the ordinary relativity we do not impose any constraint on the coordinates of a freely moving object so we should not impose any constraint in C-space, or in the subspace V_6. However, by using the projective coordinate transformation (203), without any constraint such as $\eta^a \eta_a = 0$, we arrived at the relation (215) for the line elements. If in the coordinates η^a the line element is constant, then in the coordinates x^a the line element is changing by a scale factor κ which, in general, depends on the evolution parameter τ. The line element need not be one associated between two events along a point particle's worldline: it can be between two arbitrary (space-like or time-like) events within an extended object. We may consider the line element ($\equiv$ distance squared) between two infinitesimally separated events within an extended object such that both events have the same coordinate label Λ so that $\mathrm{d}\Lambda = 0$. Then the 6-dimensional line element $\mathrm{d}x^\mu \mathrm{d}x^\nu g_{\mu\nu} - \mathrm{d}\alpha\, \mathrm{d}\Lambda$ becomes the 4-dimensional line element $\mathrm{d}x^\mu \mathrm{d}x^\nu g_{\mu\nu}$ and, because of (215) it changes with τ when κ does change. This means that the object changes its *size*, it is moving dilatationally [27, 28]. We have thus arrived at a very far reaching observation that the relativity in C-space implies *scale changes of physical objects as a result of free motion, without presence of any forces or such fields as assumed in Weyl theory*. This was advocated long time ago [27, 28], but without recurse to C-space. However, if we consider the full Clifford space C and not only the Minkowski spacetime section through C, then we arrive at a more general dilatational motion [17] related to the polyvector coordinates $x^{\mu\nu}, x^{\mu\nu\alpha}$ and $x^{0123} \equiv \tilde{\sigma}$ (also denoted s) as reviewed in section 3.

7.3 C-space Maxwell Electrodynamics

Finally, in this section we will review and complement the proposal of ref. [75] to generalize Maxwell Electrodynamics to C-spaces, namely, construct the Clifford algebra-valued extension of the Abelian field strength $F = dA$ associated with ordinary vectors A_μ. Using Clifford algebraic methods we shall describe how to generalize Maxwell's theory of Electrodynamics associated with ordinary point-charges to a generalized Maxwell theory in Clifford spaces involving *extended* charges and p-forms of arbitrary rank, not unlike the couplings of p-branes to antisymmetric tensor fields.

Based on the standard definition of the Abelian field strength $F = dA$ we shall use the same definition in terms of polyvector-valued quantities and differential operators in C-space

$$A = A_N E^N = \phi \underline{1} + A_\mu \gamma^\mu + A_{\mu\nu} \gamma^\mu \wedge \gamma^\nu + \dots \,. \tag{217}$$

The first component in the expansion ϕ is a scalar field; A_μ is the standard Maxwell field A_μ, the third component $A_{\mu\nu}$ is a rank two antisymmetric tensor field… and the last component of the expansion is a pseudo-scalar. The fact that a scalar and pseudo-scalar field appear very naturally in the expansion of the C-space polyvector valued field A_N suggests that one could attempt to identify the latter fields with a dilaton-like and axion-like field, respectively. Once again, in order to match units in the expansion (217), it requires the introduction of suitable powers of a length scale parameter, the Planck scale which is conveniently set to unity.

The differential operator is the generalized Dirac operator

$$d = E^M \partial_M = \underline{1} \partial_\sigma + \gamma^\mu \partial_{x_\mu} + \gamma^\mu \wedge \gamma^\nu \partial_{x_{\mu\nu}} + \dots \tag{218}$$

the polyvector-valued indices $M, N, \dots$ range from $1, 2 \dots 2^D$ since a Clifford algebra in D-dim has 2^D basis elements. The generalized Maxwell field strength in C-space is

$$\begin{aligned} F = dA &= E^M \partial_M (E^N A_N) = E^M E^N \partial_M A_N = \\ &= \frac{1}{2}\{E^M, E^N\} \partial_M A_N + \frac{1}{2}[E^M, E^N] \partial_M A_N = \\ &= \frac{1}{2} F_{(MN)} \{E^M, E^N\} + \frac{1}{2} F_{[MN]} [E^M, E^N]\,, \end{aligned} \tag{219}$$

where one has *decomposed* the Field strength components into a symmetric plus antisymmetric piece by simply writing the Clifford geometric product of two polyvectors $E^M E^N$ as the sum of an anticommutator plus a commutator piece respectively,

$$F_{(MN)} = \frac{1}{2}(\partial_M A_N + \partial_N A_M)\,, \tag{220}$$

$$F_{[MN]} = \frac{1}{2}(\partial_M A_N - \partial_N A_M)\,. \tag{221}$$

Let the C-space Maxwell action (up to a numerical factor) be given in terms of the antisymmetric part of the field strength:

$$I[A] = \int [\mathrm{D}X] F_{[MN]} F^{[MN]}\,, \tag{222}$$

where $[\mathrm{D}X]$ is a C-space measure comprised of all the (holographic) coordinates degrees of freedom

$$\begin{aligned} [\mathrm{D}X] \equiv (d\sigma)(dx^0 dx^1 \dots)(dx^{01} dx^{02} \dots) \dots \\ \dots (dx^{012\dots D})\,. \end{aligned} \tag{223}$$

Action (222) is invariant under the gauge transformations

$$A'_M = A_M + \partial_M \Lambda\,. \tag{224}$$

The matter-field minimal coupling (interaction term) is:

$$\int A_M dX^M = \int [\mathrm{D}X] J_M A^M \,, \tag{225}$$

where one has reabsorbed the coupling constant, the C-space analog of the electric charge, within the expression for the A field itself. Notice that this term (225) has the same form as the coupling of p-branes (whose world volume is $(p+1)$-dimensional) to antisymmetric tensor fields of rank $p+1$.

The open line integral in C-space of the matter-field interaction term in the action is taken from the polyparticle's proper time interval S ranging from $-\infty$ to $+\infty$ and can be recast via the Stokes law solely in terms of the antisymmetric part of the field strength. This requires closing off the integration contour by a semi-circle that starts at $S=+\infty$, goes all the way to C-space infinity, and comes back to the point $S=-\infty$. The field strength vanishes along the points of the semi-circle at infinity, and for this reason the net contribution to the contour integral is given by the open-line integral. Therefore, by rewriting the $\int A_M dX^M$ via the Stokes law relation, it yields

$$\int A_M dX^M = \int F_{[MN]} dS^{[MN]} = \int F_{[MN]} X^M dX^N = \\ = \int dS F_{[MN]} X^M (dX^N/dS) \,, \tag{226}$$

where in order to go from the second term to the third term in the above equation we have integrated by parts and then used the Bianchi identity for the antisymmetric component $F_{[MN]}$.

The integration by parts permits us to go from a C-space domain integral, represented by the Clifford-value hypersurface S^{MN}, to a C-space boundary-line integral

$$\int dS^{MN} = \frac{1}{2}\int (X^M dX^N - X^N dX^M) \,. \tag{227}$$

The pure matter terms in the action are given by the analog of the proper time integral spanned by the motion of a particle in spacetime:

$$\kappa \int dS = \kappa \int dS \sqrt{\frac{dX^M}{dS}\frac{dX_M}{dS}} \,, \tag{228}$$

where κ is a parameter whose dimensions are mass^{p+1} and S is the polyparticle proper time in C-space.

The Lorentz force relation in C-space is directly obtained from a variation of

$$\int dS F_{[MN]} X^M (dX^N/dS) \,, \tag{229}$$

and

$$\kappa \int dS = \kappa \int \sqrt{dX^M dX_M} \tag{230}$$

with respect to the X^M variables:

$$\kappa \frac{d^2 X_M}{dS^2} = e F_{[MN]} \frac{dX^N}{dS} \,, \tag{231}$$

where we have re-introduced the C-space charge e back into the Lorentz force equation in C-space. A variation of the terms in the action w. r. t the A_M field furnishes the following equation of motion for the A fields:

$$\partial_M F^{[MN]} = J^N \,. \tag{232}$$

By taking derivatives on both sides of the last equation with respect to the X^N coordinate, one obtains due to the symmetry condition of $\partial_M \partial_N$ versus the antisymmetry of $F^{[MN]}$ that

$$\partial_N \partial_M F^{[MN]} = 0 = \partial_N J^N = 0 \,, \tag{233}$$

which is precisely the continuity equation for the current.

The continuity equation is essential to ensure that the matter-field coupling term of the action $\int A_M dX^M = \int [\mathrm{D}X]\, J^M A_M$ is also gauge invariant, which can be readily verified after an integration by parts and setting the boundary terms to zero:

$$\delta \int [\mathrm{D}X] J^M A_M = \int [\mathrm{D}X] J^M \partial_M \Lambda = \\ = -\int [\mathrm{D}X] (\partial_M J^M) \Lambda = 0. \tag{234}$$

Gauge invariance also ensures the conservation of the energy-momentum (via Noether's theorem) defined in terms of the Lagrangian density variation. We refer to [75] for further details.

The gauge invariant C-space Maxwell action as given in eq.-(222) is in fact only a part of a more general action given by the expression

$$I[A] = \int [\mathrm{D}X]\, F^\dagger * F = \int [\mathcal{D}X] \; < F^\dagger F >_{scalar} \,. \tag{235}$$

This action can also be written in terms of components, up to dimension-dependent *numerical* coefficients, as [75]:

$$I[A] = \int [\mathcal{D}X] \; (F_{(MN)} F^{(MN)} + F_{[MN]} F^{[MN]}) \,. \tag{236}$$

For rigor, one should introduce the numerical coefficients in front of the F terms, noticing that the symmetric combination should have a different dimension-dependent coefficient than the anti-symmetric combination since the former involves contractions of $\{E^M, E^N\}^* \{E_M, E_N\}$ and the latter contractions of $[E^M, E^N]^* [E_M, E_N]$.

The latter action is strictly speaking not gauge invariant, since it contains not only the antisymmetric but also the symmetric part of F. It is invariant under a *restricted* gauge symmetry transformations. It is invariant (up to total derivatives) under *infinitesimal* gauge transformations provided the symmetric part of F is divergence-free $\partial_M F^{(MN)} = 0$ [75]. This divergence-free condition has the same effects as if one were fixing a gauge leaving a *residual* symmetry of *restricted* gauge transformations such that the gauge symmetry parameter obeys the Laplace-like equation $\partial_M \partial^M \Lambda = 0$. Such residual (restricted) symmetries are precisely those that leave invariant the divergence-free condition on the symmetric part of F. Residual, restricted symmetries occur, for example, in the light-cone gauge of p-brane actions leaving a residual symmetry of *volume*-preserving diffs. They also occur in string theory when the conformal gauge is chosen leaving a residual symmetry under conformal reparametrizations; i. e. the so-called Virasoro algebras whose symmetry transformations are given by holomorphic and anti-holomorphic reparametrizations of the string world-sheet.

This Laplace-like condition on the gauge parameter is also the one required such that the action in [75] is invariant under *finite* (restricted) gauge transformations since under such restricted finite transformations the Lagrangian changes by second-order terms of the form $(\partial_M \partial_N \Lambda)^2$, which are total derivatives if, and only if, the gauge parameter is restricted to obey the analog of Laplace equation $\partial_M \partial^M \Lambda = 0$

Therefore the action of eq-(233) is invariant under a *restricted* gauge transformation which bears a resemblance to *volume*-preserving diffeomorphisms of the p-branes action in the light-cone gauge. A lesson that we have from these considerations is that the C-space Maxwell action written in the form (235) automatically contains a gauge fixing term. Analogous result for *ordinary* Maxwell field is known from Hestenes work [1], although formulated in a slightly different way, namely by directlty considering the field equations without employing the action.

It remains to be seen if this construction of C-space generalized Maxwell Electrodynamics of p-forms can be generalized to the non-Abelian case when we replace ordinary derivatives by gauge-covariant ones:

$$F = dA \to F = DA = (dA + A \bullet A). \tag{237}$$

For example, one could define the graded-symmetric product $E_M \bullet E_N$ based on the graded commutator of Superalgebras:

$$[A, B] = AB - (-1)^{s_A s_B} BA\,, \tag{238}$$

s_A, s_B is the grade of A and B respectively. For bosons the grade is even and for fermions is odd. In this fashion the graded commutator captures both the anti-commutator of two fermions and the commutator of two bosons in one stroke. One may extend this graded bracket definition to the graded structure present in Clifford algebras, and define

$$E_M \bullet E_N = E_M E_N - (-1)^{s_M s_N} E_N E_M\,, \tag{239}$$

s_M, s_N is the grade of E_M and E_N respectively. Even or odd depending on the grade of the basis elements.

One may generalize Maxwell's theory to Born-Infeld nonlinear Electrodynamics in C-spacesbased on this extension of Maxwell Electrodynamics in C-spaces and to couple a C-space version of a Yang-Mills theory to C-space gravity, a higher derivative gravity with torsion, this will be left for a future publication. Clifford algebras have been used in the past [62] to study the Born-Infeld model in ordinary spacetime and to write a nonlinear version of the Dirac equation. The natural incorporation of monopoles in Maxwell's theory was investigated by [89] and a recent critical analysis of "unified" theories of gravity with electromagnetism has been presented by [90]. Most recently [22] has studied the covariance of Maxwell's theory from a Clifford algebraic point of view.

8 Concluding remarks

We have presented a brief review of some of the most important features of the Extended Relativity theory in Clifford-spaces (C-spaces). The "coordinates" X are non-commutating Clifford-valued quantities which incorporate the lines, areas, volumes, ... degrees of freedom associated with the collective particle, string, membrane, ... dynamics underlying the center-of-mass motion and holographic projections of the p-loops onto the embedding target spacetime backgrounds. C-space Relativity incorporates the idea of an invariant length, which upon quantization, should lead to the notion of minimal Planck scale [23]. Other relevant features are those of maximal acceleration [52], [49]; the invariance of Planck-areas under acceleration boosts; the resolution of ordering ambiguities in QFT; supersymmetry; holography [119]; the emergence of higher derivative gravity with torsion; and the inclusion of variable dimensions/signatures that allows to study the dynamics of all (closed) p-branes, for all values of p, in one single unified footing, by starting with the C-space brane action constructed in this work.

The Conformal group construction presented in sect. 7, as a natural subgroup of the Clifford group in four-dimensions, needs to be generalized to other dimensions, in particular to two dimensions where the Conformal group is infinite-dimensional. Kinani [130] has shown that the Virasoro algebra can be obtained from generalized Clifford algebras. The construction of area-preserving diffs algebras, like w_∞ and $su(\infty)$, from Clifford algebras remains an open problem. Area-preserving diffs algebras are very important in the study of membranes and gravity since Higher-dim Gravity in $(m+n)$-dim has been shown a while ago to be equivalent to a lower m-dim Yang-Mills-like gauge theory of diffs of an internal n-dim space [120] and that amounts to another explanation of the holographic principle behind the AdS/CFT duality conjecture [121]. We have shown how C-space

Relativity involves scale changes in the sizes of physical objects, in the absence of forces and Weyl'gauge field of dilations. The introduction of scale-motion degrees of freedom has recently been implemented in the wavelet-based regularization procedure of QFT by [87]. The connection to Penrose's Twistors program is another interesting project worthy of investigation.

The quantization and construction of QFTs in C-spaces remains a very daunting task since it may involve the construction of QM in Noncommutative spacetimes [136], braided Hopf quantum Clifford algebras [86], hypercomplex extensions of QM like quaternionic and octonionic QM [99], [97], [98], exceptional group extensions of the Standard Model [85], hyper-matrices and hyper-determinants [88], multi-symplectic mechanics, the de Donde-Weyl formulations of QFT [82], to cite a few, for example. The quantization program in C-spaces should share similar results as those in Loop Quantum Gravity [111], in particular the minimal Planck areas of the expectation values of the area-operator.

Spacetime at the Planck scale may be discrete, fractal, fuzzy, noncommutative. . . The original Scale Relativity theory in fractal spacetime [23] needs to be extended further to incorporate the notion of fractal "manifolds". A scale-fractal calculus and a fractal-analysis construction that are esential in building the notion of a fractal "manifold" has been initiated in the past years by [129]. It remains yet to be proven that a scale-fractal calculus in fractal spacetimes is another realization of a Connes Noncommutative Geometry. Fractal strings/branes and their spectrum have been studied by [104] that may require generalized Statistics beyond the Boltzmann-Gibbs, Bose-Einstein and Fermi-Dirac, investigated by [105], [103], among others.

Non-Archimedean geometry has been recognized long ago as the natural one operating at the minimal Planck scale and requires the use p-adic numbers instead of ordinary numbers [101]. By implementing the small/large scale, ultraviolet/infrared duality principle associated with QFTs in Noncommutative spaces, see [125] for a review, one would expect an upper maximum scale [23] and a maximum temperature [21] to be operating in Nature. Non-Archimedean Cosmologies based on an upper scale has been investigated by [94].

An upper/lower scale can be accomodated simultaneously and very naturally in the q-Gravity theory of [114], [69] based on bicovariant quantum group extensions of the Poincaré, Conformal group, where the q deformation parameter could be equated to the quantity $e^{\Lambda/L}$, such that both $\Lambda = 0$ and $L = \infty$, yield the same classical $q = 1$ limit. For a review of q-deformations of Clifford algebras and their generalizations see [86], [128].

It was advocated long ago by Wheeler and others, that information theory [106], set theory and number theory, may be the ultimate physical theory. The important role of Clifford algebras in information theory have been known for some time [95]. Wheeler's spacetime foam at the Planck scale may be the background source generation of Noise in the Parisi-Wu stochastic quantization [47] that is very relevant in Number theory [100]. The pre-geometry cellular-networks approach of [107] and the quantum-topos views based on gravitational quantum causal sets, noncommutative topology and category theory [109], [110], [124] deserves a further study within the C-space Relativity framework, since the latter theory also invokes a Category point of view to the notion of dimensions. C-space is a pandimensional continuum [14], [8]. Dimensions are topological invariants and, since the dimensions of the extended objects change in C-space, topology-change is another ingredient that needs to be addressed in C-space Relativity and which may shed some light into the physical foundations of string/M theory [118]. It has been speculated that the universal symmetries of string theory [108] may be linked to Borcherds Vertex operator algebras (the Monstruous moonshine) that underline the deep interplay between Conformal Field Theories and Number theory. A lot remains to be done to bridge together these numerous branches of physics and mathematics. Many surprises may lie ahead of us. For a most recent discussion on the path towards a Clifford-Geometric Unified Field theory of all forces see [138], [140]. The notion of a Generalized Supersymmetry in Clifford Superspaces as extensions of M, F theory algebras was recently advanced in [139].

Acknowledgements

We are indebted to M. Bowers and C. Handy at the CTSPS (Atlanta) for support. The work of M. P. has been supported by the Ministry of Education, Science and Sport of Slovenia.

References

1. Hestenes D. Spacetime algebra. Gordon and Breach, New York, 1996. Hestenes D. and Sobcyk G. Clifford algebra to geometric calculus. D. Reidel Publishing Company, Dordrecht, 1984.
2. Porteous I. R. Clifford algebras and the classical groups. Cambridge University Press, 1995.
3. Baylis W. Electrodynamics, a modern geometric approach. Boston, Birkhauser, 1999.
4. Trayling G. and Baylis W. *J. Phys.*, v. A34, 2001, 3309–3324. *Int. J. Mod. Phys.*, v. A16S1C, 2001, 909–912.
5. Jancewicz B. Multivectors and clifford algebra in electrodynamics. World Scientific, Singapore, 1989.
6. Clifford algebras and their applications in mathematical physics. Vol 1: Algebras and physics. Edited by R. Ablamowicz, B. Fauser. Vol 2: Clifford analysis. Edited by J. Ryan, W. Sprosig. Birkhauser, Boston, 2000.
7. Lounesto P. Clifford algebras and spinors. Cambridge University Press, Cambridge, 1997.

8. Castro C. *Chaos, Solitons and Fractals*, v. 10, 1999, 295. *Chaos, Solitons and Fractals*, v. 12, 2001, 1585. The search for the origins of M Theory: Loop Quantum Mechanics, loops/strings and bulk/boundary dualities. arXiv: hep-th/9809102.
9. Castro C. The programs of the Extended Relativity in C-spaces, towards physical foundations of String Theory. *Advance NATO Workshop on the Nature of Time, Geometry and the Physics of Perception*, Tatranksa Lomnica, Slovakia, May 2001, Kluwer Publishers, 2003.
10. Castro C. *Chaos, Solitons and Fractals*, v. 11, 2000, 1663. *Foundations of Physics*, v. 30, 2000, 1301.
11. Amati D., Ciafaloni M. and Veneziano G. *Phys. Letters*, v. B197, 1987, 81. Gross D., Mende P. *Phys. Letters*, v. B197, 1987, 129. Maggiore M. *Phys. Letters*, v. B304, 1993, 65.
12. Ansoldi S., Castro C. and Spallucci E. *Class. Quant. Grav.*, v. 18, 1999, 1833. Castro C. *Chaos, Solitons and Fractals*, v. 11, 2000, 1721.
13. Hosotani Y. *Phys. Rev. Letters*, v. 55, 1985, 1719. Carson L., Ho C. H. and Hosotani Y. *Phys. Rev.*, v. D37, 1988, 1519. Ho C. H. *Jour. Math. Phys.*, v. 30, 1989, 2168.
14. Pezzaglia W. Physical applications of a generalized geometric calculus. arXiv: gr-qc/9710027. Dimensionally democratic calculus and principles of polydimensional physics. arXiv: gr-qc/9912025. Classification of multivector theories and modifications of the postulates of physics. arXiv: gr-qc/9306006. Physical applications of generalized Clifford Calculus: Papatetrou equations and metamorphic curvature. arXiv: gr-qc/9710027. Classification of multivector theories and modification of the postulates of physics. arXiv: gr-qc/9306006.
15. Pavšič M. The landscape of theoretical physics: a global view; from point particle to the brane world and beyond, in search of unifying principle. Kluwer Academic, Dordrecht, 2001.
16. Pavšič M. *Foundations of Physics*, v. 31, 2001, 1185.
17. Pavšič M. *Foundations of Physics*, v. 33, 2003, 1277. See also arXiv: gr-qc/0211085.
18. Ansoldi S., Aurilia A., Castro C., Spallucci E. *Phys. Rev.*, v. D64, 026003, 2001.
19. Aurilia A., Ansoldi S. and Spallucci E. *Class. Quant. Grav.*, v. 19, 2002, 3207. See also arXiv: hep-th/0205028.
20. Castro C., Pavšič M. *Phys. Letters*, v. B539, 2002, 133.
21. Castro C. *Jour. of Entropy*, v. 3, 2001, 12–26. Castro C. and Granik A. *Foundations of Physics*, v. 33 (3), 2003, 445.
22. Ivezic T. *Foundations of Physics Letters*, v. 15, 2002, 27. Invariant Relativistic Electrodynamics, Clifford algebra approach. arXiv: hep-th/0207250.
23. Nottale L. Fractal spacetime and microphysics, towards the Theory of Scale Relativity. World Scientific, Singapore, 1992. La Relativité dans tous ses états: au-delà de l'espace-temps. Hachette Literature, 1998, Paris, 319 pages.
24. Ansoldi S., Aurilia A. and Spallucci E. *Phys. Rev.*, v. D56, No. 4, 1997, 2352. *Chaos, Solitons and Fractals*, v. 10, 1999, 197.
25. Castro C. and Pavšič M. *Int. J. Theor. Phys.*, v. 42, 2003, 1693.
26. Pavšič M. *Class. Quant. Grav.*, v. 20, 2003, 2697. See also arXiv:gr-qc/0111092.
27. Pavšič M. *Nuovo Cimento*, v. B41, 1977, 397. *International Journal of Theoretical Physics*, v. 14, 1975, 299.
28. Pavšič M. *J. Phys.* v. A13, 1980, 1367.
29. Pavšič M. *Found. Phys.*, v. 26, 1996, 159. See also arXiv:gr-qc/9506057.
30. Pavšič M. "Clifford algebra, geometry and physics. arXiv: gr-qc/0210060.
31. Terletsky J. P. *Doklady Akad. Nauk USSR*, v. 133, 1960, 329. Bilaniuk M. P., Deshpande V. K. and Sudarshan E. C. G. *American Journal of Physics*, v. 30, 1962, 718. Recami E. and Mignani R. *Rivista del Nuovo Cimento*, v. 4, 1974, 209. Recami E. *Rivista del Nuovo Cimento*, v. 9, 1986, 1.
32. Pavšič M. *J. Phys.*, v. A14, 1981, 3217.
33. Stueckelberg E. C. G. *Helv. Phys. Acta*, v. 14, 1941, 322; v. 14, 1941, 588; v. 15, 1942, 23.
34. Horwitz L. P. and Piron C. *Helv. Phys. Acta*, v. 46, 1973, 316. Horwitz L. P. and Rohrlich F. *Physical Review D*, v. 24, 1981, 1528; v. 26, 1982, 3452. Horwitz L. P., Arshansky R. I. and Elitzur A. C. *Found. Phys.*, v. 18, 1988, 1159. Arshansky R. I., Horwitz L. P. and Lavie Y. *Foundations of Physics*, v. 13, 1983, 1167.
35. Pavšič M. *Found. Phys.*, v. 21, 1991, 1005. *Nuovo Cimento*, v. A104, 1991, 1337. *Doga, Turkish Journ. Phys.*, v. 17, 1993, 768. *Found. Phys.*, v. 25, 1995, 819. *Nuovo Cimento*, v. A108, 1995, 221. *Foundations of Physics*, v. 26, 1996, 159. Pavšič M. *Nuovo Cimento*, v. A110, 1997, 369. *Found. Phys.*, v. 28, 1998, 1443. *Found. Phys.*, v. 28, 1998, 1453.
36. Misner C. W., Thorne K. S. and Wheeler J. A. Gravitation. Freeman, San Francisco, 1973, p. 1215.
37. Schild A. *Phys. Rev.*, v. D16, 1977, 1722.
38. Kastrup H. A. *Annalen der Physik (Lpz.)*, B. 7, 1962, 388.
39. Barut A. O. and Haugen R. B. *Annals of Physics*, v. 71, 1972, 519.
40. Cunningham E. *Proc. London Math. Soc.*, v. 8, 1909, 77. Bateman H. *Proc. London Math. Soc.*, v. 8, 1910, 223. Fulton T. Rohrlich F. and Witten L. *Rev. Mod. Phys.*, v. 34, 1962, 442. Wess J. *Nuovo Cimento*, v. 18, 1960, 1086. Mack G. *Nucl. Phys.*, v. B5, 1968, 499. Grillo A. F. *Riv. Nuovo Cim.*, v. 3, 1973, 146. Niederle J. and Tolar J. *Czech. J. Phys.*, v. B23, 1973, 871.
41. Hestenes D. and Ziegler R. Projective geometry with Clifford algebra. *Acta Applicandae Mathematicae*, v. 23, 1991, 25–63.
42. Ahluwalia D. V. and Burgard C. *General Relativity and Gravitation*, v. 28 (10), 1996, 1163. Ahluwalia D. V. *General Relativity and Gravitation*, v. 29 (12), 1997, 1491. Ahluwalia D. V. *Phys. Letters*, v. A275, 2000, 31. Adunas G., Rodriguez-Milla E. and Ahluwalia D. V. *Phys. Letters*, v. B485, 2000, 215.
43. Smith F. T. *Int. Jour. of Theoretical Physics*, v. 24, 1985, 155. *Int. Jour. of Theoretical Physics*, v. 25, 1985, 355. From Sets to Quarks arXiv: hep-ph/9708379.

44. Cooperstock F. I., Faraoini V. Extended Planck Scale. arXiv: hep-th/0302080.

45. Arcos H. I. and Pereira J. G. Kerr–Newman solutions as a Dirac particle. arXiv: hep-th/0210103.

46. Smilga W. Higher order terms in the contraction of $SO(3,2)$. arXiv: hep-th/0304137. Wyler A. *C. R. Acad. Sc. Paris*, v. 269, 1969, 743–745.

47. Beck C. Spatio-temporal vacuum fluctuations of quantized fields. *World Scientific Series in Advances in Nonlinear Dynamics*, v. 21, 2002.

48. Trayling G. and Baylis W. *J. Phys.*, v. A34, 2001, 3309. Chisholm J. and Farwell R. *J. Phys.*, v. A32, 1999, 2805. Chisholm J. and Farwell R. *Foundations of Physics*, v. 25, 1995, 1511.

49. Born M. *Proc. Royal Society*, v. A165, 1938, 291. *Rev. Mod. Physics*, v. 21, 1949, 463.

50. Clifford (geometric) algebras, with applications to physics, mathematics and engineering. Edited by W. E. Baylis. Birkhauser, Boston, 1997.

51. Somaro S. Higher spin and the spacetime algebra. Edited by V. Dietrich et al. *Clifford algebras and their applications in mathematical physics*, Kluwer Academic Publishers, the Netherlands, 1998, 347–368.

52. Caianiello E. Is there a maximal acceleration? *Lett. Nuovo Cimento*, v. 32, 1981, 65.

53. Nesterenko V. *Class. Quant. Grav.*, v. 9, 1992, 1101. *Phys. Lett.*, v. B327, 1994, 50.

54. Bozza V., Feoli A., Lambiase G., Papini G. and Scarpetta G. *Phys. Letters*, v. A283, 2001, 53. Nesterenko V., Feoli A., Lambiase G. and Scarpetta G. *Phys. Rev.*, v. D60, 065001, 1999.

55. Rama K. Classical velocity in kappa-deformed Poincaré algebra and maximal acceleration. arXiv: hep-th/0209129.

56. Brandt H. *Contemporary Mathematics*, v. 196, 1996, 273. *Chaos, Solitons and Fractals*, v. 10 (2–3), 1999, 267.

57. Pavšič M. *Phys. Letters*, v. B205, 1988, 231. *Phys. Letters*, v. B221, 1989, 264. Arodz H., Sitarz A., Wegrzyn P. *Acta Physica Polonica*, v. B20, 1989, 921.

58. Plyushchay M. Comment on the relativistic particle with curvature and torsion of world trajectory. arXiv: hep-th/9810101. *Mod. Phys. Lett.*, v. A10, 1995, 1463–1469.

59. Ramos E., Roca J. *Nucl. Phys.*, v. B477, 1996, 606–622.

60. Kleinert H., Chervyakov A. M. Evidence for negative stiffness of QCD Strings. arXiv: hep-th/9601030.

61. Schuller F. *Annals of Phys.*, v. 299, 2002, 174.

62. Chernitskii A. Born–Infeld electrodynamics, Clifford numbers and spinor representations. arXiv: hep-th/0009121.

63. Lukierski J., Nowicki A., Ruegg H., Tolstoy V. *Phys. Letters*, v. 264, 1991, 331. Lukierski J., Ruegg H., Zakrzewski W. *Ann. Phys.*, v. 243, 1995, 90. Lukierski J., Nowicki A. Double Special Relativity verus kappa-deformed dynamics. arXiv: hep-th/0203065.

64. Webb J., Murphy M., Flambaum V., Dzuba V., Barrow J., Churchill C., Prochaska J., and Wolfe A. Further evidence for cosmological evolution of the Fine Structure Constant. *Monthly Notices of the Royal Astronomical Society*, v. 327, 2001, 1208.

65. Uzan J. P. The fundamental constants and their variations: observational status and theoretical motivations. arXiv: hep-ph/0205340.

66. Anchordogui L., Goldberg H. *Phys. Rev.*, v. D68, 2003, 083513.

67. Lambiase G., Papini G., Scarpetta G. Maximal acceleration corrections to the Lamb shift of one electron atoms. arXiv: hep-th/9702130. Lambiase G., Papini G., Scarpetta G. *Phys. Letters*, v. A224, 1998, 349. Papini G. Shadows of a maximal acceleration. arXiv: gr-qc/0211011.

68. Castro C. *Int. J. Mod. Phys.*, v. A18, 2003, 5445. See also arXiv: hep-th/0210061.

69. Castellani L. *Phys. Letters*, v. B327, 1994, 22. *Comm. Math. Phys.*, v. 171, 1995, 383.

70. Amelino-Camelia G. *Phys. Letters*, v. B510, 2001, 255. *Int. J. Mod. Phys.*, v. D11, 2002, 35. *Int. J. Mod. Phys.*, v. D11, 2002, 1643.

71. Greisen K. *Phys. Rev. Letters*, v. 16, 1966, 748. Zatsepin G. T., Kurmin V. *Sov. Phys. JETP Letters*, v. 4, 1966, 78.

72. Ellis J., Mavromatos N. and Nanopolous D. V. *Chaos, Solitons and Fractals*, v. 10, 1999, 345.

73. Toller M. The geometry of maximal acceleration. arXiv: hep-th/0312016.

74. Low S. *Jour. Phys. A Math. Gen.* v. 35, 2002, 5711.

75. Castro C. *Mod. Phys. Letts.*, v. A19, 2004, 19.

76. Choquet-Bruhat Y., DeWitt-Morette C. and Dillard-Bleick M. Analysis, manifolds and physics. Revised edition, North Holland Publ. Co., Amsterdam, 1982.

77. Crumeyrole A. Orthogonal and sympletic Clifford algebras. Kluwer Acad. Publ., Dordrecht, 1990.

78. Blaine Lawson H. and Michelson M. L. Spin geometry. Princeton University Press, Princeton, 1980.

79. Moya A. M., Fernandez V. V. and Rodrigues W. A. *Int. J. Theor. Phys.*, v. 40, 2001, 2347–2378. See also arXiv: math-ph/0302007. Multivector functions of a multivector variable. arXiv: math.GM/0212223. Multivector functionals. arXiv: math.GM/0212224.

80. Castro C. The charge-mass-spin relationship of a Clifford polyparticle, Kerr–Newman black holes and the Fine Structure Constant. To appear in *Foundations of Physics*.

81. Vacaru S. and Vicol N. Nonlinear connections and Clifford structures. arXiv: math.DG/0205190. Vacaru S. (Non) Commutative Finsler geometry from String/M Theory. arXiv: hep-th/0211068. Vacaru S., Nadejda A. *Int. J. Math. Sci.*, v. 23, 2004, 1189–1233. Vacaru S. Clifford structures and spinors on spaces with local anisotropy. *Buletinul Academiei de Stiinte a Republicii Moldova, Fizica si Tehnica* (Izvestia Academii Nauk Respubliky Moldova, fizica i tehnika), v. 3, 1995, 53–62. Vacaru S. Superstrings in higher order extensions of Finsler superspaces. *Nucl. Phys.*, v. B434, 1997, 590–656. Vacaru S. and Stavrinos P. Spinors and space-time

anisotropy. Athens University Press, Athens, Greece, 2002, 301 pages. See also arXiv: gr-qc/0112028.

82. Kanatchikov I. *Rep. Math. Phys.*, v. 43, 1999, 157. Kisil V. arXiv: quant-ph/0306101.
83. Bjerrus-Bohr N. Quantum Gravity at large number of dimensions. arXiv: hep-th/0310263. Donoghue J. F. *Phys. Rev.*, v. D50, 1994, 3874. Strominger A. *Phys. Rev.*, v. D24, 1981, 3082.
84. Rodrigues Jr W. A., Vaz Jr J. *Adv. Appl. Clifford Algebras*, v. 7, 1997, 457–466. De Oliveira E. C. and Rodrigues Jr W. A. *Ann. der Physik*, v. 7, 1998, 654–659. *Phys. Letters*, v. A291, 2001, 367–370. Rodrigues Jr W. A., Lu J. Y. *Foundations of Physics*, v. 27, 1997, 435-508.
85. Ramond P. Exceptional groups and physics. arXiv: hep-th/0301050.
86. Fauser B. A treatise on Quantum Clifford Algebras. arXiv: math.QA/0202059.
87. Altaisky M. Wavelet based regularization for Euclidean field theory and stochastic quantization. arXiv: hep-th/0311048.
88. Tapia V. Polynomial identities for hypermatrices. arXiv: math-ph/0208010.
89. Defaria-Rosa M., Recami E. and Rodrigues Jr W. A. *Phys. Letters*, v. B173, 1986, 233.
90. Capelas de Oliveira E. and Rodrigues Jr W. A. Clifford valued differential forms, algebraic spinor fields, gravitation, electromagnetism and unified theories. arXiv: math-ph/0311001.
91. Lanczos C. *Z. Physik*, B. 57, 1929, 447–473. *Z. Physik*, B. 57, 1929, 474-483. *Z. Physik*, B. 57, 1929, 484–493. Lanczos C. *Physikalische Zeitschrift*, B. 31, 1930, 120–130.
92. Gursey F. Applications of quaternions to field equations. Ph. D. thesis, University of London, 1950, 204 pages.
93. Mosna R. A. and Rodrigues Jr W. A. *J. Math. Phys.*, v. 45, 2004, 7. See also arXiv: math-ph/0212033. Rodrigues Jr W. A. *J. Math. Phys.*, v. 45, 2004, (to appear). See also arXiv: math-ph/0212030.
94. Avinash K. and Rvachev V. I. *Foundations of Physics*, v. 30, 2000, 139.
95. Gottesman D. The Heisenberg representation of Quantum Computers. arXiv: quant-ph/9807006.
96. Baugh J., Finkelstein D., Galiautdinov A. and Shiri-Garakaui M. *Found. Phys.*, v. 33, 2003, 1267–1275.
97. Adler S. L. Quaternionic Quantum Mechanics and quantum fields. Oxford Univ. Press, New York, 1995.
98. De Leo S. and Abdel-Khalek K. Towards an octonionic world. arXiv:hep-th/9905123. De Leo S. Hypercomplex group theory. arXiv: physics/9703033.
99. Castro C. Noncommutative QM and geometry from quantization in C-spaces. arXiv: hep-th/0206181.
100. Watkins M. Number theory and physics, the website, http://www. maths.ex.ac.uk/mwatkins.
101. Vladimorov V., Volovich I. and Zelenov I. P-adic numbers in mathematical physics. World Scientific, Singapore, 1994. Brekke L. and Freund P. *Phys. Reports*, v. 231, 1991, 1. Pitkanen M. Topological geometrodynamics, the website, http://www.physics.helsinki.fi/matpitka/tgd.html.
102. Yang C. N. *Phys. Rev.*, v. 72, 1947, 874. Snyder H. S., *Phys. Rev.*, v. 71, 1947, 38. Snyder H. S. *Phys. Rev.*, v. 72, 1947, 68. Tanaka S. Yang's quantized spacetime algebra and holographic hypothesis. arXiv: hep-th/0303105.
103. Da Cruz W. Fractal von Neumann entropy. arXiv: cond-mat/0201489. Fractons and fractal statistics. arXiv: hep-th/9905229.
104. Castro C. *Chaos, Solitons and Fractals*, v. 14, 2002 1341. *Chaos, Solitons and Fractals*, v. 15, 2003, 797. Lapidus M. and Frankenhuysen M. Fractal strings, complex dimensions and the zeros of the zeta function. Birkhauser, New York, 2000.
105. Castro C. The nonextensive statistics of fractal strings and branes. To appear in *Physica A*. Havrda J. and Charvat F. *Kybernatica*, v. 3, 1967, 30. Tsallis C. *Jour. of Statistical Physics*, v. 52, 1988, 479. Tsallis C. Non-extensive statistical mechanics: a brief review of its present status. arXiv: cond-mat/0205571.
106. Frieden D. Physics from Fisher information theory. Cambridge University Press, Cambridge, 1998.
107. Nowotny T., Requardt M. *Chaos, Solitons and Fractals*, v. 10, 1999, 469.
108. Lizzi F., Szabo R. J. *Chaos, Solitons and Fractals*, v. 10, 1999, 445.
109. Raptis I. Quantum space-time as a quantum causal set. arXiv gr-qc/0201004. Presheaves, sheaves and their topoi in Quantum Gravity and Quantum Logic. arXiv: gr-qc/0110064. Non-commutative topology for Curved Quantum Causality. arXiv: gr-qc/0101082.
110. Isham C. and Butterfield J. *Found. Phys.*, v. 30, 2000, 1707–1735. Guts A. K., Grinkevich E. B. Toposes in General Theory of Relativity. arXiv: gr-qc/9610073.
111. Ashtekar A., Rovelli C. and Smolin L. *Phys. Rev. Letters*, v. 69, 1992, 237. Rovelli C. A dialog on Quantum Gravity. arXiv: hep-th/0310077. Freidel L., Livine E. and Rovelli C. *Class. Quant. Grav.*, v. 20, 2003, 1463–1478. Smolin L. How far are we from the quantum theory of gravity? arXiv: hep-th/0303185.
112. Saniga M. *Chaos, Solitons and Fractals*, v. 19, 2004, 739–741.
113. Norma Mankoc Borstnik and Nielsen H. B. *J. Math. Phys.*, v. 44, 2003, 4817–4827.
114. Castellani L. *Class. Quant. Grav.*, v. 17, 2000, 3377–3402. *Phys. Lett.*, v. B327, 1994, 22–28.
115. Guendelman E., Nissimov E. and Pacheva S. Strings, p-branes and Dp-branes with dynamical tension. arXiv: hep-th/0304269.
116. Bouwknegt P., Schouetens K. *Phys. Reports*, v. 223, 1993, 183–276. Sezgin E. Aspects of W_∞ Symmetry. arXiv: hep-th/9112025. Shen X. *Int. J. Mod. Phys.*, v. A7, 1992, 6953–6994.

117. Vasiliev M. Higher spin gauge theories, star product and AdS spaces. arXiv: hep-th/9910096. Vasiliev M., Prokushkin S. Higher-spin gauge theories with matter. arXiv: hep-th/9812242, hep-th/9806236.
118. Ne'eman Y., Eizenberg E. Membranes and other extendons (p-branes). *World Scientific Lecture Notes in Physics*, v. 39, 1995. Polchinski J. Superstrings. Cambridge University Press, Cambridge, 2000. Green M., Schwarz J. and Witten E. Superstring Theory. Cambridge University Press, Cambridge, 1986.
119. Maldacena J. *Adv. Theor. Math. Phys.*, v. 2, 1998, 231.
120. Cho Y., Soh K., Park Q., Yoon J. *Phys. Letters*, v. B286, 1992, 251. Yoon J. *Phys. Letters*, v. B308, 1993, 240. Yoon J. *Phys. Letters*, v. A292, 2001, 166. Yoon J. *Class. Quant. Grav.*, v. 16, 1999, 1863.
121. Castro C. *Europhysics Letters*, v. 61 (4), 2003, 480. *Class. Quant. Gravity*, v. 20, No. 16, 2003, 3577.
122. Zamoldchikov A. B. *Teor. Fiz.*, v. 65, 1985, 347. Pope C. et al. *Nucl. Phys.*, v. B413, 1994, 413–432. Pope C., Romans L., Shen X. *Nucl. Phys.*, v. B339, 1990, 191. Pope C., Romans L., Shen X. *Phys. Letters*, v. B236, 1990, 173. Pope C., Romans L., Shen X. *Phys. Letters*, v. B242, 1990, 401.
123. Dolan B., Tchrakian. *Phys. Letters*, v. B202 (2), 1988, 211.
124. Hawkins E., Markopoulou F. and Sahlmann H. Evolution in quantum causal histories. arXiv: hep-th/0302111. Markoupolu F., Smolin L. Quantum theory from Quantum Gravity. arXiv: gr-qc/0311059.
125. Douglas M., Nekrasov N. *Rev. Mod. Phys.*, v. 73, 2001, 977–1029.
126. Eguchi T. *Phys. Rev. Lett.*, v. 44, 1980, 126.
127. Castro C. Maximal-acceleration phase space relativity from Clifford algebras. arXiv: hep-th/0208138.
128. Abramov V., Kerner R. and Le Roy B. *J. Math. Phys.*, v. 38, 1997, 1650–1669
129. Cresson J. Scale calculus and the Schrodinger equation. arXiv: math.GM/0211071.
130. Kinani E. H. Between quantum Virasoro algebras and generalized Clifford algebras. arXiv: math-ph/0310044.
131. Capozziello S., Carloni S. and Troisi A. Quintessence without scalar fields. arXiv: astro-ph/0303041. Carroll S., Duvvuri V., Trodden M. and Turner M. Is cosmic speed-up due to new gravitational physics? arXiv: astro-ph/0306438. Lue A., Scoccimarro R. and Strakman G. Differentiating between Modified Gravity and Dark Energy. arXiv: astro-ph/0307034.
132. Aurilia A., Smailagic A. and Spallucci E. *Physical Review*, v. D47, 1993, 2536. Aurilia A. and Spallucci E. *Classical and Quantum Gravity*, v. 10, 1993, 1217. Aurilia A., Spallucci E. and Vanzetta I. *Physical Review*, v. D50, 1994, 6490. Ansoldi S., Aurilia A. and Spallucci E. *Physical Review*, v. D53, 1996, 870. Ansoldi S., Aurilia A. and Spallucci E. *Physical Review*, v. D56, 1997, 2352.
133. Mosna R. A., Miralles D. and Vaz Jr J. Z_2-gradings of Clifford algebras and multivector structures. arXiv: math-ph/0212020.
134. Castro C. On noncommutative Yang's space-time algebra, holography, area quantization and C-space Relativity. Submitted to *Eur. Physics Journal C*, see also CERN–CDS EXT–2004–090. Noncommutative branes in Clifford space backgrounds and Moyal–Yang star products with UV-IR cutoffs. Submitted to the *Jour. Math. Phys.*, 2005.
135. Armenta J., Nieto J. A. The de Sitter relativistic top theory. arXiv: 0405254. Nieto J. A. Chirotope concept in various scenarios of physics. arXiv: hep-th/0407093. Nieto J. A. Matroids and p-branes. *Adv. Theor. Math. Phys.*, v. 8, 2004, 177–188.
136. Yang C. N. *Phys. Rev.*, v. 72, 1947, 874. *Proc. of the Intern. Conference on Elementary Particles*, Kyoto, 1965, 322–323.
137. Tanaka S. *Nuovo Cimento*, v. 114B, 1999, 49. Tanaka S. Noncommutative field theory on Yang's space-time algebra, covariant Moyal star products and matrix model. arXiv: hep-th/0406166. Space-time quantization and nonlocal field theory. arXiv: hep-th/0002001. Yang's quantized space-time algebra and holographic hypothesis. arXiv: hep-th/0303105.
138. Castro C. On Dual Phase Space Relativity, the Machian Principle and Modified Newtonian Dynamics. *Progress in Physics*, 2005, v. 1, 20–30. The Extended Relativity Theory in Born–Clifford phase spaces with a lower and upper length scales and Clifford group geometric unification. Submitted to *Foundations of Physics*, 2004, see also CERN–CDS EXT–2004–128.
139. Castro C. Polyvector super-Poincaré algebras, M, F Theory algebras and generalized supersymmetry in Clifford spaces. Submitted to the *Int. Jour. Mod. Phys. A*, 2005.
140. Pavšič M. Kaluza–Klein theory without extra dimensions: curved Clifford space. arXiv: hep-th/0412255. Clifford spaces as a generalization of spacetime: prospects for QFT of point particles and strings. arXiv: hep-th/0501222. An alternative approach to the relation between bosons and fermions: employing Clifford space. arXiv: hep-th/0502067.

Rational Numbers Distribution and Resonance

Kyril Dombrowski*

This study solves a problem on the distribution of rational numbers along the number plane and number line. It is shown that the distribution is linked to resonance phenomena and also to stability of oscillating systems.

"God created numbers, all the rest has been created by Man...". With greatest esteem to Leopold Kronecker, one of the founders of the contemporary theory of numbers, it is impossible to agree with him in both the divine origin of number and Man's creation of mathematics. I propound herein the idea that numbers, their relations, and all mathematics in general are objective realities of our world. A part of science is not only understanding things, but also studying the relations that are objective realities in nature.

In this work I am going to consider a problem concerning the distribution of rational numbers along the number line and also in the number plane, and the relation of this distribution to resonance phenomena and stability of oscillating systems in low linear perturbations.

Any oscillating process involving at least two interacting oscillators is necessarily linked to abstract numbers — ratios between the oscillation periods. This fact displays a close relationship between such sections of science as the physical theory of oscillations and the abstract theory of numbers.

As is well known, the rational numbers are distributed on the number line everywhere compactly, so this problem statement that a function of their distribution exists might be thought false, as the case of prime numbers. But, as we will see below, it is not false — a rational numbers distribution function has an objective reality, manifest in numerous physical phenomena of Nature. This thesis will become clearer if we consider the "number lattice" introduced by Minkowski (Fig. 1). Therein are given all points of coordinates p and q which are related to numerators and denominators, respectively. If we exclude all points of the Minkowski lattice with coordinates have a common divisor different from unity, this plane will contain only "rational points" p/q (the non-cancelled fractions). Their distribution in the plane is defined by a sequence of numbers forming a rational series (Fig. 1).

This simplest drawing shows that rational numbers are distributed *inhomogeneously* in the Minkowski number plane. It is easy to see that this distribution is symmetric with respect to the axis $p = q$. Numbers of columns (and rows) in intervals, limited by this axis and one of the coordinate axes, are equal to Euler functions — the numbers less than m and relatively prime with m. Therefore, if we expand the number lattice infinitely, the *average density* of rational numbers in the plane (the ratio between the number of rational numbers and the

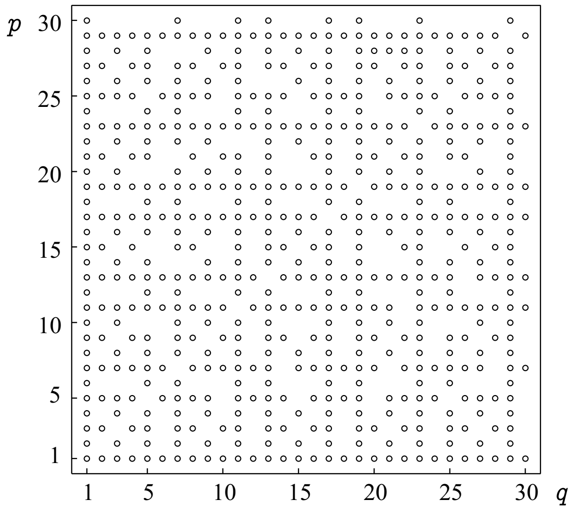


Fig. 1: The lattice of numbers (Minkowski's lattice).

number of all possible pairs of natural numbers the points of the lattice) approaches the limit

$$\lim_{N\to\infty} \frac{\mathrm{Ra}\left(N^2\right)}{N^2} = \lim_{N\to\infty} \frac{2}{N^2} \sum_{m=1}^{\infty} \varphi\left(m\right) = $$
$$= \frac{1}{\zeta\left(2\right)} = \frac{6}{\pi^2} = \left(\sum_{m=1}^{\infty} \frac{1}{n^2}\right)^{-1},$$

where N is the number of rational numbers, $\mathrm{Ra}\left(N^2\right)$ is the number of rational numbers located inside the square whose elements are of length equal to N, $\varphi\left(m\right)$ is Euler's function, $\zeta\left(n\right)$ is Riemann's zeta function, m and n are natural numbers. In particular, we can conclude from this that when $N < \infty$ the average density of rational numbers located in the plane is restricted to a very narrow interval of numerical values. It is possible to this verify by very simple calculations.

To study the problem of what is common to the rational number distribution and resonance phenomena it is necessary to have a one-dimensional picture of the function $\mathrm{Ra}\left(x\right)$ on the number line. In this problem, because the set of rational numbers is infinitely dense, we need to give a criterion for selecting a finite number of rational numbers which could give an objective picture of their distribution on the number line. We can do this in two ways. First, we can study, for instance, the distribution of rational number rays, drawn from the origin of coordinates in the Minkowski lattice. This

*Translated from the Russian by D. Rabounski and S. J. Crothers.

is a contemporary development of the method created by Klein [1]. Second, we can employ continued fractions, taking into account Khinchin's remark that "continued fractions... in their pure form display properties of the numbers they represent" [2]. So we can employ the mathematical apparatus of continued fractions as a systematic ground in order to find an analogous result that had been previously obtained by a purely arithmetical way.

We will use the second option because it is easier (although it is more difficult to imagine). So, let us plot points by writing a single-term continued fraction $1/n$ (so these are the numbers $1/1$, $1/2$, $1/3$, ...) inside an interval of unit length. We obtain thereby the best approximations of these numbers. This could be done inside every interval $1/(n+1) < x < 1/n$ by plotting points which are numerical values of a two-term continued fraction

$$\frac{1}{m+\frac{1}{n}} = \frac{n}{mn+1}\,.$$

These points, according to the theory of continued fractions, are the best approximations of the numbers $1/n$ from the left side.We then get the best approximations of the numbers $1/n$ from the right side, expressed by the fractions

$$\frac{1}{l+\frac{1}{1+\frac{1}{n}}} = \frac{n+1}{l\,(n+1)+n}\,, \qquad l, n, m = 1, 2, 3\dots\ .$$

We will call the approximation obtained the *first order approximation* (the second and third rank approximation in Khinchin's terminology). It is evident that every rational point of k-th order obtained in this way has analogous sequences of the $(k+1)$-th rank and higher. Such sequences fill the whole set of rational numbers.

To consider the simplest cases of resonance it would be enough to take the first order approximation, but to consider numerous processes such as colour vision, musical harmony, or Bohr's orbit distribution in atoms, requires a high order distribution function for rational numbers.

To obtain the function $\mathrm{Ra}\,(x)$ as a regular diagram we define this function (meaning the finite approximation order, the first order in this case) as a quantity in reverse to the interval between the neighbouring rational points located on the number line, where the points are plotted in the fashion of Khinchin, mentioned above. If the numerical values of the numbers l, m, n are limited, this interval is finite (see Fig. 2a). Such a drawing gives a possibility for estimating the structure of rational number distribution along the number line. In Fig. 2a we consider the distribution structure of rational numbers derived from a three-component continued fraction. For the purpose of comparison, Fig. 2b depicts a voltage function dependent on the stimulating frequency in an oscillating contour (drawn in the same scale as that in Fig. 2a). In this case an alternating signal frequency at a constant voltage was applied to the input of a resonance amplifier (an active LC-filter having a frequency of 1 kHz and the quality $Q = 17$). The frequency at the input was varied within the interval 200–1000 Hz through steps of 25 Hz. The average numerical value of the outgoing voltage was measured for two different voltages of the incoming signal — 0.75V and 1.25V.

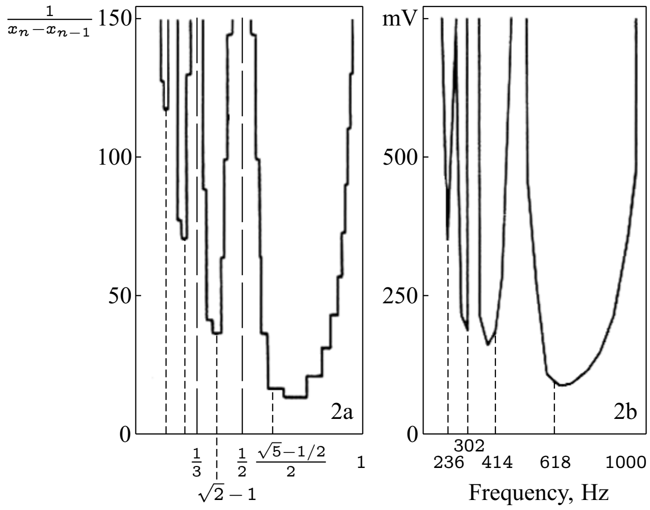


Fig. 2: The rational numbers distribution.

The apexes of both functions shown in the diagrams are located at the points plotted by the fractions $1/n$. This fact is trivial, because both apexes are actually analogous to Fourier-series expansions of white noise. Such experimental diagrams could be obtained in a purely theoretical way.

Much more interesting is the problem of the minimum numerical values of both functions. The classical theory of oscillations predicts that the minimum points should coincide with the minimum amplitude of forced oscillations, while according to the theory of continued fractions the minimum points should coincide with irrational numbers which, being the roots of the equation $x^2 \pm px - 1 = 0$ for all p, are approximated by rational numbers less accurately than by other numbers [2].

Direct calculations give the following numbers

$$M_1 = \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}} = \frac{\sqrt{5}\mp 1}{2} = (0.6180339\dots)^{\pm 1},$$

$$M_2 = \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}} = \frac{\sqrt{8}\mp 1}{2} = (0.4142135\dots)^{\pm 1},$$

....

$$M_n = \frac{1}{n+\frac{1}{n+\frac{1}{n+\dots}}} = \frac{\sqrt{n^2+4}\mp n}{2}\,.$$

In other words, the first conclusion is that the distribution of rational numbers, represented by continued fractions with

Table 1: Orbital radii of planets in the solar system in comparison with the calculated values of the radii $R_k = \left(\sqrt{n^2+4} \mp n\right)/2$

Planet	Real R_k	Calculated R_k	n	$\dfrac{R_{k(\mathrm{calc})}}{R_{k(\mathrm{real})}}$
Mercury	0.0744	0.0765	-13	1.0282
Venus	0.1390	0.1401	-7	1.0079
Earth	0.1922	0.1926	-5	1.0021
Mars	0.2929	0.3028	-3	1.0338
Asteroids	0.6180	0.6180	-1	1.0000
Jupiter	1.0000	1.0000	0	1.0000
Saturn	1.8334	1.6180	1	0.8825
Uranus	3.6883	3.3028	3	0.8955
Neptune	5.7774	5.1926	5	0.8988
Pluto	7.6398	7.1401	7	0.9346

Table 2: Orbital periods of planets in the solar system in comparison with the calculated values of the periods $T_k = \left(\sqrt{n^2+4} \mp n\right)/2$

Planet	Real T_k	Calculated T_k	n	$\dfrac{T_{k(\mathrm{calc})}}{T_{k(\mathrm{real})}}$
Mercury	0.0203	0.0203	-49	1.0000
Venus	0.0519	0.0524	-19	1.0096
Earth	0.0843	0.0828	-12	0.9822
Mars	0.1586	0.1623	-6	1.0233
Asteroids	0.4877	0.4142	-2	0.8493
Jupiter	1.0000	1.0000	0	1.0000
Saturn	2.4834	2.4142	2	0.9721
Uranus	7.0827	7.1378	7	1.0077
Neptune	13.8922	14.0711	13	1.0129
Pluto	21.1166	21.0475	21	0.9967

Note: Here the measurement units are the orbital radius and period of Jupiter. For asteroids the overall average orbit is taken, its radius 3.215 astronomical units and period 5.75 years are the average values between asteroids.

a limited number of elements, takes its minimum density at the points of a unit interval on number line as shown by the aforementioned numbers. The second conclusion is that if these numbers express ratios between interacting frequencies, the amplitude of the forced oscillations takes its minimum numerical value.

It is evident that an oscillating system, where the oscillation parameters undergo changes due to interactions inside the system, will be maximally stable in that case where the forced oscillation amplitude will be a minimum.

The simplest verification of this thesis is given by the solar system. As we know it Laplace's classic works, the whole solar system (the planet orbits on the average) are stable under periodic gravitational perturbations only if the ratios between the orbital parameters are expressed by irrational numbers. If we will take this problem forward, proceeding from the viewpoint proposed above, the ratios between the orbital periods T_k/T_0 or, alternatively, the ratios between their functions (the average orbital radii R_k/R_0) will be close to those numbers that correspond to the minima of the rational numbers density on number line

$$\frac{T_k}{T_0};\frac{R_k}{R_0} \approx M_n = \frac{\left(\sqrt{n^2+4} \mp n\right)}{2}\,.$$

The truth or falsity of this can be decided by using Table 1 and Table 2.

As a matter of fact, all that has been said on the distribution of rational numbers on a unit interval could be extrapolated for the entire number line (proceeding from the above mentioned concept).

All that has been said gives a possibility to formulate the next conclusions:

1. Rational numbers having limited numerator and denominator are distributed inhomogeneously along the number line;
2. Oscillating systems, having a peculiarity to change their own parameters because of interactions inside the systems, have a tendency to reach a stable state where the separate oscillators frequencies are interrelated by specific numbers — minima of the rational number density on number line.

References

1. Klein F. Elementarmathematik vom höheren Standpunkte aus erster Band. Dritte Auflage, Springer Verlag, Berlin, 1924.
2. Khinchin A. Ya. Continued fractions. Nauka, Moscow, 1978.
3. Dombrowski K. I. *Bulletin of Soviet Atron. Geodesical Society*, 1956, No. 17 (24), 46–50.

On the General Solution to Einstein's Vacuum Field and Its Implications for Relativistic Degeneracy

Stephen J. Crothers

Australian Pacific College, Lower Ground, 189 Kent St., Sydney, 2000, Australia

E-mail: thenarmis@yahoo.com

The general solution to Einstein's vacuum field equations for the point-mass in all its configurations must be determined in such a way as to provide a means by which an infinite sequence of particular solutions can be readily constructed. It is from such a solution that the underlying geometry of Einstein's universe can be rightly explored. I report here on the determination of the general solution and its consequences for the theoretical basis of relativistic degeneracy, i. e. gravitational collapse and the black hole.

1 Introduction

A serious misconception prevails that the so-called "Schwarzschild solution" is a solution for the vacuum field. Not only is this incorrect, it is not even Schwarzschild's solution. The aforesaid solution was obtained by David Hilbert [1], a full year after Karl Schwarzschild [2] obtained his original solution. Moreover, Hilbert's metric is a corruption of the solution first found by Johannes Droste [3], and subsequently by Hermann Weyl [4] by a different method.

The orthodox concepts of gravitational collapse and the black hole owe their existence to a confusion as to the true nature of the r-parameter in the metric tensor for the gravitational field.

The error in the conventional analysis of Hilbert's solution is twofold in that two tacit and invalid assumptions are made:

(a) r is a coordinate and radius (of some kind) in the gravitational field;

(b) The regions $0 < r < \alpha = 2m$ and $\alpha < r < \infty$ are valid.

Contrary to the conventional analysis the nature and range or the r-parameter must be determined by rigorous mathematical means, *not* by mere assumption, tacit or otherwise. When the required mathematical rigour is applied it is revealed that $r_0 = \alpha$ denotes a point, not a 2-sphere, and that $0 < r < \alpha$ is undefined on the Hilbert metric. The consequence of this is that gravitational collapse, if it occurs in Nature at all, cannot produce a relativistic black hole under any circumstances. Since the Michell-Laplace dark body is not a black hole either, there is no theoretical basis for it whatsoever. Furthermore, the conventional conception of gravitational collapse is demonstrably false.

The sought for general solution must not only result in a means for construction of an infinite sequence of particular solutions, it must also naturally produce the solutions due to Schwarzschild, Droste and Weyl, and M. Brillouin [5]. To obtain the general solution the general conditions that the required solution must satisfy must be established. Abrams [9] has determined these conditions. I obtain them by other arguments, and therefrom construct the general solution, from which the original Schwarzschild solution, the Droste/Weyl solution, and the Brillouin solution all arise quite naturally. It will be evident that the black hole is theoretically unsound. Indeed, it never arose in the solutions of Schwarzschild, Droste and Weyl, and Brillouin. It comes solely from the mathematically inadmissible assumptions conventionally imposed upon the Hilbert metric.

I provide herein a derivation of the general solution for the simple point-mass and briefly discuss its geometry. Although I have obtained the complete solution up to the rotating point-charge I reserve its derivation to a subsequent paper and similarly a full discussion of the geometry to a third paper. However, I include the expression for the overall general solution as a prelude to my following papers.

2 The general solution for the simple point-mass and its basic geometry

A general metric for the static, time-symmetric, cento-symmetric configuration of energy or matter in quasi-Cartesian coordinates is,

$$\begin{aligned} ds^2 &= L(r)dt^2 - M(r)(dx^2 + dy^2 + dz^2) - \\ &\quad - N(r)(xdx + ydy + zdz)^2, \\ r &= \sqrt{x^2 + y^2 + z^2}\,, \end{aligned} \tag{1}$$

where, $\forall\ t$, L, M, N are analytic functions such that,

$$L, M, N > 0\,. \tag{2}$$

In polar coordinates (1) becomes,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{3}$$

where analytic $A, B, C > 0$ owing to (2).

Transform (3) by setting

$$r^* = \sqrt{C(r)}\,, \tag{4}$$

then

$$\begin{aligned} ds^2 &= A^*(r^*)dt^2 - B^*(r^*)dr^{*2} - \\ &\quad - r^{*2}(d\theta^2 + \sin^2\theta d\varphi^2)\,, \end{aligned} \tag{5}$$

from which one obtains in the usual way,

$$\begin{aligned} ds^2 &= \left(\frac{r^*-\alpha}{r^*}\right)dt^2 - \left(\frac{r^*}{r^*-\alpha}\right)dr^{*2} - \\ &\quad - r^{*2}(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \tag{6}$$

Substituting (4) gives

$$\begin{aligned} ds^2 &= \left(\frac{\sqrt{C}-\alpha}{\sqrt{C}}\right)dt^2 - \left(\frac{\sqrt{C}}{\sqrt{C}-\alpha}\right)\frac{C'^2}{4C}dr^2 - \\ &\quad - C(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \tag{7}$$

Thus, (7) is a general metric in terms of one unknown function $C(r)$. The following arguments are coordinate independent since $C(r)$ in (7) is an arbitrary function.

The general metric for Special Relativity is,

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2 d\varphi^2\right), \tag{8}$$

and the radial distance (the *proper distance*) between two *points* is,

$$d = \int_{r_0}^{r} dr = r - r_0\,. \tag{9}$$

Let a test particle be located at each of the points r_0 and $r > r_0$ (owing to the isotropy of space there is no loss of generality in taking $r \geqslant r_0 \geqslant 0$). Then by (9) the distance between them is given by

$$d = r - r_0\,,$$

and if $r_0 = 0$, $d \equiv r$ in which case the distance from $r_0 = 0$ is the same as the radius (the *curvature radius*) of a great circle, the circumference χ of which is from (8),

$$\chi = 2\pi\sqrt{r^2} = 2\pi r\,. \tag{10}$$

In other words, the curvature radius and the proper radius are identical, owing to the pseudo-Euclidean nature of (8). Furthermore, d gives the radius of a sphere centred at the *point* r_0. Let the test particle at r_0 acquire mass. This produces a gravitational field centred at the *point* $r_0 \geqslant 0$. The geometrical relations between the components of the metric tensor of General Relativity must be precisely the same in the metric of Special Relativity. Therefore the distance between r_0 and $r > r_0$ is no longer given by (9) and the curvature radius no longer by (10). Indeed, the proper radius R_p, in keeping with the geometrical relations on (8), is now given by,

$$R_p = \int_{r_0}^{r} \sqrt{-g_{11}}dr\,, \tag{11}$$

where from (7),

$$-g_{11} = \left(1 - \frac{\alpha}{\sqrt{C(r)}}\right)^{-1} \frac{[C'(r)]^2}{4C(r)}\,. \tag{12}$$

Equation (11) with (12) gives the mapping of d from the flat spacetime of Special Relativity into the curved spacetime of General Relativity, thus,

$$\begin{aligned} R_p(r) &= \int \sqrt{\frac{\sqrt{C}}{\sqrt{C}-\alpha}\,\frac{C'}{2\sqrt{C}}}\,dr = \\ &= \sqrt{\sqrt{C(r)}\left(\sqrt{C(r)}-\alpha\right)} + \\ &+ \alpha\ln\left|\frac{\sqrt{\sqrt{C(r)}} + \sqrt{\sqrt{C(r)}-\alpha}}{K}\right|, \\ K &= \text{const}\,. \end{aligned} \tag{13}$$

The relationship between r and R_p is

$$r \to r_0 \Rightarrow R_p \to 0\,,$$

so from (13) it follows,

$$r \to r_0 \;\Rightarrow C(r_0) = \alpha^2,\; K = \sqrt{\alpha}\,.$$

So (13) becomes,

$$\begin{aligned} R_p(r) &= \sqrt{\sqrt{C(r)}\left(\sqrt{C(r)}-\alpha\right)} + \\ &+ \alpha\ln\left|\frac{\sqrt{\sqrt{C(r)}} + \sqrt{\sqrt{C(r)}-\alpha}}{\sqrt{\alpha}}\right|. \end{aligned} \tag{14}$$

Therefore (7) is singular only at $r = r_0$, where $C(r_0) = \alpha^2$ and $g_{00} = 0$ $\forall$ r_0, irrespective of the value of r_0. $C(r_0) = \alpha^2$ emphasizes the true meaning of α, viz., α is a scalar invariant which fixes the spacetime for the point-mass from an infinite number of mathematically possible forms, as pointed out by Abrams. Moreover, α embodies the effective gravitational mass of the source of the field, and fixes a boundary to an otherwise incomplete spacetime. Furthermore, one can see from (13) and (14) that r_0 is arbitrary, i.e. the point-mass can be located at any point and its location has no intrinsic meaning. Furthermore, the condition $g_{00} = 0$ is clearly equivalent to the boundary condition $r \to r_0 \Rightarrow R_p \to 0$, from which it follows that $g_{00} = 0$ is the *end result* of gravitational collapse. There exists no value of r making $g_{11} = 0$.

If $C' = 0$ for $r > r_0$ the structure of (7) is destroyed: $g_{11} = 0$ for $r > r_0 \Rightarrow B(r) = 0$ for $r > r_0$ in violation of (3). Therefore $C' \neq 0$. For (7) to be spatially asymptotically flat,

$$\lim_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1\,. \tag{15}$$

Since $C(r)$ must behave like $\left(r - r_o\right)^2$ and make (7) singular only at $r = r_0$, $C(r)$ must be a strictly monotonically increasing function. Then by virtue of (15) and the fact that $C' \neq 0$, it follows that $C' > 0$ for $r > r_0$. Thus the necessary conditions that must be imposed upon $C(r)$ to render a solution to (3) are:

1. $C'(r) > 0$ for $r > r_0$;
2. $\displaystyle\lim_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1$;
3. $C(r_0) = \alpha^2$.

I call the foregoing the Metric Conditions of Abrams for the point-mass (MCA) since when $r_0 = 0$ they are precisely the conditions he determined by his use of (3) and the field equations. In addition to MCA any admissible function $C(r)$ must reduce (7) to the metric of Special Relativity when $\alpha = 2m = 0$.

The invalid conventional assumptions that $0 < r < \alpha$ and that r is a radius of sorts in the gravitational field lead to the incorrect conclusion that $r = \alpha$ is a 2-sphere in the gravitational field of the point-mass. The quantity $r = \alpha$ does not describe a 2-sphere; it does not yield a Schwarzschild sphere; it is actually a *point*. Stavroulakis [10, 8, 9] has also remarked upon the true nature of the r-parameter (coordinate radius). Since MCA must be satisfied, admissible systems of coordinates are restricted to a particular (infinite) class. To satisfy MCA, and therefore (3), and (7), the form that $C(r)$ can take must be restricted to,

$$C_n(r) = \left[(r - r_0)^n + \alpha^n\right]^{\frac{2}{n}}\,, \tag{16}$$

$$r_0 \in (\Re - \Re^-),\; n \in \Re^+,$$

where n and r_0 are arbitrary. I call equations (16) Schwarzschild forms. The value of n in (16) fixes a set of coordinates, and the infinitude of such reflects the fact that no set of coordinates is privileged in General Relativity.

The general solution for the simple point-mass is therefore,

$$ds^2 = \left(\frac{\sqrt{C_n} - \alpha}{\sqrt{C_n}}\right) dt^2 - \left(\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}\right) \frac{{C'_n}^2}{4C_n} dr^2 - - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{17}$$

$$C_n(r) = \left[(r - r_0)^n + \alpha^n\right]^{\frac{2}{n}}\,,\; n \in \Re^+,$$

$$r_0 \in (\Re - \Re^-)\,,$$

$$r_0 < r < \infty\,,$$

where n and r_0 are arbitrary. Therefore with r_0 arbitrary, (17) reduces to the metric of Special Relativity when $\alpha = 2m = 0$.

From (17), with $r_0 = 0$ and n taking integer values, the following infinite sequence obtains:

$C_1(r) = (r + \alpha)^2$ (Brillouin's solution)

$C_2(r) = (r^2 + \alpha^2)$

$C_3(r) = (r^3 + \alpha^3)^{\frac{2}{3}}$ (Schwarzschild's solution)

$C_4(r) = (r^4 + \alpha^4)^{\frac{1}{2}}$, etc.

Hilbert's solution is rightly obtained when $r_0 = \alpha$, i. e. when $r_0 = \alpha$ and the values of n take integers, the infinite sequence of particular solutions is then given by,

$C_1(r) = r^2$ [Droste/Weyl/(Hilbert) solution]

$C_2(r) = (r - \alpha)^2 + \alpha^2$,

$C_3(r) = \left[(r - \alpha)^3 + \alpha^3\right]^{\frac{2}{3}}$,

$C_4(r) = \left[(r - \alpha)^4 + \alpha^4\right]^{\frac{1}{2}}$, etc.

The curvature $f = R^{ijkm}R_{ijkm}$ is finite everywhere, including $r = r_0$. Indeed, for metric (17) the Kretschmann scalar is,

$$f = \frac{12\alpha^2}{C_n^3} = \frac{12\alpha^2}{\left[\left(r - r_0\right)^n + \alpha^n\right]^{\frac{6}{n}}}\,. \tag{18}$$

Gravitational collapse does not produce a curvature singularity in the gravitational field of the point-mass. The scalar invariance of $f(r_0) = \frac{12}{\alpha^4}$ is evident from (18).

All the particular solutions of (17) are inextendible, since the singularity when $r = r_0$ is quasiregular, irrespective of the values of n and r_0. Indeed, the circumference χ of a great circle becomes,

$$\chi = 2\pi\sqrt{C(r)}. \tag{19}$$

Then the ratio

$$\lim_{r\to r_0} \frac{\chi}{R_p} \to \infty, \tag{20}$$

shows that $R_p(r_0) \equiv 0$ is a quasiregular singularity and cannot be extended.

Equation (19) shows that $\chi = 2\pi\alpha$ is also a scalar invariant for the point-mass.

It is plain from the foregoing that the Kruskal-Szekeres extension is meaningless, that the "Schwarzschild radius" is meaningless, that the orthodox conception of gravitational collapse is incorrect, and that the black hole is not consistent at all with General Relativity. All arise wholely from a bungled analysis of Hilbert's solution.

3 Implications for gravitational collapse

As is well known the gravitational potential Φ for an arbitrary metric is

$$g_{00} = (1 - \Phi)^2\,, \tag{21}$$

from which it is concluded that gravitational collapse occurs at $\Phi = 1$. Physically, the conventional process of collapse involves Newtonian gravitation down to the so-called "gravitational radius". Far from the source, the alleged weak field potential is,

$$\Phi = \frac{m}{r},$$

and so

$$g_{00} = 1 - \frac{\alpha}{r}, \tag{22}$$

$$\alpha = 2m.$$

The scalar α is conventionally called the "gravitational radius", or the "Schwarzschild radius", or the "event horizon". However, as I have shown, neither α nor the coordinate radius r are radii in the gravitational field. In the case of the Hilbert metric, $r_0 = \alpha$ is a *point*, not a 2-sphere. It is the location of the point-mass. In consequence of this $g_{00} = 0$ is the end result of gravitational collapse. It therefore follows that in the vacuum field,

$$0 < g_{00} < 1, \qquad 1 < |g_{11}| < \infty,$$

$$\alpha < \sqrt{C(r)}.$$

In the case of the Hilbert metric, $C(r) = r^2$, so

$$0 < g_{00} < 1, \qquad 1 < |g_{11}| < \infty,$$

$$\alpha < r.$$

In the case of Schwarzschild's metric we have $C(r) = \left(r^3 + \alpha^3\right)^{\frac{2}{3}}$, so

$$0 < g_{00} < 1, \qquad 1 < |g_{11}| < \infty,$$

$$0 < r.$$

It is unreasonable to expect the weak field potential function to be strictly Newtonian. Only in the infinitely far field is Newton's potential function to be recovered. Consequently, the conventional weak field expression (22) cannot be admitted with the conventional interpretation thereof. The correct potential function must contain the arbitrary location of the point-mass. From (21),

$$\Phi = 1 - \sqrt{g_{00}} = 1 - \sqrt{1 - \frac{\alpha}{\sqrt{C(r)}}},$$

so in the weak far field,

$$\Phi \approx 1 - \left(1 - \frac{\alpha}{2\sqrt{C}}\right) = \frac{m}{\sqrt{C}},$$

and so

$$g_{00} = 1 - \frac{\alpha}{\sqrt{C(r)}} = 1 - \frac{\alpha}{\left[(r - r_0)^n + \alpha^n\right]^{\frac{1}{n}}}, \tag{23}$$

$$r_0 \in (\Re - \Re^-), \; n \in \Re^+.$$

Then

$$\text{as } r \to \infty, \; g_{00} \to 1 - \frac{\alpha}{r - r_0},$$

and Newton is recovered at infinity.

According to (23), at $r = r_0$, $g_{00} = 0$ and $\Phi = \frac{1}{2}$. The weak field potential approaches a finite maximum of $\frac{1}{2}$ (i. e. $\frac{1}{2}c^2$), in contrast to Newton's potential. The conventional concept of gravitational collapse at $r_s = \alpha$ is therefore meaningless.

Similarly, it is unreasonable to expect Kepler's 3rd Law to be unaffected by general relativity, contrary to the conventional analysis. Consider the Lagrangian,

$$\begin{aligned} L &= \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{d\tau}\right)^2\right] - \\ &- \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2\right] - \\ &- \frac{1}{2}\left[C_n\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right)\right], \\ C_n(r) &= \left[\left(r - r_0\right)^n + \alpha^n\right]^{\frac{2}{n}}, \; n \in \Re^+, \\ r_0 &\in (\Re - \Re^-), \; r_0 < r < \infty, \end{aligned} \tag{24}$$

where τ is the proper time.

Restricting motion, without loss of generality, to the equatorial plane, $\theta = \frac{\pi}{2}$, the Euler-Lagrange equations for (24) are,

$$\begin{aligned} &\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\frac{d^2\sqrt{C_n}}{d\tau^2} + \frac{\alpha}{2C_n}\left(\frac{dt}{d\tau}\right)^2 - \\ &-\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-2}\frac{\alpha}{2C_n}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - \sqrt{C_n}\left(\frac{d\varphi}{d\tau}\right)^2 = 0, \end{aligned} \tag{25}$$

$$\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\frac{dt}{d\tau} = \text{const} = k, \tag{26}$$

$$C_n\frac{d\varphi}{d\tau} = \text{const} = h, \tag{27}$$

and $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ becomes,

$$\begin{aligned} &\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{d\tau}\right)^2 - \\ &-\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - C_n\left(\frac{d\varphi}{d\tau}\right)^2 = 1. \end{aligned} \tag{28}$$

Using the foregoing equations it readily follows that the angular velocity is,

$$\omega = \sqrt{\frac{\alpha}{2C_n^{\frac{3}{2}}}}. \tag{29}$$

Then,

$$\lim_{r\to r_0} \omega = \frac{1}{\alpha\sqrt{2}} \tag{30}$$

is a scalar invariant which shows that the angular velocity approaches a finite limit, in contrast to Newton's theory where it becomes unbounded. Schwarzschild obtained this result for his particular solution. Equation (29) is the General Relativistic modification of Kepler's 3rd Law.

For a falling particle in a true Schwarzschild field,

$$d\tau = \sqrt{g_{00}}\, dt = \sqrt{1 - \frac{\alpha}{\sqrt{C(r)}}}\, dt\,.$$

Therefore, as a neutral test particle approaches the field source at r_0 along a radial geodesic, $d\tau \to 0$. Thus, according to an external observer, it takes an infinite amount of coordinate time for a test particle to reach the source. Time stops at the Schwarzschild point-mass. The conventional concepts of the Schwarzschild sphere and its interior are meaningless.

Doughty [10] has shown that the acceleration of a test particle approaching the point-mass along a radial geodesic is given by,

$$a = \frac{\sqrt{-g_{11}}\left(-g^{11}\right)|g_{00,1}|}{2g_{00}}. \tag{31}$$

By (17),

$$a = \frac{\alpha}{2C^{\frac{3}{4}}\left(\sqrt{C}-\alpha\right)^{\frac{1}{2}}}\,.$$

Clearly, as $r \to r_0$, $a \to \infty$, independently of the value of r_0. In the case of $C(r) = r^2$, where $r_0 = \alpha$,

$$a = \frac{\alpha}{2r^{\frac{3}{2}}\sqrt{r-\alpha}}\,, \tag{32}$$

so $a \to \infty$ as $r \to r_0 = \alpha$.

Applying (31) to the Kruskal-Szekeres extension gives rise to the absurdity of an infinite acceleration at $r = \alpha$ where it is conventionally claimed that there is no matter and no singularity. It is plainly evident that gravitational collapse terminates at a Schwarzschild simple point-mass, not in a black hole. Also, one can readily see that the alleged interchange of the spatial and time coordinates "inside" the "Schwarzschild sphere" is nonsensical. To amplify this, in (17), suppose $\sqrt{C(r)} < \alpha$, then

$$ds^2 = -\left(\frac{\alpha}{\sqrt{C}}-1\right)dt^2 + \left(\frac{\alpha}{\sqrt{C}}-1\right)^{-1}\frac{C'^2}{4C}dr^2 - C\left(d\theta^2 + \sin^2 d\varphi^2\right). \tag{33}$$

Let $r = \tilde{t}$ and $t = \tilde{r}$, then

$$ds^2 = \left(\frac{\alpha-\sqrt{C}}{\sqrt{C}}\right)^{-1}\frac{\dot{C}^2}{4C}d\tilde{t}^2 - \left(\frac{\alpha-\sqrt{C}}{\sqrt{C}}\right)d\tilde{r}^2 - C(\tilde{t})\left(d\theta^2 + \sin^2 d\varphi^2\right). \tag{34}$$

This is a time dependent metric which does not have any relationship to the original static problem. It does not extend (17) at all, as also noted by Brillouin in the particular solution given by him. Equation (34) is meaningless.

It is noteworthy that Hagihara [11] has shown that all geodesics that do not run into the Hilbert boundary at $r_0 = \alpha$ are complete. His result is easily extended to any $r_0 \geqslant 0$ in (17).

The correct conclusion is that gravitational collapse terminates at the point-mass without the formation of a black hole in all general relativistic circumstances.

4 Generalization of the vacuum solution for charge and angular momentum

The foregoing analysis can be readily extended to include the charged and rotating point-mass. In similar fashion it follows that the Reissner-Nordstrom, Carter, Graves-Brill, Kerr, and Kerr-Newman black holes are all inconsistent with General Relativity.

In a subsequent paper I shall derive the following overall general solution for the point-mass when $\Lambda = 0$,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(C_n + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}\frac{C_n'^{\,2}}{4C_n}dr^2 - \rho^2 d\theta^2\,,$$

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}},\ r_0 \in \left(\Re - \Re^-\right),$$

$$n \in \Re^+,\ a = \frac{L}{m},\ \rho^2 = C_n + a^2\cos^2\theta\,,$$

$$\Delta = C_n - \alpha\sqrt{C_n} + q^2 + a^2\,,$$

$$\beta = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta},\ a^2 + q^2 < m^2,$$

$$r_0 < r < \infty\,.$$

The different configurations for the point-mass are easily extracted from this set of equations by the setting of the values of the parameters in the obvious way.

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

Epilogue

My interest in the problem of the black hole was aroused by coming across the papers of the American physicist Leonard S. Abrams, and subsequently to the original papers of Schwarzschild, Droste, Weyl, Hilbert, and Brillouin. I was drawn to the logic of Abrams' approach in his determination of the required metric in terms of a single generalised function and the conditions that this function must satisfy to render a solution for the point-mass. It was not until I read Abrams that I became aware of the startling facts that the "Schwarzschild solution" is not due to Schwarzschild, that Schwarzschild did not predict the black hole and made none of the claims about black holes that are invariably attributed to him in the textbooks and almost invariably in the literature. These facts alone give cause for disquiet and reading of the original papers gives cause for serious concern about how modern science is reported.

Dr. Leonard S. Abrams was born in Chicago in 1924 and died on December 28, 2001, in Los Angeles at the age of 77. He received a B. S. in Mathematics from the California Institute of Technology and a Ph. D. in physics from the University of California at Los Angeles at the age of 45. He spent almost all of his career working in the private sector, although he taught at a variety of institutions including California State University at Dominguez Hills and at the University of Southern California. He was a pioneer in applying game theory to business problems and was an expert in noise theory, but his first love always was general relativity. His principle theoretical contributions focused on non-black hole solutions to Einstein's equations and on the inextendability of the "Schwarzschild" solution. Dr. Abrams is survived by his wife and two children.

Dr. Abrams encountered great resistance to publication of his work on General Relativity. Nonetheless he continued with his work and managed to publish several important papers despite the obstacles placed in his way by the mainstream authorities.

I extend my thanks to Diana Abrams for providing me with information about her late husband.

References

1. Hilbert D. *Nachr. Ges. Wiss. Gottingen, Math. Phys. Kl.*, 1917, 53, (see this Ref. in arXiv: physics/0310104).
2. Schwarzschild K. On the gravitational field of a mass point according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 189 (see this item also in arXiv: physics/9905030.
3. Droste J. The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field. *Ned. Acad. Wet., S. A.*, 1917, v. 19, 197 (see also in www.geocities.com/theometria/Droste.pdf).
4. Weyl H. Zur Gravitationstheorie. *Ann. Phys.* (*Leipzig*), 1917, v. 54, 117.
5. Brillouin M. The singular points of Einstein's Universe. *Journ. Phys. Radium*, 1923, v. 23, 43 (see also in arXiv: physics/0002009).
6. Abrams L. S. Black holes: the legacy of Hilbert's error. *Can. J. Phys.*, 1989, v. 67, 919 (see also in arXiv: gr-qc/0102055).
7. Stavroulakis N. A statical smooth extension of Schwarzschild's metric. *Lettere al Nuovo Cimento*, 1974, v. 11, 8.
8. Stavroulakis N. On the Principles of General Relativity and the S$\Theta(4)$-invariant metrics. *Proc. 3rd Panhellenic Congr. Geometry*, Athens, 1997, 169.
9. Stavroulakis N. On a paper by J. Smoller and B. Temple. *Annales de la Fondation Louis de Broglie*, 2002, v. 27, 3 (see also in www.geocities.com/theometria/Stavroulakis-1.pdf).
10. Doughty N. *Am. J. Phys.*, 1981, v. 49, 720.
11. Hagihara Y. *Jpn. J. Astron. Geophys.*, 1931, v. 8, 67.

On the Ramifications of the Schwarzschild Space-Time Metric

Stephen J. Crothers

Australian Pacific College, Lower Ground, 189 Kent St., Sydney, 2000, Australia

E-mail: thenarmis@yahoo.com

In a previous paper I derived the general solution for the simple point-mass in a true Schwarzschild space. I extend that solution to the point-charge, the rotating point-mass, and the rotating point-charge, culminating in a single expression for the general solution for the point-mass in all its configurations when $\Lambda = 0$. The general exact solution is proved regular everywhere except at the arbitrary location of the source of the gravitational field. In no case does the black hole manifest. The conventional solutions giving rise to various black holes are shown to be inconsistent with General Relativity.

1 Introduction

In a previous paper [1] I showed that the general solution of the vacuum field for the simple point-mass is regular everywhere except at the arbitrary location of the source of the field, $r = r_0$, $r_0 \in (\Re - \Re^-)$, where there is a quasiregular singularity. I extend herein the general solution to the rotating and charged configurations of the point-mass and show that they too are regular everywhere except at $r = r_0$, obviating the formation of the Reissner-Nordstrom, Kerr, and Kerr-Newman black holes. Consequently, there is no basis in General Relativity for the black hole.

The sought for complete solution for the point-mass must reduce to the general solution for the simple point-mass in a natural way, give rise to an infinite sequence of particular solutions in each particular configuration, and contain a scalar invariant which embodies all the factors that contribute to the effective gravitational mass of the field's source for the respective configurations.

2 The vacuum field of the point-charge

The general metric, in polar coordinates, for the vacuum field is, in relativistic units,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad (1)$$

where analytic $A, B, C > 0$. The general solution to (1) for the simple point-mass is,

$$ds^2 = \left[\frac{(\sqrt{C_n}-\alpha)}{\sqrt{C_n}}\right]dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n}-\alpha)}\right]\frac{{C_n'}^2}{4C_n}dr^2 - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad (2)$$

$$C_n(r) = \left[\left(r - r_0\right)^n + \alpha^n\right]^{\frac{2}{n}}\,,\ \alpha = 2m\,,\ \ r_0 \in (\Re - \Re^-)\,,$$

$$n \in \Re^+\,, \quad r_0 < r < \infty\,,$$

where $C_n(r)$ satisfies the Metric conditions of Abrams (MCA) [2]* for the simple point-mass,

1. $C_n'(r) > 0$, $r > r_0$;
2. $\lim\limits_{r\to\infty} \dfrac{C_n(r)}{\left(r - r_0\right)^2} = 1$;
3. $C_n(r_0) = \alpha^2$.

The Reissner-Nordstrom [3] solution is,

$$ds^2 = \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad (3)$$

which is conventionally taken to be valid for all $\frac{q^2}{m^2}$. It is also alleged that (3) can be extended down to $r = 0$, giving rise to the so-called Reissner-Nordstrom black hole. These conventional allegations are demonstrably false.

The conventional analysis simply looks at (3) and makes two mathematically invalid assumptions, viz.,

1. *The parameter r is a radius of some kind in the gravitational field*;
2. *r down to $r = 0$ is valid*.

The nature and range of the r-parameter must be established by mathematical rigour, *not* by mere assumption.

Transform (1) by the substitution

$$r^* = \sqrt{C(r)}. \quad (4)$$

---

*Abrams' equation (A.1) should read:

$$-8\pi T_1^1 = \frac{-1}{C} + \frac{C'^2}{4BC^2} + \frac{A'C'}{2ABC} = 0\,,$$

and his equation (A.6),

$$\frac{2C''}{C'} - [\ln(ABC)]' = 0\,.$$

The errors are apparently escapees from the proof reading.

Equation (4) carries (1) into

$$ds^2 = A^*(r^*)dt^2 - B^*(r^*)dr^{*2} - r^{*2}(d\theta^2 + \sin^2\theta d\varphi^2)\,. \quad (5)$$

Using (5) to determine the Maxwell stress-energy tensor, and substituting the latter into the Einstein-Maxwell field equations in the usual way, yields,

$$\begin{aligned} ds^2 &= \left(1 - \frac{\alpha}{r^*} + \frac{q^2}{r^{*2}}\right) dt^2 - \\ &- \left(1 - \frac{\alpha}{r^*} + \frac{q^2}{r^{*2}}\right)^{-1} dr^{*2} - r^{*2}(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \quad (6)$$

Substituting (4) into (6),

$$\begin{aligned} ds^2 &= \left(1 - \frac{\alpha}{\sqrt{C}} + \frac{q^2}{C}\right) dt^2 - \left(1 - \frac{\alpha}{\sqrt{C}} + \frac{q^2}{C}\right)^{-1} \times \\ &\times \frac{C'^2}{4C} dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \quad (7)$$

The proper radius R_p on (1) is,

$$R_p(r) = \int \sqrt{B(r)}\, dr\,. \quad (8)$$

The parameter r therefore does not lie in the spacetime M_q of the point-charge.

Taking $B(r)$ from (7) into (8) gives the proper distance in M_q,

$$\begin{aligned} R_p(r) &= \int \left(1 - \frac{\alpha}{\sqrt{C(r)}} + \frac{q^2}{C(r)}\right)^{-\frac{1}{2}} \frac{C'(r)}{2\sqrt{C(r)}} dr = \\ &= \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2} + \\ &+ m \ln \left| \frac{\sqrt{C(r)} - m + \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2}}{K} \right|, \end{aligned} \quad (9)$$

$$K = \text{const.}$$

The valid relationship between r and $R_p(r)$ is,

$$\text{as } \; r \to r_0\,, \quad R_p(r) \to 0\,,$$

so by (9),

$$r \to r_0 \;\Rightarrow \sqrt{C(r_0)} = m \pm \sqrt{m^2 - q^2}\,,$$

$$K = \pm\sqrt{m^2 - q^2}\,.$$

When $q = 0$, (9) must reduce to the Droste/Weyl [4, 5] solution, so it requires,

$$\sqrt{C(r_0)} = m + \sqrt{m^2 - q^2}\,. \quad (10)$$

Then by (9),

$$K = \sqrt{m^2 - q^2}\,, \; q^2 < m^2\,. \quad (11)$$

Clearly, r_0 is the lower bound on r.

Putting (11) into (9) gives,

$$\begin{aligned} R_p(r) &= \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2} + \\ &+ m \ln \left| \frac{\sqrt{C(r)} - m + \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2}}{\sqrt{m^2 - q^2}} \right|. \end{aligned} \quad (12)$$

Equation (7) is therefore singular *only* when $r = r_0$ in which case $g_{00} = 0$. Hence, the condition $r \to r_0 \Rightarrow R_p \to 0$ is equivalent to $r = r_0 \Rightarrow g_{00} = 0$.

If $C' = 0$ the structure of (7) is destroyed, since $g_{11} = 0\ \forall\ r > r_0 \Rightarrow B(r) = 0\ \forall\ r > r_0$ in violation of (1). Therefore $C'(r) \neq 0$ for $r > r_0$.

For (7) to be asymptotically flat,

$$r \to \infty \Rightarrow \frac{C(r)}{\left(r - r_0\right)^2} \to 1\,. \quad (13)$$

Therefore,

$$\lim_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1\,. \quad (14)$$

Since $C(r)$ behaves like $\left(r - r_0\right)^2$, must make (7) singular only at $r = r_0$, and $C'(r) \neq 0$ for $r > r_0$, $C(r)$ is strictly monotonically increasing, therefore, $C'(r) > 0$ for $r > r_0$. Thus, to satisfy (1) and (7), $C(r)$ must satisfy,

1. $C'(r) > 0$, $r > r_0$;
2. $\lim_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1$;
3. $\sqrt{C(r_0)} = \beta = m + \sqrt{m^2 - q^2}$, $q^2 < m^2$.

I call the foregoing the Metric Conditions of Abrams (MCA) for the point-charge. Abrams [6] obtained them by a different method — using (1) and the field equations directly.

In the absence of charge (7) must reduce to the general Schwarzschild solution for the simple point-mass (2). The only functions that satisfy this requirement and MCA are,

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,, \quad q^2 < m^2\,,$$

$$n \in \Re^+\,, \quad r_0 \in (\Re - \Re^-)\,,$$

where n and r_0 are arbitrary. Therefore, the general solution for the point-charge is,

$$\begin{aligned} ds^2 &= \left(1 - \frac{\alpha}{\sqrt{C}} + \frac{q^2}{C}\right) dt^2 - \left(1 - \frac{\alpha}{\sqrt{C}} + \frac{q^2}{C}\right)^{-1} \times \\ &\times \frac{C'^2}{4C} dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,, \end{aligned} \quad (15)$$

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}},$$

$$\beta = m + \sqrt{m^2 - q^2}, \quad q^2 < m^2,$$

$$n \in \Re^+, \quad r_0 \in (\Re - \Re^-),$$

$$r_0 < r < \infty.$$

When $n = 1$ and $r_0 = 0$, Abrams' [6] solution for the point-charge results.

Equation (15) is regular $\forall\ r > r_0$. There is no event horizon and therefore no Reissner-Nordstrom black hole. Furthermore, the Graves-Brill black hole and the Carter black hole are also invalid.

By (15) the correct rendering of (3) is,

$$ds^2 = \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \tag{16}$$

$$q^2 < m^2, \quad m + \sqrt{m^2 - q^2} < r < \infty,$$

so Nordstrom's assumption that $\sqrt{C(0)} = 0$ is invalid.

The scalar curvature $f = R_{ijkm} R^{ijkm}$ for (1) with charge included is,

$$f = \frac{8\left[6\left(m\sqrt{C} - q^2\right)^2 + q^4\right]}{C^4}.$$

Using (15) the curvature is,

$$f = \frac{8\left[6\left(m\left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{1}{n}} - q^2\right)^2 + q^4\right]}{\left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{8}{n}}}.$$

The curvature is always finite, even at r_0. No curvature singularity can arise in the gravitational field of the point-charge. Furthermore,

$$f(r_0) = \frac{8\left[6\left(m\beta - q^2\right)^2 + q^4\right]}{\beta^8},$$

where $\beta = m + \sqrt{m^2 - q^2}$. Thus, $f(r_0)$ is a scalar invariant for the point-charge. When $q = 0$, $f(r_0) = \frac{12}{\alpha^4}$, which is the scalar curvature invariant for the simple point-mass.

From (15) the circumference χ of a great circle is given by,

$$\chi = 2\pi\sqrt{C(r)}.$$

The proper radius is given by (12). Then the ratio $\frac{\chi}{R_p} > 2\pi$ for finite r and,

$$\lim_{r\to\infty} \frac{\chi}{R_p} = 2\pi,$$

$$\lim_{r\to r_0} \frac{\chi}{R_p} \to \infty,$$

which shows that $R_p(r_0)$ is a quasiregular singularity and cannot be extended.

Consider the Lagrangian,

$$L = \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)\left(\frac{dt}{d\tau}\right)^2\right] - - \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2\right] - - \frac{1}{2}\left[C_n\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right)\right]. \tag{17}$$

Restricting motion to the equatorial plane without loss of generality, the Euler-Lagrange equations from (17) are,

$$\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)\frac{d^2\sqrt{C_n}}{d\tau^2} + \left(\frac{\alpha}{2C_n} - \frac{q^2}{C_n^{\frac{3}{2}}}\right)\left(\frac{dt}{d\tau}\right)^2 - - \left(\frac{\alpha}{2C_n} - \frac{q^2}{C_n^{\frac{3}{2}}}\right)\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)^{-2}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - - \sqrt{C_n}\left(\frac{d\varphi}{d\tau}\right)^2 = 0, \tag{18}$$

$$\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)\frac{dt}{d\tau} = k = \text{const}, \tag{19}$$

$$C_n \frac{d\varphi}{d\tau} = h = \text{const}. \tag{20}$$

Also, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ becomes,

$$\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)\left(\frac{dt}{d\tau}\right)^2 - - \left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - - C_n\left(\frac{d\varphi}{d\tau}\right)^2 = 1. \tag{21}$$

It follows from these equations that the angular velocity ω of a test particle is,

$$\omega^2 = \left(\frac{\alpha}{2C_n^{\frac{3}{2}}} - \frac{q^2}{C_n^2}\right) = = \left[\frac{\alpha}{2\left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{3}{n}}} - \frac{q^2}{\left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{4}{n}}}\right]. \tag{22}$$

Then,

$$\lim_{r\to r_0} \omega = \sqrt{\frac{\alpha}{2\beta^3} - \frac{q^2}{\beta^4}}, \tag{23}$$

where $\beta = m + \sqrt{m^2 - q^2}$, $q^2 < m^2$.

Equation (22) is Kepler's 3rd Law for the point-charge. It obtains the finite limit given in (23), which is a scalar invariant for the point-charge. When $q = 0$, equations (22) and (23) reduce to those for the simple point-mass,

$$\omega = \sqrt{\frac{\alpha}{2C_n^{\frac{3}{2}}}}\,,$$

$$\lim_{r\to r_0} \omega = \frac{1}{\alpha\sqrt{2}}\,.$$

In the case of a photon in circular orbit about the point-charge, (21) yields,

$$\omega^2 = \frac{1}{C_n}\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right), \tag{24}$$

and (18) yields,

$$\omega^2 = \frac{1}{\sqrt{C_n}}\left(\frac{\alpha}{2C_n} - \frac{q^2}{C_n^{\frac{3}{2}}}\right). \tag{25}$$

Equating the two, denoting the stable photon radial coordinate by r_{ph}, and solving for the curvature radius $\sqrt{C_{ph}} = \sqrt{C_n(r_{(ph)})}$, gives (since when $q = 0$, $\sqrt{C_{ph}} \neq 0$),

$$\sqrt{C_{ph}} = \sqrt{C_n(r_{(ph)})} = \frac{3\alpha + \sqrt{9\alpha^2 - 32q^2}}{4}\,, \tag{26}$$

which is a scalar invariant. In terms of coordinate radii,

$$r_{ph} = \left[\frac{\left(3\alpha + \sqrt{9\alpha^2 - 32q^2}\right)^n}{4^n} - \beta^n\right]^{\frac{1}{n}} + r_0\,, \tag{27}$$

which depends upon the values of n and r_0.

When $q = 0$ equations (26) and (27) reduce to the corresponding equations for the simple point-mass,

$$\sqrt{C_n(r_{ph})} = \frac{3\alpha}{2}, \tag{28}$$

$$r_{ph} = \left[\left(\frac{3\alpha}{2}\right)^n - \alpha^n\right]^{\frac{1}{n}} + r_0\,. \tag{29}$$

The proper radius associated with (28) and (29) is,

$$R_{p(ph)} = \frac{\alpha\sqrt{3}}{2} + \alpha\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right), \tag{30}$$

which is a scalar invariant for the simple point-mass. Putting (26) into (12) gives the invariant proper radius for a stable photon orbit about the point-charge.

3 The vacuum field of the rotating point-mass

The Kerr solution, in Boyer-Lindquist coordinates and relativistic units is,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \end{aligned} \tag{31}$$

$$a = \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta\,,$$

$$\Delta = r^2 - r\alpha + a^2, \quad 0 < r < \infty\,,$$

where L is the angular momentum.

If $a = 0$, equation (31) reduces to Hilbert's [7] solution for the simple point-mass,

$$\begin{aligned} ds^2 &= \left(1 - \frac{\alpha}{r}\right)dt^2 - \left(1 - \frac{\alpha}{r}\right)^{-1}dr^2 - \\ &- r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \end{aligned} \tag{32}$$

$$0 < r < \infty\,.$$

However, according to the general formula (2) the correct range for r in (32) is,

$$\sqrt{C(r_0)} < r < \infty\,,$$

where $\sqrt{C(r_0)} = \alpha$. Therefore (32) should be,

$$\begin{aligned} ds^2 &= \left(1 - \frac{\alpha}{r}\right)dt^2 - \left(1 - \frac{\alpha}{r}\right)^{-1}dr^2 - \\ &- r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \end{aligned} \tag{33}$$

$$\alpha < r < \infty\,.$$

Equation (33) is the Droste/Weyl solution.

Since the r that appears in (32) is the same r appearing in (31) and (33), taking (4) into account, the correct general form of (31) is,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(C + a^2\right)d\varphi - a dt\right]^2 - \\ &- \frac{\rho^2}{\Delta}\frac{C'^2}{4C}dr^2 - \rho^2 d\theta^2\,, \end{aligned} \tag{34}$$

$$a = \frac{L}{m}, \quad \rho^2 = C + a^2\cos^2\theta\,,$$

$$\Delta = C - \alpha\sqrt{C} + a^2\,, \quad r_0 < r < \infty\,.$$

When $a = 0$, (34) must reduce to (2).

If $C' = 0$ the structure of (34) is destroyed, since then $g_{11} = 0 \ \forall\ r > r_0 \Rightarrow B(r) = 0$ in violation of (1). Therefore $C' \neq 0$. Equation (34) must have a global arrow for time, whereupon $g_{00}(r_0 = 0$, so

$$\Delta(r_0) = C(r_0) - \alpha\sqrt{C(r_0)} + a^2 = a^2\sin^2\theta\,. \tag{35}$$

Solving (35) for $\sqrt{C(r_0)}$ gives,

$$\beta = \sqrt{C(r_0)} = m \pm \sqrt{m^2 - a^2 \cos^2\theta}\,, \tag{36}$$

having used $\alpha = 2m$. When $a = 0$ (36) must reduce to the value for Schwarzschild's [8] original solution, i. e. $\sqrt{C(r_0)} = \alpha = 2m$, therefore the plus sign must be taken in (36). Since the angular momentum increases the gravitational mass, and since there can be no angular momentum without mass, $a^2 < m^2$. Thus, there exists no spacetime for $a^2 \geqslant m^2$. To reduce to (2) equation (36) becomes,

$$\beta = \sqrt{C(r_0)} = m + \sqrt{m^2 - a^2 \cos^2}\,, \tag{37}$$

$$a^2 < m^2\,.$$

Equation (34) must be asymptotically flat, so

$$r \to \infty \Rightarrow \frac{C(r)}{\left(r - r_0\right)^2} \to 1\,. \tag{38}$$

Therefore,

$$\lim_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1\,. \tag{39}$$

Since $C(r)$ behaves like $\left(r - r_0\right)^2$, must make (34) singular only at $r = r_0$, and $C'(r) > 0 \;\forall\; r > r_0$, $C(r)$ is strictly monotonically increasing, so

$$C'(r) > 0,\; r > r_0\,. \tag{40}$$

Consequently, the conditions that $C(r)$ must satisfy to render a solution to (34) are:

1. $C'(r) > 0,\; r > r_0$;
2. $\lim\limits_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1$;
3. $\sqrt{C(r_0)} = \beta = m + \sqrt{m^2 - a^2\cos^2\theta},\; a^2 < m^2$.

I call the foregoing the Metric Conditions of Abrams (MCA) for the rotating point-mass.

The only form admissible for $C(r)$ in (34) that satisfies MCA and is reducible to (2) is,

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}}\,, \tag{41}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,, \quad a^2 < m^2\,,$$

$$r_0 \in (\Re - \Re^-) \quad n \in \Re^+\,.$$

Associated with (31) are the so-called "horizons" and "static limits" given respectively by,

$$r_h = m \pm \sqrt{m^2 - a^2}\,, \quad r_b = m \pm \sqrt{m^2 - a^2\cos^2\theta}\,, \tag{42}$$

where r_h is obtained from (31) by setting its $\Delta = 0$, and r_b by setting its $g_{00} = 0$. Conventionally equations (42) are rather arbitrarily restricted to,

$$r_h = m + \sqrt{m^2 - a^2}\,, \quad r_b = m + \sqrt{m^2 - a^2\cos^2\theta}\,, \tag{43}$$

$$a^2 < m^2\,.$$

For (34), $\Delta \geq 0$ and so there is no static limit, since by (41),

$$C_n(r_0) = \beta^2 \Rightarrow\,, \tag{44}$$

$$\Rightarrow \Delta(r_0) = \beta^2 - \alpha\beta + a^2\,.$$

Solving (41) i .e .

$$\sqrt{C_n(r)} = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{1}{n}}\,, \tag{45}$$

gives the r-parameter location of a spacetime event,

$$r = \left[C_n(r)^{\frac{1}{2n}} - \beta^n\right]^{\frac{1}{n}} + r_0\,. \tag{46}$$

When $a = 0$, equation (46) reduces to $r_0 = \alpha$, as expected for the non-rotating point-mass.

From (46) it is concluded that there exists no spacetime drag effect for the rotating point-mass and no ergosphere.

The generalisation of equation (34) is then,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(C + a^2\right)d\varphi - a dt\right]^2 - \\ &- \frac{\rho^2}{\Delta}\frac{C'^2}{4C}dr^2 - \rho^2 d\theta^2\,, \end{aligned} \tag{47}$$

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}}, \quad n \in \Re^+,$$

$$r_0 \in (\Re - \Re^-)\,, \quad \beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,, \quad a^2 < m^2\,,$$

$$a = \frac{L}{m}\,, \quad \rho^2 = C_n + a^2\cos^2\theta\,,$$

$$\Delta = C_n - \alpha\sqrt{C_n} + a^2\,,$$

$$r_0 < r < \infty\,.$$

Equation (47) is regular $\forall\; r > r_0$, and $g_{00} = 0$ only when $r = r_0$. There is no event horizon and therefore no Kerr black hole.

By (47) the correct expression for the Kerr solution (31) is,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2\,, \end{aligned} \tag{48}$$

$$\Delta = r^2 - r\alpha + a^2 , \quad a = \frac{L}{m} , \quad \rho^2 = r^2 + a^2 \cos^2\theta ,$$
$$a^2 < m^2 , \quad m + \sqrt{m^2 - a^2\cos^2\theta} < r < \infty .$$

When $a = 0$ in (48) the Droste/Weyl solution (33) is recovered.

4 The vacuum field of the rotating point-charge

The Kerr-Newman solution is, in relativistic units,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 , \end{aligned} \tag{49}$$
$$a = \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta , \quad \Delta = r^2 - r\alpha + a^2 + q^2 ,$$
$$0 < r < \infty .$$

By applying the analytic technique of section 3, the general solution for the rotating point-charge is found to be,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(C_n + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}\frac{C_n'^{\,2}}{4C_n}dr^2 - \rho^2 d\theta^2 , \\ C_n(r) &= \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}} , \quad n \in \Re^+ , \\ r_0 &\in (\Re - \Re^-), \quad \beta = m + \sqrt{m^2 - (q^2 + a^2\cos^2\theta)} , \\ a^2 + q^2 &< m^2 , \quad a = \frac{L}{m}, \quad \rho^2 = C_n + a^2\cos^2\theta , \\ \Delta &= C_n - \alpha\sqrt{C_n} + q^2 + a^2 , \\ r_0 &< r < \infty . \end{aligned} \tag{50}$$

Equations (50) give the overall general solution to Einstein's vacuum field when $\Lambda = 0$. The associated Metric Conditions of Abrams (MCA) for the rotating point-charge are,

1. $C_n'(r) > 0, \ r > r_0$;
2. $\lim\limits_{r\to\infty} \dfrac{C_n(r)}{\left(r - r_0\right)^2} = 1$;
3. $\sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - (q^2 + a^2\cos^2\theta)}$, $a^2 + q^2 < m^2$.

From (50) it is concluded that there exists no spacetime drag effect for the rotating point-charge, and no ergosphere.

Equation (50) is regular $\forall\, r > r_0$, and $g_{00} = 0$ only when $r = r_0$; $r_h \equiv r_0$. When $a = 0$ in (50) the general solution for the point-charge (15) is recovered. If both $a = 0$ and $q = 0$ in (50) the general solution (2) for the simple Schwarzschild point-mass is recovered. There is no event horizon and therefore no Kerr-Newman black hole.

By (50) the correct expression for the Kerr-Newman solution (49) is,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 , \\ a &= \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - r\alpha + a^2 + q^2 , \\ q^2 + a^2 &< m^2, \quad m + \sqrt{m^2 - (q^2 + a^2\cos^2\theta)} < r < \infty . \end{aligned} \tag{51}$$

If $a = 0$ in (51) the correct expression for the Reissner-Nordstrom solution (16) is recovered. If $q = 0$ in (51) the correct expression for the Kerr solution (48) is recovered. If both $a = 0$ and $q = 0$ in (51) the correct expression for Hilbert's (i. e. the Droste/Weyl) solution (33) is recovered.

5 The Einstein-Rosen Bridge

The Einstein-Rosen Bridge [9] is obtained by substituting into the Droste/Weyl solution (33) the transformation,

$$u^2 + \alpha = r , \tag{52}$$

which carries (33) into,

$$\begin{aligned} ds^2 &= \left[\frac{u^2}{(u^2 + \alpha)}\right]dt^2 - \\ &- 4\left(u^2 + \alpha\right)du^2 - \left(u^2 + \alpha\right)^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \\ &-\infty < u < \infty . \end{aligned} \tag{53}$$

Metric (53) is singular nowhere, and as u runs $-\infty$ to 0 and 0 to $+\infty$, r runs $+\infty$ to α then α to $+\infty$, thereby allegedly removing the singularity at $r = \alpha$. However, (53) is inadmissible by (2): (52) is not a valid form for $C_n(r)$ for the simple point-mass. This manifests in a violation of MCA. Indeed,

$$\lim_{u\to\infty}\frac{C(u)}{u^2} = \lim_{u\to\infty}\frac{\left(u^2 + \alpha\right)^2}{u^2} \to \infty , \tag{54}$$

so the far field is not flat. The Einstein-Rosen Bridge is therefore invalid.

6 Interacting black holes and the Michell-Laplace dark body

It is quite commonplace for black holes to be posited as members of binary systems, either as a hole and a star, or as two holes. Even colliding black holes are frequently alleged (see e. g. [10]). Such ideas are inadmissible, even if the existence of black holes were allowed. All solutions to the Einstein field equations involve a single gravitating body and a test particle. No solutions are known that address two bodies of comparable mass. It is not even known if solutions to such configurations exist. One simply cannot talk of black hole binaries or colliding black holes unless it can be shown, as pointed out by McVittie [11], that Einstein's field equations admit of solutions for such configurations. Without such an existence theorem these ideas are without any theoretical basis. McVittie's existence theorem however, does not exist, because the black hole does not exist in the formalism of General Relativity. It is also commonly claimed that the Michell-Laplace dark body is a kind of black hole or an anticipation of the black hole [10, 12]. This claim is utterly false as there always exists a class of observers who can see a Michell-Laplace dark body [11]: ipso facto, it is not a black hole. Consequently, there is no theoretical basis whatsoever for the existence of black holes. If such an object is ever detected then both Newton and Einstein would be invalidated.

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

References

1. Crothers S. J. On the general solution to Einstein's vacuum field and it implications for relativistic degeneracy. *Progress in Physics*, v. 1, 2005, 68–73.
2. Abrams L. S. Black holes: the legacy of Hilbert's error. *Can. J. Phys.*, v. 67, 1989, 919 (see also in arXiv: gr-qc/0102055).
3. Nordstrom G. *K. Nederlandse Akad. Van Wetenschappen, Proceedings*, v. 20, 1918, 1238.
4. Droste J. The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field. *Ned. Acad. Wet., S. A.*, v. 19, 1917, 197 (see also in www.geocities.com/theometria/Droste.pdf).
5. Weyl H. Zur Gravitationstheorie. *Ann. d. Phys. (Leipzig)*, v. 54, 1917, 117.
6. Abrams L. S. The total space-time of a point charge and its consequences for black holes. *Int. J. Theor. Phys.*, v. 35, 1996, 2661 (see also in arXiv: gr-qc/0102054).
7. Hilbert, D. *Nachr. Ges. Wiss. Gottingen, Math. Phys. Kl.*, v. 53, 1917 (see also in arXiv: physics/0310104).
8. Schwarzschild K. On the gravitational field of a mass point according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 189 (see also in arXiv: physics/9905030).
9. Einstein A., Rosen, N. The particle problem in the General Theory of Relativity. *Phys. Rev.* v. 48, 1935, 73.
10. Misner C. W., Thorne K. S., Wheeler J. A. Gravitation. W. H. Freeman and Company, New York, 1973.
11. McVittie G. C. Laplace's alleged "black hole". *The Observatory*, v. 98, 1978, 272 (see also in www.geocities.com/theometria/McVittie.pdf).
12. Hawking S., Ellis G. F. R. The large scale structure of space-time. Cambridge University Press, Cambridge, 1973.

Experiments with Rotating Collimators Cutting out Pencil of α-Particles at Radioactive Decay of ^{239}Pu Evidence Sharp Anisotropy of Space

Simon E. Shnoll*, Ilia A. Rubinshtein†, Konstantin I. Zenchenko*, Vladimir A. Shlekhtarev*, Alexander V. Kaminsky, Alexander A. Konradov‡, Natalia V. Udaltsova*

*Institute of Theoretical and Experimental Biophysics, Russian Academy of Sciences, Pushchino, Moscow Region, 142290, Russia
E-mail: shnoll@iteb.ru

†Skobeltsyn Inst. of Nuclear Physics, Moscow State Univ., Moscow, 119992, Russia
‡Institute of Biochemical Physics, Russian Academy of Sciences, Moscow, Russia

As shown in our previous experiments fine structure of histograms of α-activity measurements serve as a sensitive tool for investigation of cosmo-physical influences. Particularly, the histograms structure is changed with the period equal to sidereal (1436 min) and solar (1440) day. It is similar with the high probability in different geographic points at the same local (longitude) time. More recently investigations were carried out with collimators, cutting out separate flows of total α-particles flying out at radioactive decay of ^{239}Pu. These experiments revealed sharp dependence the histogram structure on the direction of α-particles flow.
In the presented work measurements were made with collimators rotating in the plane of sky equator. It was shown that during rotation the shape of histograms changes with periods determined by number of revolution. These results correspond to the assumption that the histogram shapes are determined by a picture of the celestial sphere, and also by interposition of the Earth, the Sun and the Moon.

1 Introduction

It has been earlier shown, that the fine structure of statistical distributions of measurement results of processes of various nature depends on cosmo-physical factors. The shape of corresponding histograms changes with the period equal to sidereal and solar day, i. e. 1436 and 1440 minutes [1, 2, 3, 4].

These periods disappeared at measurements of alpha-activity of ^{239}Pu samples near the North Pole [5]. These results corresponded to the assumption of association of the histogram shapes with a picture of the celestial sphere, and also with interposition of the Earth, the Sun and the Moon.

However, at measurements at latitude 54° N (in Pushchino), absence of the daily period [9] also was revealed when using collimators restricting a flow of the alpha particles of radioactive decay at the direction to the north celestial pole. This result meant, that the question is not about dependence on a picture of the celestial sphere above a place of measurements, but about a direction of alpha particles flow.

In experiments with two collimators, directed one to the East and another to the West, it was revealed, that histograms of the similar shape at measurements with west collimator appear at 718 minutes (half of sidereal day) later then ones registered with East collimator [9]. Therefore, as acquired, the space surrounding the Earth is highly anisotropic, and this anisotropy is connected basically to a picture of the celestial sphere (sphere of distant stars).

This suggestion has been confirmed in experiments with collimators, rotated counter-clockwise, west to east (i. e. in a direction of rotation of the Earth), as well as clockwise (east to west). The description of these experiments is given further.

2 Methods

As well as earlier, the basic object of these of research was a set of histograms constructed by results of measurements of alpha-activity of samples ^{239}Pu.

Experimental methods, the devices for alpha-radioactivity measurements of ^{239}Pu samples with collimators, and also construction of histograms and analysis of its shapes, are described in details in the earlier publications [2, 3, 8]. Measurements of number of events of radioactive decay were completed by device designed by one of the authors (I. A. R.). In this device the semi-conductor detector (photo diode) is placed after collimator, restricting a flow of the alpha particles in a certain direction. Results of measurements, consecutive numbers of events of the decay registered by the detector in 1-second intervals, are stored in computer archive.

Depending on specific targets, a time sequence of 1-second measurements was summarized to consecutive values of activity for 6, 15 or 60 seconds. Obtained time series were separated into consecutive pieces of 60 numbers in each. A histogram was built for each piece of 60 numbers. Histograms were smoothed using the method of moving averages for the greater convenience of a visual estimation of similarity of their shapes (more details see in [8, 9]). Comparison of histograms was performed using auxiliary computer program by Edwin Pozharski [8].

A mechanical device designed by one of the authors (V. A. Sh.) was used in experiments with rotation of collimators. In this device the measuring piece of equipment with collimator was attached to the platform rotated in a plane of Celestial Equator.

3 Results

Three revolutions of collimator counter-clockwise in a day.

The diurnal period of increase in frequency of histograms with similar shape means dependence of an observable picture on rotation of the Earth.

The period of approximately 24 hours or with higher resolution 1436 minutes is also observed at measurements using collimators restricting a flow of alpha particles in a certain direction [9, 10]. Therefore, the fine structure of distribution of results of measurements depends on what site of celestial sphere the flow of alpha particles is directed to. Studies of shapes of histograms constructed by results of measurements using rotated collimators testify to the benefit of this assumption.

The number of the "diurnal" cycles at clockwise rotation should be one less then numbers of collimator revolutions because of compensation of the Earth rotation.

At May 28 through June 10, 2004, we have performed measurements of alpha-activity of a sample ^{239}Pu at 3 collimator revolutions a day, and also, for the control, simultaneous measurements with motionless collimator, directed to the West. Results of these measurements are presented on Fig. 1–4. At these figures a dependence of frequency histograms of the same shape on size of time interval between similar histograms is shown.

Fig.1 shows results of comparison of 60-minute histograms, constructed at measurements with motionless collimator. A typical dependence repeatedly obtained in earlier studies is visible at the Fig. 1: histograms of the same shape most likely appear at the nearest intervals of time ("effect of a near zone") and in one day (24 hours).

Fig. 2 presents the result of comparison of 60-minute histograms constructed at measurements with collimator rotated 3 times a day counter-clockwise in a plane of celestial equator.

As you can see at the Fig. 2, at three revolutions of collimator counter-clockwise, the frequency of similar histograms fluctuates with the period of 6 hours: peaks correspond to the intervals of 6, 12, 18 and 24 hours.

24-hour period at a higher resolution consists of two components. It is visible by comparison of one-minute histograms shown at Fig. 3 for measurements with motionless collimator and at Fig. 4 for measurements at 3 collimator revolutions counter-clockwise. At measurements with motionless collimator (Fig. 3) there are two peaks — one corresponds to sidereal day (1436 minutes), the second, which is less expressed, corresponds to solar day (1440 minutes).

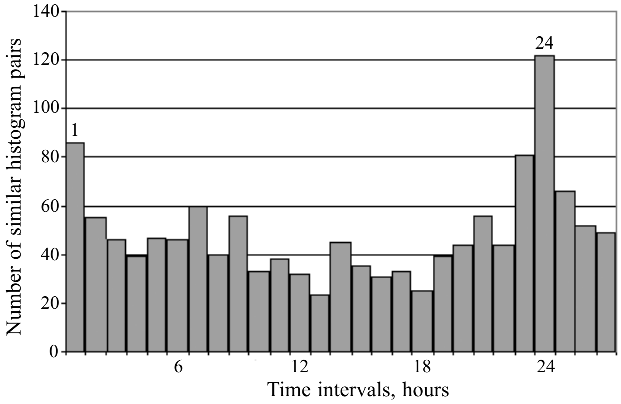


Fig. 1: Frequency of similar 60-minute histograms against the time interval between histograms. Measurements of alpha-activity of a ^{239}Pu sample by detector with motionless collimator directed to the West, June 8–30, 2004.

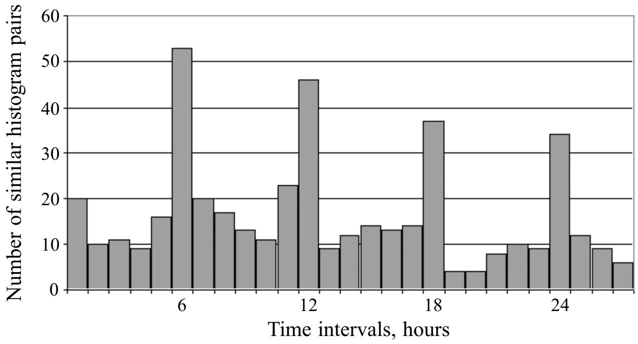


Fig. 2: Frequency of similar 60-minute histograms against the time interval between histograms. Measurements of alphaactivity of a ^{239}Pu sample by detector with collimator, making three revolutions counter-clockwise (west to east) in a day.

You can see at Fig. 4 that 6-hour period at measurements with three revolutions of collimator also has two components. The first 6-hour maximum has two joint peaks of 359 and 360 minutes. The second 12-hour maximum has two peaks of 718 and 720 minutes. The third maximum (18 hours) has two peaks of 1077 and 1080 minutes. And the fourth one (24 hours) has two peaks of 1436 and 1440 minutes.

Results of these experiments confirm a conclusion according to which a change in histogram shape is caused by change in direction of alpha particles flow in relation to distant stars and the Sun (and other space objects). This conclusion is supported also by results of experiments with rotation of collimator clockwise.

In these experiments collimator made one revolution a day clockwise, east to west, i. e. against daily rotation of the

Earth. As a result, the flow of alpha particles all the time was directed to the same point of celestial sphere. We expected in this case disappearance the diurnal period of frequency of similar histograms. This expectation was proved to be true.

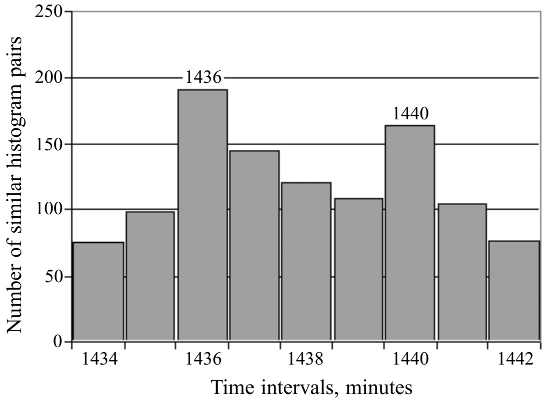


Fig. 3: 24-hour period of frequency of similar histograms with the one-minute resolution. Measurements of May 29 — June 1, 2004 by detector with motionless collimator directed to the West.

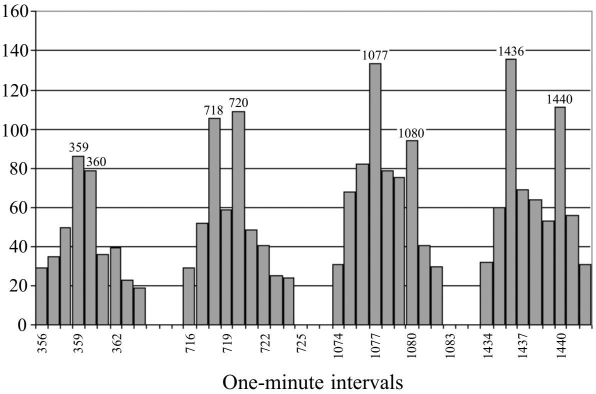


Fig. 4: Experiments with rotated collimators. Frequency of similar 1-minute histograms by time interval between them. Three revolutions a day counter-clockwise. Two components of the 6-hour period: sidereal and solar.

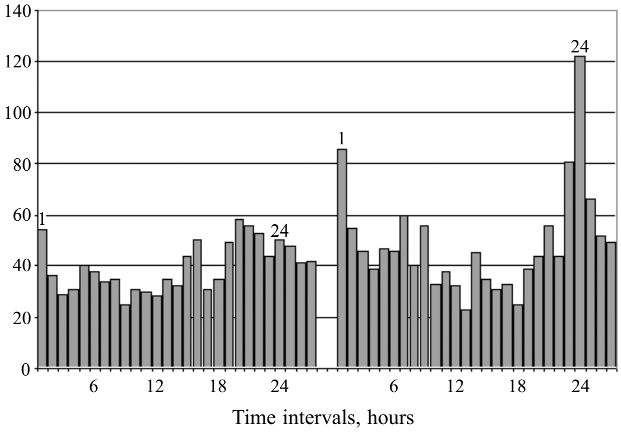


Fig. 5: 60-minutes histograms. Left: 1 revolution clockwise. Right: control, motionless collimator.

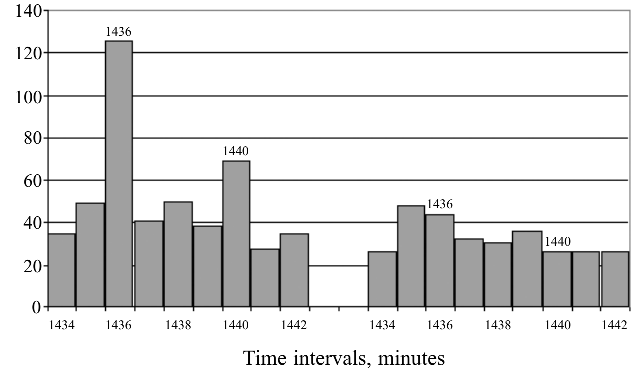


Fig. 6: One-minute histograms. Left: control, motionless collimator. Right: rotation 1 revolution clockwise (east to west).

On Fig. 5 and 6 one can see that in such experiments frequency of appearance of similar 60 minute and 1-minute histograms does not depend on time. At the same time at synchronous measurements with motionless collimator the usual dependence with the diurnal period and near zone effect is observed.

4 Discussion

Results of measurements with rotated collimator confirm a conclusion about dependence of fine structure of statistical distributions on a direction in space. This fine structure is defined by a spectrum of amplitudes of fluctuations of measured values. Presence of "peaks" and "hollows" at corresponding histograms suggests presence of the primary, allocated, "forbidden" and "permissible" values of amplitudes of fluctuations in each given moment [4]. Thus, a fine structure of statistical distributions presents a spectrum of the permissible amplitudes of fluctuations, and dependence of it on a direction in space shows sharp anisotropy of space.

It is necessary to emphasize, that the question is not about influence on the subject of measurement (in this case on radioactive decay). With accuracy of traditional statistical criteria, overall characteristics of distribution of radioactive decay measurements compliant with Poisson distribution [3]. Only the shape of histogram constructed for small sample size varies regularly. This regularity emerges in precise sidereal and solar periods of increase of frequency of similar histograms.

As shown above, the shape of histograms constructed by results of measurements of alpha-activity of samples ^{239}Pu, varies with the period determined by number of revolutions in relation to celestial sphere and the Sun. In experiments with collimator, which made three revolutions counter-clockwise, the "diurnal" period was equal to 6 hours (three revolutions of collimator and one revolution of the Earth was observed — in total 4 revolutions in relation to celestial sphere and the Sun give the period equal $24/4 = 6$ hours).

The result obtained in experiments with one revolution of collimator clockwise is not less important. The Earth rotation is compensated and a flow of alpha particles is directed all the time to the same point of celestial sphere. In these experiments the diurnal period was not observed at all.

The obtained results, though very clear ones, cause natural bewilderment.

Really, it is completely not obvious, by virtue of what reasons the spectrum of amplitudes of fluctuations of number of alpha particles, may depend on a direction of their flow in relation to celestial sphere and the Sun. The explanation of these phenomena probably demands essential change in general physical conceptions.

In such situation a dominant problem is to validate a reliability of the discussed phenomena. In aggregate of performed studies, we believe this task was completed .

5 Acknowledgements

S. E. Shnoll is indebted to M. N. Kondrashova and L. A. Blumenfeld for mutual understanding.

The authors are grateful to our colleagues, T. A. Zenchenko, D. P. Kharakoz, B. M. Vladimirsky, B. V. Komberg, V. K. Lyapidevskii for collaboration and for valuable discussions.

The vivid interest of V. P. Tikhonov to the problem studied and his generous financial support were essential.

References

1. Shnoll S. E., Kolombet V. A., Pozharski E. V., Zenchenko T. A., Zvereva I. M. and Konradov A. A. Realization of discrete states during fluctuations in macroscopic processes. *Physics-Uspehi*, 1998, v. 162(10), 1129–1140.
2. Shnoll S. E., Pozharski E. V., Zenchenko T. A., Kolombet V. A., Zvereva I. M. and Konradov A. A. Fine structure of distributions in measurements of different processes as affected by geophysical and cosmophysical factors. *Phys. and Chem. Earth A: Solid Earth and Geod.*, 1999, v. 24(8), 711–714.
3. Shnoll S. E., Zenchenko T. A., Zenchenko K. I., Pozharski E. V., Kolombet V. A. and Konradov A. A. Regular variation of the fine structure of statistical distributions as a consequence of cosmophysical agents. *Physics-Uspehi*, 2000, v. 43(2), 205–209.
4. Shnoll S. E. Discrete distribution patterns: arithmetic and cosmophysical origins of their macroscopic fluctuations. *Biophysics*, 2001, v. 46(5), 733–741.
5. Shnoll S. E., Rubinstein I. A., Zenchenko K. I., Zenchenko T. A., Konradov A. A., Shapovalov S. N., Makarevich A. V., Gorshkov E. S. and Troshichev O. A. Dependence of "macroscopic fluctuations" on geographic coordinates (by materials of Arctic and Antarctic expeditions). *Biophysics*, 2003, v. 48(5), 1123–1131.
6. Shnoll S. E., Zenchenko K. I., Berulis I. I., Udaltsova N. V., Zhirkov S. S. and Rubinstein I. A. Dependence of "macroscopic fluctuations" on cosmophysical factors. Spatial anisotropy. *Biophysics*, 2004, v. 49(1), 129–139.
7. Shnoll S. E., Zenchenko K. I., Berulis I. I., Udaltsova N. V. and Rubinstein I. A. Fine structure of histograms of alpha-activity measurements depends on direction of alpha particles flow and the Earth rotation: experiments with collimators. arXiv: physics/0412007.
8. Shnoll S.E., Kolombet V.A., Zenchenko T.A., Pozharskii E.V., Zvereva I.M. and Konradov A.A. Cosmophysical origin of "macroscopic fluctations". *Biophysics*, 1998, v. 43(5), 909–915.
9. Fedorov M. V., Belousov L. V., Voeikov V. L., Zenchenko T. A., Zenchenko K. I., Pozharskii E. V., Konradov A. A. and Shnoll S. E. Synchronous changes in dark current fluctuations in two separate photomultipliers in relation to Earth rotation. *Astrophysics and Space Science*, 2003, v. 283, 3–10.

There Is No Speed Barrier for a Wave Phase Nor for Entangled Particles

Florentin Smarandache
Dept. of Mathematics, University of New Mexico, 200 College Road, Gallup, NM 87301, USA
E-mail: fsmarandache@yahoo.com; smarand@unm.edu

In this short paper, as an extension and consequence of Einstein-Podolski-Rosen paradox and Bell's inequality, one promotes the hypothesis (it has been called the Smarandache Hypothesis [1, 2, 3]) that: There is no speed barrier in the Universe and one can construct arbitrary speeds, and also one asks if it is possible to have an infinite speed (instantaneous transmission)? Future research: to study the composition of faster-than-light velocities and what happens with the laws of physics at faster-than-light velocities?

This is the new version of an early article. That early version, based on a 1972 paper [4], was presented at the Universidad de Blumenau, Brazil, May–June 1993, in the Conference on "Paradoxism in Literature and Science"; and at the University of Kishinev, in December 1994. See that early version in [5].

1 Introduction

What is new in science (physics)?

According to researchers from the common group of the University of Innsbruck in Austria and US National Institute of Standards and Technology (starting from December 1997, Rainer Blatt, David Wineland et al.):

- Photon is a bit of light, the quantum of electromagnetic radiation (quantum is the smallest amount of energy that a system can gain or lose);
- Polarization refers to the direction and characteristics of the light wave vibration;
- If one uses the entanglement phenomenon, in order to transfer the polarization between two photons, then: whatever happens to one is the opposite of what happens to the other; hence, their polarizations are opposite of each other;
- In quantum mechanics, objects such as subatomic particles do not have specific, fixed characteristic at any given instant in time until they are measured;
- Suppose a certain physical process produces a pair of entangled particles A and B (having opposite or complementary characteristics), which fly off into space in the opposite direction and, when they are billions of miles apart, one measures particle A; because B is the opposite, the act of measuring A instantaneously tells B what to be; therefore those instructions would somehow have to travel between A and B faster than the speed of light; hence, one can extend the Einstein-Podolsky-Rosen paradox and Bell's inequality and assert that the light speed is not a speed barrier in the Universe.

Such results were also obtained by: Nicolas Gisin at the University of Geneva, Switzerland, who successfully teleported quantum bits, or qubits, between two labs over 2 km of coiled cable. But the actual distance between the two labs was about 55 m; researchers from the University of Vienna and the Austrian Academy of Science (Rupert Ursin et al. have carried out successful teleportation with particles of light over a distance of 600 m across the River Danube in Austria); researchers from Australia National University and many others [6, 7, 8].

2 Scientific hypothesis

We even promote the hypothesis that:

> There is no speed barrier in the Universe, which would theoretically be proved by increasing, in the previous example, the distance between particles A and B as much as the Universe allows it, and then measuring particle A.

It has been called the *Smarandache Hypotesis* [1, 2, 3].

3 An open question now

If the space is infinite, is the maximum speed infinite?

"This Smarandache hypothesis is controversially interpreted by scientists. Some say that it violates the theory of relativity and the principle of causality, others support the ideas that this hypothesis works for particles with no mass or imaginary mass, in non-locality, through tunneling effect, or in other (extra-) dimension(s)." Kamla John, [9].

Scott Owens' answer [10] to Hans Gunter in an e-mail from January 22, 2001 (the last one forwarded it to the author): "It appears that the only things the Smarandache hypothesis can be applied to are entities that do not have real mass or energy or information. The best example I can come up with is the difference between the wavefront velocity of a photon and the phase velocity. It is common for the phase velocity to exceed the wavefront velocity c, but that does not mean that any real energy is traveling faster than c. So, while it is possible to construct arbitrary speeds from zero in infinite, the superluminal speeds can only apply to purely imaginary entities or components."

Would be possible to accelerate a photon (or another particle traveling at, say, $0.99c$ and thus to get speed greater than c (where c is the speed of light)?

4 Future possible research

It would be interesting to study the composition of two velocities v and u in the cases when:

$$\mathrm{v} < c \text{ and } u = c;$$
$$\mathrm{v} = c \text{ and } u = c;$$
$$\mathrm{v} > c \text{ and } u = c;$$
$$\mathrm{v} > c \text{ and } u > c;$$
$$\mathrm{v} < c \text{ and } u = \infty;$$
$$\mathrm{v} = c \text{ and } u = \infty;$$
$$\mathrm{v} > c \text{ and } u = \infty;$$
$$\mathrm{v} = \infty \text{ and } u = \infty.$$

What happens with the laws of physics in each of these cases?

References

1. Motta L. Smarandache Hypothesis: Evidences, Implications, and Applications. *Proceedings of the Second International Conference on Smarandache Type Notions In Mathematics and Quantum Physics*, University of Craiova, Craiova, Romania, December 21–24, 2000 (see the e-print version in the web site at York University, Canada, http://at.yorku.ca/cgi-bin/amca/caft-03).
2. Motta L. and Niculescu G., editors. *Proceedings of the Second International Conference on Smarandache Type Notions In Mathematics and Quantum Physics*, American Research Press, 2000.
3. Weisstein E. W. Smarandache Hypothesis. *The Encyclopedia of Physics*, Wolfram Research (http://scienceworld.wolfram.com/physics/SmarandacheHypothesis.htm).
4. Smarandache F. Collected Papers. Vol. III, Abaddaba Publ. Hse., Oradea, Romania, 2000, 158.
5. Smarandache F. There is no speed barrier in the Universe. *Bulletin of Pure and Applied Sciences*, Delhi, India, v. 17D (Physics), No. 1, 1998, 61.
6. Rincon P. Teleportation breakthrough made. BBC News Online, 2004/06/16.
7. Rincon P. Teleportation goes long distance. BBC News Online, 2004/08/18.
8. Whitehouse D. Australian teleport breakthrough. BBC News Online, 2002/06/17.
9. Kamla J. Private communications. 2001.
10. Owens S. Private communications. 2001.

The Journal on Advanced Studies in Theoretical and Experimental Physics, including Related Themes from Mathematics

PROGRESS IN PHYSICS

A quarterly issue scientific journal, registered with the Library of Congress (DC).
This journal is peer reviewed and included in the abstracting and indexing coverage of:
Mathematical Reviews and MathSciNet (AMS, USA), DOAJ of Lund University (Sweden),
Referativnyi Zhurnal VINITI (Russia), etc.

ISSN: 1555-5534 (print)
ISSN: 1555-5615 (online)
4 issues per annum

Electronic version of this journal:
http://www.geocities.com/ptep_online

To order printed issues of this journal, contact the Editor in Chief.

This journal is powered by BaKoMa-TEX

A variety of books can be downloaded free from the E-library of Science:
http://www.gallup.unm.edu/∼smarandache

ISSN: 1555-5534 (print)
ISSN: 1555-5615 (online)

Standard Address Number: 297-5092
Printed in the United States of America

JULY 2005 **VOLUME 2**

CONTENTS

Information for Authors and Subscribers

Progress in Physics has been created for publications on advanced studies in theoretical and experimental physics, including related themes from mathematics. All submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

All submissions should be designed in LaTeX format using *Progress in Physics* template. This template can be downloaded from *Progress in Physics* home page http://www.geocities.com/ptep_online. Abstract and the necessary information about author(s) should be included into the papers. To submit a paper, mail the file(s) to Chief Editor.

All submitted papers should be as brief as possible. Short articles are preferable.

All that has been accepted for the online issue of *Progress in Physics* is printed in the paper version of the journal. To order printed issues, contact Chief Editor.

This journal is non-commercial, academic edition. It is printed from private donations.

On the Geometry of the General Solution for the Vacuum Field of the Point-Mass

Stephen J. Crothers
Sydney, Australia
E-mail: thenarmis@yahoo.com

The black hole, which arises solely from an incorrect analysis of the Hilbert solution, is based upon a misunderstanding of the significance of the coordinate radius r. This quantity is neither a coordinate nor a radius in the gravitational field and cannot of itself be used directly to determine features of the field from its metric. The appropriate quantities on the metric for the gravitational field are the proper radius and the curvature radius, both of which are functions of r. The variable r is actually a Euclidean parameter which is mapped to non-Euclidean quantities describing the gravitational field, namely, the proper radius and the curvature radius.

1 Introduction

The variable r has given rise to much confusion. In the conventional analysis, based upon the Hilbert metric, which is almost invariably and incorrectly called the "Schwarzschild" solution, r is taken both as a coordinate and a radius in the spacetime manifold of the point-mass. In my previous papers [1, 2] on the general solution for the vacuum field, I proved that r is neither a radius nor a coordinate in the gravitational field (M_g, g_g), as Stavroulakis [3, 4, 5] has also noted. In the context of (M_g, g_g) r is a Euclidean parameter in the flat spacetime manifold (M_s, g_s) of Special Relativity. Insofar as the point-mass is concerned, r specifies positions on the real number line, the radial line in (M_s, g_s), not in the spacetime manifold of the gravitational field, (M_g, g_g). The gravitational field gives rise to a mapping of the *distance* $D = |r - r_0|$ between two *points* $r,\ r_0 \in \Re$ into (M_g, g_g). Thus, r becomes a *parameter* for the spacetime manifold associated with the gravitational field. If $R_p \in (M_g, g_g)$ is the proper radius, then the gravitational field gives rise to a mapping ψ,

$$\psi : |r - r_0| \in (\Re - \Re^-) \to R_p \in (M_g, g_g)\,, \qquad (A)$$

where $0 \leqslant R_p < \infty$ in the gravitational field, on account of R_p being a *distance* from the point-mass located at the point $R_p(r_0) \equiv 0$.

The mapping ψ must be obtained from the geometrical properties of the metric tensor of the solution to the vacuum field. The r-parameter location of the point-mass does not have to be at $r_0 = 0$. The point-mass can be located at any point $r_0 \in \Re$. A test particle can be located at any point $r \in \Re$. The point-mass and the test particle are located at the end points of an interval along the *real line* through r_0 and r. The distance between these points is $D = |r - r_0|$. In (M_s, g_s), r_0 and r may be thought of as describing 2-spheres about an origin $r_c = 0$, but only the distance between these 2-spheres enters into consideration. Therefore, if two test particles are located, one at any point on the 2-sphere $r_0 \neq 0$ and one at a point on the 2-sphere $r \neq r_0$ on the radial line through r_0 and r, the distance between them is the length of the radial interval between the 2-spheres, $D = |r - r_0|$. Consequently, the domain of both r_0 and r is the real number line. In this sense, (M_s, g_s) may be thought of as a parameter space for (M_g, g_g), because ψ maps the *Euclidean distance* $D = |r - r_0| \in (M_s, g_s)$ into the *non-Euclidean proper distance* $R_p \in (M_g, g_g)$: the radial line in (M_s, g_s) is precisely the real number line. Therefore, the required mapping is appropriately written as,

$$\psi : |r - r_0| \in (M_s, g_s) \to R_p \in (M_g, g_g)\,. \qquad (B)$$

In the pseudo-Euclidean (M_s, g_s) the polar coordinates are $r,\ \theta, \varphi$, but in the pseudo-Riemannian manifold (M_g, g_g) of the point-mass and point-charge, r is *not* the radial coordinate. Conventionally there is the persistent misconception that what are polar coordinates in Minkowski space must also be polar coordinates in Einstein space. This however, does not follow in any rigorous way. In (M_g, g_g) the variable r is nothing more than a real-valued parameter, of no physical significance, for the true radial quantities in (M_g, g_g). The parameter r *never* enters into (M_g, g_g) directly. Only in Minkowski space does r have a direct physical meaning, as mapping (B) indicates, where it is a radial coordinate. Henceforth, when I refer to the radial coordinate or r-parameter I always mean $r \in (M_s, g_s)$.

The solution for the gravitational field of the simple configurations of matter and charge requires the determination of the mapping ψ. The orthodox analysis has completely failed to understand this and has consequently failed to solve the problem.

The conventional analysis simply looks at the Hilbert metric and makes the following unjustified assumptions, tacitly or otherwise;

(a) *The variable r is a radius and/or coordinate of some kind in the gravitational field.*

(b) *The regions $0 < r < 2m$ and $2m < r < \infty$ are both valid.*

(c) *A singularity in the gravitational field must occur only where the Riemann tensor scalar curvature invariant (Kretschmann scalar) $f = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded.*

The orthodox analysis has never proved these assumptions, but nonetheless simply takes them as given, finds for itself a curvature singularity at $r = 0$ in terms of f, and with legerdemain reaches it by means of an *ad hoc* extension in the ludicrous Kruskal-Szekeres formulation. However, the standard assumptions are incorrect, which I shall demonstrate with the required mathematical rigour.

Contrary to the usual practise, one *cannot* talk about extensions into the region $0 < r < 2m$ or division into R and T regions until it has been rigorously established that the said regions are valid to begin with. One *cannot* treat the r-parameter as a radius or coordinate of any sort in the gravitational field without first demonstrating that it is such. Similarly, one *cannot* claim that the scalar curvature must be unbounded at a singularity in the gravitational field until it has been demonstrated that this is truly required by Einstein's theory. Mere *assumption* is *not* permissible.

2 The basic geometry of the simple point-mass

The usual metric g_s of the spacetime manifold (M_s, g_s) of Special Relativity is,

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \tag{1}$$

The foregoing metric can be statically generalised for the simple (i. e. non-rotating) point-mass as follows,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \tag{2a}$$

$$A, B, C > 0 ,$$

where A, B, C are analytic functions. I emphatically remark that *the geometric relations between the components of the metric tensor of (2a) are precisely the same as those of (1)*.

The standard analysis writes (2a) as,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \tag{2b}$$

and claims it the most general, which is incorrect. The form of $C(r)$ cannot be pre-empted, and must in fact be rigorously determined from the general solution to (2a). The physical features of (M_g, g_g) must be determined exclusively by means of the resulting $g_{\mu\mu} \in (M_g, g_g)$, *not* by foisting upon (M_g, g_g) the interpretation of elements of (M_s, g_s) in the misguided fashion of the orthodox relativists who, having written (2b), incorrectly treat r in (M_g, g_g) precisely as the r in (M_s, g_s).

With respect to (2a) I identify the coordinate radius, the r-parameter, the radius of curvature, and the proper radius as follows:

(a) The coordinate radius is $D = |r - r_0|$.

(b) The r-parameter is the variable r.

(c) The radius of curvature is $R_c = \sqrt{C(r)}$.

(d) The proper radius is $R_p = \int \sqrt{B(r)}dr$.

The orthodox motivation to equation (2b) is to evidently obtain the circumference χ of a great circle, $\chi \in (M_g, g_g)$ as,

$$\chi = 2\pi r ,$$

to satisfy its unproven assumptions about r. But this equation is only formally the same as the equation of a circle in the Euclidean plane, because in (M_g, g_g) it describes a non-Euclidean great circle and therefore does not have the same meaning as the equation for the ordinary circle in the Euclidean plane. The orthodox assumptions distort the fact that r is only a real parameter in the gravitational field and therefore that (2b) is not a general, but a particular expression, in which case the form of $C(r)$ has been fixed to $C(r) = r^2$. Thus, the solution to (2b) can only produce a particular solution, not a general solution in terms of $C(r)$, for the gravitational field. Coupled with its invalid assumptions, the orthodox relativists obtain the Hilbert solution, a correct *particular form* for the metric tensor of the gravitational field, but interpret it incorrectly with such a great thoroughness that it defies rational belief.

Obviously, the spatial component of (1) describes a sphere of radius r, centred at the point $r_0 = 0$. On this metric $r \geqslant r_0$ is usually assumed. Now in (1) the distance D between two points on a radial line is given by,

$$D = |r_2 - r_1| = r_2 - r_1 . \tag{3}$$

Furthermore, owing to the "origin" being usually fixed at $r_1 = r_0 = 0$, there is no distinction between D and r. Hence r is both a coordinate *and* a radius (distance). However, the correct description of points by the spatial part of (1) must still be given in terms of *distance*. Any point in any direction is specified by its *distance* from the "origin". It is this distance which is the important quantity, not the coordinate. It is simply the case that on (1), in the usual sense, the distance and the coordinate are identical. Nonetheless, the distance from the designated "origin" is still the important quantity, not the coordinate. It is therefore clear that the designation of an origin is arbitrary and one can select *any* $r_0 \in \Re$ as the origin of coordinates. Thus, (1) is a special case of a general expression in which the origin of coordinates is arbitrary and the distance from the origin to another point does not take the same value as the coordinate designating it. The "origin" $r_0 = 0$ has *no intrinsic meaning*. The relativists and the mathematicians have evidently failed to understand this elementary geometrical fact. Consequently, they have managed to attribute to $r_0 = 0$ miraculous qualities of which it is not worthy, one of which is the formation of the black hole.

Equations (1) and (2a) are not sufficiently general and so their forms suppress their true geometrical characteristics. Consider two points P_1 and P_2 on a radial line in Euclidean 3-space. With the usual Cartesian coordinates let P_1 and P_2 have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. The distance between these points is,

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \\ = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2 + |z_1 - z_2|^2} \geqslant 0\,. \tag{4}$$

If $x_1 = y_1 = z_1 = 0$, D is usually called a radius and so written $D \equiv r$. However, one may take P_1 or P_2 as an origin for a sphere of radius D as given in (4). Clearly, a general description of 3-space must rightly take this feature into account. Therefore, the most general line-element for the gravitational field in quasi-Cartesian coordinates is,

$$ds^2 = F dt^2 - G\left(dx^2 + dy^2 + dz^2\right) - \\ - H\left(|x - x_0|dx + |y - y_0|dy + |z - z_0|dz\right)^2, \tag{5}$$

where $F, G, H > 0$ are functions of

$$D = \sqrt{|x - x_0|^2 + |y - y_0|^2 + |z - z_0|^2} = |r - r_0|\,,$$

and $P_0\,(x_0, y_0, z_0)$ is an arbitrary origin of coordinates for a sphere of radius D centred on P_0.

Transforming to spherical-polar coordinates, equation (5) becomes,

$$ds^2 = -H|r - r_0|^2 dr^2 + F dt^2 - \\ - G\left(dr^2 + |r - r_0|^2 d\theta^2 + |r - r_0|^2 \sin^2\theta d\varphi^2\right) = \\ = A(D)dt^2 - B(D)dr^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{6}$$

where $A, B, C > 0$ are functions of $D = |r - r_0|$. Equation (6) is just equation (2a), but equation (2a) has suppressed the significance of distance and the arbitrary origin and is therefore invariably taken with $D \equiv r \geqslant 0$, $r_0 = 0$.

In view of (6) the most general expression for (1) for a sphere of radius $D = |r - r_0|$, centred at some $r_0 \in \Re$, is therefore,

$$ds^2 = dt^2 - dr^2 - \left(r - r_0\right)^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) = \tag{7a}$$

$$= dt^2 - \frac{(r - r_0)^2}{|r - r_0|^2} dr^2 - \left|r - r_0\right|^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) = \tag{7b}$$

$$= dt^2 - \left(d|r - r_0|\right)^2 - \left|r - r_0\right|^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{7c}$$

The spatial part of (7) describes a sphere of radius $D = \left|r - r_0\right|$, centred at the arbitrary point r_0 and reaching to some point $r \in \Re$. Indeed, the curvature radius R_c of (7) is,

$$R_c = \sqrt{\left(r - r_0\right)^2} = \left|r - r_0\right|, \tag{8}$$

and the circumference χ of a great circle centred at r_0 and reaching to r is,

$$\chi = 2\pi\left|r - r_0\right|. \tag{9}$$

The proper radius (distance) R_p from r_0 to r on (7) is,

$$R_p = \int_0^{|r - r_0|} d\left|r - r_0\right| = \int_{r_0}^{r} \left[\frac{r - r_0}{\left|r - r_0\right|}\right] dr = \left|r - r_0\right|. \tag{10}$$

Thus $R_p \equiv R_c \equiv D$ on (7), owing to its pseudo-Euclidean nature.

It is evident by similar calculation that $r \equiv R_c \equiv R_p$ in (1). Indeed, (1) is obtained from (7) when $r_0 = 0$ and $r \geqslant r_0$ (although the absolute value is suppressed in (1) and (7a)). The geometrical relations between the components of the metric tensor are *inviolable*. Therefore, in the case of (1), the following obtain,

$$D = |r| = r\,,$$

$$R_c = \sqrt{|r|^2} = \sqrt{r^2} = r\,,$$

$$\chi = 2\pi|r| = 2\pi r\,, \tag{11}$$

$$R_p = \int_0^{|r|} d\,|r| = \int_0^{r} dr = r\,.$$

However, equation (1) hides the true arbitrary nature of the origin r_0. Therefore, the correct geometrical relations have gone unrecognized by the orthodox analysis. I note, for instance, that G. Szekeres [6], in his well-known paper of 1960, considered the line-element,

$$ds^2 = dr^2 + r^2 d\omega^2\,, \tag{12}$$

and proposed the transformation $\overline{r} = r - 2m$, to allegedly carry (12) into,

$$ds^2 = d\overline{r}^2 + \left(\overline{r} - 2m\right)^2 d\omega^2\,. \tag{13}$$

The transformation to (13) by $\overline{r} = r - 2m$ is incorrect: by it Szekeres should have obtained,

$$ds^2 = d\overline{r}^2 + \left(\overline{r} + 2m\right)^2 d\omega^2\,. \tag{14}$$

If one sets $r = \overline{r} - 2m$, then (13) obtains from (12). Szekeres then claims on (13),

> "*Here we have an apparent singularity on the sphere $\overline{r} = 2m$, due to a spreading out of the origin over a sphere of radius $2m$. Since the exterior region $\overline{r} > 2m$ represents the whole of Euclidean space (except the origin), the interior $\overline{r} < 2m$ is entirely disconnected from it and represents a distinct manifold.*"

His claims about (13) are completely false. He has made an incorrect assumption about the origin. His equation (12) describes a sphere of radius r centred at $r = 0$, being identical to the spatial component of (1). His equation (13) is precisely the spatial component of equation (7) with $r_0 = 2m$ and $r \geqslant r_0$, and therefore actually describes a sphere of radius $D = \overline{r} - 2m$ centred at $\overline{r}_0 = 2m$. His claim that $\overline{r} = 2m$ describes a sphere is due to his invalid assumption that $\overline{r} = 0$ has some intrinsic meaning. It did not come from his transformation. The claim is false. Consequently there is no interior region at all and no distinct manifold anywhere. All Szekeres did unwittingly was to move the origin for a sphere from the coordinate value $r_0 = 0$ to the coordinate value $r_0 = 2m$. In fact, he effectively repeated the same error committed by Hilbert [8] in 1916, an error, which in one guise or another, has been repeated relentlessly by the orthodox theorists.

It is now plain that r is neither a radius nor a coordinate in the metric (6), but instead gives rise to a *parameterization* of the relevant radii R_c and R_p on (6).

Consider (7) and introduce a test particle at each of the *points* r_0 and r. Let the particle located at r_0 acquire mass. The coordinates r_0 and r do not change, however in the gravitational field (M_g, g_g) the distance between the point-mass and the test particle, and the radius of curvature of a great circle, centred at r_0 and reaching to r in the parameter space (M_s, g_s), will no longer be given by (11).

The solution of (6) for the vacuum field of a point-mass will yield a mapping of the Euclidean *distance* $D = |r - r_0|$ into a non-Euclidean proper radius $R_P(r)$ in the pseudo-Riemannian manifold (M_g, g_g), locally generated by the presence of matter at the r-parameter $r_0 \in (M_s, g_s)$, i. e. at the invariant point $R_p(r_0) \equiv 0$ in (M_g, g_g).

Transform (6) by setting,

$$R_c = \sqrt{C(D(r))} = \frac{\chi}{2\pi}\,, \qquad (15)$$

$$D = |r - r_0|\,.$$

Then (6) becomes,

$$ds^2 = A^*(R_c)dt^2 - B^*(R_c)dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \qquad (16)$$

In the usual way one obtains the solution to (16) as,

$$ds^2 = \left(\frac{R_c - \alpha}{R_c}\right)dt^2 - \left(\frac{R_c}{R_c - \alpha}\right)dR_c^2 - R_c^2(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

$$\alpha = 2m\,,$$

which by using (15) becomes,

$$ds^2 = \left(\frac{\sqrt{C} - \alpha}{\sqrt{C}}\right)dt^2 - \left(\frac{\sqrt{C}}{\sqrt{C} - \alpha}\right)\frac{C'^2}{4C}\left[\frac{r - r_0}{|r - r_0|}\right]^2 dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

that is,

$$ds^2 = \left(\frac{\sqrt{C} - \alpha}{\sqrt{C}}\right)dt^2 - \left(\frac{\sqrt{C}}{\sqrt{C} - \alpha}\right)\frac{C'^2}{4C}dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,, \qquad (17)$$

which is the line-element derived by Abrams [7] by a different method. Alternatively one could set $r = R_c$ in (6), as Hilbert in his work [8] effectively did, to obtain the familiar Droste/Weyl/(Hilbert) line-element,

$$ds^2 = \left(\frac{r - \alpha}{r}\right)dt^2 - \left(\frac{r}{r - \alpha}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \qquad (18)$$

and then noting, as did J. Droste [9] and A. Eddington [10], that r^2 can be replaced by a general analytic function of r without destroying the spherical symmetry of (18). Let that function be $C(D(r))$, $D = |r - r_0|$, and so equation (17) is again obtained. Equation (18) taken literally is an *incomplete* particular solution since the boundary on the r-parameter has not yet been rigorously established, but equation (17) provides a way by which the form of $C(D(r))$ might be determined to obtain a means by which all particular solutions, in terms of an infinite sequence, may be constructed, according to the general prescription of Eddington. Clearly, the correct form of $C(D(r))$ must naturally yield the Droste/Weyl/(Hilbert) solution, as well as the true Schwarzschild solution [11], and the Brillouin solution [12], amongst the infinitude of particular solutions that the field equations admit. (Fiziev [13] has also shown that there exists an infinite number of solutions for the point-mass and that the Hilbert black hole is not consistent with general relativity.)

In the gravitational field only the circumference χ of a great circle is a measurable quantity, from which R_c and R_p

are calculated. To obtain the metric for the field in terms of χ, use (15) in (17) to yield,

$$ds^2 = \left(1 - \frac{2\pi\alpha}{\chi}\right) dt^2 - \left(1 - \frac{2\pi\alpha}{\chi}\right)^{-1} \frac{d\chi^2}{4\pi^2} - \\ - \frac{\chi^2}{4\pi^2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \\ \alpha = 2m. \tag{19}$$

Equation (19) is independent of the r-parameter entirely. Since only χ is a measurable quantity in the gravitational field, (19) constitutes the correct solution for the gravitational field of the simple point-mass. In this way (19) is truely the *only* solution to Einstein's field equations for the simple point-mass.

The only assumptions about r that I make are that the point-mass is to be located somewhere, and that somewhere is r_0 in parameter space (M_s, g_s), the value of which must be obtained rigorously from the geometry of equation (17), and that a test particle is located at some $r \neq r_0$ in parameter space, where $r,\ r_0 \in \Re$.

The geometrical relationships between the components of the metric tensor of (1) must be precisely the same in (6), (17), (18), and (19). Therefore, the circumference χ of a great circle on (17) is given by,

$$\chi = 2\pi\sqrt{C(D(r))},$$

and the proper distance (proper radius) $R_p(r)$ on (6) is,

$$R_p(r) = \int \sqrt{B(D(r))}dr.$$

Taking $B(D(r))$ from (17) gives,

$$R_p(D) = \int \sqrt{\frac{\sqrt{C}}{\sqrt{C}-\alpha}\,\frac{C'}{2\sqrt{C}}}\,dr = \\ = \sqrt{\sqrt{C(D)}\left(\sqrt{C(D)}-\alpha\right)} + \\ + \alpha\ln\left|\frac{\sqrt{\sqrt{C(D)}} + \sqrt{\sqrt{C(D)}-\alpha}}{K}\right|, \\ D = |r - r_0|, \\ K = \text{const.} \tag{20}$$

The relationship between r and R_p is,

$$as\ r \to r_0^{\pm},\ R_p(r) \to 0^+,$$

or equivalently,

$$as\ D \to 0^+,\ R_p(r) \to 0^+,$$

where r_0 is the parameter space location of the point-mass. Clearly $0 \leqslant R_p < \infty$ always and the point-mass is invariantly located at $R_p(r_0) \equiv 0$ in (M_g, g_g), a manifold with boundary.

From (20),

$$R_p(r_0) \equiv 0 = \sqrt{\sqrt{C(r_0)}\left(\sqrt{C(r_0)}-\alpha\right)} + \\ + \alpha\ln\left|\frac{\sqrt{\sqrt{C(r_0)}} + \sqrt{\sqrt{C(r_0)}-\alpha}}{K}\right|,$$

and so,

$$\sqrt{C(r_0)} \equiv \alpha,\ K = \sqrt{\alpha}.$$

Therefore (20) becomes

$$R_p(r) = \sqrt{\sqrt{C(|r-r_0|)}\left(\sqrt{C(|r-r_0|)}-\alpha\right)} + \\ + \alpha\ln\left|\frac{\sqrt{\sqrt{C(|r-r_0|)}} + \sqrt{\sqrt{C(|r-r_0|)}-\alpha}}{\sqrt{\alpha}}\right|, \\ r, r_0 \in \Re, \tag{21}$$

and consequently for (19),

$$2\pi\alpha < \chi < \infty.$$

Equation (21) is the required mapping. One can see that r_0 cannot be determined: in other words, r_0 is entirely arbitrary. One also notes that (17) is consequently singular only when $r = r_0$ in which case $g_{00} = 0$, $\sqrt{C_n(r_0)} \equiv \alpha$, and $R_p(r_0) \equiv 0$. There is no value of r that makes $g_{11} = 0$. One therefore sees that the condition for singularity in the gravitational field is $g_{00} = 0$; indeed $g_{00}(r_0) \equiv 0$.

Clearly, contrary to the orthodox claims, r does not determine the geometry of the gravitational field directly. It is not a radius in the gravitational field. The quantity $R_p(r)$ is the non-Euclidean radial coordinate in the pseudo-Riemannian manifold of the gravitational field around the *point* $R_p = 0$, which corresponds to the parameter *point* r_0.

Now in addition to the established fact that, in the case of the simple (i .e . non-rotating) point-mass, the lower bound on the radius of curvature $\sqrt{C(D(r_0))} \equiv \alpha$, $C(D(r))$ must also satisfy the no matter condition so that when $\alpha = 0$, $C(D(r))$ must reduce to,

$$C(D(r)) \equiv \left|r - r_0\right|^2 = (r - r_0)^2; \tag{22}$$

and it must also satisfy the far-field condition (spatially asympotically flat),

$$\lim_{r\to\pm\infty} \frac{C(D(r))}{\left(r - r_0\right)^2} \to 1. \tag{23}$$

When $r_0 = 0$ equation (22) reduces to,

$$C(|r|) \equiv r^2 \,,$$

and equation (23) reduces to,

$$\lim_{r \to \pm\infty} \frac{C(|r|)}{r^2} \to 1.$$

Furthermore, $C(r)$ must be a strictly monotonically increasing function of r to satisfy (15) and (21), and $C'(r) \neq 0$ $\forall\ r \neq r_0$ to satisfy (17) from (2a). The only general form for $C(D(r))$ satisfying all the required conditions (the Metric Conditions of Abrams [7]), from which an infinite sequence of particular solutions can be obtained [1] is,

$$C_n(D(r)) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}} \,, \qquad (24)$$

$$n \in \Re^+, \ \ r \in \Re, \ r_0 \in \Re \,,$$

where n and r_0 are arbitrary. Then clearly, when $\alpha = 0$, equations (7) are recovered from equation (17) with (24), and when $r_0 = 0$ and $\alpha = 0$, equation (1) is recovered.

According to (24), when $r_0 = 0$ and $r \geqslant r_0$, and n is taken in integers, the following infinite sequence of particular solutions obtains,

$C_1(r) = (r + \alpha)^2$ (Brillouin's solution [12])
$C_2(r) = r^2 + \alpha^2$
$C_3(r) = (r^3 + \alpha^3)^{\frac{2}{3}}$ (Schwarzschild's solution [11])
$C_4(r) = (r^4 + \alpha^4)^{\frac{1}{2}}$, etc.

When $r_0 = \alpha$ and $r \in \Re^+$, and n is taken in integers, the following infinite sequence of particular solutions is obtained,

$C_1(r) = r^2$ (Droste/Weyl/(Hilbert) [9, 14, 8])
$C_2(r) = (r - \alpha)^2 + \alpha^2$
$C_3(r) = [(r - \alpha)^3 + \alpha^3]^{\frac{2}{3}}$
$C_4(r) = [(r - \alpha)^4 + \alpha^4]^{\frac{1}{2}}$, etc.

The Schwarzschild forms obtained from (24) satisfy Eddington's prescription for a general solution.

By (17) and (24) the circumference χ of a great circle in the gravitational field is,

$$\chi = 2\pi\sqrt{C_n(r)} = 2\pi\left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}} \,, \qquad (25)$$

and the proper radius $R_p(r)$ is, from (21),

$$R_p(r) = \sqrt{\left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}}\left[\left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}} - \alpha\right]} + \alpha \ln\left|\frac{\left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{2n}} + \sqrt{\left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}} - \alpha}}{\sqrt{\alpha}}\right| . \qquad (26)$$

According to (24), $\sqrt{C_n(D(r_0))} \equiv \alpha$ is a scalar invariant, being independent of the value of r_0. Nevertheless the field is singular at the point-mass. By (21),

$$\lim_{r \to \pm\infty} \frac{R_p^2}{|r - r_0|^2} = 1 \,,$$

and so,

$$\lim_{r \to \pm\infty} \frac{R_p^2}{C_n(D(r))} = \lim_{r \to \pm\infty} \frac{\frac{R_p^2}{|r - r_0|^2}}{\frac{C_n(D(r))}{|r - r_0|^2}} = 1 \,.$$

Now the ratio $\frac{\chi}{R_p} > 2\pi$ for *all* finite R_p, and

$$\lim_{r \to \pm\infty} \frac{\chi}{R_p} = 2\pi \,,$$

$$\lim_{r \to r_0^{\pm}} \frac{\chi}{R_p} = \infty \,,$$

so $R_p(r_0) \equiv 0$ is a quasiregular singularity and cannot be extended. The singularity occurs when parameter $r = r_0$, irrespective of the values of n and r_0. Thus, there is no sense in the orthodox notion that the region $0 < r < \alpha$ is an *interior* region on the Hilbert metric, since $r_0 \neq 0$ on that metric. Indeed, by (21) and (24) $r_0 = \alpha$ on the Hilbert metric. Equation (26) amplifies the fact that it is the *distance* $D = |r - r_0|$ that is mapped from parameter space into the proper radius (distance) in the gravitational field, and a distance *must* be $\geqslant 0$.

Consequently, strictly speaking, r_0 is not a singular point in the gravitational field because r is merely a parameter for the radial quantities in (M_g, g_g); r is neither a radius nor a coordinate in the gravitational field. No value of r can really be a singular point in the gravitational field. However, r_0 is mapped invariantly to $R_p = 0$, so $r = r_0$ *always* gives rise to a quasiregular singularity in the gravitational field, at $R_p(r_0) \equiv 0$, reflecting the fact that r_0 is the boundary on the r-parameter. Only in this sense should r_0 be considered a singular point.

The Kretschmann scalar $f = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ for equation (17) with equation (24) is,

$$f = \frac{12\alpha^2}{[C_n(D(r))]^3} = \frac{12\alpha^2}{\left(|r - r_0|^n + \alpha^n\right)^{\frac{6}{n}}} \,. \qquad (27)$$

Taking the near-field limit on (27),

$$\lim_{r \to r_0^{\pm}} f = \frac{12}{\alpha^4} \,,$$

so $f(r_0) \equiv \frac{12}{\alpha^4}$ is a scalar invariant, irrespective of the values of n and r_0, invalidating the orthodox assumption that the singularity must occur where the curvature is unbounded. Indeed, no curvature singularity can arise in the gravitational field. The orthodox analysis claims an unbounded curvature singularity at $r_0 = 0$ in (18) purely and simply by its invalid initial assumptions, *not* by mathematical imperative. It incorrectly assumes $\sqrt{C_n(r)} \equiv R_p(r) \equiv r$, then with its additional invalid assumption that $0 < r < \alpha$ is valid on the Hilbert metric, finds from (27),

$$\lim_{r \to 0^+} f(r) = \infty \,,$$

thereby satisfying its third invalid assumption, by *ad hoc* construction, that a singularity occurs only where the curvature invariant is unbounded.

The Kruskal-Szekeres form has no meaning since the r-parameter is not the radial coordinate in the gravitational field at all. Furthermore, the value of r_0 being entirely arbitrary, $r_0 = 0$ has no particular significance, in contrast to the mainstream claims on (18).

The value of the r-parameter of a certain spacetime event depends upon the coordinate system chosen. However, the proper radius $R_p(D(r))$ and the curvature radius $\sqrt{C_n(D(r))}$ of that event are independent of the coordinate system. This is easily seen as follows. Consider a great circle centred at the point-mass and passing through a spacetime event. Its circumference is measured at χ. Dividing χ by 2π gives,

$$\frac{\chi}{2\pi} = \sqrt{C_n(D(r))}\,.$$

Putting $\frac{\chi}{2\pi} = \sqrt{C_n(D(r))}$ into (21) gives the proper radius of the spacetime event,

$$R_p(r) = \sqrt{\frac{\chi}{2\pi}\left(\frac{\chi}{2\pi} - \alpha\right)} + \alpha \ln \left| \frac{\sqrt{\frac{\chi}{2\pi}} + \sqrt{\frac{\chi}{2\pi} - \alpha}}{\sqrt{\alpha}} \right| \,,$$

$$2\pi\alpha \leqslant \chi < \infty,$$

which is independent of the coordinate system chosen. To find the r-parameter in terms of a particular coordinate system set,

$$\frac{\chi}{2\pi} = \sqrt{C_n(D(r))} = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{1}{n}} \,,$$

so

$$|r - r_0| = \left[\left(\frac{\chi}{2\pi}\right)^n - \alpha^n\right]^{\frac{1}{n}}.$$

Thus r for any particular spacetime event depends upon the arbitrary values n and r_0, which establish a coordinate system. Then when $r = r_0$, $R_p = 0$, and the great circumference $\chi = 2\pi\alpha$, irrespective of the values of n and r_0. A truly coordinate independent description of spacetime events has been attained.

The mainstream insistence, on the Hilbert solution (18), without proof, that the r-parameter is a radius of sorts in the gravitational field, the insistence that its r can, without proof, go down to zero, and the insistence, without proof, that a singularity in the field must occur only where the curvature is unbounded, have produced the irrational notion of the black hole. The fact is, the radius *always* does go down to zero in the gravitational field, but that radius is the *proper radius* R_p ($R_p = 0$ corresponding to a coordinate radius $D = 0$), not the curvature radius R_c, and certainly not the r-parameter.

There is no escaping the fact that $r_0 = \alpha \neq 0$ in (18). Indeed, if $\alpha = 0$, (18) *must* give,

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

the metric of Special Relativity when $r_0 = 0$. One cannot set the lower bound $r_0 = \alpha = 0$ in (18) and simultaneously keep $\alpha \neq 0$ in the components of the metric tensor, which is effectively what the orthodox analysis has done to obtain the black hole. The result is unmitigated nonsense. The correct form of the metric (18) is obtained from the associated Schwarzschild form (24): $C(r) = r^2$, $r_0 = \alpha$. Furthermore, the proper radius of (18) is,

$$R_p(r) = \int_\alpha^r \sqrt{\frac{r}{r - \alpha}}\,dr \,,$$

and so

$$R_p(r) = \sqrt{r\,(r - \alpha)} + \alpha \ln \left| \frac{\sqrt{r} + \sqrt{r - \alpha}}{\sqrt{\alpha}} \right| .$$

Then,

$$r \to \alpha^+ \Rightarrow D = |r - \alpha| = (r - \alpha) \to 0\,,$$

and in (M_g, g_g),

$$r^2 \equiv C(r) \to C(\alpha) = \alpha^2 \Rightarrow R_p(r) \to R_p(\alpha) = 0\,.$$

Thus, the r-parameter is mapped to the radius of curvature $\sqrt{C(r)} = \frac{\chi}{2\pi}$ by ψ_1, and the radius of curvature is mapped to the proper radius R_p by ψ_2. With the mappings established the r-parameter can be mapped directly to R_p by $\psi(r) = \psi_2 \circ \psi_1(r)$. In the case of the simple point-mass the mapping ψ_1 is just equation (24), and the mapping ψ_2 is given by (21).

The local acceleration of a test particle approaching the point-mass along a radial geodesic has been determined by N. Doughty [15] at,

$$a = \frac{\sqrt{-g_{rr}}\,\left(-g^{rr}\right)|g_{tt,r}|}{2g_{tt}}\,. \tag{28}$$

For (17) the acceleration is,

$$a = \frac{\alpha}{2C_n^{\frac{3}{4}}\left(C_n^{\frac{1}{2}} - \alpha\right)^{\frac{1}{2}}}.$$

Then,

$$\lim_{r \to r_0^{\pm}} a = \infty,$$

since $C_n(r_0) \equiv \alpha^2$; thereby confirming that matter is indeed present at the point $R_p(r_0) \equiv 0$.

In the case of (18), where $r \in \Re^+$,

$$a = \frac{\alpha}{2r^{\frac{3}{2}} \left(r - \alpha\right)^{\frac{1}{2}}},$$

and $r_0 = \alpha$ by (24), so,

$$\lim_{r \to \alpha^+} a = \infty\,.$$

Y. Hagihara [16] has shown that all those geodesics which do not run into the boundary at $r = \alpha$ on (18) are complete. Now (18) with $\alpha < r < \infty$ is a particular solution by (24), and $r_0 = \alpha$ is an arbitrary point at which the point-mass is located in parameter space, therefore all those geodesics in (M_g, g_g) not running into the point $R_p(r_0) \equiv 0$ are complete, irrespective of the value of r_0.

Modern relativists do not interpret the Hilbert solution over $0 < r < \infty$ as Hilbert did, instead making an arbitrary distinction between $0 < r < \alpha$ and $\alpha < r < \infty$. The modern relativist maintains that one is entitled to just "choose" a region. However, as I have shown, this claim is inadmissible. J. L. Synge [17] made the same unjustified assumptions on the Hilbert line-element. He remarks,

> "*This line-element is usually regarded as having a singularity at $r = \alpha$, and appears to be valid only for $r > \alpha$. This limitation is not commonly regarded as serious, and certainly is not so if the general theory of relativity is thought of solely as a macroscopic theory to be applied to astronomical problems, for then the singularity $r = \alpha$ is buried inside the body, i. e. outside the domain of the field equations $R_{mn} = 0$. But if we accord to these equations an importance comparable to that which we attach to Laplace's equation, we can hardly remain satisfied by an appeal to the known sizes of astronomical bodies. We have a right to ask whether the general theory of relativity actually denies the existence of a gravitating particle, or whether the form (1.1) may not in fact lead to the field of a particle in spite of the apparent singularity at $r = \alpha$.*"

M. Kruskal [18] remarks on his proposed extension of the Hilbert solution into $0 < r < 2m$,

> "*That this extension is possible was already indicated by the fact that the curvature invariants of the Schwarzschild metric are perfectly finite and well behaved at $r = 2m$∗.*"

which betrays the very same unproven assumptions.

G. Szekeres [6] says of the Hilbert line-element,

> "*. . . it consists of two disjoint regions, $0 < r < 2m$, and $r > 2m$, separated by the singular hypercylinder $r = 2m$.*"

which again betrays the same unproven assumptions.

I now draw attention to the following additional problems with the Kruskal-Szekeres form.

(a) Applying Doughty's acceleration formula (28) to the Kruskal-Szekeres form, it is easily found that,

$$\lim_{r \to 2m^-} a = \infty.$$

But according to Kruskal-Szekeres there is *no matter* at $r = 2m$. Contra-hype.

(b) As $r \to 0$, $u^2 - v^2 \to -1$. These loci are spacelike, and therefore *cannot* describe *any* configuration of matter or energy.

Both of these anomalies have also been noted by Abrams in his work [7]. Either of these features alone proves the Kruskal-Szekeres form inadmissible.

The correct geometrical analysis excludes the interior Hilbert region on the grounds that it is not a region at all, and invalidates the assumption that the r-parameter is some kind of radius and/or coordinate in the gravitational field. Consequently, the Kruskal-Szekeres formulation is meaningless, both physically and mathematically. In addition, the so-called "Schwarzschild radius" (*not* due to Schwarzschild) is also a meaningless concept - it is not a radius in the gravitational field. Hilbert's $r = 2m$ is indeed a point, i. e. the "Schwarzschild radius" is a point, in both parameter space and the gravitational field: by (21), $R_p(2m) = 0$.

The *form* of the Hilbert line-element is given by Karl Schwarzschild in his 1916 paper, where it occurs there in the equation he numbers (14), in terms of his "auxiliary parameter" R. However Schwarzschild also includes there the equation $R = \left(r^3 + \alpha^3\right)^{\frac{1}{3}}$, having previously established the range $0 < r < \infty$. Consequently, Schwarzschild's auxiliary parameter R (which is actually a curvature radius) has the lower bound $R_0 = \alpha = 2m$. Schwarzschild's R^2 and Hilbert's r^2 can be replaced with any appropriate analytic function $C_n(r)$ as given by (24), so the range and the boundary on r will depend upon the function chosen. In the case of Schwarzschild's particular solution the range is $0 < r < \infty$ (since $r_0 = 0$, $C_3(r) = \left(r^3 + \alpha^3\right)^{\frac{2}{3}}$) and in Hilbert's particular solution the range is $2m < r < \infty$ (since $r_0 = 2m$, $C_1(r) = r^2$).

The geometry and the invariants are the important properties, but the conventional analysis has shockingly erred in its geometrical analysis and identification of the invariants, as a direct consequence of its initial invalidated assumptions about the r-parameter, and clings irrationally to these assumptions to preserve the now sacrosanct, but nonetheless ridiculous, black hole.

The only reason that the Hilbert solution conventionally breaks down at $r = \alpha$ is because of the initial arbitrary and incorrect assumptions made about the parameter r. There is no *pathology of coordinates* at $r = \alpha$. If there is anything pathological about the Hilbert metric it has nothing to do with coordinates: the etiology of a pathology must therefore be found elewhere.

There is no doubt that the Kruskal-Szekeres form is a solution of the Einstein vacuum field equations, however that does not guarantee that it is a solution to the problem. There exists an infinite number of solutions to the vacuum field equations which do not yield a solution for the gravitational field of the point-mass. Satisfaction of the field equations is a necessary but insufficient condition for a potential solution to the problem. It is evident that the conventional conditions (see [19]) that must be met are inadequate, viz.,

1. *be analytic*;
2. *be Lorentz signature*;
3. *be a solution to Einstein's free-space field equations*;
4. *be invariant under time translations*;
5. *be invariant under spatial rotations*;
6. *be (spatially) asymptotically flat*;
7. *be inextendible to a wordline L*;
8. *be invariant under spatial reflections*;
9. *be invariant under time reflection*;
10. *have a global time coordinate*.

This list must be augmented by a boundary condition at the location of the point-mass, which is, in my formulation of the solution, $r \to r_0^{\pm} \Rightarrow R_p(r) \to 0$. Schwarzschild actually applied a form of this boundary condition in his analysis. Marcel Brillouin [12] also pointed out the necessity of such a boundary condition in 1923, as did Abrams [7] in more recent years, who stated it equivalently as, $r \to r_0 \Rightarrow C(r) \to \alpha^2$. The condition has been disregarded or gone unrecognised by the mainstream authorities. Oddly, the orthodox analysis violates its own stipulated condition for a global time coordinate, but quietly disregards this inconsistency as well.

Any constants appearing in a valid solution must appear in an invariant derived from the solution. The solution I obtain meets this condition in the invariance, at $r = r_0$, of the circumference of a great circle, of Kepler's 3rd Law [1, 2], of the Kretschmann scalar, of the radius of curvature $C(r_0) = \alpha^2$, of $R_p(r_0) \equiv 0$, and not only in the case of the point-mass, but also in all the relevant configurations, with or without charge.

The fact that the circumference of a great circle approaches the finite value $2\pi\alpha$ is no more odd than the conventional oddity of the change in the arrow of time in the "interior" Hilbert region. Indeed, the latter is an even more violent oddity: inconsistent with Einstein's theory. The finite limit of the said circumference is consistent with the geometry resulting from Einstein's gravitational tensor. The variations of θ and φ displace the proper radius vector, $R_p(r_0) \equiv 0$, over the spherical surface of finite area $4\pi\alpha^2$, as noted by Brillouin. Einstein's theory admits nothing more pointlike.

Objections to Einstein's formulation of the gravitational tensor were raised as long ago as 1917, by T. Levi-Civita [20], on the grounds that, from the mathematical standpoint, it lacks the invariant character actually required of General Relativity, and further, produces an unacceptable consequence concerning gravitational waves (i.e they carry neither energy nor momentum), a solution for which Einstein vaguely appealed *ad hoc* to quantum theory, a last resort obviated by Levi-Civita's reformulation of the gravitational tensor (which extinguishes the gravitational wave), of which the conventional analysis is evidently completely ignorant: but it is not pertinent to the issue of whether or not the black hole is consistent with the theory as it currently stands on Einstein's gravitational tensor.

3 The geometry of the simple point-charge

The fundamental geometry developed in section **2** is the same for all the configurations of the point-mass and the point-charge. The general solution for the simple point-charge [2] is,

$$ds^2 = \left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right) dt^2 - \left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)^{-1} \times \times \frac{C_n'^{\,2}}{4C_n} dr^2 - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{29}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}},$$

$$\beta = m + \sqrt{m^2 - q^2}\,, \quad q^2 < m^2\,,$$

$$n \in \Re^+, \quad r, r_0 \in \Re\,.$$

where n and r_0 are arbitrary.

From (29), the radius of curvature is given by,

$$R_c = \sqrt{C_n(r)} = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{1}{n}},$$

which gives for the near-field limit,

$$\lim_{r \to r_0^{\pm}} \sqrt{C_n(r)} = \sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - q^2}\,.$$

The expression for the proper radius is,

$$R_p(r) = \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2} + + m \ln\left|\frac{\sqrt{C(r)} - m + \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2}}{\sqrt{m^2 - q^2}}\right|.$$

Then

$$\lim_{r\to r_0^\pm} R_p(r) = R_p(r_0) \equiv 0\,.$$

The ratio $\frac{\chi}{R_p} > 2\pi$ for all finite R_p, and

$$\lim_{r\to\pm\infty} \frac{\chi}{R_p(r)} = 2\pi\,,$$

$$\lim_{r\to r_0^\pm} \frac{\chi}{R_p(r)} = \infty\,,$$

so $R_p(r_0) \equiv 0$ is a quasiregular singularity and cannot be extended.

Now, since the circumference χ of a great circle is the only measurable quantity in the gravitational field, the unique solution for the field of the simple point-charge is,

$$\begin{aligned} ds^2 &= \left(1-\frac{2\pi\alpha}{\chi}+\frac{4\pi^2q^2}{\chi^2}\right)dt^2 - \\ &- \left(1-\frac{2\pi\alpha}{\chi}+\frac{4\pi^2q^2}{\chi^2}\right)^{-1}\frac{d\chi^2}{4\pi^2} - \\ &-\frac{\chi^2}{4\pi^2}\left(d\theta^2+\sin^2\theta d\varphi^2\right), \end{aligned} \qquad (30)$$

$$2\pi\left(m+\sqrt{m^2-q^2}\right) < \chi < \infty\,.$$

Equation (30) is entirely independent of the r-parameter.

In terms of equation (29), the Kretschmann scalar takes the form [21],

$$f(r) = \frac{8\left[6\left(m\sqrt{C_n(r)}-q^2\right)^2+q^4\right]}{C_n^4(r)}\,, \qquad (31)$$

so

$$\begin{aligned} \lim_{r\to r_0^\pm} f(r) = f(r_0) &= \frac{8\left[6\left(m\beta-q^2\right)^2+q^4\right]}{\beta^8} \\ &= \frac{8\left[6\left(m^2+m\sqrt{m^2-q^2}-q^2\right)^2+q^4\right]}{(m+\sqrt{m^2-q^2})^8}\,, \end{aligned}$$

which is a scalar invariant. Thus, no curvature singularity can arise in the gravitational field of the simple point-charge.

The standard analysis incorrectly takes $\sqrt{C_n(r)} \equiv R_p(r) \equiv r$, then with this assumption, and the additional invalid assumption that $0 < r < \infty$ is true on the Reissner-Nordstrom solution, obtains from equation (31) a curvature singularity at $r = 0$, satisfying, by an *ad hoc* construction, its third invalid assumption that a singularity can only arise at a point where the curvature invariant is unbounded.

Equation (29) is singular *only* when $g_{00} = 0$; indeed $g_{00}(r_0) \equiv 0$. Hence, $0 \leqslant g_{00} \leqslant 1$.

Applying Doughty's acceleration formula (28) to equation (29) gives,

$$a = \frac{\left|m\sqrt{C_n(r)}-q^2\right|}{C_n(r)\sqrt{C_n(r)-\alpha\sqrt{C_n(r)}+q^2}}\,.$$

Then,

$$\lim_{r\to r_0^\pm} a = \frac{\left|m\beta-q^2\right|}{\beta^2\sqrt{\beta^2-\alpha\beta+q^2}} = \infty\,,$$

confirming that matter is indeed present at $R_p(r_0) \equiv 0$.

4 The geometry of the rotating point-charge

The usual expression for the Kerr-Newman solution is, in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \end{aligned} \qquad (32)$$

$$a = \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta\,,$$

$$\Delta = r^2 - r\alpha + a^2 + q^2, \quad 0 < r < \infty\,.$$

This metric is alleged to have an event horizon r_h and a static limit r_b, obtained by setting $\Delta = 0$ and $g_{00} = 0$ respectively, to yield,

$$r_h = m \pm \sqrt{m^2 - a^2 - q^2}$$

$$r_b = m \pm \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,.$$

These expressions are conventionally quite arbitrarily taken to be,

$$r_h = m + \sqrt{m^2 - a^2 - q^2}$$

$$r_b = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,,$$

apparently because no-one has been able to explain away the meaning of the the "inner" horizon and the "inner" static limit. This in itself is rather disquieting, but nonetheless accepted with furtive whispers by the orthodox theorists. It is conventionally alleged that the "region" between r_h and r_b is an ergosphere, in which spacetime is dragged in the direction of the of rotation of the point-charge.

The conventional taking of the r-parameter for a radius in the gravitational field is manifest. However, as I have shown, the r-parameter is neither a coordinate nor a radius in the gravitational field. Consequently, the standard analysis is erroneous.

I have already derived elsewhere [2] the general solution for the rotating point-charge, which I write in most general form as,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 -$$
$$-\frac{\sin^2\theta}{\rho^2}\left[\left(C_n + a^2\right)d\varphi - adt\right]^2 - \frac{\rho^2}{\Delta}\frac{{C'_n}^2}{4C_n}dr^2 - \rho^2 d\theta^2\,,$$
$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}\,, \quad n \in \Re^+\,, \tag{33}$$
$$r,\ r_0 \in \Re, \quad \beta = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,,$$
$$a^2 + q^2 < m^2, \quad a = \frac{L}{m}, \quad \rho^2 = C_n + a^2\cos^2\theta\,,$$
$$\Delta = C_n - \alpha\sqrt{C_n} + q^2 + a^2\,,$$

where n and r_0 are arbitrary.

Once again, since only the circumference of a great circle is a measurable quantity in the gravitational field, the unique general solution for all configurations of the point-mass is,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 -$$
$$-\frac{\sin^2\theta}{\rho^2}\left[\left(\frac{\chi^2}{4\pi^2} + a^2\right)d\varphi - adt\right]^2 - \frac{\rho^2}{\Delta}\frac{d\chi^2}{4\pi^2} - \rho^2 d\theta^2\,,$$
$$a^2 + q^2 < m^2, \quad a = \frac{L}{m}, \quad \rho^2 = \frac{\chi^2}{4\pi^2} + a^2\cos^2\theta\,, \tag{34}$$
$$\Delta = \frac{\chi^2}{4\pi^2} - \frac{\alpha\chi}{2\pi} + q^2 + a^2\,,$$
$$2\pi\left(m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\right) < \chi < \infty\,.$$

Equation (34) is entirely independent of the r-parameter.

Equation (34) emphasizes the fact that the concept of a point in pseudo-Euclidean Minkowski space is not attainable in the pseudo-Riemannian gravitational field. A point-mass (or point-charge) is characterised by a proper radius of zero and a finite, non-zero radius of curvature. Einstein's universe admits of nothing more pointlike. The relativists have assumed that, insofar as the point-mass is concerned, the Minkowski point can be achieved in Einstein space, which is not correct.

The radius of curvature of (33) is,

$$\sqrt{C_n(r)} = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{1}{n}}\,, \tag{35}$$

which goes down to the limit,

$$\lim_{r \to r_0^{\pm}} \sqrt{C_n(r)} = \sqrt{C_n(r_0)} = \beta = = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,, \tag{36}$$

where the proper radius $R_p(r_0) \equiv 0$. The standard analysis incorrectly takes (36) for the "radius" of its static limit.

It is evident from (35) and (36) that the radius of curvature depends upon the direction of radial approach. Therefore, the spacetime is not isotropic. Only when $a = 0$ is spacetime isotropic. The point-charge is always located at $R_p(r_0) \equiv 0$ in $(M_g,\ g_g)$, irrespective of the value of n, and irrespective of the value of r_0. The conventional analysis has failed to realise that its r_b is actually a varying radius of curvature, and so incorrectly takes it as a measurable radius in the gravitational field. It has also failed to realise that the location of the point-mass in the gravitational field is not uniquely specified by the r-coordinate at all. The point-mass is *always* located just where $R_p = 0$ in (M_g, g_g) and its "position" in (M_g, g_g) is otherwise meaningless. The test particle has already encountered the source of the gravitational field when the radius of curvature has the value $C_n(r_0) = \beta$. The so-called ergosphere also arises from the aforesaid misconceptions.

When $\theta = 0$ the limiting radius of curvature is,

$$\sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - q^2 - a^2}\,, \tag{37}$$

and when $\theta = \frac{\pi}{2}$, the limiting radius of curvature is,

$$\sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - q^2}\,,$$

which is the limiting radius of curvature for the simple point-charge (i. e. no rotation) [2].

The standard analysis incorrectly takes (37) as the "radius" of its event horizon.

If $q = 0$, then the limiting radius of curvature when $\theta = 0$ is,

$$\sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - a^2}\,, \tag{38}$$

and the limiting radius of curvature when $\theta = \frac{\pi}{2}$ is,

$$\sqrt{C_n(r_0)} = \beta = 2m = \alpha\,,$$

which is the radius of curvature for the simple point-mass.

The radii of curvature at intermediate azimuth are given generally by (36). In all cases the near-field limits of the radii of curvature give $R_p(r_0) \equiv 0$.

Clearly, the limiting radius of curvature is minimum at the poles and maximum at the equator. At the equator the effects of rotation are not present. A test particle approaching the rotating point-charge or the rotating point-mass equatorially experiences the effects only of the non-rotating situation of each configuration respectively. The effects of the rotation manifest only in the values of azimuth other than $\frac{\pi}{2}$. There is no rotational drag on spacetime, no ergosphere and no event horizon, i. e. no black hole.

The effects of rotation on the radius of curvature will necessarily manifest in the associated form of Kepler's 3rd Law, and the Kretschmann scalar [22].

I finally remark that the fact that a singularity arises in the gravitational field of the point-mass is an indication that a material body cannot collapse to a point, and therefore such a model is inadequate. A more realistic model must be sought in terms of a non-singular metric, of which I treat elsewhere [23].

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

References

1. Crothers S. J. On the general solution to Einstein's vacuum field and it implications for relativistic degeneracy. *Progress in Physics*, 2005, v. 1, 68–73.
2. Crothers S. J. On the ramifications of the Schwarzschild spacetime metric. *Progress in Physics*, 2005, v. 1, 74–80.
3. Stavroulakis N. A statical smooth extension of Schwarzschild's metric. *Lettere al Nuovo Cimento*, 1974, v. 11, 8 (see also in www.geocities.com/theometria/Stavroulakis-3.pdf).
4. Stavroulakis N. On the Principles of General Relativity and the SΘ(4)-invariant metrics. *Proc. 3rd Panhellenic Congr. Geometry*, Athens, 1997, 169 (see also in www.geocities.com/theometria/Stavroulakis-2.pdf).
5. Stavroulakis N. On a paper by J. Smoller and B. Temple. *Annales de la Fondation Louis de Broglie*, 2002, v. 27, 3 (see also in www.geocities.com/theometria/Stavroulakis-1.pdf).
6. Szekeres, G. On the singularities of a Riemannian manifold. *Math. Debreca.*, 1960, v. 7, 285.
7. Abrams L. S. Black holes: the legacy of Hilbert's error. *Can. J. Phys.*, 1989, v. 67, 919 (see also in arXiv: gr-qc/0102055).
8. Hilbert, D. *Nachr. Ges. Wiss. Gottingen, Math. Phys. Kl.*, v. 53, 1917 (see also in arXiv: physics/0310104).
9. Droste J. The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field. *Ned. Acad. Wet., S. A.*, 1917, v. 19, 197 (see also in www.geocities.com/theometria/Droste.pdf).
10. Eddington A. S. The mathematical theory of relativity. Cambridge University Press, Cambridge, 2nd edition, 1960.
11. Schwarzschild K. On the gravitational field of a mass point according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 189 (see this item also in arXiv: physics/9905030).
12. Brillouin M. The singular points of Einstein's Universe. *Journ. Phys. Radium*, 1923, v. 23, 43 (see also in arXiv: physics/0002009).
13. Fiziev P. P. Gravitational field of massive point particle in general relativity. arXiv: gr-gc/0306088.
14. Weyl H. *Ann. Phys. (Leipzig)*, 1917, v. 54, 117.
15. Doughty N. *Am. J. Phys.*, 1981, v. 49, 720.
16. Hagihara Y. *Jpn. J. Astron. Geophys.*, 1931, v. 8, 67.
17. Synge J. L. The gravitational field of a particle. *Proc. Roy. Irish Acad.*, 1950, v. 53, 83.
18. Kruskal M. D. Maximal extension of Schwarzschild metric. *Phys. Rev.*, 1960, v. 119, 1743.
19. Finkelstein D. Past-future asymmetry of the gravitational field of a point particle. *Phys. Rev.*, 1958, v. 110, 965.
20. Levi-Civita T. Mechanics. — On the analytical expression that must be given to the gravitational tensor in Einstein's theory. *Rendiconti della Reale Accademia dei Lincei*, v. 26, 1917, 381 (see also in arXiv: physics/9906004).
21. Abrams L. S. The total space-time of a point charge and its consequences for black holes. *Int. J. Theor. Phys.*, 1996, v. 35, 2661 (see also in arXiv: gr-qc/0102054).
22. Crothers S. J. On the Generalisation of Kepler's 3rd Law for the Vacuum Field of the Point-Mass. *Progress in Physics*, 2005, v. 2, 37–42.
23. Crothers S. J. On the vacuum field of a sphere of incompressible fluid. *Progress in Physics*, 2005, v. 2, 43–47.

A Theory of Gravity Like Electrodynamics

Dmitri Rabounski
E-mail: rabounski@yahoo.com

This study looks at the field of inhomogeneities of time coordinates. Equations of motion, expressed through the field tensor, show that particles move along time lines because of rotation of the space itself. Maxwell-like equations of the field display its sources, which are derived from gravitation, rotations, and inhomogeneity of the space. The energy-momentum tensor of the field sets up an inhomogeneous viscous media, which is in the state of an ultrarelativistic gas. Waves of the field are transverse, and the wave pressure is derived from mainly sub-atomic processes — excitation/relaxation of atoms produces the positive/negative wave pressures, which leads to a test of the whole theory.

Contents

1 Inhomogeneity of observable time. Defining the field

The meaning of Einstein's General Theory of Relativity consists of his idea that all properties of the world are derived from the geometrical structure of space-time, from the world-geometry, in other words. This is a way to geometrize physics. The introduction of his artificial postulates became only of historical concern subsequent to his setting up of the meaning of the theory — all the postulates are naturally contained in the geometry of a four-dimensional pseudo-Riemannian space with the sign-alternating signature $(-+++)$ or $(+---)$ he assigned to the basic space-time of the theory.

Verification of the theory by experiments has shown that the four-dimensional pseudo-Riemannian space satisfies our observable world in most cases. In general we can say that all that everything we can obtain theoretically in this space geometry must have a physical interpretation.

Here we take a pseudo-Riemannian space with the signature $(+---)$, where time is real and spatial coordinates are imaginary, because the observable projection of a four-dimensional impulse on the spatial section of any given observer is positive in this case. We also assign to space-time Greek indices, while spatial indices are Latin*.

As it is well-known [1], $dS = m_0 c\,ds$ is an elementary action to displace a free mass-bearing particle of rest-mass m_0 through a four-dimensional interval of length ds. What happens to matter during this action? To answer this question let us substitute the square of the interval $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$ into the action. As a result we see that

$$dS = m_0 c\,ds = m_0 c\sqrt{g_{\alpha\beta}\,dx^\alpha dx^\beta}\,, \qquad (1)$$

so the particle moves in space-time along geodesic lines (free motion), because the field carries the fundamental metric tensor $g_{\alpha\beta}$. At the same time Einstein's equations link the metric tensor $g_{\alpha\beta}$ to the energy-momentum tensor of matter through the four-dimensional curvature of space-time. This implies that the gravitational field is linked to the field of the space-time metric in the frames of the General Theory of Relativity. For this reason one regularly concludes that the action (1) displacing free mass-bearing particles is produced by the gravitational field.

Let us find which field will manifest by the action (1) as a source of free motion, if the space-time interval ds therein is written with quantities which would be observable by a real observer located in the four-dimensional pseudo-Riemannian space.

A formal basis here is the mathematical apparatus of physically observable quantities (the theory of chronometric invariants), developed by Zelmanov in the 1940's [2, 3]. Its essence is that if an observer accompanies his reference body,

*Alternatively, Landau and Lifshitz in their *The Classical Theory of Fields* [1] use the space signature $(-+++)$, which gives an advantage in certain cases. They also use other notations for tensor indices: in their book space-time indices are Latin, while spatial indices are Greek.

his observable quantities are projections of four-dimensional quantities on his time line and the spatial section — *chronometrically invariant quantities*, made by projecting operators $b^\alpha = \frac{dx^\alpha}{ds}$ and $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$ which fully define his real reference space (here b^α is his velocity with respect to his real references). Thus, the chr.inv.-projections of a world-vector Q^α are $b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}}$ and $h^i_\alpha Q^\alpha = Q^i$, while chr.inv.-projections of a world-tensor of the 2nd rank $Q^{\alpha\beta}$ are $b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}$, $h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}$, $h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}$. Physically observable properties of the space are derived from the fact that chr.inv.-differential operators $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2} v_i \frac{{}^*\partial}{\partial t}$ are non-commutative $\frac{{}^*\partial^2}{\partial x^i \partial t} - \frac{{}^*\partial^2}{\partial t\, \partial x^i} = \frac{1}{c^2} F_i \frac{{}^*\partial}{\partial t}$ and $\frac{{}^*\partial^2}{\partial x^i \partial x^k} - \frac{{}^*\partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{{}^*\partial}{\partial t}$, and also from the fact that the chr.inv.-metric tensor h_{ik} may not be stationary. The observable characteristics are the chr.inv.-vector of gravitational inertial force F_i, the chr.inv.-tensor of angular velocities of the space rotation A_{ik}, and the chr.inv.-tensor of rates of the space deformations D_{ik}, namely

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} \tag{2}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \tag{3}$$

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}, \quad D^k_k = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}, \tag{4}$$

where w is gravitational potential, $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation, $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ is the chr.inv.-metric tensor, and also $h = \det\|h_{ik}\|$, $h g_{00} = -g$, $g = \det\|g_{\alpha\beta}\|$. Observable inhomogeneity of the space is set up by the chr.inv.-Christoffel symbols $\Delta^i_{jk} = h^{im}\Delta_{jk,m}$, which are built just like Christoffel's usual symbols $\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$ using h_{ik} instead of $g_{\alpha\beta}$.

A four-dimensional generalization of the main chr.inv.-quantities F_i, A_{ik}, and D_{ik} (by Zelmanov, the 1960's [4]) is: $F_\alpha = -2c^2 b^\beta a_{\beta\alpha}$, $A_{\alpha\beta} = c h^\mu_\alpha h^\nu_\beta a_{\mu\nu}$, $D_{\alpha\beta} = c h^\mu_\alpha h^\nu_\beta d_{\mu\nu}$, where $a_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha b_\beta - \nabla_\beta b_\alpha)$, $d_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha b_\beta + \nabla_\beta b_\alpha)$.

In this way, for any equations obtained using general covariant methods, we can calculate their physically observable projections on the time line and the spatial section of any particular reference body and formulate the projections in terms of their real physically observable properties, from which we obtain equations containing only quantities measurable in practice.

Expressing $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ through the observable time interval

$$d\tau = \frac{1}{c}\, b_\alpha dx^\alpha = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i \tag{5}$$

and also the observable spatial interval $d\sigma^2 = h_{\alpha\beta}\, dx^\alpha dx^\beta = h_{ik}\, dx^i dx^k$ (note, $b^i = 0$ for an observer who accompanies his reference body), we come to the formula

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{6}$$

Using this formula, we can write down the action (1) to displace a free mass-bearing particle in the form

$$dS = m_0 c \sqrt{b_\alpha b_\beta\, dx^\alpha dx^\beta - h_{\alpha\beta}\, dx^\alpha dx^\beta}\,. \tag{7}$$

If the particle is at rest with respect to the observer's reference body, then its observable displacement along his spatial section is $dx^i = 0$, so its observable chr.inv.-velocity vector equals zero; $\mathrm{v}^i = \frac{dx^i}{d\tau} = 0$. Such a particle moves only along time lines. In this case, in the accompanying reference frame, we have $h_{\alpha\beta}\, dx^\alpha dx^\beta = h_{ik}\, dx^i dx^k = 0$ hence the action is

$$dS = m_0 c\, b_\alpha dx^\alpha, \tag{8}$$

so the mass-bearing particle moves freely along time lines because it is carried solely by the vector field b^α.

What is the physical meaning of this field? The vector b^α is the operator of projection on time lines (non-uniform, in general case) of a real observer, who accompanies his reference body. This implies that the vector field b^α defines the geometrical structure of the real space-time along time lines. Projecting an interval of four-dimensional coordinates dx^α onto the time line of a real observer in his accompanying reference frame, we obtain the interval of real physical time $d\tau = \frac{1}{c} b_\alpha dx^\alpha = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2} v_i dx^i$ he observes. For his measurements in the same spatial point, in other words, along the same time line, $d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt$. This formula and the previous one lead us to the conclusion that the components of the observer's vector b^α define a "density" of physically observable time in his accompanying reference frame. As it is easy to see, the observable time density depends on the gravitational potential and, in the general case, on the rotation of the space. Hence, the vector field b^α in the accompanying reference frame is the field of inhomogeneity of observable time references. For this reason we will call it the *field of density of observable time*.

In the same way, a field of the tensor $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$ projecting four-dimensional quantities on the observer's spatial section is the *field of density of the spatial section*.

From the geometric viewpoint, we can illustrate the conclusions in this way. The vector field b^α and the tensor field $h_{\alpha\beta}$ of the accompanying reference frame of an observer, located in a four-dimensional pseudo-Riemannian space, "split" the space into time lines and a spatial section, properties of which (such as inhomogeneity, anisotropy, curvature, etc.) depend on the physical properties of the observer's reference body. Owing to this "splitting" process, the field of the fundamental metric tensor $g_{\alpha\beta}$, containing the geometrical structure of this space, "splits" as well (7). Its "transverse component" is the time density field, a four-dimensional potential of which is the monad vector b^α. The "longitudinal component" of this splitting is the field of density of the spatial section.

2 The field tensor. Its observable components: gravitational inertial force and the space rotation tensor

Chr.inv.-projections of the four-dimensional vector potential b^α of a time density field are, respectively

$$\varphi = \frac{b_0}{\sqrt{g_{00}}} = 1\,, \qquad q^i = b^i = 0\,. \tag{9}$$

Emulating the way that Maxwell's electromagnetic field tensor is introduced, we introduce the *tensor of a time density field* as the rotor of its four-dimensional vector potential

$$F_{\alpha\beta} = \nabla_\alpha\, b_\beta - \nabla_\beta\, b_\alpha = \frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\,. \tag{10}$$

Taking into account that $F_{00} = F^{00} = 0$, as for any antisymmetric tensor of the 2nd rank, after some algebra we obtain the other components of the field tensor $F_{\alpha\beta}$

$$F_{0i} = \frac{1}{c^2}\sqrt{g_{00}}\,F_i\,, \qquad F_{ik} = \frac{1}{c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right), \tag{11}$$

$$F_{0\cdot}^{\cdot 0} = -\frac{1}{c^3}\,v_k F^k, \qquad F_{0\cdot}^{\cdot i} = -\frac{1}{c^2}\sqrt{g_{00}}\,F^i, \tag{12}$$

$$F_{k\cdot}^{\cdot 0} = -\frac{1}{\sqrt{g_{00}}}\left(\frac{1}{c^2}\,F_k + \frac{2}{c^2}\,v^m A_{mk} - \frac{1}{c^4}\,v_k v_m F^m\right), \tag{13}$$

$$F_{k\cdot}^{\cdot i} = \frac{1}{c^3}\,v_k F^i + \frac{2}{c}\,A_{k\cdot}^{\cdot i}\,, \qquad F^{ik} = -\frac{2}{c}\,A^{ik}, \tag{14}$$

$$F^{0k} = -\frac{1}{\sqrt{g_{00}}}\left(\frac{1}{c^2}\,F^k + \frac{2}{c^2}\,v_m A^{mk}\right). \tag{15}$$

We denote chr.inv.-projections of the field tensor just like the chr.inv.-projections of the Maxwell tensor [5], to display their physical sense. We will refer to the time projection

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = -\frac{1}{c^2}\,F^i, \qquad E_i = h_{ik}E^k = -\frac{1}{c^2}\,F_i \tag{16}$$

of the field tensor $F_{\alpha\beta}$ as "electric". The spatial projection

$$H^{ik} = F^{ik} = -\frac{2}{c}\,A^{ik}, \quad H_{ik} = h_{im}h_{kn}F^{mn} = -\frac{2}{c}\,A_{ik} \tag{17}$$

of the field tensor will be referred to as "magnetic". So, we arrive at physical definitions of the components:

> The "electric" observable component of a time density field manifests as the gravitational inertial force F_i. The "magnetic" observable component of a time density field manifests as the angular velocity A_{ik} of the space rotation.

In accordance with the above, two particular cases of time density fields are possible. These are:

1. If a time density field has $H_{ik} = 0$ and $E^i \neq 0$, then the field is strictly of the "electric" kind. This particular case corresponds to a holonomic (non-rotating) space filled with gravitational force fields;
2. A time density field is of the "magnetic" kind, if therein $E^i = 0$ and $H_{ik} \neq 0$. This is a non-holonomic space, where fields of gravitational inertial forces are homogeneous or absent. This case is possible also if, according to the chr.inv.-definition of the force
$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{18}$$
where the first term — a force of gravity would be reduced by the second term — is a centrifugal force of inertia.

In addition to the field tensor $F_{\alpha\beta}$, we introduce the field pseudotensor $F^{*\alpha\beta}$ dual and in the usual way [1]

$$F^{*\alpha\beta} = \frac{1}{2}\,E^{\alpha\beta\mu\nu}F_{\mu\nu}\,, \quad F_{*\alpha\beta} = \frac{1}{2}\,E_{\alpha\beta\mu\nu}F^{\mu\nu}, \tag{19}$$

where the four-dimensional completely antisymmetric discriminant tensors $E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}$ and $E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}$, transforming regular tensors into pseudotensors in inhomogeneous anisotropic pseudo-Riemannian spaces, are not physically observable quantities. The completely antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$, being defined in a Galilean reference frame in Minkowski space [1], does not have this quality either. Therefore we employ Zelmanov's chr.inv.-discriminant tensors $\varepsilon^{\alpha\beta\gamma} = b_\sigma E^{\sigma\alpha\beta\gamma}$ and $\varepsilon_{\alpha\beta\gamma} = b^\sigma E_{\sigma\alpha\beta\gamma}$ [2], which in the accompanying reference frame are

$$\varepsilon^{ikm} = \frac{e^{ikm}}{\sqrt{h}}\,, \qquad \varepsilon_{ikm} = e_{ikm}\sqrt{h}\,. \tag{20}$$

Using components of the field tensor $F_{\alpha\beta}$, we obtain chr.inv.-projections of the field pseudotensor, which are

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = -\frac{1}{c}\,\varepsilon^{ikm}A_{km} = -\frac{2}{c}\,\Omega^{*i}, \tag{21}$$

$$E^{*ik} = F^{*ik} = \frac{1}{c^2}\,\varepsilon^{ikm}F_m\,, \tag{22}$$

where $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{ikm}A_{km}$ is the chr.inv.-pseudovector of angular velocities of the space rotation. Their relations to the field tensor chr.inv.-projections express themselves just like any chr.inv.-pseudotensors [5, 6], by the formulae

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}H_{mn}\,, \qquad H_{*i} = \frac{1}{2}\,\varepsilon_{imn}H^{mn}, \tag{23}$$

$$\varepsilon^{ipq}H_{*i} = \frac{1}{2}\,\varepsilon^{ipq}\varepsilon_{imn}H^{mn} = H^{pq}, \tag{24}$$

$$\varepsilon_{ikp}H^{*p} = E_{ik}\,, \qquad E^{*ik} = -\varepsilon^{ikm}E_m\,, \tag{25}$$

where $\varepsilon^{ipq}\varepsilon_{imn} = \delta^p_m\delta^q_n - \delta^q_m\delta^p_n$, see [1, 5, 6] for details.

We introduce the invariants $J_1 = F_{\alpha\beta}F^{\alpha\beta}$ and $J_2 = F_{\alpha\beta}F^{*\alpha\beta}$ for a time density field. Their formulae are

$$J_1 = F_{\alpha\beta}F^{\alpha\beta} = \frac{4}{c^2}A_{ik}A^{ik} - \frac{2}{c^4}F_iF^i, \tag{26}$$

$$J_2 = F_{\alpha\beta}F^{*\alpha\beta} = -\frac{8}{c^3}F_i\Omega^{*i}, \tag{27}$$

so the time density field can be *spatially isotropic* (one of the invariants becomes zero) under the conditions:

- the invariant $A_{ik}A^{ik}$ of the space rotation field and the invariant F_iF^i of the gravitational inertial force field are proportional one to another $A_{ik}A^{ik} = \frac{1}{2c^2}F_iF^i$;
- $F_i\Omega^{*i} = 0$, so the acting gravitational inertial force F_i is orthogonal to the space rotation pseudovector Ω^{*i};
- both of the conditions are realized together.

3 Equations of free motion. Putting the acting force into a form like Lorentz's force

Time lines are geodesics by definition. In accordance with the *least action principle*, an action replacing a particle along a geodesic line is minimum. Actually, the least action principle implies that geodesic lines are also lines of the least action. This is the physical viewpoint.

We are going to consider first a free mass-bearing particle, which is at rest with respect to an observer and his reference body. Such a particle moves only along a time line, so it moves solely because of the action of the inhomogeneity of time coordinates along the time line — a time density field.

The action that a time density field expends in displacing a free mass-bearing particle of rest-mass m_0 at dx^α has the value $dS = m_0 c\, b_\alpha dx^\alpha$ (8). Because of the least action, variation of the action integral along geodesic lines equals zero

$$\delta\int_a^b dS = 0\,, \tag{28}$$

which, after substituting $dS = m_0 c b_\alpha dx^\alpha$, becomes $\delta\int_a^b dS = m_0 c\,\delta\int_a^b b_\alpha dx^\alpha = m_0 c\int_a^b \delta b_\alpha dx^\alpha + m_0 c\int_a^b b_\alpha d\delta x^\alpha$ where $\int_a^b b_\alpha d\delta x^\alpha = b_\alpha\delta x^\alpha\big|_a^b - \int_a^b db_\alpha \delta x^\alpha = -\int_a^b db_\alpha\delta x^\alpha$. Because $\delta b_\alpha = \frac{\partial b_\alpha}{\partial x^\beta}\delta x^\beta$ and $db_\alpha = \frac{\partial b_\alpha}{\partial x^\beta}dx^\beta$,

$$\delta\int_a^b dS = m_0 c\,\delta\int_a^b\left(\frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\right)dx^\beta\delta x^\alpha. \tag{29}$$

This variation is zero, so along time lines we have

$$m_0 c\left(\frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\right)dx^\beta = 0\,. \tag{30}$$

This condition, being divided by the interval ds, gives general covariant equations of motion of the particle

$$m_0 c\,F_{\alpha\beta}U^\beta = 0\,, \tag{31}$$

wherein $F_{\alpha\beta}$ is the time density field tensor and U^β is the particle's four-dimensional velocity*.

Taking chr.inv.-projections of (31) multiplied by c^2, we obtain chr.inv.-equations of motion of the particle

$$m_0 c^3\frac{F_{0\sigma}U^\sigma}{\sqrt{g_{00}}} = 0\,, \qquad m_0 c^2 F_{\cdot\sigma}^{i\cdot}U^\sigma = 0\,, \tag{32}$$

where the scalar equation gives the work to displace the particle, and the vector equations its observable acceleration.

It is interesting to note that the left side of the equations, which is the acting force, both in the general covariant form and its chr.inv.-projections we have obtained, has the same form as Lorentz's force, which displaces charged particles in electromagnetic fields [5]. From the mathematical viewpoint this fact implies that the time density field acts on mass-bearing particles as the electromagnetic field moves electric charge.

Taking $ds^2 = c^2 d\tau^2 - d\sigma^2 = c^2 d\tau^2\left(1 - \frac{\mathrm{v}^2}{c^2}\right)$, that is formula (4) into account, we obtain

$$U^\alpha = \frac{dx^\alpha}{ds} = \frac{1}{c\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\frac{dx^\alpha}{d\tau}\,, \tag{33}$$

$$U^0 = \frac{\frac{1}{c^2}v_k\mathrm{v}^k + 1}{\sqrt{g_{00}}\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,, \qquad U^i = \frac{1}{c\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\mathrm{v}^i. \tag{34}$$

Using the components obtained for the field tensor $F_{\alpha\beta}$ (11–15) and taking into account that the observable velocity of the particle we are considering is $\mathrm{v}^i = 0$, we transform the chr.inv.-equations of motion (32) into the final form. The scalar equation becomes zero, while the vector equations become $m_0 F^i = 0$ or, substituting $E^i = -\frac{1}{c^2}F^i$ (16),

$$m_0 c^2 E^i = 0\,, \tag{35}$$

leading us to the following conclusions:

1. The "electric" and the "magnetic" components of a time density field do not produce work to displace a free mass-bearing particle along time lines. Such a particle falls freely along its own time line under the time density field;
2. In this case $E^i = 0$, so the particle falls freely along its own time line, being carryied solely by the "magnetic" component $H_{ik} = -\frac{2}{c}A_{ik} \neq 0$ of the time density field;
3. Inhomogeneity of the spatial section (the chr.inv.-Christoffel symbols Δ^i_{jk}) or its deformations (the chr. inv.-deformation rate tensor D_{ik}) do not have an effect on free motion along time lines.

*Do not confound this vector $U^\alpha = \frac{dx^\alpha}{ds}$ with the vector $b^\alpha = \frac{dx^\alpha}{ds}$: they are built on different dx^α. The vector b^α contains displacement of the observer with respect to his reference body, while the vector U^α contains displacement of the particle.

In other words, the "magnetic" component $H_{ik} = -\frac{2}{c} A_{ik}$ of a time density field "screws" particles into time lines (a very rough analogy). There are no other sources which could cause particles to move along time lines, because observable particles with the whole spatial section move from past into future, hence $H_{ik} \neq 0$ everywhere in our real world. So, our real space is strictly non-holonomic, $A_{ik} \neq 0$.

This purely mathematical result brings us to the very important conclusion that under any conditions a real space is non-holonomic at the "start", that is, a "primordial non-orthogonality" of the real spatial section to time lines. Conditions such as three-dimensional rotations of the reference body, are only additions, intensifying or reducing this start-rotation of the space, depending on their relative directions*.

We are now going to consider the second case of free motion — the general case, where a free mass-bearing particle moves freely not only along time lines, but also along the spatial section with respect to the observer and his reference body. Chr.inv.-equations of motion in this general case had been deduced by Zelmanov [2]. They have the form

$$\begin{gathered} \frac{dE}{d\tau} - mF_i \mathrm{v}^i + mD_{ik} \mathrm{v}^i \mathrm{v}^k = 0, \qquad E = mc^2 \\ \frac{d(m\mathrm{v}^i)}{d\tau} - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0. \end{gathered} \tag{36}$$

Let us express the equations through the "electric" and the "magnetic" observable components of the acting field of time density. Substituting $E^i = -\frac{1}{c^2} F^i$ and $H_{ik} = -\frac{2}{c} A_{ik}$ into the Zelmanov equations (36), we obtain

$$\begin{gathered} \frac{dE}{d\tau} + mc^2 E_i \mathrm{v}^i + mD_{ik} \mathrm{v}^i \mathrm{v}^k = 0, \\ \frac{d(m\mathrm{v}^i)}{d\tau} + mc^2\left(E^i + \frac{1}{c} H^{ik}\mathrm{v}_k\right) + \\ + 2mD^i_k \mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0. \end{gathered} \tag{37}$$

From this we see that a free mass-bearing particle moves freely along the spatial section because of the factors:

1. The particle is carried with a time density field by its "electric" $E^i \neq 0$ and "magnetic" $H_{ik} \neq 0$ components;
2. The particle is also moved by forces which manifest as an effect of inhomogeneity Δ^i_{nk} and deformations D_{ik} of the spatial section. As we can see from the scalar equation, the field of the space inhomogeneities does not produce any work to displace free mass-bearing particles, only the space deformation field produces the work.

In particular, a mass-bearing particle can be moved freely along the spatial section, solely because of the field of time density. As it easy to see from equations (37), this is possible under the following conditions

$$D_{ik}\mathrm{v}^i\mathrm{v}^k = 0\,, \qquad D^i_k = -\frac{1}{2}\,\Delta^i_{nk}\mathrm{v}^n, \tag{38}$$

so it is possible in the following particular cases:

- if the spatial section has no deformations, $D_{ik} = 0$;
- if, besides the absence of the deformations ($D_{ik} = 0$), the spatial section is homogeneous, $\Delta^i_{nk} = 0$.†

The scalar equations of motion (37) also show that, under the particular conditions (38), the energy dE to displace the particle at dx^i equals the work

$$dE = -mc^2 E_i dx^i \tag{39}$$

the "electric" field component E_i expends for this displacement. The vector equations of motion in this particular case show that the "electric" and the "magnetic" components of the acting field of time density accelerate the particle just like external forces‡

$$\frac{dp^i}{d\tau} = -mc^2\left(E^i + \frac{1}{c} H^{ik}\mathrm{v}_k\right). \tag{40}$$

Looking at the right sides of equations (39, 40), we see that they have a form identical to the right sides of the chr.inv.-equations of motion of a charged particle in the electromagnetic field [5]. This implies also that the field of time density acts on mass-bearing particles as an electromagnetic field moves electric charge.

4 The field equations like electrodynamics

As is well-known, the theory of the electromagnetic field, in a pseudo-Riemannian space, characterizes the field by a system of equations known also as the *field equations*:

- Lorentz's condition stipulates that the four-dimensional vector potential A^α of the field remains unchanged just like any four-dimensional vector in a four-dimensional pseudo-Riemannian space

$$\nabla_\sigma A^\sigma = 0\,; \tag{41}$$

- the charge conservation law (the continuity equation) shows that the field-inducing charge cannot be destroyed, but merely re-distributed in the space

$$\nabla_\sigma j^\sigma = 0\,, \tag{42}$$

*A similar conclusion had also been given by the astronomer Kozyrev [7], from his studies of the interior of stars. In particular, besides the "start" self-rotation of the space, he had come to the conclusion that additional rotations will produce an inhomogeneity of observable time around rotating bulky bodies like stars or planets. The consequences should be more pronounced in the interaction of the components of bulky double stars [8]. He was the first to used the term "time density field". It is interesting that his arguments, derived from a purely phenomenological analysis of astronomical observations, did not link to Riemannian geometry and the mathematical apparatus of the General Theory of Relativity.

†However the first condition $D_{ik} = 0$ would be sufficient.

‡Here the chr.inv.-vector $p^i = m\mathrm{v}^i$ is the particle's observable impulse.

where j^α is the four-dimensional current vector; its observable projections are the chr.inv.-charge density scalar $\rho = \frac{1}{c\sqrt{g_{00}}}\, j_0$ and the chr.inv.-current density vector j^i, which are sources inducing the field;

- Maxwell's equations show properties of the field, expressed by components of the field tensor $F_{\alpha\beta}$ and its dual pseudotensor $F^{*\alpha\beta}$. The first group of the Maxwell equations contains the field sources ρ and j^i, the second group does not contain the sources

$$\nabla_\sigma F^{\alpha\sigma} = \frac{4\pi}{c}\, j^\alpha, \qquad \nabla_\sigma F^{*\alpha\sigma} = 0\,. \tag{43}$$

We can put all the equations into chr.inv.-form, employing Zelmanov's formula [2] for the divergence of a vector Q^α, where he expressed the divergence through chr.inv.-projections $\varphi = \frac{Q_0}{\sqrt{g_{00}}}$ and $q^i = Q^i$ of this vector

$$\nabla_\sigma Q^\sigma = \frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t} + \varphi D\right) + {}^*\nabla_i\, q^i - \frac{1}{c^2} F_i q^i, \tag{44}$$

where we use his notation for *chr.inv.-divergence*

$${}^*\nabla_i\, q^i = \frac{^*\partial q^i}{\partial x^i} + q^i\,\frac{^*\partial \ln\sqrt{h}}{\partial x^i} = \frac{^*\partial q^i}{\partial x^i} + q^i \Delta^j_{ji}\,. \tag{45}$$

In particular, the chr.inv.-Maxwell equations, which are chr.inv.-projections of the Maxwell general covariant equations (43), had first been obtained for an arbitrary field potential by del Prado and Pavlov [9], Zelmanov's students, at Zelmanov's request. The equations are

$$\left.\begin{aligned} &{}^*\nabla_i\, E^i - \frac{1}{c}\, H^{ik} A_{ik} = 4\pi\rho \\ &{}^*\nabla_k\, H^{ik} - \frac{1}{c^2} F_k H^{ik} - \frac{1}{c}\left(\frac{^*\partial E^i}{\partial t} + E^i D\right) = \frac{4\pi}{c}\, j^i \end{aligned}\right\}\text{I}, \tag{46}$$

$$\left.\begin{aligned} &{}^*\nabla_i\, H^{*i} - \frac{1}{c}\, E^{*ik} A_{ik} = 0 \\ &{}^*\nabla_k\, E^{*ik} - \frac{1}{c^2} F_k E^{*ik} - \frac{1}{c}\left(\frac{^*\partial H^{*i}}{\partial t} + H^{*i} D\right) = 0 \end{aligned}\right\}\text{II}, \tag{47}$$

From the mathematical viewpoint, equations of the field are a system of 10 equations in 10 unknowns (the Lorentz condition, the charge conservation law, and two groups of the Maxwell equations), which define the given vector field A^α and its inducing sources in a pseudo-Riemannian space. Actually, equations like these should exist for any four-dimensional vector field, a time density field included. The only difference should be that the equations should be changed according to a formula for the specific vector potential.

We are going to deduce such equations for the field b^α we are considering — *equations of a time density field*.

Because $\varphi = 1$ and $q^i = 0$ are chr.inv.-projections of the potential b^α of a time density field, the *Lorentz condition* $\nabla_\sigma\, b^\sigma = 0$ for a time density field b^α becomes the equality

$$D = 0\,, \tag{48}$$

where $D = h^{ik} D_{ik}$, being the spur of the deformation rate tensor, is the rate of expansion of an elementary volume. Actually, the obtained Lorentz condition (48) implies that the value of an elementary volume filled with a time density field remains unchanged under its deformations.

We now collect chr.inv.-projections of the tensor of a time density field $F_{\alpha\beta}$ and of the field pseudotensor $F^{*\alpha\beta}$ together: $E_i = -\frac{1}{c^2}\, F_i$, $H^{ik} = -\frac{2}{c}\, A^{ik}$, $H^{*i} = -\frac{2}{c}\, \Omega^{*i}$, $E^{*ik} = \frac{1}{c^2}\, \varepsilon^{ikm} F_m$. We also take Zelmanov's identities for the chr.inv.-discriminant tensors [2] into account

$$\frac{^*\partial\varepsilon_{imn}}{\partial t} = \varepsilon_{imn}\, D\,, \qquad \frac{^*\partial\varepsilon^{imn}}{\partial t} = -\varepsilon^{imn} D\,, \tag{49}$$

$${}^*\nabla_k\, \varepsilon_{imn} = 0\,, \qquad {}^*\nabla_k\, \varepsilon^{imn} = 0\,. \tag{50}$$

Substituting the chr.inv.-projections into (46, 47) along with the obtained Lorentz condition $D = 0$ (48), we arrive at *Maxwell-like chr.inv.-equations* for a time density field

$$\left.\begin{aligned} &\frac{1}{c^2}\, {}^*\nabla_i\, F^i - \frac{2}{c^2} A_{ik} A^{ik} = -\,4\pi\rho \\ &\frac{2}{c}\, {}^*\nabla_k\, A^{ik} - \frac{2}{c^3} F_k A^{ik} - \frac{1}{c^3}\frac{^*\partial F^i}{\partial t} = -\frac{4\pi}{c}\, j^i \end{aligned}\right\}\text{I}, \tag{51}$$

$$\left.\begin{aligned} &{}^*\nabla_i\, \Omega^{*i} + \frac{1}{c^2}\, F_i\, \Omega^{*i} = 0 \\ &{}^*\nabla_k \left(\varepsilon^{ikm} F_m\right) - \frac{1}{c^2}\, \varepsilon^{ikm} F_k F_m + 2\,\frac{^*\partial\Omega^{*i}}{\partial t} = 0 \end{aligned}\right\}\text{II}, \tag{52}$$

so that the field-inducing sources ρ and j^i are

$$\rho = -\frac{1}{4\pi c^2}\left({}^*\nabla_i\, F^i - 2 A_{ik} A^{ik}\right), \tag{53}$$

$$j^i = -\frac{1}{2\pi}\, {}^*\nabla_k\, A^{ik} - \frac{1}{2\pi c^2} F_k A^{ik} - \frac{1}{4\pi c^2}\frac{^*\partial F^i}{\partial t}\,. \tag{54}$$

The "*charge*" *conservation law* $\nabla_\sigma\, j^\sigma = 0$ (the continuity equation), after substituting chr.inv.-projections $\varphi = c\rho$ and $q^i = j^i$ of the "current" vector j^α, takes the chr.inv.-form

$$\begin{aligned} &\frac{1}{c^2}\frac{^*\partial}{\partial t}\left(A_{ik} A^{ik}\right) + \frac{1}{c^2} F_i \frac{^*\partial A^{ik}}{\partial x^k} - \frac{^*\partial^2 A^{ik}}{\partial x^i \partial x^k} + \\ &\quad + \left(\frac{1}{c^2} F_i \Delta^j_{jk} + \frac{^*\partial \Delta^j_{jk}}{\partial x^i} + \Delta^j_{ji}\Delta^l_{lk}\right) A^{ik} - \\ &\quad - \frac{1}{2c^2}\, F^i \frac{^*\partial\Delta^j_{ji}}{\partial t} - \frac{1}{c^4} F_i F_k A^{ik} = 0\,, \end{aligned} \tag{55}$$

The Lorentz condition (48), the Maxwell-like equations (51, 52), and the continuity equation (55) we have obtained are *chr.inv.-equations of a time density field*.

5 Waves of the field

Let us turn now to d'Alembert's equations. We are going to obtain the equations for a time density field.

d'Alembert's operator $\Box = g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta$, being applied to a field, may or may not be zero. The second case is known as the d'Alembert equations with field-inducing sources, while the first case is known as the d'Alembert equations without sources. If the field has no sources, then the field is free. This is a wave. So, the d'Alembert equations without sources are equations of propagation of waves of the field.

From this reason, the d'Alembert equations for the vector potential b^α of a time density field without the sources

$$\Box b^\alpha = 0 \tag{56}$$

are the equations of propagation of waves of the time density field. Chr.inv.-projections of the equations are

$$b_\sigma \Box b^\sigma = 0\,, \qquad h^i_\sigma \Box b^\sigma = 0\,. \tag{57}$$

We substitute chr.inv.-projections $\varphi = 1$ and $q^i = 0$ of the field potential b^α into this. Then, taking into account that the Lorentz condition for the field b^α is $D = 0$ (48), after some algebra we obtain the *chr.inv.-d'Alembert equations* for the time density field without sources

$$\begin{gathered}\frac{1}{c^2}\,F_iF^i - D_{ik}D^{ik} = 0\,,\\ \frac{1}{c^2}\frac{{}^*\partial F^i}{\partial t} + h^{km}\Big\{\frac{{}^*\partial D^i_m}{\partial x^k} + \frac{{}^*\partial A^{\cdot i}_{m\cdot}}{\partial x^k} + \\ + \Delta^i_{kn}\big(D^n_m - A^{\cdot n}_{m\cdot}\big) - \Delta^n_{km}\big(D^i_n - A^{\cdot i}_{n\cdot}\big)\Big\} = 0\,.\end{gathered} \tag{58}$$

Unfortunately, a term like $\frac{1}{a^2}\frac{\partial^2 q^i}{\partial t^2}$ containing the linear speed a of the waves is not present, because of $q^i = 0$. For this reason we have no possibility of saying anything about the speed of waves traveling in time density fields. At the same time the obtained equations (58) display numerous specific peculiarities of a space filled with the waves:

1. The rate of deformations of a surface element in waves of a time density field is powered by the value of the acting gravitational inertial force F_i. If $F_i = 0$, the observable spatial metric h_{ik} is stationary;
2. If a space, filled with waves of a time density field, is homogeneous $\Delta^i_{kn} = 0$ and also the acting force field is stationary $F_i = \text{const}$, the spatial structure of the space deformations is the same as that of the space rotation field.

6 Energy-momentum tensor of the field

Proceeding from the general covariant equations of motion along only time lines, we are going to deduce the energy-momentum tensor for time density fields. It is possible to do this in the following way.

The aforementioned equations $m_0 c\,F_{\alpha\beta}U^\beta = 0$ (31), being taken in contravariant (upper-index) form, are

$$m_0 c F^{\alpha\cdot}_{\cdot\sigma}U^\sigma = 0\,, \tag{59}$$

where U^σ is the four-dimensional velocity of the particle. The left side of the equations has the dimensions $[\text{gramme}/\text{sec}]$ as well as a four-dimensional force. Because of motion along only time lines, such particle moves solely under the action of a time density field whose tensor is $F_{\alpha\beta}$.

If this free-moving particle is not a point-mass, then it can be represented by a current j^α of the time density field. On the other hand, such currents are defined by the 1st group $\nabla_\sigma F^{\alpha\sigma} = \frac{4\pi}{c}\,j^\alpha$ of the Maxwell-like equations of the field. In this case equations of motion (59), drawing an analogy with an electromagnetic field current, take the form

$$\mu F^{\alpha\cdot}_{\cdot\sigma}\,j^\sigma = 0\,. \tag{60}$$

The numerical coefficient μ here is a new fundamental constant. This new constant having the dimension $[\text{gramme}/\text{sec}]$ gives the dimensions $[\text{gramme}/\text{cm}^2\times\text{sec}^2]$ to the left side of the equations, making the left side a current of the acting four-dimensional force (59) through $1\,\text{cm}^2$ per 1 second. The numerical value of this constant μ can be found from measurements of the wave pressure of a time density field, see formula (101) below. However it does not exclude that future studies of the problem will yield an analytic formula for μ, linking it to other fundamental constants.

Chr.inv.-projections of the equations (60)

$$\frac{\mu F_{0\sigma}\,j^\sigma}{\sqrt{g_{00}}} = 0\,, \qquad \mu F^{i\cdot}_{\cdot\sigma}\,j^\sigma = 0\,, \tag{61}$$

after substituting the $F_{\alpha\beta}$ components (11–15) take the form

$$\mu E_k\,j^k = 0\,, \qquad \mu c\left(\rho E^i - \frac{1}{c}\,H^{i\cdot}_{\cdot k}\,j^k\right) = 0\,, \tag{62}$$

where E^i is the "electric" observable component and H_{ik} is the "magnetic" observable component of the time density field. Sources ρ and j^i inducing the field are defined by the 1st group of the Maxwell-like chr.inv.-equations (51).

Actually, the term*

$$f^\alpha = \mu F^{\alpha\cdot}_{\cdot\sigma}\,j^\sigma \tag{63}$$

on the left side of the general covariant equations of motion (60) can be transformed with the 1st Maxwell-like group $\nabla_\beta F^{\sigma\beta} = \frac{4\pi}{c}\,j^\sigma$ to the form $f_\alpha = \frac{\mu c}{4\pi}\,F_{\alpha\sigma}\nabla_\beta F^{\sigma\beta}$ which is

$$f_\alpha = \frac{\mu c}{4\pi}\Big[\nabla_\beta\big(F_{\alpha\sigma}F^{\sigma\beta}\big) - F^{\sigma\beta}\,\nabla_\beta F_{\alpha\sigma}\Big]\,, \tag{64}$$

where we express the second term in the form $F^{\sigma\beta}\nabla_\beta F_{\alpha\sigma} = \frac{1}{2}\,F^{\sigma\beta}\left(\nabla_\beta F_{\alpha\sigma} + \nabla_\sigma F_{\beta\alpha}\right) = -\frac{1}{2}\,F^{\sigma\beta}\left(\nabla_\beta F_{\sigma\alpha} + \nabla_\sigma F_{\alpha\beta}\right) = -\frac{1}{2}\,F^{\sigma\beta}\nabla_\sigma F_{\alpha\beta} = \frac{1}{2}\,F^{\sigma\beta}\nabla_\alpha F_{\sigma\beta}$. Using this formula, we transform the current f^α (63) to the form

$$f_\alpha = \frac{\mu c}{4\pi}\,\nabla_\beta\left(-F_{\alpha\sigma}F^{\beta\sigma} + \frac{1}{4}\,\delta^\beta_\alpha F_{pq}F^{pq}\right), \tag{65}$$

*From the physical viewpoint, this term is a current of the acting four-dimensional force, produced by the time density field.

so we write the current f^α in the form

$$f^\alpha = \nabla_\beta T^{\alpha\beta} \tag{66}$$

just as electrodynamics does to deduce the energy-momentum tensor $T^{\alpha\beta}$ of electromagnetic fields. In this way, we obtain the *energy-momentum tensor* of a time density field, which is

$$T^{\alpha\beta} = \frac{\mu c}{4\pi}\left(-F^{\alpha\cdot}_{\cdot\sigma}F^{\beta\sigma} + \frac{1}{4}\,g^{\alpha\beta}F_{pq}F^{pq}\right), \tag{67}$$

the form of which is the same as the energy-momentum tensor of electromagnetic fields [1, 5] to within the coefficient of its dimension. It is easy to see that the tensor is symmetric, so its spur is zero, $T^\sigma_\sigma = g_{\alpha\beta}T^{\alpha\beta} = 0$.

So forth we deduce the chr.inv.-projections of the energy-momentum tensor of a time density field

$$q = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\,T^i_0}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik}. \tag{68}$$

After substituting the required components of the field tensor $F_{\alpha\beta}$ (11–15), we obtain

$$q = \frac{\mu}{4\pi c}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_kF^k\right), \tag{69}$$

$$J^i = -\frac{\mu}{2\pi c}F_kA^{ik}, \tag{70}$$

$$U^{ik} = -\frac{\mu c}{4\pi}\left(4A^{i\cdot}_{\cdot m}A^{mk} + \frac{1}{c^2}F^iF^k + \right. \left. + A_{pq}A^{pq}h^{ik} - \frac{1}{2c^2}F_pF^ph^{ik}\right). \tag{71}$$

In accordance with dimensions, the chr.inv.-projections have the following physical meanings:

- the time observable projection q [gramme/cm×sec²] is the energy [gm×cm²/sec²] this time density field contains in 1 cm³. Actually, the chr.inv.-scalar q is the *observable density of the field*;
- the mixed observable projection J^i [gramme/sec³] is the energy the time density field transfers through 1 cm² per second, in other words, this is the *observable density of the field momentum*;
- the spatial observable projection U^{ik} [gm×cm/sec⁴] is the tensor of the field momentum flux observable density, in other words, the *field strength tensor*.

7 Physical properties of the field

It has been proven by Zelmanov [10], that the chr.inv.-field strength tensor U^{ik}, can be written in covariant (lower index) form as follows

$$U_{ik} = p_0h_{ik} - \alpha_{ik} = ph_{ik} - \beta_{ik}\,, \tag{72}$$

where $\alpha_{ik} = \beta_{ik} + \frac{1}{3}\,\alpha h_{ik}$ is the viscous strength tensor of the field. Zelmanov called α_{ik} the *viscosity of the 2nd kind* (here $\alpha = h^{ik}\alpha_{ik} = \alpha^n_n$ is its spur). Its anisotropic part β_{ik}, called the *viscosity of the 1st kind*, manifests as anisotropic deformations of the space. The quantity p_0 is that pressure inside the medium, which equalizes its density in the absence of viscosity, p is the true pressure of the medium*. It is easy to see that the viscous strength tensors α_{ik} and β_{ik} are chr.inv.-quantities by their definitions.

By extracting the viscous strength tensors α_{ik} and β_{ik} from the formula of the strength tensor U_{ik} of a time density field, we are going to deduce the equation of state of the field.

Transforming U^{ik} (71) into covariant form and also keeping the formula for q (69) in the mind, we write

$$U_{ik} = -qc^2h_{ik} - \left. - \frac{\mu c}{4\pi}\left(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2}F_iF_k - \frac{1}{c^2}F_mF^mh_{ik}\right), \right. \tag{73}$$

which, after equating to $U_{ik} = p_0h_{ik} - \alpha_{ik}$ (72), gives the equilibrium pressure in the field

$$p_0 = -qc^2, \tag{74}$$

while the *viscous strength tensor* of the field is

$$\alpha_{ik} = \frac{\mu c}{4\pi}\left(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2}F_iF_k - \frac{1}{c^2}F_mF^mh_{ik}\right). \tag{75}$$

Because the spur of this tensor α_{ik}, as it is easy to see, is not zero, $\alpha = h^{ik}\alpha_{ik} = -\frac{\mu c}{\pi}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_kF^k\right) \neq 0$, the tensor $\alpha_{ik} = \beta_{ik} + \frac{1}{3}\,\alpha h_{ik}$ has the non-zero anisotropic part

$$\beta_{ik} = \frac{\mu c}{4\pi}\left(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2}F_iF_k - \right. \left. - \frac{1}{3c^2}F_mF^mh_{ik} + \frac{4}{3}A_{mn}A^{mn}h_{ik}\right), \tag{76}$$

so viscous strengths of time density fields are anisotropic. It is also easy to see that this anisotropy increases with the value $A_{pq}A^{pq}$ of the space rotation.

Because the viscous strengths α_{ik} are anisotropic, the equilibrium pressure $p_0 = -qc^2$ and the true pressure p inside the medium are different. The true pressure is

$$p = \frac{\mu c}{12\pi}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_kF^k\right), \tag{77}$$

*The equation of state of a medium is the relation between the pressure p inside the medium and its density q. In a non-viscous medium or where the viscous strengths are isotropic, the true pressure p is the same as the equilibrium pressure p_0. The equation of state of a dust medium has the form $p = 0$. Ultra-relativistic gases have the equation of state $p = \frac{1}{3}qc^2$. The equation of state of matter inside atomic nuclei is $p = qc^2$. Vacuum and μ-vacuum have the equation of state $p = -qc^2$, see [5].

which gives the *equation of state* for time density fields

$$p = \frac{1}{3}\,qc^2. \tag{78}$$

Finally, we write the strength tensor $U_{ik} = ph_{ik} - \beta_{ik}$ of a time density field in the form

$$U_{ik} = \frac{1}{3}\,qc^2 h_{ik} - \beta_{ik}\,. \tag{79}$$

So, we can conclude for the physical properties of time density fields:

1. In general, a time density field is a non-stationary distributed medium, because its density may be $q \neq$ const. The field becomes stationary $q =$ const under stationary space rotation $A_{ik} =$ const, and stationary gravitational inertial force $F_i =$ const;
2. A time density field bears momentum, because $J^i = -\frac{\mu}{2\pi c}\,F_k A^{ik} \neq 0$. So, the field can transfer impulse. The field does not transfer impulse $J^i = 0$, if the space does not rotate $A_{ik} = 0$. The absence of gravitation does not affect the field's transfer of impulse, because the "inertial" part of the force F_i remains unchanged even in the absence of gravitational fields;
3. A time density field is an emitting medium $J^i \neq 0$ in a non-holonomic (rotating) space. In a holonomic (non-rotating) space the field does not produce radiations;
4. A time density field is a viscous medium. The viscosity α_{ik} (75), derived from non-zero rotation of the space or from gravitational inertial force, is anisotropic. The anisotropy β_{ik} increases with the space rotation speed. The field is viscous anisotropic anyhow, because its viscous strengths would be $\alpha_{ik} = 0$ and $\beta_{ik} = 0$ only if both $A_{ik} = 0$ and $F_i = 0$. But in this case the field density would be $q = 0$, so the field itself is not there;
5. Therefore the equilibrium pressure p_0 does not possess a physical sense for time density fields; only the true pressure is real $p = p_0 - \frac{1}{3}\alpha$;
6. The equation of state for time density fields is $p = \frac{1}{3}qc^2$ (78) indicating that such fields are in the *state of an ultrarelativistic gas* — at positive density of the medium its inner pressure becomes positive, the medium is compressed.

8 Action of the field without sources

According to §27 of *The Classical Theory of Fields* [1], an elementary action for a whole system consisting of an electromagnetic field and a single charged particle, which are located in a pseudo-Riemannian space, contains three parts*

$$\begin{aligned} dS &= dS_{\text{m}} + dS_{\text{mf}} + dS_{\text{f}} = \\ &= m_0 c\,ds + \frac{e}{c}\,\mathcal{A}_\alpha dx^\alpha + a\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} dV dt\,, \end{aligned} \tag{80}$$

where $\mathcal{A}^\alpha$ is the four-dimensional electromagnetic field potential, $\mathcal{F}_{\alpha\beta} = \nabla_\alpha \mathcal{A}_\beta - \nabla_\beta \mathcal{A}_\alpha$ is the electromagnetic field tensor, $dV = dx\,dy\,dz$ is an elementary thee-dimensional volume filled with this field.

The first term S_{m} is "that part of the action which depends only on the properties of the particles, that is, just the action for free particles. . . . The quantity S_{mf} is that part of the action which depends on the interaction between the particles and the field. . . . Finally S_{f} is that part of the action which depends only on the properties of the field itself, that is, S_{f} is the action for a field in the absence of charges".

Because the action S_{f} must depend only on the field properties, the action must be taken over the space volume, filled with the field. The action must be scalar: only the 1st field invariant $J_1 = \mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta}$ has this property. The 2nd field invariant $J_2 = \mathcal{F}_{\alpha\beta}\,\mathcal{F}^{*\alpha\beta}$ is pseudoscalar, not scalar, leading to the detailed discussion in Landau and Lifshitz.

"The numerical value of a depends on the choice of units for measurement of the field. . . . From now on we shall use the Gaussian system of units; in this system a is a dimensionless quantity equal to $\frac{1}{16\pi}$".

According to §27 of *The Classical Theory of Fields* we have $dS_{\text{f}} = a\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} dV dt = \frac{1}{16\pi c}\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} d\Omega$, where $d\Omega = c\,dt\,dV = c\,dt\,dx\,dy\,dz$ is an elementary space (four-dimensional) volume. So the action (80) takes the final form

$$dS = m_0 c\,ds + \frac{e}{c}\,\mathcal{A}_\alpha dx^\alpha + \frac{1}{16\pi c}\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} d\Omega\,. \tag{81}$$

According to this consideration, we write an elementary action for the whole system consisting of a time density field and a single mass-bearing particle, which falls freely along time lines in a pseudo-Riemannian space, as follows

$$\begin{aligned} dS = dS_{\text{m}} + dS_{\text{mt}} &= m_0 c\,ds + a_{\text{mt}}\,F_{\alpha\beta}\,F^{\alpha\beta} d\Omega = \\ &= m_0 c\,b_\alpha dx^\alpha + a_{\text{mt}}\,F_{\alpha\beta}\,F^{\alpha\beta} d\Omega\,, \end{aligned} \tag{82}$$

where $F_{\alpha\beta}$ is the time density field tensor, a_{mt} is a constant consisting of other fundamental constants.

The first term S_{m} is that part of the action for the interaction between the particle and the time density field carrying it into motion along time lines. The second term

*In accordance with the least action principle, this action must have a minimum, so the integral of the action between a pair of world-points and the action itself must be positive. A negative action could give rise to a quantity with arbitrarily "large" negative values, which cannot have a minimum. Because in *The Classical Theory of Fields* Landau and Lifshitz take a pseudo-Riemannian space with the signature $(-+++)$, they write in §3 that ". . . the clock at rest always indicates a greater time interval than the moving one". Therefore they put "minus" before the action. To the contrary, we stick to a pseudo-Riemannian space with Zelmanov's signature $(+---)$, because in this case three-dimensional observable impulse is positive. In a space with such a signature, a regular observer takes his own flow of observable time positive always, $d\tau > 0$. Any particle, moving from past into future, has also a positive count of its own time coordinate $dt > 0$ with respect to the observer's clock. Therefore we put "plus" before the action.

$S_{\rm mt}$, depending only on the field properties, is the action for the field in the absence of its sources. In the absence of time density fields the second term $S_{\rm mt}$ is zero, so only $S_{\rm m} = m_0 cds$ remains here. A time density field is absent if the space is free of rotation $A_{ik} = 0$ and gravitational inertial forces $F_i = 0$, therefore if the conditions $g_{0i} = 0$ and $g_{00} = 1$ are true. This situation is possible in a pseudo-Riemannian space with a unit diagonal metric, which is the Minkowski space of the Special Theory of Relativity, where there is no gravitational field and no rotation. But in considering real space, we are forced to take a time density field into account. So we need to consider the terms $S_{\rm m}$ and $S_{\rm mt}$ together.

The constant $a_{\rm mt}$, according to its dimension, is the same as the constant μ in the energy-momentum tensor of time density fields, taken with the numerical coefficient $a = \frac{1}{16\pi}$, in the Gaussian system of units.

As a result, we obtain the action (82) in the final form

$$dS = dS_{\rm m} + dS_{\rm mt} = m_0 c\, b_\alpha dx^\alpha + \frac{\mu}{16\pi} F_{\alpha\beta} F^{\alpha\beta} d\Omega\,. \quad (83)$$

Because an action for a system is expressed through Lagrange's function L of the system as $dS = Ldt$, we take the action $dS_{\rm mt}$ in the form $dS_{\rm mt} = \frac{\mu c}{16\pi} F_{\alpha\beta} F^{\alpha\beta} dV dt = Ldt$, for the Lagrangian of an elementary volume $dV = dxdydz$ of the field. We therefore obtain the *Lagrangian density* in time density fields

$$\Lambda = \frac{\mu c}{16\pi} F_{\alpha\beta} F^{\alpha\beta} = \frac{\mu}{4\pi c} \left(A_{ik} A^{ik} - \frac{1}{2c^2} F_i F^i \right). \quad (84)$$

The term $A_{ik}A^{ik}$ here, being expressed through the space rotation angular velocity pseudovector Ω^{*i}, is

$$A_{km} A^{km} = \varepsilon_{kmn} \Omega^{*n} A^{km} = 2\Omega_{*n} \Omega^{*n}, \quad (85)$$

because $\varepsilon_{nkm}\Omega^{*n} = \frac{1}{2}\varepsilon^{npq}\varepsilon_{nkm}A_{pq} = \frac{1}{2}\left(\delta^p_k\delta^q_m - \delta^q_k\delta^p_m\right)A_{pq} = A_{km}$ and $\Omega_{*n} = \frac{1}{2}\varepsilon_{nkm}A^{km}$. So the space rotation plays the first violin, defining the Lagrangian density in time density fields. Rotation velocities in macro-processes are incommensurably small in comparison with rotations of atoms and particles. For instance, in the 1st Bohr orbit in an atom of hydrogen, measuring the value of Λ in the units of the energy-momentum constant μ, we have $\Lambda \simeq 9.1\times10^{21}\mu$. On the Earth's surface near the equator the value is $\Lambda \simeq 2.8\times10^{-20}\mu$, so it is in order of 10^{42} less than in atoms. Therefore, because the Lagrangian of a system is the difference between its kinetic and potential energies, we conclude that time density fields produce their main energy flux in atoms and sub-atomic interactions, while the energy flux produced by the fields of macro-processes is negligible.

9 Plane waves of the field under gravitation is neglected. The wave pressure

In general, because the electric and the magnetic strengths of a time density field are $E_i = -\frac{1}{c^2}F_i$ and $H^{ik} = -\frac{1}{c}A^{ik}$, the chr.inv.-vector of its momentum density J^i (70) can be written as follows

$$J^i = -\frac{\mu}{2\pi c} F_k A^{ik} = -\frac{\mu c}{4\pi} E_k H^{ik}. \quad (86)$$

We are going to consider a particular case, where the field depends on only one coordinate. Waves of such a field traveling in one direction are known as *plane waves*.

We assume the field depends only on the axis $x^1 = x$, so only the component $J^1 = -\frac{\mu}{2\pi c} F_k A^{1k}$ of the field's chr.inv.-momentum density vector is non-zero. Then a plane wave of the field travels along the axis $x^1 = x$. Assuming the space rotating in xy plane (only the components $A^{12} = -A^{21}$ are non-zeroes) and replacing the tensor A^{ik} with the space rotation angular velocity pseudovector Ω_{*m} in the form $\varepsilon^{mik}\Omega_{*m} = \frac{1}{2}\varepsilon^{mik}\varepsilon_{mpq}A^{pq} = \frac{1}{2}\left(\delta^i_p\delta^k_q - \delta^k_p\delta^i_q\right)A^{pq} = A^{ik}$, we obtain

$$J^1 = -\frac{\mu}{2\pi c} F_2 A^{12} = -\frac{\mu}{2\pi c} F_2\, \varepsilon^{123}\Omega_{*3}\,. \quad (87)$$

It is easy to see that while a plane wave of the field travels along the axis $x^1 = x$, the field's "electric" and "magnetic" strengths are directed along the axes $x^2 = y$ and $x^3 = z$, i. e. orthogonal to the direction the wave travels. Therefore waves travelling in time density fields are *transverse waves*.

Following the arguments of Landau and Lifshitz in §47 of *The Classical Theory of Fields* [1], we define the *wave pressure* of a field as the total flux of the field energy-momentum, passing through a unit area of a wall. So the pressure $\mathfrak{F}_i$ is the sum

$$\mathfrak{F}_i = T_{ik} n^k + T'_{ik} n^k \quad (88)$$

of the spatial components of the energy-momentum tensor $T_{\alpha\beta}$ in a wave, falling on the wall, and of the energy-momentum tensor $T'_{\alpha\beta}$ in the reflected wave, projected onto the unit spatial vector $\vec{n}_{(k)}$ orthogonal to the wall surface.

Because the chr.inv.-strength tensor of a field is $U_{ik} = c^2 h_{i\alpha} h_{k\beta} T^{\alpha\beta} = c^2 T_{ik}$ [2], we obtain

$$\mathfrak{F}_i = \frac{1}{c^2} \left(U_{ik} n^k + U'_{ik} n^k \right), \quad (89)$$

where $U_{ik} = c^2 T_{ik}$ and $U'_{ik} = c^2 T'_{ik}$ are the chr.inv.-strength tensors in the falling wave and in the reflected wave. So the three-dimensional wave pressure vector $\mathfrak{F}_i$ has the property of chronometric invariance.

Using our formulae for the density q (68) and the strength tensor U_{ik} (71) obtained for time density fields, we are going to find the pressure a wave of such field exerts on a wall.

We consider the problem in a weak gravitational field, assuming its potential w and the attracting force of gravity negligible. We can do this because formulae (68) and (71) contain gravitation in only higher order terms. So the space rotation plays the first violin in the wave pressure $\mathfrak{F}_i$ in time density fields, gravitational inertial forces act there only because of their inertial part.

A plane wave travels along a single spatial direction: we assume axis $x^1 = x$. In this case the chr.inv.-field strength tensor U_{ik} has the sole non-zero component U_{11}. All the other components of the strength tensor U_{ik} are zero, which simplifies this consideration.

We assume the space rotating around the axis $x^3 = z$ (the rotation is in the xy-plane) at a constant angular velocity Ω. In this case $A_{12} = -A_{21} = -\Omega$, $A_{13} = 0$, $A_{23} = 0$, so the components of the rotation linear velocity $v_i = A_{ik}x^k$ are $v_1 = -\Omega y$, $v_2 = \Omega x$, $v_3 = 0$. Then the components of the acting gravitational inertial force will be $F_1 = -\frac{\partial v_1}{\partial t} = \Omega \frac{\partial y}{\partial t} = \Omega v_2 = \Omega^2 x$, $F_2 = -\frac{\partial v_2}{\partial t} = \Omega^2 y$, $F_3 = 0$. Because in this case $A_{ik}A^{ik} = 2A_{12}A^{12} = 2\Omega^2$ and $A_{1m}A^{m\cdot}_{\cdot 1} = A_{1m}A^{mn}h_{1n} = A_{12}A^{21}h_{11} = -\Omega^2 h_{11}$, we obtain

$$q = \frac{\mu}{4\pi c}\left[2\Omega^2 + \frac{1}{2c^2}\,\Omega^4\left(x^2+y^2\right)\right], \tag{90}$$

$$U_{11} = \frac{\mu c}{4\pi}\left[2\Omega^2 h_{11} - \frac{1}{c^2}\,\Omega^4 x^2 + \frac{1}{2c^2}\,\Omega^4\left(x^2+y^2\right)h_{11}\right]. \tag{91}$$

We assume a coefficient of the reflection $\Re$ as the ratio between the density of the field energy q' in the reflected wave to the energy density q in the falling wave. Actually, because $q' = \Re q$, the reflection coefficient $\Re$ is the energy loss of the field after the reflection.

We assume $x = x_0 = 0$ at the reflection point on the surface of the wall. Then we have $U_{11} = qc^2 h_{11}$, which, after substituting into (89), gives the pressure

$$\mathfrak{F}_1 = (1+\Re)\,q\,h_{11}n^1 \tag{92}$$

that a plane wave of a time density field exerts on the wall.

To bring this formula into final form in a Riemannian space becomes a problem, because the coordinate axes are curved there, and inhomogeneous. For this reason we cannot define the angles between directions in a Riemannian space itself, the angle of incidence and the angle of reflection of a wave for instance. At the same time, to consider this problem in the Minkowski space of the Special Theory of Relativity, as done by Landau and Lifshitz for the pressure of plane electromagnetic waves [1], would be senseless — because in Minkowski space we have $g_{00} = 1$ and $g_{0i} = 0$, then $F_i = 0$ and $A_{ik} = 0$, which implies no time density fields there.

To solve this problem correctly for a Riemannian space, let us introduce a *locally geodesic reference frame*, following Zelmanov. We therefore introduce a locally geodesic reference frame at the point of reflection of a wave on the surface of a wall. Within infinitesimal vicinities of any point of such a reference frame the fundamental metric tensor is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{\alpha\beta}}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \ldots, \tag{93}$$

i. e. its components at a point, located in the vicinities, are different from those at the point of reflection to within only the higher order terms, values of which can be neglected. Therefore, at any point of a locally geodesic reference frame the fundamental metric tensor can be considered constant, while the first derivatives of the metric (the Christoffel symbols) are zero.

As a matter of fact, within infinitesimal vicinities of any point located in a Riemannian space, a locally geodesic reference frame can be set up. At the same time, at any point of this locally geodesic reference frame a tangential flat Euclidean space can be set up so that this reference frame, being locally geodesic for the Riemannian space, is the global geodesic for that tangential flat space.

The fundamental metric tensor of a flat Euclidean space is constant, so the values of $\tilde{g}_{\mu\nu}$, taken in the vicinities of a point of the Riemannian space, converge to the values of the tensor $g_{\mu\nu}$ in the flat space tangential at this point. Actually, this means that we can build a system of basis vectors $\vec{e}_{(\alpha)}$, located in this flat space, tangential to curved coordinate lines of the Riemannian space.

In general, coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other (if the space is non-holonomic). So the lengths of the basis vectors may sometimes be very different from unity.

We denote a four-dimensional vector of infinitesimal displacement by $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$, so that $d\vec{r} = \vec{e}_{(\alpha)}dx^\alpha$, where components of the basis vectors $\vec{e}_{(\alpha)}$ tangential to the coordinate lines are $\vec{e}_{(0)} = \{e^0_{(0)}, 0, 0, 0\}$, $\vec{e}_{(1)} = \{0, e^1_{(1)}, 0, 0\}$, $\vec{e}_{(2)} = \{0, 0, e^2_{(2)}, 0\}$, $\vec{e}_{(3)} = \{0, 0, 0, e^2_{(3)}\}$. The scalar product of the vector $d\vec{r}$ with itself is $d\vec{r}d\vec{r} = ds^2$. On the other hand, the same quantity is $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$. As a result we have $g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)}e_{(\beta)}\cos\,(x^\alpha; x^\beta)$. So we obtain

$$g_{00} = e^2_{(0)}\,, \qquad g_{0i} = e_{(0)}e_{(i)}\cos\,(x^0; x^i)\,, \tag{94}$$

$$g_{ik} = e_{(i)}e_{(k)}\cos\,(x^i; x^k)\,. \tag{95}$$

The gravitational potential is $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$. So the time basis vector $\vec{e}_{(0)}$ tangential to the time line $x^0 = ct$, having the length $e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$, is smaller than unity the greater is the gravitational potential w.

The space rotation linear velocity v_i and, according to it, the chr.inv.-metric tensor h_{ik} are

$$v_i = -c\,e_{(i)}\cos\,(x^0; x^i)\,, \tag{96}$$

$$h_{ik} = e_{(i)}e_{(k)}\Big[\cos(x^0; x^i)\cos(x^0; x^k) - \cos(x^i; x^k)\Big]. \tag{97}$$

Harking back to the formula for the pressure $\mathfrak{F}_1$ (92), that a plane wave of a time density field traveling along the axis $x^1 = x$ exerts on a wall, we have

$$\mathfrak{F}_1 = (1+\Re)\,q\Big[\cos^2(x^0; x^1) + 1\Big]\,n_{(1)}e^2_{(1)}\cos(x^1; n^1), \tag{98}$$

because according to the signature $(+---)$, the spatial coordinate axes in the pseudo-Riemannian space are directed opposite to the same axes x^i in the tangential flat Euclidean space.

We denote $\cos(x^1; n^1) = \cos\theta$, where θ is the angle of reflection. Assuming $e_{(1)} = 1$, $n_{(1)} = 1$, $v_{(1)} = v$ we obtain the field density $q = \frac{\mu}{2\pi c}\Omega^2\left(1+\frac{v^2}{4c^2}\right)$, so that the wave pressure $\mathfrak{F}_{\text{N}} = \mathfrak{F}_1 \cos\theta$ normal to the wall surface is

$$\mathfrak{F}_{\text{N}} = (1+\mathfrak{R})\left(1+\frac{v^2}{c^2}\right) q\cos^2\theta\,, \tag{99}$$

which, for low rotational velocities gives*

$$\mathfrak{F}_{\text{N}} = (1+\mathfrak{R})\, q\cos^2\theta\,, \qquad q = \frac{\mu}{2\pi c}\,\Omega^2. \tag{100}$$

Most of rotations we observe are slow. The maximum of the known velocities is that for an electron in the 1st Bohr orbit ($v_{\text{b}} = 2.18\times10^8$ cm/sec). Therefore the ratio $\frac{v^2}{c^2}$, taking reaches a maximum numerical value of only 5.3×10^{-5}.

The presence of wave pressure in time density fields provides a way of measuring the numerical value of the energy-momentum constant μ, specific for such fields. For instance, a gyroscope, rotating around the axis $x^3 = z$, will be a source of circular waves of the field of time density propagating in the xy-plane. In this case the chr.inv.-field strength tensor U_{ik} has the non-zero components U_{11}, U_{12}, U_{21}. It is easy to calculate that the normal wave pressure of a circular wave will be different from the pressure of a plane wave (99) in only higher order terms. The same situation applies for spherical waves†. Therefore the normal pressure exerted by the waves on a wall orthogonal to the direction $x^1 = x$, shall be

$$\mathfrak{F}_{\text{N}} = \frac{\mu}{2\pi c}\,(1+\mathfrak{R})\,\Omega^2 \tag{101}$$

to within the higher order terms withheld. Rotations at 6×10^3 rpm ($\Omega = 100$ rps) are achievable in modern gyroscopes, rotations in atoms are much greater, taking their maximum angular velocity to 4.1×10^{16} rps in the 1st Bohr orbit. A torsion balance registers forces, values of which are about 10^{-5} dynes. Then in accordance with the formula (101), if the wave pressure in an experiment is $\mathfrak{F}_{\text{N}} \approx 10^{-5}$ din/cm², derived from atomic transformations, the constant's numerical value will be in the order of $\mu \approx 10^{-28}$ gramme/sec.

Of course this is a crude supposition, based on the precision limits of measurement. Anyhow, the exact numerical value of the energy-momentum constant μ will be ascertained from special measurements with a torsion balance.

*Formula (100) is the same as $\mathfrak{F}_{\text{N}} = (1+\mathfrak{R})\,q\cos^2\theta$ — the normal pressure exerted by a plane electromagnetic wave in Minkowski space, (see §47 in *The Classical Theory of Fields* [1]). So the wave pressure of a time density field depends on the reflection coefficient $0 \leqslant \mathfrak{R} \leqslant 1$ in the same way as the pressure of electromagnetic waves.

†In a real experiment such a gyroscope, being an arbitrarily thin disc, will be a source of spherical waves of a time density field which propagates in all spatial directions. The waves will merely have a maximum amplitude in the gyroscope's rotation plane xy.

10 Physical conditions in atoms

So we have obtained formulae for chr.inv.-projections of the energy-momentum tensor of time density fields, which are physically observable characteristics of such fields — the energy density q (69), the momentum density J^i (70), and the strength tensor U_{ik} (71).

The formulae must be valid everywhere, the inside of atoms included. At the same time, physical conditions in atoms are subject to Bohr's quantum postulates. For an external observer, an atom can be represented as a tiny gyroscope, the rotations of which are ruled by the quantum laws. The quantised rotations of electrons are sources of a time density field, which shall be perceptible, because of the super-rapid angular velocities up to the maximum value in the 1st Bohr orbit $\Omega_{\text{b}} = 4.1\times10^{16}$ rps. This is a way of formulating the physical conditions under which a time density field exists in atoms.

Taking the above into account, we formulate the physical conditions with postulates, which result from the application of Bohr's postulates to a time density field in atoms.

POSTULATE I *A time density field in an atom remains unchanged in the absence of external influences. The atom radiates or absorbs waves of the time density field only in transitions of the electrons between their stationary orbits.*

Naturally, when an atom is in a stable state, all its electrons are located in their orbits. Such a stable atom, having a set of quantum orbital angular velocities, must possess numerous quantum states of the time density field. The quantum states are set up with the second postulate‡.

POSTULATE II *A time density field is quantised in atoms. Its energy density and the momentum density take quantum numerical values which, in accordance with the quantization of electron orbits, in n-th stationary orbit are*

$$q_{\text{n}} = \frac{\mu}{2\pi c}\left(1+\frac{v_{\text{n}}^2}{4c^2}\right)\frac{v_{\text{n}}^2}{R_{\text{n}}^2}\,, \tag{102}$$

‡To introduce the second postulate we assume a reference frame in an atom, where an electron rotates around the nucleus at the angular velocity Ω in the xy-plane. Then $A_{12} = -A_{21} = -\Omega$, $A_{13} = 0$, $A_{23} = 0$. So out of all components of Ω^{*i} only Ω^{*3} is non-zero: $\Omega^{*3} = = \frac{1}{2}\varepsilon^{3mn}A_{mn} = \frac{1}{2}\left(\varepsilon^{312}A_{12} + \varepsilon^{321}A_{21}\right) = \varepsilon^{312}A_{12} = \frac{e^{312}}{\sqrt{h}}A_{12} = -\frac{\Omega}{\sqrt{h}}$ and $\Omega_{*3} = \frac{1}{2}\varepsilon_{3mn}A^{mn} = \varepsilon_{312}A^{12} = e_{312}\sqrt{h}\,A_{12} = -\sqrt{h}\,\Omega$. In calculating $h = \det\|h_{ik}\|$, it should be noted that the components of the space rotation linear velocity $v_i = A_{ik}x^k$ in this reference frame are $v_1 = -\Omega y$, $v_2 = \Omega x$, $v_3 = 0$. We obtain $h_{11} = 1+\frac{1}{c^2}\Omega^2 y^2$, $h_{22} = 1+\frac{1}{c^2}\Omega^2 x^2$, $h_{12} = -\frac{1}{c^2}\Omega^2 xy$, $h_{33} = 1$. Then $h = \det\|h_{ik}\| = = h_{11}h_{22} - (h_{12})^2 = 1 + \frac{1}{c^2}\Omega^2(x^2+y^2)$. In the 1st Bohr orbit we have $\frac{1}{c^2}\Omega^2(x^2+y^2) = \frac{1}{c^2}\Omega^2 R^2 = 5.3\times10^{-5}$, so we can set $h \approx 1$ to within the higher order terms withheld. Harking back to the formulae for Ω^{*3} and Ω_{*3}, we see that the space rotates in atoms at a constant angular velocity $\Omega^{*3} = -\Omega$, $\Omega_{*3} = -\Omega$, then in the assumed reference frame we have $A_{ik}A^{ik} = 2A_{12}A^{12} = 2\Omega_{*3}\Omega^{*3} = 2\Omega^2$, and also $F_1 = -\frac{\partial v_1}{\partial t} = \Omega^2 x$, $F_2 = -\frac{\partial v_2}{\partial t} = \Omega^2 y$, $F_3 = 0$, which is taken into account in Postulate II.

$$J_{\rm n} = \sqrt{(J_i J^i)_{\rm n}} = \frac{\mu}{2\pi c}\,\Omega^3_{\rm n} R_{\rm n} = \frac{\mu}{2\pi c}\frac{v^3_{\rm n}}{R^2_{\rm n}}\,. \tag{103}$$

Calculating the field density in neighbouring levels n and n+1, we take into account that the n-th orbital radius relates to the 1st Bohr radius as $R_{\rm n} = {\rm n}^2 R_{\rm b}$. As a result we obtain

$$\bar{q} = q_{\rm n} - q_{\rm n+1} = \\ = \frac{\mu}{2\pi c}\,\Omega^2_{\rm b}\left\{\left[\frac{1}{{\rm n}^6} - \frac{1}{({\rm n}+1)^6}\right] + \frac{v^2_{\rm b}}{4c^2}\left[\frac{1}{{\rm n}^8} - \frac{1}{({\rm n}+1)^8}\right]\right\}, \tag{104}$$

so the difference between the field density in the neighbour levels is inversely proportional to ${\rm n}^7$, and ${\rm n} \gg 1$ gives

$$\bar{q} = q_{\rm n} - q_{\rm n+1} \approx \frac{1}{{\rm n}^7}\frac{3\mu}{\pi c}\,\Omega^2_{\rm b}\,, \tag{105}$$

and $\bar{q} \to 0$ for quantum numbers ${\rm n} \to \infty$.

Theoretically, the non-zero field density, $q \neq 0$, must result in a flux of the field momentum (this flux is set up by the field strength tensor $U_{ik} = \frac{1}{3}\,qc^2 h_{ik} - \beta_{ik}$). So an electron, moving in its orbit, should be radiating a momentum flux of the time density field (waves of the field). Because of the momentum loss in the radiation, the electron's own angular velocity would decrease, contradicting the experimental facts on the stability of atoms in the absence of external influences. To obviate this contradiction the third postulate is,

POSTULATE III *An atom radiates a quantum portion of momentum flux of a time density field, when an electron transits from the n-th quantum level to the (n+1)-th level in the atom. When an electron transits from the (n+1)-th level to the n-th level, the atom absorbs the same portion of the momentum flux, which is*

$$\overline{U}_{11} = U^{\rm n}_{11} - U^{\rm n+1}_{11} = \\ = \frac{\mu c}{2\pi}\,\Omega^2_{\rm b}\left\{\left[\frac{1}{{\rm n}^6} - \frac{1}{({\rm n}+1)^6}\right] - \frac{v^2_{\rm b}}{4c^2}\left[\frac{1}{{\rm n}^8} - \frac{1}{({\rm n}+1)^8}\right]\right\}. \tag{106}$$

We assume in this formula that the atom radiates/absorbs a plane wave of a time density field, which travels along the $x^1 = x$ axis. Taking this formula with ${\rm n} \gg 1$, we have

$$\overline{U}_{11} = U^{\rm n}_{11} - U^{\rm n+1}_{11} \approx \frac{1}{{\rm n}^7}\frac{3\mu c}{\pi}\,\Omega^2_{\rm b}\,, \tag{107}$$

which, for quantum numbers ${\rm n} \to \infty$, gives $\overline{U}_{11} \to 0$. So for quantum numbers ${\rm n} \gg 1$ we have the ratio

$$\overline{U}_{11} = \bar{q}c^2. \tag{108}$$

In accordance with the correspondence principle, any result of quantum theory at high quantum numbers must coincide with the relevant classical results; any difference being imperceptible. We therefore take into consideration the formulae for q (69) and U_{ik} (71) in atoms, obtained by the methods of the classical theory of fields, under the condition $h \approx 1$. As a result we get the formulae $q = \frac{\mu}{2\pi c}\,\Omega^2\left(1 + \frac{v^2}{4c^2}\right) \approx \frac{\mu}{2\pi c}\,\Omega^2$ and $U_{ik} = \frac{\mu c}{2\pi}\,\Omega^2\left(h_{11} - \frac{v^2}{2c^2} + \frac{v^2}{4c^2}h_{11}\right) \approx \frac{\mu c}{2\pi}\,\Omega^2$, leading to the same relationship $U_{11} = qc^2$ that quantum theory has given (108). So the correspondence principle is valid for time density fields in atoms.

Postulate III has two consequences:

CONSEQUENCE I *An atom undergoing excitation radiates the momentum flux of a time density field, producing a positive wave pressure in the field.*

Calculating this positive pressure, orthogonal to the surface of a wall (here θ is the angle of reflection, $\Re$ is the reflection coefficient) for quantum numbers ${\rm n} \gg 1$, we obtain

$$\overline{\mathfrak{F}}_{\rm N} = (1 + \Re)\,\bar{q}\cos^2\theta\,. \tag{109}$$

CONSEQUENCE II *An atom undergoing relaxation absorbs the momentum flux of a time density field. In this case the wave pressure in a time density field near the atom becomes negative.*

As a matter of fact, this negative pressure around a relaxing atom should be

$$\overline{\mathfrak{F}}_{\rm N} = -\,(1 + \Re)\,\bar{q}\cos^2\theta\,. \tag{110}$$

That is, in accordance with this theory, excitation of atoms causes radiation of waves of the time density field. An effect derived from the radiation should be the positive pressure of the waves. On the other hand, relaxing atoms, absorbing waves of the time density field, should be sources of negative wave pressure.

It is interesting that this effect is opposite to that which atoms produce in an electromagnetic field — it is well-known that relaxing atoms radiate electromagnetic waves, so they produce a positive wave pressure in an electromagnetic field.

The predicted repulsion/attraction produced by atomic processes, being outside the actions of electromagnetic or gravitational fields, are peculiarities of only the theory of the time density field herein. So the given conclusions open up wide possibilities for checking the whole theory in practice.

In particular, for instance, if a torsion balance registered the repulsing/attracting wave pressure $\overline{\mathfrak{F}}_{\rm N}$ derived from subatomic excitation/relaxation processes, we will have obtained the numerical value of the energy-momentum constant μ for time density fields. After substituting $\bar{q}$ (105) into the wave pressure $\overline{\mathfrak{F}}_{\rm N}$, assuming $\cos\theta = 1$, we arrive at a formula for experimental calculations

$$\mu = \frac{\pi c{\rm n}^7}{3\Omega^2_{\rm b}}\frac{\overline{\mathfrak{F}}_{\rm N}}{(1 + \Re)}\,. \tag{111}$$

A torsion balance registered such forces at $\sim 10^{-5}$ dynes in prior experiments. The torsion balance had a 2-in long nylon thread, 15 μm in diameter, and a 3-in long wooden balance suspended in the ratio 8:1 of the length. The balance had a reflecting shield at the end of the long arm and a lead load on the short arm. The torsion balance was located inside a box isolated from air convection and light radiation. Chemical reactions of the opposite directions, processes of crystallization and dissolution were sources of a time density field acting on the torsion balance. Prof. Kyril Stanyukovich and Dr. Larissa Borissova assisted me in the experiments that were repeated a number of time during a period of 2 years in Moscow (Russia). The balance underwent deviations of up to 90° in directions predicted by this theory.

Even heating up bodies and cooling down bodies gave the same thermal influence, moving the balance in opposite directions, according to the theory, so the discovered phenomenon is outside thermal influences on torsion balance.

The techniques and measurements are very simply, and could therefore be reproduced in any physical laboratory. Anyway the experiments should be continued, with the aim of determining the exact numerical value of the energy-momentum constant μ for time density fields through formula (111).

11 Conclusions

Let us collect the main results of this analysis.

By projecting an interval of four-dimensional coordinates dx^α onto the time line of an observer, who accompanies his references ($b^i = 0$), we obtain an interval of physical time $d\tau = \frac{1}{c}\,b_\alpha dx^\alpha$ he observes. Observations at the same spatial point give $d\tau = \sqrt{g_{00}}\,dt$, so the operator of projection on time lines b^α defines observable inhomogeneity of time references in the accompanying reference frame.

So, observable inhomogeneities of time references can be represented as a field of "density" of observable time τ. The projecting operator b^α is the field "potential", chr.inv.-projections of which are $\varphi = 1$ and $q^i = 0$.

The field tensor $F_{\alpha\beta} = \nabla_\alpha b_\beta - \nabla_\beta b_\alpha$ for time density fields was introduced as well as Maxwell's electromagnetic field tensor. Its chr.inv.-projections $E^i = -\frac{1}{c^2}F^i$ and $H_{ik} = -\frac{2}{c}A_{ik}$ are derived from the gravitational inertial force and rotation of the space. We referred to the E^i and H_{ik} as the "electric" and "magnetic" observable components of the time density field, respectively. We also introduced the field pseudotensor $F^{*\alpha\beta}$, dual of the $F_{\alpha\beta}$, and also the field invariants.

Equations of motion of a free mass-bearing particle, being expressed through the E^i and H_{ik}, group them into an acting force of a form similar to the Lorentz force. In particular if the particle moves only along time lines, it moves solely because of the "magnetic" component $H_{ik} \neq 0$ of a time density field. In other words, the space rotation A_{ik} effectively "screws" particles into the time lines. Because observable particles with the whole spatial section move from past into future, a "starting" non-holonomity, $A_{ik} \neq 0$, will exist in our real space that is a "primordial non-orthogonality" of the real spatial section to the time lines. Other physical conditions (gravitation, rotation, etc.) are only augmentations that intensify or reduce this starting-rotation of the space.

A system of equations of a time density field consists of Lorentz's condition $\nabla_\sigma b^\sigma = 0$, two groups of Maxwell-like equations, $\nabla_\sigma F^{\alpha\sigma} = \frac{4\pi}{c}\,j^\alpha$ and $\nabla_\sigma F^{*\alpha\sigma} = 0$, and the continuity equation $\nabla_\sigma j^\sigma = 0$, which define the main properties of the field and its-inducing sources. All the equations have been deduced here in chr.inv.-form.

The energy-momentum tensor $T^{\alpha\beta}$ we have deduced for time density fields has the following observable projections: chr.inv.-scalar q of the field density; chr.inv.-vector J^i of the field momentum density, and chr.inv.-tensor U^{ik} of the field strengths. Their specific formulas define physical properties of such fields:

1. A time density field is non-stationary distributed medium $q \neq \text{const}$, it becomes stationary, $q = \text{const}$, under stationary rotation, $A_{ik} = \text{const}$, of the space and stationary gravitational inertial force $F_i = \text{const}$;
2. The field bears momentum ($J^i \neq 0$ in the general case), so it can transfer impulse;
3. In a rotating space, $A_{ik} \neq 0$, the field is an emitting medium;
4. The field is viscous. The viscosity α_{ik} is anisotropic. The anisotropy increases with the space rotation speed;
5. The equation of state of the field is $p = \frac{1}{3}qc^2$, so the field is like an ultrarelativistic gas: at positive density the pressure is positive — the medium compresses.

For a plane wave of the field considered, we have concluded that waves of the time density fields are transverse. The wave pressure in the fields is derived from atomic and sub-atomic transformations mainly, because of huge rotational velocities. Exciting atoms produces a positive wave pressure in the time density field, while the wave pressure resulting from relaxing atoms is negative. This effect is opposite to that of the electromagnetic field — relaxing atoms radiate γ-quanta, producing a positive pressure of light waves.

Experimental tests have a basis in the predicted repulsion/attraction, produced by sub-atomic processes, being outside of known effects of electromagnetic or gravitational fields, which are peculiarities only of this theory. A torsion balance registered such forces at $\sim 10^{-5}$ dynes in prior experiments. The registered repulsion/attraction is outside thermal effects on the torsion balance.

The results we have obtained in this study imply that even if inhomogeneity of time references is a tiny correction to ideal time, a field of the inhomogeneities that is a time density field, manifest as gravitational and inertial forces, has a more fundamental effect on observable phenomena, than those previouly supposed.

Acknowledgements

I began this research in 1984 when I, a young scientist in those years, started my scientific studies under the direction of Prof. Kyril Stanyukovich, who in the 1940's was already a well-known expert in the General Theory of Relativity. At that time we found an article of 1971, wherein Prof. Nikolai Kozyrev (1908–1983) reported on his experiments with a torsion balance [7].

His high precision torsion balance, having unequal arms, registered weak forces of attraction/repulsion at 10^{-5}–10^{-6} dynes; the forces derived from creative/destructive processes in his laboratory. Proceeding from his considerations, redistribution of energy should produce a non-uniformity of time that generates a force field of attraction or repulsion depending on the creative/destructive direction of the redistributing energy process.

Kozyrev did not put this idea into any mathematical form. So his propositions, having a purely phenomenological a basis, remained without a theory. Kozyrev was the famous experimental physicist and astronomer of the 20th century who discovered volcanic activity in the Moon (1958), the atmosphere of Mercury (1963), and many other phenomena. His authority in physical experiment was beyond any doubt.

In those years Stanyukovich was head of the Department of Fundamental Theoretical Metrology, Surface and Vacuum Scientific Centre, State Committee for Standards (Moscow, Russia). Dr. Larissa Borissova worked in his Department. We were all close friends, despite our age difference. We had a good time working with Stanyukovich, who was friendly in his conversations on different scientific problems.

I was interested by Kozyrev's experiments with the torsion balance, therefore Stanyukovich proposed me that I investigate them, meaning that it would be helpful for me to build the required theory. During 1984–1985 I twice visited the laboratory of late Kozyrev in Pulkovo Astronomical Observatory, near Leningrad. Then, in Moscow, we made a copy of his torsion balance, and modified it to make it more sensitive.

Stanyukovich was right — the experiments were approbated with strictly positive result. We repeated the experiments for some of his colleagues, in particular for Dr. Vitaly Schelest.

More than 15 years were required for the development of this theory. I finished the whole theory only in 2004. Stanyukovich had died; only Larissa Borissova and I remain. Our years of friendly conversations with Stanyukovich, and his patient personal instructions reached us by all his experience in theoretical physics. Actually, we are beholden to him. Today I would like to do only one thing — satisfy the hopes of my teacher.

Finally I am grateful to my colleagues: Stephen J. Crothers for some editing this paper, and Larissa Borissova who checked all my calculations that here.

References

1. Landau L. D. and Lifshitz E. M. The classical theory of fields. GITTL, Moscow, 1939 (ref. with the 4th final exp. edition, Butterworth–Heinemann, 1980, 428 pages).
2. Zelmanov A. L. Chronometric invariants. Dissertation, 1944. First published: CERN, EXT-2004-117, 236 pages.
3. Zelmanov A. L. Chronometric invariants and co-moving coordinates in the general relativity theory. *Doklady Acad. Nauk USSR*, 1956, v. 107 (6), 815–818.
4. Zelmanov A. L. Orthometric form of monad formalism and its relations to chronometric and kinemetric invariants. *Doklady Acad. Nauk USSR*, 1976, v. 227 (1), 78–81.
5. Borissova L. B. and Rabounski D. D. Fields, vacuum, and the mirror Universe. Editorial URSS, Moscow, 2001, 272 pages (the 2nd revised ed.: CERN, EXT-2003-025).
6. Rabounski D. D. The new aspects of General Relativity. CERN, EXT-2004-025, 117 pages.
7. Kozyrev N. A. On the possibility of experimental investigation of the properties of time. *Time in Science and Philosophy*, Academia, Prague, 1971, 111–132.
8. Kozyrev N. A. Physical peculiarities of the components of double stars. *Colloque "On the Evolution of Double Stars"*, *Comptes rendus*, Communications du Observatoire Royal de Belgique, ser. B, no. 17, Bruxelles, 1967, 197–202.
9. Del Prado J. and Pavlov N. V. Private communications to A. L. Zelmanov, Moscow, 1968–1969.
10. Zelmanov A. L. To relativistic theory of anisotropic inhomogeneous Universe. *Proceedings of the 6th Soviet Conference on Cosmogony*, Nauka, Moscow, 1959, 144–174.

Gravitational Waves and Gravitational Inertial Waves in the General Theory of Relativity: A Theory and Experiments

Larissa Borissova
E-mail: lborissova@yahoo.com

This research shows that gravitational waves and gravitational inertial waves are linked to a special structure of the Riemann-Christoffel curvature tensor. Proceeding from this a classification of the waves is given, according to Petrov's classification of Einstein spaces and gravitational fields located therein. The world-lines deviation equation for two free particles (the Synge equation) is deduced and that for two force-interacting particles (the Synge-Weber equation) in the terms of chronometric invariants — physical observable quantities in the General Theory of Relativity. The main result drawn from the deduced equations is that in the field of a falling gravitational wave there are not only spatial deviations between the particles but also deviations in the time flow. Therefore an effect from a falling gravitational wave can manifest only if the particles located on the neighbouring world-lines (both geodesics and non-geodesics) are in motion at the initial moment of time: gravitational waves can act only on moving neighbouring particles. This effect is purely parametric, not of a resonance kind. Neither free-mass detectors nor solid-body detectors (the Weber pigs) used in current experiments can register gravitational waves, because the experimental statement (freezing the pigs etc.) forces the particles of which they consist to be at rest. In aiming to detect gravitational waves other devices should be employed, where neighbouring particles are in relative motion at high speeds. Such a device could, for instance, consist of two parallel laser beams.

Contents

1 Introduction and advanced results

The fact that gravitational waves have not yet been discovered has attracted the attention of experimental physicists over the last decade. Initial interest in gravitational waves arose in 1968–1971 when Joseph Weber, professor at Maryland University (USA), carried out his first experiments with gravitational antennae. He registered weak signals, in common with all his independent antennae, which were separated by up to 1000 km [1]. He supposed that some processes at the centre of the Galaxy were the origin of the registered signals. However such an interpretation had a significant drawback: the frequency of the observed signals (more than 5 per month) meant that the energy spent by the signal's source, located at the centre of the Galaxy, should be more than $M_{\odot}c^2{\times}10^3$ per annum ($M_{\odot}$ is the mass of the Sun, c is the velocity of light). This energy expenditure is a fantastic value, if we accept today's bounds on the age of the Galaxy [2, 3, 4].

In 1972 the experiments were approbated by the a common group of researchers working at Moscow University and the Institute of Space Research (Moscow, Russia). Their antennae were similar to Weber's antennae, but they were separated by 20 km. The registering system in their antennae was better than that for the Weber detectors, making the whole system more sensitive. But. . . 20 days of observations gave no signals that would be more than noise [5].

The experiments were continued in 1973–1974 at laboratories in Rochester University, Bell Company, and IBM in USA [6, 7], Frascati, Münich, Meudon (Italy, Germany, France) [8], Glasgow University (Scotland) [9] and other laboratories around the world. The experimental systems used in these attempts were more sensitive than those of the Weber detectors, but none registered the Weber effect.

Because theoretical considerations showed that huge gravitational waves should be accompanied by other radiations, the researchers conducted a search for radio outbreaks [10] and neutron outbreaks [11]. The result was negative. At the same time it was found that Weber's registered effects were related to solar and geomagnetic activities, and also to outbreaks of space beams [12, 13].

The search for gravitational waves has continued. Higher precision and more sensitive modifications of the Weber antennae (solid detectors of the resonance kind) are used in this search. But even the second generation of Weber detectors have not led scientists to the expected results. Besides gravitational antennae of the Weber kind, there are antennae based on free masses. Such detectors consist of two freely suspended masses located far from one another, within the visibility of a laser range-finder. Supposed deviations of the masses, derived from a gravitational wave, should be registered by the laser beam.

So gravitational waves have not been discovered in experiments. Nonetheless it is accepted by most physicists that the discovery of gravitational waves should be one of the main verifications of the General Theory of Relativity. The main arguments in support of this thesis are:

1. Gravitational fields bear an energy described by the energy-momentum pseudotensor [14, 15];
2. A linearized form of the equations of Einstein's equations permits a solution describing weak plane gravitational waves, which are transverse;
3. An energy flux, radiated by gravitational waves, can be calculated through the energy-momentum pseudotensor of the field [14, 15];
4. Such waves, because of their physical nature, are derived from instability of components of the fundamental metric tensor (this tensor plays the part of the four-dimensional gravitational potential).

These theoretical considerations were placed into the foreground of the theory for detecting gravitational waves, the main part in the theory being played by the theoretical works of Joseph Weber, the pioneer and famous expert in the detection of gravitational waves [16]. His main theoretical claim was that he deduced equations of deviation of world-lines — equations that describe relative oscillations of *two non-free particles* in a gravitational field, particles which are connected by a force of non-gravitational nature. Equations of deviation of geodesic lines, describing relative oscillations of *two free particles*, was obtained earlier by Synge [17]. In general, relative oscillations of test-particles, both free particles and linked (interacting) particles, are derived from the space curvature*, given by the Riemann-Christoffel four-dimensional tensor. Equality to zero of all its components in an area is the necessary and sufficient condition for the four-dimensional space (space-time) to be flat in the area under consideration, so no gravitational fields exist in the area.

Thus the Synge-Weber equation provides a means for the calculation of the relative oscillations of test-particles, derived from the presence of the space curvature (gravitational fields). Weber proposed a gravitational wave detector consisting of two particles connected by a spring that imitates a non-gravitational interaction between them. In his analysis he made the substantial supposition that the under action of gravitational waves the model will behave like a harmonic oscillator where the forcing power is in the Riemann-Christoffel curvature tensor. Weber made calculations and theoretical propositions for the behaviour of this model. This model is known as the *quadrupole mass-detector* [17].

The Weber calculations served as theoretical grounds for creating a whole industry, the main task of which has been the building of resonance type detectors, known as the Weber detectors (the Weber pigs). It is supposed that the body of a Weber detector, having cylindrical form, should be deformed under the action of a gravitational wave. This deformation should lead to a piezoelectric effect. Thus, oscillations of atoms in the cylindrical pig, resulting from a gravitational wave, could be registered. To amplify the effect in measurements, the level of noise was lowered by cooling the cylinder pigs down to temperature close to 0 K.

But the fact that gravitational waves have not yet been discovered does not imply that the waves do not exist in Nature. The corner-stone of this problem is that the Weber theory of detection is linked to a search for waves of only a specific kind — weak transverse waves of the space deformation (*weak deformation transverse waves*). However, besides the Weber theory, there is the *theory of strong gravitational waves*, which is independent of the Weber theory. Studies of the theory of strong gravitational waves reached its peak in the 1950's.

Generally speaking, all theoretical studies of gravitational waves can be split into three main groups:

1. The first group consists of studies whose task is to give an invariant definition for gravitational waves. These are studies made by Pirani [18, 19], Lichnerowicz [20, 21], Bel [22, 23, 24], Debever [25, 26, 27], Hély [28], Trautman [29], Bondi [19, 30], and others.
2. The second group joins studies around a search for such solutions to the Einstein equations for gravitational fields, which, proceeding from physical considerations, could describe gravitational radiations. These are studies made by Bondi [31], Einstein and Rosen [32, 33], Peres [34], Takeno [35, 36], Petrov [37], Kompaneetz [38], Robinson and Trautman [39], and others.
3. The task of works related to the third group is to study gravitational inertial waves, covariant with respect of transformations of spatial coordinates and also invariant with respect of transformations of time [40, 41].

*As it is well-known, the space curvature is linked to the gravitational field by the Einstein equations.

The studies are based on the theory of physically observable quantities — Zelmanov's theory of chronometric invariants [42, 43].

Most criteria for gravitational waves are linked to the structure of the Riemann-Christoffel curvature tensor, hence one assumes space curvature the source of such waves.

Besides these three main considerations, the theory of gravitational waves is directly linked to the algebraical classification of spaces given by Petrov [37] (*Petrov classification*), according to which three kinds for spaces (gravitational fields) exist. They are dependent on the structure of the Riemann-Christoffel curvature tensor:

1. Fields of gravitation of the 1st kind are derived from island distributions of masses. An instance of such a field is the that of a spherical distribution of matter (a spherical mass island) described by the Schwarzschild metric [44]. Spaces containing such fields approach a flat space at an infinite distance from the gravitating island.

2–3. Spaces containing gravitational fields of the 2nd and 3rd kinds cannot asymptotically approach a flat space even, if they are empty. Such spaces can be curved themselves, independently of the presence of gravitating matter. Such fields satisfy most of the invariant definitions given to gravitational waves [40, 45, 46, 47].

It should be noted that the well-known solution that gives weak plane gravitational waves [14, 15] is related to fields of the sub-kind N of the 2nd kind by Petrov's classification (see p. 38). Hence the theory of weak plane gravitational waves is a particular case of the theory of strong gravitational waves. But, bseides this well-studied particular case, the theory of strong gravitational waves contains many other approaches to the problem and give other methods for the detection of gravitational waves, different to the Weber detectors in principle (see [48], for instance).

We need to look at the gravitational wave problem from another viewpoint, by studying other cases of the theory of strong gravitational waves not considered before. Exploring such new approaches to the theory of gravitational waves is the main task of this research.

At the present time there are many solutions of the gravitational wave problem, but none of them are satisfactory. The principal objective of this research is to extract that which is common to every one of the theoretical approaches.

We will see further that this analysis shows, according to most definitions given for gravitational waves, that a gravitational field is assumed a wave field if the space where it is located has the specific curvature described by numerous particular cases of the Riemann-Christoffel curvature tensor.

Note that we mean the Riemannian (four-dimensional) curvature, whose formula contains accelerations, rotations, and deformations of the observer's reference space. Analysis of most wave solutions to the gravitational field equations (Einstein's equations) shows that such gravitational waves have a *deformation nature* — they are waves of the space deformations. The true nature of gravitational waves can be found by employing the mathematical methods of chronometric invariants (the theory of physically observable quantities in the General Theory of Relativity), which show that the space deformation (non-stationarity of the spatial observable metric) consists of two factors:

1. Changes of the observer's scale of distance with time (deformations of the 1st kind);
2. Possible vortical properties of the acting gravitational inertial force field (deformations of the 2nd kind).

Waves of the space deformations (of the 1st or 2nd kind) underlie the detection attempts of the experimental physicists.

Because such gravitational waves are expected to be weak, one usually uses the metric for weak plane gravitational waves of the 1st kind (which are derived from changes of the distance scale with time).

The basis for all the experiments is the Synge-Weber equation (the world-lines deviation equation), which sets up a relation between relative oscillations of test-particles and the Riemann-Christoffel curvature tensor. Unfortunately Joseph Weber himself gave only a rough analysis of his equation, aiming to describe the behaviour of a quadrupole mass-detector in the field of weak plane gravitational waves. In his analysis he assumed (without substantial reasons) that space deformation waves of the 1st kind must produce a resonance effect in a quadrupole mass-detector.

However, it would be more logical way, making no assumptions or propositions, to solve the Synge-Weber equation aiming exactly. Weber did not do this, limiting himself instead to only rough bounds on possible solutions.

In this research we obtain exact solutions to the Synge-Weber equation in the fields of weak plane gravitational waves. As a result we conclude that the expected relative oscillations of test-particles, which originate in the space deformation waves of the 1st kind, *are not of the resonance kind* as Weber alleged from his analysis, but are instead *parametric oscillations*.

This deviation between our conclusion and Weber's false conclusion is very important, because oscillations of a parametric kind appear only if test-particles are moving*, whilst in Weber's statement of the experiment the particles are at rest in the observer's laboratory reference frame. All activities in search of gravitational waves using the Weber pigs are concentrated around attempts to isolate the bulk pigs from external affects — experimental physicists place them in mines in the depths of mountains and cool them to 2 K,

*In other words, if their velocities are different from zero. Parametric oscillations merely add their effect to the relative motion of the moving particles. Parametric oscillations cannot be excited in a system of particles which are at rest with respect to each other and the observer.

so particles of matter in the pigs can be assumed at rest with respect to one another and to the observer. At present dozens of Weber pigs are used in such experiments all around the world. Experimental physicists spend billions and billions of dollars yearly on their experiments with the Weber pigs.

Parametric oscillations do not appear in resting particles, so the space deformation waves of the 1st kind can not excite parametric oscillations in the Weber pigs. Therefore the *gravitational waves expected by scientists cannot be registered by solid-body detectors of the resonance kind* (*the Weber pigs*).

Even so, everything said so far does not mean rejection of the experimental search for gravitational waves. We merely need to look at the problem from another viewpoint. We need to remember the fact that our world is not a three-dimensional space, but a four-dimensional space-time. For this reason we need to turn our attention to the fact that relative deviations of particles in the field of gravitational waves have both spatial components and a time component. Therefore it would be reasonable to propose an experiment by which, having a detector under the influence of gravitational waves, we could register both relative displacements of particles in the detector and also corrections to time flow in the detector due to the waves (the second task is much easier from the technical viewpoint).

Here are two aspects for consideration. First, in solving the Synge-Weber equations we must take its time component into account; we must not neglect the time component. Second, we should turn our attention to possible experimental effects derived from gravitational waves of the 2nd (deformation) kind, which appear if the acting gravitational inertial force field is vortical, as it will be shown further that in this case there is a field of the space rotation (stationary or non-stationary)*. Such experiments, aiming to register gravitational waves of the 2nd kind are progressive because they are much simpler and cheaper than the search for waves of the 1st kind.

2 Theoretical bases for the possibility of registering gravitational waves

Gravitational waves were already predicted by Einstein [37], but what space objects could be sources of the waves is not a trivial problem. Some link the possibility of gravitational radiations to clusters of black holes. Others await powerful gravitational radiations from super-dense compact stars of radii close to their gravitational radii† $r \sim r_g$. Although the "black hole solution", being under substantial criticism from the purely mathematical viewpoint [53, 54, 55], makes objects like black holes very doubtful, the existence of super-dense neutron stars is outside of doubt between astronomers. Gravitational waves at frequencies of 10^2–10^4 Hz should also be radiated in super-nova explosions by explosion of their super-dense remains [56].

The search for gravitational waves, beginning with Weber's observations of 1968–1971, is realized by using gravitational antennae, the most promising of which are:

1. Solid-body detectors (the Weber cylinder pigs);
2. Antennae built on free masses.

A solid-body detector of the Weber kind is a massive cylindrical pig of 1–3 metres in length, made with high precision. This experiment supposes that gravitational waves are waves of the space deformation. For this reason the waves cause a piezoelectic effect in the pig, one consequence of which is mechanical oscillations at low frequencies that can be registered in the experiment. It is supposed that such oscillations have a resonance nature. An immediate problem is that such resonance in massive pigs can be caused by very different external processes, not only waves of the space deformation. To remove other effects, experimental physicists locate the pigs in deep tunnels in mountains and cool the pigs down to temperature close to 0 K.

An antenna of the second kind consists of two masses, separated by $\Delta l \sim 10^3$–10^4 metres, and a laser range-finder which should register small changes of Δl. Both masses are freely suspended. This experiment supposes that waves of the space deformation should change the distance between the free masses, and should be registered by the laser range-finder. It is possible to use two satellites located in the same orbit near the Earth, having a range-finder in each of the satellites. Such satellites, being in free fall along the orbit, should be an ideal system for measurements, if it were not for effects due to the terrestrial globe. In practice it would be very difficult to divorce the effect derived from waves of the space deformation (supposed gravitational waves) and many other factors derived from the inhomogeneity of the Earth's gravitational field (purely geophysical factors).

The mathematical model for such an antenna consists of two free test-particles moving on neighbouring geodesic lines located infinitely close to one another. The mathematical model for a solid-body detector (a Weber pig) consists of two test-masses connected by a spring that gives a model for elastic interactions inside a real cylindrical pig, in which changes reveal the presence of a wave of the space deformation.

From the theoretical perspective, we see that the possibility of registering waves of the space deformation (supposed gravitational waves) is based on the supposition that particles which encounter such a wave should be set into relative oscillations, the origin of which is the space curvature. The

*There are well-known Hafele-Keating experiments concerned with displacing standard clocks around the terrestrial globe, where rotation of the Earth space sensibly changes the measured time flow [49, 50, 51, 52].

†According to today's mainstream concepts, the gravitational radius r_g of an object is that minimal distance from its centre to its surface, starting from which this object is in a special state — *collapse*. One means that any object going into collapse becomes a "black hole". From the purely mathematical viewpoint, under collapse, the potential w of the gravitational field of the object merely reaches its upper ultimate numerical value $\mathrm{w} = c^2$.

strong solution for this problem had been given by Synge for free particles [17]. He considered a two-parameter family of geodesic lines $x^\alpha = x^\alpha(s, v)$, where s is a parameter along the geodesics, v is a parameter along the direction orthogonal to the geodesics (it is taken in the plane normal to the geodesics). Along each geodesic line $v = \text{const}$.

He introduced two vectors

$$U^\alpha = \frac{\partial x^\alpha}{\partial s}\,, \qquad V^\alpha = \frac{\partial x^\alpha}{\partial v}\,, \tag{2.1}$$

where $\alpha = 0, 1, 2, 3$ denotes four-dimensional (space-time) indexes. The vectors satisfy the condition

$$\frac{\mathrm{D}U^\alpha}{\partial v} = \frac{\mathrm{D}V^\alpha}{\partial s}\,, \tag{2.2}$$

(where D is the absolute derivative operator) that can be easy verified by checking the calculation. The parameter v is different for neighbouring geodesics; the difference is dv. Therefore, studying relative displacements of two geodesics $\Gamma(v)$ and $\Gamma(v + dv)$, we shall study the vector of their infinitesimal relative displacement

$$\eta^\alpha = \frac{\partial x^\alpha}{\partial v}\,dv = V^\alpha dv\,. \tag{2.3}$$

The deviation of the geodesic line $\Gamma(v + dv)$ from the geodesic line $\Gamma(v)$ can be found by solving the equation [17]

$$\begin{aligned}\frac{\mathrm{D}^2 V^\alpha}{ds^2} &= \frac{\mathrm{D}}{ds}\frac{\mathrm{D}V^\alpha}{ds} = \frac{\mathrm{D}}{ds}\frac{\mathrm{D}U^\alpha}{dv} = \\ &= \frac{\mathrm{D}}{dv}\frac{\mathrm{D}U^\alpha}{ds} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma,\end{aligned} \tag{2.4}$$

where $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ is the Riemann-Christoffel curvature tensor. This equality has been obtained using the relation [17]

$$\frac{\mathrm{D}^2 V^\alpha}{ds dv} - \frac{\mathrm{D}^2 V^\alpha}{dv ds} = R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma. \tag{2.5}$$

For two neighbour geodesic lines, the following relation is obviously true

$$\frac{\mathrm{D}U^\alpha}{ds} = \frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = 0\,, \tag{2.6}$$

where $\Gamma^\alpha_{\beta\gamma}$ are Christoffel's symbols of the 2nd kind. Then (2.4) takes the form

$$\frac{\mathrm{D}^2 V^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma = 0\,, \tag{2.7}$$

or equivalently,

$$\frac{\mathrm{D}^2 \eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta \eta^\gamma = 0\,. \tag{2.8}$$

It can be shown [17] that,

$$\frac{\partial}{\partial s}(U_\alpha V^\alpha) = U_\alpha\frac{\mathrm{D}V^\alpha}{ds} = U_\alpha\frac{\mathrm{D}U^\alpha}{dv} = \frac{1}{2}\frac{\partial}{\partial v}(U_\alpha U^\alpha)\,. \tag{2.9}$$

The quantity $U_\alpha U^\alpha = g_{\alpha\beta}U^\alpha U^\beta$ takes the numerical value +1 for non-isotropic geodesics (substantial particles) or 0 for isotropic geodesics (massless light-like particles). Therefore

$$U_\alpha V^\alpha = \text{const}\,. \tag{2.10}$$

In the particular case where the vectors U_α and V^α are orthogonal to each to other at a point, where $U_\alpha V^\alpha$ is true, the orthogonality remains true everywhere along the $\Gamma(v)$.

Thus relative accelerations of free test-particles are caused by the presence the space curvature ($R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta} \neq 0$), and linear velocities of the particles are determined by the geodesic equations (2.6).

Relative accelerations of test-particles, connected by a force Φ^α of non-gravitational nature, are determined by the Synge-Weber equation [16]. The Synge-Weber equation is the generalization of equation (2.8) for that case where the particles, each having the rest-mass m_0, are moved along non-geodesic world-lines, determined by the equation

$$\frac{\mathrm{D}U^\alpha}{ds} = \frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = \frac{\Phi^\alpha}{m_0 c^2}\,. \tag{2.11}$$

In this case the world-lines deviation equation takes the form

$$\frac{\mathrm{D}^2 \eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta \eta^\gamma = \frac{1}{m_0 c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}\,dv\,, \tag{2.12}$$

which describes relative accelerations of the interacting masses. In this case

$$\frac{\partial}{\partial s}(U_\alpha \eta^\alpha) = \frac{1}{m_0 c^2}\,\Phi_\alpha \eta^\alpha, \tag{2.13}$$

so the angle between the vectors U^α and η^α does not remain constant for the interacting particles.

Equations (2.8) and (2.12) describe relative accelerations of free particles and interacting particles, respectively. Then, to obtain formulae for the velocity U^α it is necessarily to solve the geodesic equations for free particles (2.6) and the world-line equations for interacting particles (2.11). We consider the equations (2.8) and (2.12) as a mathematical base, with which we aim to calculate gravitational wave detectors: (1) antennae built on free particles, and (2) solid-body detectors of the resonance kind (the Weber detectors).

3 Invariant criteria for gravitational waves and their link to Petrov's classification

From the discussion in the previous paragraphs, one concludes that a physical factor enforcing relative displacements of test-particles (both free particles and interacting particles) is the space curvature — a gravitational field wherein the particles are located.

Here the next question arises. How well justified is the statement of the gravitational wave problem?

Generally speaking, in the General Theory of Relativity, there is a problem in describing gravitational waves in a mathematically correct way. This is a purely mathematical problem, not solved until now, because of numerous difficulties. In particular, the General Theory of Relativity does not contain a satisfactory general covariant definition for the energy of gravitational fields. This difficulty gives no possibility of describing gravitational waves as traveling energy of gravitational fields.

The next difficulty is that when one attempts to solve the gravitational wave problem using the classical theory of differential equations, he sees that the gravitational field equations (the Einstein equations) are a system of 10 nonlinear equations of the 2nd order written with partial derivatives. No universal boundary conditions exist for such equations.

The gravitational field equations (the Einstein equations) are

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\kappa T_{\alpha\beta} + \lambda g_{\alpha\beta}\,, \qquad (3.1)$$

where $R_{\alpha\beta} = R^{\sigma\cdots}_{\cdot\alpha\sigma\beta}$ is Ricci's tensor, $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, $\kappa = \frac{8\pi G}{c^2}$ is Einstein's constant for gravitational fields, G is Gauss' constant of gravitation, λ is the cosmological constant (λ-term).

When studying gravitational waves, one assumes $\lambda = 0$. Sometimes one uses a particular case of the Einstein equations (3.1)

$$R_{\alpha\beta} = \kappa g_{\alpha\beta}\,, \qquad (3.2)$$

in which case the space, where the gravitational field is located, is called an *Einstein space*. If $\kappa = 0$, we have an *empty space* (without gravitating matter). But even in empty spaces ($\kappa = 0$) gravitational fields can exist, if the spaces are of the 2nd and 3rd kinds by Petrov's classification.

In accordance with the classical theory of differential equations, those gravitational fields that describe gravitational waves are determined by solutions of the Einstein equations with initial conditions located in a characteristic surface. A wave is a Hadamard break in the initial characteristic surface; such a surface is known as the *wave front*. The wave front is determined as the characteristic isotropic surface $S\{\Phi(x^\alpha) = 0\}$ for the Einstein equations. Here the scalar function Φ satisfies the eikonal equation [20, 21]

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta = 0\,, \qquad (3.3)$$

where ∇_α denotes covariant differentiation with respect to Riemannian coherence with the metric $g_{\alpha\beta}$. The trajectories along which gravitational waves travel (gravitational rays) are bicharacteristics of the field equations, having the form

$$\frac{dx^\alpha}{d\tau} = g^{\alpha\sigma} \nabla_\sigma \Phi\,, \qquad (3.4)$$

where τ is a parameter along lines of the geodesic family.

But the general solution of the Einstein equations with initial conditions in the hypersurface is unknown. For this reason the next problem arises: it is necessary to formulate an effective criterion which could determine solutions to the Einstein equations with initial conditions in the characteristic hypersurface.

There is another difficulty: there is no general covariant d'Alembertian which, being in its clear form, could be included into the Einstein equations.

Therefore, solving the gravitational wave problem reduces to the problem of formulating an invariant criterion which could determine this family of the field equations as wave equations.

Following this approach, analogous the classical theory of differential equations, we encounter an essential problem. Are functions $g_{\alpha\beta}(x^\sigma)$ smooth when we set up the Cauchy problem for the Einstein equation? A gravitational wave is interpreted as Hadamard break for the curvature tensor field in the initial characteristic hypersurface. The curvature tensor field permits a Hadamard break only if the functions $g_{\alpha\beta}(x^\sigma)$ permit breaks in their first derivatives. In accordance with Hadamard himself [20], the second derivatives of $g_{\alpha\beta}$ can have a break in a surface $S\{\Phi(x^\alpha) = 0\}$

$$[\partial_{\rho\sigma}\, g_{\alpha\beta}] = a_{\alpha\beta}\, l_\rho\, l_\sigma\,, \qquad (l_\alpha \equiv \partial_\alpha \Phi) \qquad (3.5)$$

only if a Hadamard break in the curvature tensor field $[R_{\alpha\beta\gamma\delta}]$ satisfies the equations [21]

$$l_\lambda [R_{\mu\alpha\beta\nu}] + l_\alpha [R_{\mu\beta\lambda\nu}] + l_\beta [R_{\mu\lambda\alpha\nu}] = 0\,. \qquad (3.6)$$

Proceeding from such an interpretation of the characteristic hypersurface for the Einstein equations, and also supposing that a break $[R_{\alpha\beta\gamma\delta}]$ in the curvature tensor $R_{\alpha\beta\gamma\delta}$ located in the front of a gravitational wave is proportional to the tensor itself, Lichnerowitz [20, 21] formulated this criterion for gravitational waves:

Lichnerowitz' criterion The space curvature $R_{\alpha\beta\gamma\delta} \neq 0$ determines the state of "full gravitational radiations", only if there is a vector $l^\alpha = 0$ satisfying the equations

$$\begin{aligned} & l^\mu R_{\mu\alpha\beta\nu} = 0\,, \\ & l_\lambda R_{\mu\alpha\beta\nu} + l_\alpha R_{\mu\beta\lambda\nu} + l_\beta R_{\mu\lambda\alpha\nu} = 0\,, \end{aligned} \qquad (3.7)$$

and thus the vector l^α is isotropic ($l_\alpha l^\alpha = 0$). If $R_{\alpha\beta} = 0$ (the space is free of masses, so it is empty), the equations (3.7) determine the state of "clear gravitational radiations".

There is also Zelmanov's invariant criterion for gravitational waves [40]*, it is linked to the Lichnerowitz criterion.

*This criterion is named for Abraham Zelmanov, although it had been published by Zakharov [40]. This happened because Zelmanov gave many of his unpublished results, his unpublished criterion included, to Zakharov, who completed his dissertation under Zelmanov's leadership at that time.

Zelmanov proceeded from the general covariant generalization given for the d'Alembert wave operator

$$\Box^{\sigma}_{\sigma} \equiv g^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}\,. \tag{3.8}$$

Zelmanov's criterion The space determines the state of gravitational radiations, only if the curvature tensor: (a) is not a covariant constant quantity ($\nabla_{\sigma} R_{\mu\alpha\beta\gamma} = 0$); (b) satisfies the general covariant condition

$$\Box^{\sigma}_{\sigma} R_{\mu\alpha\beta\nu} = 0. \tag{3.9}$$

Thus, as it was shown in [40], any empty space that satisfies the Zelmanov criterion also satisfies the Lichnerowitz criterion. On the other hand, any empty space that satisfies the Lichnerowitz criterion (excluding that trivial case where $\nabla_{\sigma} R_{\mu\alpha\beta\gamma} = 0$) also satisfies the Zelmanov criterion.

There are also other criteria for gravitational waves, introduced by Bel, Pirani, Debever, Maldybaeva and others [58]. Each of the criteria has its own advantages and drawbacks, therefore none of the criteria can be considered as the final solution of this problem. Consequently, it would be a good idea to consider those characteristics of gravitational wave fields which are common to most of the criteria. Such an integrating factor is Petrov's classification — the algebraic classification of Einstein spaces given by Petrov [37], in the frame of which those gravitational fields that satisfy the condition (3.2) are classified by their relation to the algebraic structure of the Riemann-Christoffel curvature tensor.

As is well known, the components of the Riemann-Christoffel tensor satisfy the identities

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}\,, \quad R_{\alpha[\beta\gamma\delta]} = 0\,. \tag{3.10}$$

Because of (3.10), the curvature tensor is related to tensors of a special family, known as *bitensors*. They satisfy two conditions:

1. Their covariant and contravariant valencies are even;
2. Both covariant and contravariant indices of the tensors are split into pairs and inside each pair the tensor $R_{\alpha\beta\gamma\delta}$ is antisymmetric.

A set of tensor fields located in an n-dimensional Riemannian space is known as a *bivector set*, and its representation at a point is known as a *local bivector set*. Every antisymmetric pair of indices $\alpha\beta$ is denoted by a common index a, and the number of the common indices is $N = \frac{n(n-1)}{2}$. It is evident that if $n = 4$ we have $N = 6$. Hence a bitensor $R_{\alpha\beta\gamma\delta} \rightarrow R_{ab}$, located in a four-dimensional space, maps itself into a six-dimensional bivector space. It can be metrised by introducing the specific metric tensor

$$g_{ab} \rightarrow g_{\alpha\beta\gamma\delta} \equiv g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\,. \tag{3.11}$$

The tensor g_{ab} $(a, b = 1, 2, \dots N)$ is symmetric and non-degenerate. The metric g_{ab}, given for the sign-alternating $g_{\alpha\beta}$, can be sign-alternating, having a signature dependent on the signature of the $g_{\alpha\beta}$. So, for Minkowski's signature $(+---)$, the signature of the g_{ab} is $(+++---)$.

Mapping the curvature tensor $R_{\alpha\beta\gamma\delta}$ onto the metric bivector space R_N, we obtain the symmetric tensor R_{ab} $(a, b = 1, 2, \dots N)$ which can be associated with a lambda-matrix

$$\left(R_{ab} - \Lambda\, g_{ab}\right). \tag{3.12}$$

Solving the classic problem of linear algebra (reducing the lambda-matrix to its canonical form along a real distance), we can find a classificaton for V_n under a given n. Here the *specific kind* of an Einstein space we are considering is set up by a *characteristic* of the lambda-matrix. This kind remains unchanged in that area where this characteristic remains unchanged.

Bases of elementary divisors of the lambda-matrix for any V_n have an ordinary geometric meaning as *stationary curvatures*. Naturally, the Riemannian curvature V_n in a two-dimensional direction is determined by an ordinary (single-sheet) bivector $V^{\alpha\beta} = V^{\alpha}_{(1)} V^{\beta}_{(2)}$, of the form

$$K = \frac{R_{\alpha\beta\gamma\delta}V^{\alpha\beta}V^{\gamma\delta}}{g_{\alpha\beta\gamma\delta}V^{\alpha\beta}V^{\gamma\delta}}\,. \tag{3.13}$$

If $V^{\alpha\beta}$ is not ordinary, the invariant K is known as the *bivector curvature in the given vector's direction*. Mapping K onto the bivector space, we obtain

$$K = \frac{R_{ab}V^{a}V^{b}}{g_{ab}V^{a}V^{b}}\,, \qquad a, b = 1, 2, \dots N. \tag{3.14}$$

Ultimate numerical values of the K are known as *stationary curvatures* taken at a given point, and the vectors V^a corresponding to the ultimate values are known as *stationary not simple bivectors*. In this case

$$V^{\alpha\beta} = V^{\alpha}_{(1)} V^{\beta}_{(2)}\,, \tag{3.15}$$

so the stationary curvature coincides with the Riemannian curvature V_n in the given two-dimension direction.

The problem of finding the ultimate values of K is the same as finding those vectors V^a where the K takes the ultimate values, that is, the same as finding *undoubtedly stationary directions*. The necessary and sufficient condition of stationary state of the V^a is

$$\frac{\partial}{\partial V^a}\, K = 0\,. \tag{3.16}$$

The problem of finding the stationary curvatures for Einstein spaces had been solved by Petrov [40]. If the space signature is sign-alternating, generally speaking, the stationary curvatures are complex as well as the stationary bivectors relating to them in the V_n.

For four-dimensional Einstein spaces with Minkowski signature, we have the following theorem [40]:

THEOREM *Given an ortho-frame $g_{\alpha\beta} = \{+1, -1, -1, -1\}$, there is a symmetric paired matrix (R_{ab})*

$$R_{ab} = \left(\begin{array}{c|c} M & N \\ \hline N & -M \end{array} \right), \tag{3.17}$$

where M and N are two symmetric square matrices of the 3rd order, whose components satisfy the relationships

$$m_{11}+m_{22}+m_{33} = -\kappa\,, \qquad n_{11}+n_{22}+n_{33} = 0\,. \tag{3.18}$$

After transformations, the lambda-matrix $(R_{ab} - \Lambda g_{ab})$ where $g_{\alpha\beta} = \{+1, +1, +1, -1, -1, -1\}$ takes the form

$$\begin{aligned}(R_{ab} - \Lambda g_{ab}) &= \\ &= \left(\begin{array}{c|c} M + iN + \Lambda\varepsilon & 0 \\ \hline 0 & M - iN + \Lambda\varepsilon \end{array} \right) \equiv \\ &\equiv \begin{pmatrix} Q(\Lambda) & 0 \\ 0 & \bar{Q}(\Lambda) \end{pmatrix},\end{aligned} \tag{3.19}$$

where $Q(\Lambda)$ and $\bar{Q}(\Lambda)$ are three-dimensional matrices, the elements of which are complex conjugates, ε is the three-dimensional unit matrix. The matrix $Q(\Lambda)$ can have only one of the following types of characteristics:

(1) [111]; (2) [21]; (3) [3]. It is evident that the initial lambda-matrix can have only one characteristic drawn from:

(1) $[111, \overline{111}]$; (2) $[21, \overline{21}]$; (3) $[3, 3]$.

The bar in the second half of a characteristic implies that elementary divisors in both matrices are complex conjugates. There is no bar in the third kind because the elementary divisors there are always real.

Taking a lambda-matrix of each of the three possible kinds, Petrov deduced the canonical form of the matrix (R_{ab}) in a non-holonomic ortho-frame [40]

<u>The 1st Kind</u>

$$\begin{gathered}(R_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \\ M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \\ N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},\end{gathered} \tag{3.20}$$

where $\sum_{s=1}^{3} \alpha_s = -\kappa\,,\ \sum_{s=1}^{3} \beta_s = 0$ (so in this case there are 4 independent parameters, determining the space structure by an invariant form),

<u>The 2nd Kind</u>

$$\begin{gathered}(R_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \\ M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2+1 & 0 \\ 0 & 0 & \alpha_2-1 \end{pmatrix}, \\ N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 1 \\ 0 & 1 & \beta_2 \end{pmatrix},\end{gathered} \tag{3.21}$$

where $\alpha_1 + 2\alpha_2 = -\kappa\,,\ \beta_1 + 2\beta_2 = 0$ (so in this case there are 2 independent parameters determining the space structure by an invariant form),

<u>The 3rd Kind</u>

$$\begin{gathered}(R_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \\ M = \begin{pmatrix} -\frac{\kappa}{3} & 1 & 0 \\ 1 & -\frac{\kappa}{3} & 0 \\ 0 & 0 & -\frac{\kappa}{3} \end{pmatrix}, \\ N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},\end{gathered} \tag{3.22}$$

so no independent parameters determining the space structure by an invariant form exist in this case.

Thus Petrov had solved the problem of reducing a lambda-matrix to its canonical form along a real path in a space of the sign-alternating metric. Although this solution is obtained only at given point, the classification obtained is invariant because the results are applicable to any point in the space.

Real curvatures take the form

$$\Lambda_s = \alpha_s + i\beta_s\,, \tag{3.23}$$

in gravitational fields (spaces) of the 3rd kind, where the quantities Λ_s are real: $\Lambda_1 = \Lambda_2 = \Lambda_3 = -\frac{\kappa}{3}$.

Values of some stationary curvatures in gravitational fields (spaces) of the 1st and 2nd kinds can be coincident. If they coincide, we have sub-kinds of the fields (spaces). The 1st kind has 3 sub-kinds: I ($\Lambda_1 \neq \Lambda_2 \neq \Lambda_3$); D ($\Lambda_2 = \Lambda_3$); O ($\Lambda_1 = \Lambda_2 = \Lambda_3$). If the space is empty ($\kappa = 0$) the kind O means the flat space. The 2nd kind has 2 sub-kinds: II ($\Lambda_1 \neq \Lambda_2$); N ($\Lambda_1 = \Lambda_2$). Kinds I and II are called basic kinds.

In empty spaces (empty gravitational fields) the stationary curvatures become the unit value $\Lambda = 0$, so the spaces (fields) are called *degenerate*.

Studying the algebraic structure of the curvature tensor for known solutions to the Einstein equations, it was shown that the most of the solutions are of the 1st kind by Petrov's classification. The curvature decreases with distance from a gravitating mass. In the extreme case where the distance becomes infinite the space approaches the Minkowski flat space. The well-known Schwarzschid solution, describing a spherically symmetric gravitational field derived from a spherically symmetric island of mass located in an empty space, is classified as the sub-kind D of the 1st kind [44].

Invariant criteria for gravitational waves are linked to the algebraic structure of the curvature tensor, which should be associated with a given criterion from the aforementioned types. The most well-known solutions, which are interpreted as gravitational waves, are attributed to the sub-kind N (of the 1st kind). Other solutions are attributed to the 2nd kind and the 3rd kind. It should be noted that spaces of the 2nd and 3rd kinds cannot be flat anywhere, because components of the curvature tensor matrix $\|R_{ab}\|$ contain $+1$ and -1. This makes asymptotical approach to a curvature of zero impossible, i.e. excludes asymptotical approach to Minkowski space. Therefore, because of the structure of such fields, gravitational fields in a space of the 2nd kind (the sub-kind N) or the 3rd kind, are gravitational waves of the curvature traveling everywhere in the space. Pirani [18] holds that gravitational waves are solutions to gravitational fields in spaces of the 2nd kind (the sub-kind N) or the 3rd kind by Petrov's classification. The following solutions are classified as sub-kind N: Peres' solution [34] where he describes flat gravitational waves

$$ds^2 = (dx^0)^2 - 2\alpha(dx^0 + dx^3)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2; \tag{3.24}$$

Takeno's solution [35]

$$ds^2 = (\gamma + \rho)(dx^0)^2 - 2\rho dx^0 dx^3 - \alpha(dx^1)^2 - 2\delta dx^1 dx^2 - \beta(dx^2)^2 + (\rho - \gamma)(dx^3)^2, \tag{3.25}$$

where $\alpha = \alpha(x^1 - x^0)$, and γ, ρ, β, δ are functions of $(x^3 = x^0)$; Petrov's solution [37], studied also by Bondi, Pirani and Robertson in another coordinate system [19]

$$ds^2 = (dx^0)^2 - (dx^1)^2 + \alpha(dx^2)^2 + 2\beta dx^2 dx^3 + \gamma(dx^3)^2, \tag{3.26}$$

where α, β, γ are functions of $(x^1 + x^0)$.

A detailed study of relations between the invariant criteria for gravitational waves and Petrov's classification had been undertaken by Zakharov [40]. He proved:

THEOREM *In order that a given space satisfies the state of "pure gravitational radiations" (in the Lichnerowicz sense), it is a necessary and sufficient condition that the space should be of the sub-kind N by Petrov's algebraical classification, characterized by equality to zero of the values of the curvature tensor matrix $\|R_{ab}\|$ in the bivector space.*

THEOREM *An Einstein space that satisfies Zelmanov's criterion can only be an empty space ($\varkappa = 0$) of the sub-kind N. And conversely, any empty space V_4 of the sub-kind N (excluding the sole symmetric space* of this kind), that is described by the metric*

$$ds^2 = 2dx^0 dx^1 - \mathrm{sh}^2 dx^0 (dx^2)^2 - \sin^2 dx^0 (dx^3)^2, \tag{3.27}$$

satisfies the Zelmanov criterion.

With these theorems we obtain the general relation between the Zelmanov criterion for gravitational wave fields located in empty spaces and the Lichnerowicz criterion for "pure gravitational radiations":

> An empty V_4, satisfying the Zelmanov criterion for gravitational wave fields, also satisfies the Lichnerowicz criterion for "pure gravitational radiations". Conversely, any empty V_n, satisfying the Lichnerowicz criterion (excluding the sole trivial V_n described by the metric 3.27), satisfies the Zelmanov criterion. The relation between the criteria in the general case is still an open problem.

In [40] it was shown that all known solutions to the Einstein equations in vacuum, which satisfy the Zelmanov and Lichnerowicz criteria, can be obtained as particular cases of the more generalized metric whose space permits a covariant constant vector field l^α

$$\nabla_\sigma l^\alpha = 0\,. \tag{3.28}$$

It is evident that condition (3.10) leads automatically to the first condition (3.7), hence this empty V_4 is classified as sub-kind N by Petrov's classification and, also, there the vector l^α, playing a part of the gravitational field wave vector, is isotropic $l_\alpha l^\alpha = 0$ and unique. According to Eisenhart's theorem [60], the space V_4 containing the unique isotropic covariant constant vector l^α (the absolute parallel vector field l^α, in other words), has the metric

$$ds^2 = \varepsilon(dx^0)^2 + 2dx^0 dx^1 + 2\varphi dx^0 dx^2 + 2\psi dx^0 dx^3 + \alpha(dx^2)^2 + 2\gamma dx^2 dx^3 + \beta(dx^3)^2, \tag{3.29}$$

where ε, φ, ψ, α, β, γ are functions of x^0, x^2, x^3, and $l^\alpha = \delta^\alpha_1$. The metric (3.29), satisfying equations (3.2), is the exact solution to the Einstein equations for vacuum, and satisfies the Zelmanov and Lichnerowicz gravitational wave criteria. This solution generalizes well-known solutions deduced by Takeno, Peres, Bondi, Petrov and others, that satisfy the aforementioned criteria [40].

The metric (3.29), taken under some additional conditions [30], satisfies the Einstein equations in their general form (3.1) in the case where $\lambda = 0$ and the energy-momentum tensor $T_{\alpha\beta}$ describes an isotropic electromagnetic field where Maxwell's tensor $F_{\mu\nu}$ satisfies the conditions

$$F_{\mu\nu}F^{\mu\nu} = 0\,, \qquad F_{\mu\nu}F^{*\mu\nu} = 0\,, \tag{3.30}$$

$F^{*\mu\nu} = \frac{1}{2}\eta^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the pseudotensor dual of the Maxwell tensor, $\eta^{\mu\nu\rho\sigma}$ is the discriminant tensor. Direct substitution shows that this metric satisfies the following requirements: the Riner-Wheeler condition [61]

$$R = 0\,, \qquad R_{\alpha\rho}R^{\rho\beta} = \frac{1}{4}\,\delta^\beta_\alpha\,(R_{\rho\sigma}R^{\rho\sigma}) = 0\,, \tag{3.31}$$

*A space is called *symmetric*, if its curvature tensor is a covariant constant, i.e. if it satisfies the condition $\nabla_\sigma R_{\alpha\beta\gamma\delta} = 0$.

and also the Nordtvedt-Pagels condition [62]

$$\eta_{\mu\varepsilon\gamma\sigma}\left(R^{\delta\gamma,\sigma}R^{\varepsilon\tau}-R^{\delta\varepsilon,\sigma}R^{\gamma\tau}\right), \qquad (3.32)$$

where $R^{\delta\gamma,\sigma}=g^{\sigma\mu}\nabla_{\mu}R^{\delta\gamma}$, $\delta^{\alpha}_{\beta}=g^{\alpha}_{\beta}$.

From the physical viewpoint we have an interest in isotropic electromagnetic fields because an observer who accompanies it should be moving at the velocity of light [18, 21]. Hence, isotropic electromagnetic fields can be interpreted as fields of electromagnetic radiation without sources. On the other hand, according to Eisenhart theorem [60], a space V_4 with the metric (3.29) permits an absolute parallel vector field $l^{\alpha}=\delta^{\alpha}_{1}$. Taking this fact and also the Einstein equations into account, we conclude that the vector l^{α} considered in this case satisfies the Lichnerowicz criterion for "full gravitational radiations".

Thus the metric (3.29), satisfying the conditions

$$\begin{gathered} R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-\kappa T_{\alpha\beta}\,, \\ T_{\alpha\beta}=\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\alpha\beta}-F_{\alpha\sigma}F_{\beta\cdot}^{\cdot\sigma}\,, \\ F_{\alpha\beta}F^{\alpha\beta}=0\,, \qquad F_{\alpha\beta}F^{*\alpha\beta}=0 \end{gathered} \qquad (3.33)$$

and under the additional condition [30]

$$R_{2323}=R_{0232}=R_{0323}=0\,, \qquad (3.34)$$

is the exact solution to the Einstein equations which describes co-existence of both gravitational waves and electromagnetic waves. This solution does not satisfy the Zelmanov criterion in the general case, but the solution satisfies it in some particular cases where $T_{\alpha\beta}\neq 0$, and also under $R_{\alpha\beta}=0$.

Wave properties of recursion curvature spaces were studied in [63]. A recursion curvature space is a Riemannian space having a curvature which satisfies the relationship

$$\nabla_{\sigma}R_{\alpha\beta\gamma\delta}=l_{\sigma}R_{\alpha\beta\gamma\delta}\,. \qquad (3.35)$$

Because of Bianchi's identity, such spaces satisfy

$$l_{\sigma}R_{\alpha\beta\gamma\delta}+l_{\alpha}R_{\beta\sigma\gamma\delta}+l_{\beta}R_{\sigma\alpha\gamma\delta}=0\,. \qquad (3.36)$$

Total classification for recursion curvature spaces had been given by Walker [64]. His results [64] were applied to the basic space-time of the General Theory of Relativity, see [65] for the results. For the class of prime recursion spaces*, we are particularly interested in the two metrics

$$ds^2=\psi(x^0,x^2)(dx^0)^2+2dx^0dx^1-(dx^2)^2-(dx^3)^2, \qquad (3.37)$$

$$\begin{aligned} ds^2 &= 2dx^0dx^1+\psi(x^1,x^2)(dx^1)^2- \\ &\qquad -(dx^2)^2-(dx^3)^2, \quad \psi>0\,. \end{aligned} \qquad (3.38)$$

*A recursion curvature space is known as prime or simple, if it contains $n-2$ parallel vector fields, which could be isotropic or non-isotropic. Here n is the dimension of the space.

For the metric (3.37) there is only one component of the Ricci tensor that is not zero, $R_{00}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^{2\,2}}$, in the metric (3.38) only $R_{11}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^{2\,2}}$ is not zero. Einstein spaces with such metrics can only be empty ($\kappa=0$) and flat ($R_{\alpha\beta\gamma\delta}=0$). This can be proven by checking that both metrics satisfy conditions (3.31) and (3.32), which describe isotropic electromagnetic fields.

Both metrics are interesting from the physical viewpoint: in these cases the origin of the space curvature is an isotropic electromagnetic field. Moreover, if we remove this field from the space, the space becomes flat. Besides these there are few metrics which are exact solutions to the Einstein-Maxwell equations, related to the class of isotropic electromagnetic fields. Neither of the said metrics satisfy the Zelmanov and Lichnerowicz criteria.

Minkowski's signature permits only two metrics for non-simple recursion curvature spaces. They are the metric

$$\begin{gathered} ds^2=\psi(x^0,x^2,x^3)(dx^0)^2+2dx^0dx^1+ \\ +K_{22}(dx^2)^2+2K_{23}dx^2dx^3+K_{33}(dx^3)^2, \\ K_{22}<0\,, \qquad K_{22}K_{33}-K_{23}^2<0\,, \end{gathered} \qquad (3.39)$$

wherein $\psi=\chi_1(x_0)(a_{22}(x^2)^2+2a_{23}x^2x^3+a_{33}(x^3)^2)+$ $+\chi_2(x^0)x^2+\chi_3(x^0)x^3$, and the metric

$$\begin{gathered} ds^2=2dx^0dx^1+\psi(x^1,x^2,x^3)(dx^1)^2+ \\ +K_{22}(dx^2)^2+2K_{23}dx^2dx^3+K_{33}(dx^3)^2, \end{gathered} \qquad (3.40)$$

wherein $\psi=\chi_1(x_1)(a_{22}(x^2)^2+2a_{23}x^2x^3+a_{33}(x^3)^2)+$ $+\chi_2(x^1)x^2+\chi_3(x^1)x^3$. Here a_{ij}, K_{ij} ($i,j=2,3$) are constants.

Both metrics satisfy the conditions $R_{\alpha\beta}=\kappa g_{\alpha\beta}$ only if $\kappa=0$, reducing to the single relationship

$$K_{33}a_{22}+K_{22}a_{33}-2K_{23}a_{23}=0\,. \qquad (3.41)$$

In this case both metrics are of the sub-kind N by Petrov's classification. It is interesting to note that the metric (3.40) is stationary and, at the same time, describes "pure gravitational radiation" by Lichnerowicz. Such a solution was also obtained in [65].

In the general case ($R_{\alpha\beta}\neq\kappa g_{\alpha\beta}$) the metrics (3.39) and (3.40) satisfy conditions (3.32) and (3.33), so the metrics are solutions to the Einstein-Maxwell equations that describe co-existing gravitational waves and electromagnetic waves without sources. In this general case both metrics satisfy the Zelmanov and Lichnerowicz invariant criteria. The solution (3.40) is stationary.

All that has been detailed above applies to gravitational waves as waves of the space curvature, which exist in any reference frame.

Additionally it would be interesting to study another approach to the gravitational radiation problem, where the main issue is gravitational inertial waves, connected to the given reference frame of an observer. This new approach is linked directly to the mathematical apparatus of physically observable quantities (the theory of chronometric invariants), introduced by Zelmanov in 1944 [42, 43]. In order to understand the true results given by gravitational wave experiments it is necessary to master this mathematical apparatus, which is described concisely in the in the next section.

4 Basics of the theory of physical observable quantities

In brief, the essence of the mathematical apparatus of physically observable quantities (the theory of chronometric invariants), developed by Zelmanov in 1940's [42, 43] is that, if an observer accompanies his reference body, his observable quantities are projections of four-dimensional quantities on his time line and the spatial section — *chronometrically invariant quantities*, made by the projecting operators $b^\alpha = \frac{dx^\alpha}{ds}$ and $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$ which fully define his real reference space (here b^α is his velocity with respect to his real references). The chr.inv.-projections of a world-vector Q^α are $b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}}$ and $h^i_\alpha Q^\alpha = Q^i$, while chr.inv.-projections of a world-tensor of the 2nd rank $Q^{\alpha\beta}$ are $b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}$, $h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}$, $h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}$. Physically observable properties of the space are derived from the fact that the chr. inv.-differential operators $\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2} v_i \frac{^*\partial}{\partial t}$ are non-commutative, so that $\frac{^*\partial^2}{\partial x^i \partial t} - \frac{^*\partial^2}{\partial t\,\partial x^i} = \frac{1}{c^2} F_i \frac{^*\partial}{\partial t}$ and $\frac{^*\partial^2}{\partial x^i \partial x^k} - \frac{^*\partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{^*\partial}{\partial t}$, and also from the fact that the chr.inv.-metric tensor h_{ik} may not be stationary. The observable characteristics are the chr.inv.-vector of gravitational inertial force F_i, the chr.inv.-tensor of angular velocities of the space rotation A_{ik}, and the chr.inv.-tensor of rates of the space deformations D_{ik}, namely

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} \tag{4.1}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \tag{4.2}$$

$$D_{ik} = \frac{1}{2}\frac{^*\partial h_{ik}}{\partial t}, \quad D^{ik} = -\frac{1}{2}\frac{^*\partial h^{ik}}{\partial t}, \quad D^k_k = \frac{^*\partial \ln\sqrt{h}}{\partial t}, \tag{4.3}$$

where w is the gravitational potential, $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation, $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ is the metric chr.inv.-tensor, and $h = \det\|h_{ik}\|$, $h g_{00} = -g$, $g = \det\|g_{\alpha\beta}\|$. Observable inhomogeneity of the space is set up by the chr.inv.-Christoffel symbols $\Delta^i_{jk} = h^{im}\Delta_{jk,m}$, which are built just like Christoffel's regular symbols $\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$, but using h_{ik} instead of $g_{\alpha\beta}$.

In this way, any equations obtained using general covariant methods we can calculate their physically observable projections on the time line and the spatial section of any particular reference body and formulate the projections with its real physically observable properties. From this we arrive at equations containing only quantities measurable in practice. Expressing $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$ through the observable time interval

$$d\tau = \frac{1}{c}\, b_\alpha dx^\alpha = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i \tag{4.4}$$

and also the observable spatial interval $d\sigma^2 = h_{\alpha\beta}\, dx^\alpha dx^\beta = h_{ik}\, dx^i dx^k$ (note that $b^i = 0$ for an observer who accompanies his reference body). We arrive at the formula

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{4.5}$$

From an"external" viewpoint, an observer's three-dimensional space is the *spatial section* $x^0 = ct = \text{const}$. At any point of the space-time a local spatial section (a local space) can be placed orthogonal to the *time line*. If there exists a space-time enveloping curve for such local spaces, then it is a spatial section everywhere orthogonal to the time lines. Such a space is called *holonomic*. If no enveloping curve exists for such local spaces, so there only exist spatial sections locally orthogonal to the time lines, such a space is called *non-holonomic*. A spatial section, placed in a holonomic space, is everywhere orthogonal to the time lines, i. e. $g_{0i} = 0$ is true there. In the presence of $g_{0i} = 0$ we have $v_i = 0$, hence $A_{ik} = 0$. This implies that non-holonomity of the space and its three-dimensional rotation are the same. In a non-holonomic space $g_{0i} \neq 0$ and $A_{ik} \neq 0$. Hence $A_{ik} = 0$ is the necessary and sufficient condition of holonomity of the space. So A_{ik} is the *tensor of the space non-holonomity*.

Zelmanov had also found that the chr.inv.-quantities F_i and A_{ik} are linked to one another by two identities

$$\frac{^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{^*\partial F_k}{\partial x^i} - \frac{^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{4.6}$$

$$\begin{gathered}\frac{^*\partial A_{km}}{\partial x^i} + \frac{^*\partial A_{mi}}{\partial x^k} + \frac{^*\partial A_{ik}}{\partial x^m} + \\ + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0\,,\end{gathered} \tag{4.7}$$

which are known as *Zelmanov's identities*.

Components of the usual Christoffel symbols

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\, g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right). \tag{4.8}$$

are linked to the chr.inv.-Christoffel symbols

$$\Delta^i_{jk} = \frac{1}{2}\, h^{im}\left(\frac{^*\partial h_{jm}}{\partial x^k} + \frac{^*\partial h_{km}}{\partial x^j} - \frac{^*\partial h_{jk}}{\partial x^m}\right), \tag{4.9}$$

and other chr.inv.-chractersitics of the accompanying reference space of the given observer by the relations

$$D_k^i + A_{k\cdot}^{\cdot i} = \frac{c}{\sqrt{g_{00}}}\left(\Gamma_{0k}^i - \frac{g_{0k}\Gamma_{00}^i}{g_{00}}\right), \qquad (4.10)$$

$$F^k = -\frac{c^2\,\Gamma_{00}^k}{g_{00}}\,, \qquad g^{i\alpha}g^{k\beta}\,\Gamma_{\alpha\beta}^m = h^{iq}h^{ks}\Delta_{qs}^m. \qquad (4.11)$$

Here is the four-dimensional generalization of the chr.inv.-quantities F_i, A_{ik}, and D_{ik} (by Zelmanov, the 1960's [57]): $F_\alpha = -2c^2 b^\beta a_{\beta\alpha}$, $A_{\alpha\beta} = ch_\alpha^\mu h_\beta^\nu a_{\mu\nu}$, $D_{\alpha\beta} = ch_\alpha^\mu h_\beta^\nu d_{\mu\nu}$, where $a_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha b_\beta - \nabla_\beta b_\alpha\right)$, $d_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha b_\beta + \nabla_\beta b_\alpha\right)$.

Zelmanov also deduced formulae for chr.inv.-projections of the Riemann-Christoffel tensor [42]. He followed the same procedure by which the Riemann-Christoffel tensor was built, proceeding from the non-commutativity of the second derivatives of an arbitrary vector taken in the given space. Taking the second chr.inv.-derivatives of an arbitrary vector

$${}^*\nabla_i{}^*\nabla_k\, Q_l - {}^*\nabla_k{}^*\nabla_i\, Q_l = \frac{2A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdots j}Q_j\,, \quad (4.12)$$

he obtained the chr.inv.-tensor

$$H_{lki\cdot}^{\cdots j} = \frac{{}^*\partial\Delta_{il}^j}{\partial x^k} - \frac{{}^*\partial\Delta_{kl}^j}{\partial x^i} + \Delta_{il}^m\Delta_{km}^j - \Delta_{kl}^m\Delta_{im}^j\,, \quad (4.13)$$

which is like Schouten's tensor from the theory of non-holonomic manifolds [59]. The tensor $H_{lki}^{\cdots j}$ differs algebraically from the Riemann-Christoffel tensor because of the presence of rotation of the space A_{ik} in the formula (4). Nevertheless its generalization gives the chr.inv.-tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right), \qquad (4.14)$$

which possesses all the algebraic properties of the Riemann-Christoffel tensor in this three-dimensional space. Therefore Zelmanov called C_{iklj} the *chr.inv.-curvature tensor*, which actually is the tensor of the observable curvature of the observer's spatial section. This tensor, describing the observable curvature of the three-dimensional space of an observer, possesses all the properties of the Riemann-Christoffel curvature tensor in the three-dimensional space and, at the same time, the property of chronometric invariance. Its contraction

$$C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im}C_{kimj}\,, \qquad C = C_j^j = h^{lj}C_{lj} \quad (4.15)$$

gives the chr.inv.-scalar C whose sense is the *observable three-dimensional curvature* of this space.

Substituting the necessary components of the Riemann-Christoffel tensor into the formulae for its chr.inv.-projections $X^{ik} = -c^2\frac{R_{\cdot 0\cdot 0\cdot}^{\cdot i\cdot k}}{g_{00}}$, $Y^{ijk} = -c\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}$, $Z^{ijkl} = c^2R^{ijkl}$, and by lowering indices Zelmanov obtained the formulae

$$\begin{aligned} X_{ij} = \frac{{}^*\partial D_{ij}}{\partial t} &- \left(D_i^l + A_{i\cdot}^{\cdot l}\right)\left(D_{jl} + A_{jl}\right) + \\ &+ \frac{1}{2}\left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2}\,F_iF_j\,, \end{aligned} \qquad (4.16)$$

$$\begin{aligned} Y_{ijk} = {}^*\nabla_i\left(D_{jk} + A_{jk}\right) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \\ + \frac{2}{c^2}A_{ij}F_k\,, \end{aligned} \qquad (4.17)$$

$$\begin{aligned} Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - A_{il}A_{kj} + \\ + 2A_{ij}A_{kl} - c^2C_{iklj}\,, \end{aligned} \qquad (4.18)$$

where we have $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$ just like the Riemann-Christoffel tensor. Contraction of the spatial observable projection Z_{iklj} step-by-step gives

$$Z_{il} = D_{ik}D_l^k - D_{il}D + A_{ik}A_{l\cdot}^{\cdot k} + 2A_{ik}A_{\cdot l}^{k\cdot} - c^2C_{il}\,, \quad (4.19)$$

$$Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2C\,. \quad (4.20)$$

Besides these considerations, taken in an observer's accompanying reference frame, Zelmanov considered a *locally geodesic reference frame* that can be introduced at any point of the pseudo-Riemannian space. Within infinitesimal vicinities of any point of such a reference frame the fundamental metric tensor is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2\tilde{g}_{\alpha\beta}}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \ldots, \quad (4.21)$$

i. e. its components at a point, located in the vicinities, are different to those at the point of reflection to within only the higher order terms, values of which can be neglected. Therefore, at any point of a locally geodesic reference frame the fundamental metric tensor can betaken as constant, while the first derivatives of the metric (the Christoffel symbols) are zero.

As a matter of fact, within infinitesimal vicinities of any point located in a Riemannian space, a locally geodesic reference frame can be defined. At the same time, at any point of this locally geodesic reference frame, a tangential flat Euclidean space can be defined so that this reference frame, being locally geodesic for the Riemannian space, is the global geodesic for that tangential flat space.

The fundamental metric tensor of a flat Euclidean space is constant, so values of $\tilde{g}_{\mu\nu}$, taken in the vicinities of a point of the Riemannian space converge to values of the tensor $g_{\mu\nu}$ in the flat space tangential at this point. Actually, this means that we can build a system of basis vectors $\vec{e}_{(\alpha)}$, located in this flat space, tangential to curved coordinate lines of the Riemannian space.

In general, coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other (if the space is non-holonomic). So the lengths of the basis vectors may be sometimes very different from unity.

We denote a four-dimensional vector of infinitesimal displacement by $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$. Then $d\vec{r} = \vec{e}_{(\alpha)}dx^\alpha$, where components of the basis vectors $\vec{e}_{(\alpha)}$ tangential to the coordinate lines are $\vec{e}_{(0)} = \{e_{(0)}^0, 0, 0, 0\}$, $\vec{e}_{(1)} = \{0, e_{(1)}^1, 0, 0\}$, $\vec{e}_{(2)} = \{0, 0, e_{(2)}^2, 0\}$, $\vec{e}_{(3)} = \{0, 0, 0, e_{(3)}^2\}$. The scalar product of the vector $d\vec{r}$ with itself is $d\vec{r}d\vec{r} = ds^2$. On the other hand, the same quantity is $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$. As a result we have $g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)}e_{(\beta)}\cos(x^\alpha; x^\beta)$. So we obtain

$$g_{00} = e^2_{(0)}\,, \qquad g_{0i} = e_{(0)}e_{(i)}\cos(x^0; x^i)\,, \tag{4.22}$$

$$g_{ik} = e_{(i)}e_{(k)}\cos(x^i; x^k)\,. \tag{4.23}$$

The gravitational potential is $\mathrm{w} = c^2(1-\sqrt{g_{00}})$. So, the time basis vector $\vec{e}_{(0)}$ tangential to the time line $x^0 = ct$, having the length $e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$ is smaller than unity the greater is the gravitational potential w.

The space rotation linear velocity v_i and, according to it, the chr.inv.-metric tensor h_{ik} are

$$v_i = -c\,e_{(i)}\cos(x^0; x^i)\,, \tag{4.24}$$

$$h_{ik} = e_{(i)}e_{(k)}\Big[\cos(x^0; x^i)\cos(x^0; x^k) - \cos(x^i; x^k)\Big]\,. \tag{4.25}$$

This representation enablkes us to see the geometric sense of physical quantities measurable in experiments, because we represent them through pure geometric characteristics of the observer's space — the angles between coordinate axes etc.

This completes the basics of Zelmanov's mathematical apparatus of chronometric invariants (physically observable quantities) that will be employed below with the aim of studying the gravitational wave problem.

5 Gravitational inertial waves and their link to the chronometrically invariant representation of Petrov's classification

Of all the experimental statements on the General Theory of Relativity, including the search for gravitational wave experiments, the most important case is that where the observer is at rest with respect to his laboratory reference frame and all physical standards located in it. Quantities measured by the observer in an *accompanying reference frame* are *chronometrically invariant quantities* (see the previous paragraph for the details). Keeping this fact in mind, Zelmanov formulated his *chronometrically invariant criterion for gravitational waves*. This criterion is invariant only for transformations of coordinates of that reference system which is at rest with respect to the laboratory references (the body of reference). Such an approach, in contrast to the invariant approach, permits us to interpret the results of measurement in terms of physically observable quantities, providing thereby a means of comparing results given by the theory of gravitational waves to results obtained from real physical experiments.

In order to solve the problem of interpretation of experimental data on gravitational waves it is appropriate to consider a more general case — fields of gravitational inertial waves. Such fields are more general because they are applicable to both gravitational fields and the inertial field of the observer's reference frame. The mathematical method that we propose to apply to this problem joins both fields into a common field. The method itself does not differ for each field: to set an invariant difference between gravitational fields and the observer's inertial field would be possible only by introducing an additional invariant criterion.

Gravitational waves are determined independently of both spatial coordinate frames and space-time reference frames. In contrast to gravitational waves, gravitational inertial waves are determined only in the reference frame of an observer, who observes them. They are determined with precision to within so-called "inner" transformations of coordinates

$$\left.\begin{aligned} &\text{(a)}\quad \tilde{x}^0 = \tilde{x}^0(x^0, x^1, x^2, x^3) \\ &\text{(b)}\quad \tilde{x}^i = \tilde{x}^i(x^1, x^2, x^3)\,, \qquad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \end{aligned}\right\} \tag{5.1}$$

which does not change the space-time reference frame itself.

Invariance with respect to (5.1) splits into invariance with respect to (5.1a), so-called *chronometric invariance*, and also invariance with respect to (5.1b), so-called *spatial invariance*. Therefore a definition given for gravitational inertial waves should be:

(1) chronometrically invariant;

(2) spatially covariant.

We then have a basis by which we introduce the chronometrically invariant spatially covariant d'Alembert operator [40]*

$$^*\Box = h^{ik}\,{}^*\nabla_i{}^*\nabla_k - \frac{1}{a^2}\frac{^*\partial^2}{\partial t^2}\,, \tag{5.2}$$

where $h^{ik} = -g^{ik}$ is the chr.inv.-metric tensor (the physically observable metric tensor) in its contravariant (upper-index) form, $^*\nabla_i$ is the symbol for the chr.inv.-derivative (the chr.inv.-analogue to the covariant derivative symbol ∇_σ), a is the linear velocity at which attraction of gravity spreads, $\frac{^*\partial}{\partial t}$ is the symbol for the chr.inv.-derivative with respect to time.

A chronometrically invariant criterion for gravitational inertial waves, formulated according to Zelmanov's idea, is:

Zelmanov's chr.inv.-criterion Chr.inv.-quantities f, characterising the observer's reference space, such as the gravitational inertial force vector F_i, the space non-holonomity (self-rotation) tensor A_{ik}, the space deformation rate tensor D_{ik}, the spatial curvature tensor C_{iklj}, and also scalar quantities, built on them, and also the Riemann-Christoffel curvature tensor's chr.inv.-components X^{ij}, Y^{ijk}, Z^{iklj} must satisfy equations of the form

$$^*\Box f = A, \tag{5.3}$$

*This approach to the gravitational inertial wave problem was developed by Zelmanov, although it had first been published by Zakharov because the latter prepared his dissertation under Zelmanov's leadership: see footnote on page 35.

where A is an arbitrary function of four-dimensional world-coordinates, which has no more than first order derivatives of the f.

The Zelmanov chr.inv.-criterion (5.3) was applied in analyzing well-known solutions to the Einstein equations in emptiness [40]. This criterion is true for the metrics (3.25) in that case where the gravitational inertial force vector F^i is the wave function. But, at the same time, most of the invariant criteria for gravitational waves are related to some conditions and limitations imposed on the curvature tensor. Therefore it would be most interesting to study relations between gravitational wave criteria and gravitational inertial wave criteria in that case where the Riemann-Christoffel curvature tensor's chr.inv.-components X^{ij}, Y^{ijk}, Z^{iklj} are the wave functions.

What is the relation between the Zelmanov invariant criterion (3.9) and his chr.inv.-criterion (5.3)? This problem was solved by Zakharov [40, 58]. His method was to express equation (3.9) in chr.inv.-form. In chr.inv..-form (in the terms of physically observable quantities) equation (3.9) takes the form

$$^*\Box X^{ij} = A^{ij}_{(1)}\,, \quad ^*\Box Y^{ijk} = A^{ijk}_{(2)}\,, \quad ^*\Box Z^{iklj} = A^{iklj}_{(3)}\,, \tag{5.4}$$

where $A^{ij}_{(1)}$, $A^{ijk}_{(2)}$, $A^{iklj}_{(3)}$ are chronometrically invariant and spatially invariant tensors, which have no more than first order derivatives of the wave functions X^{ij}, Y^{ijk}, Z^{iklj}. Thus those gravitational fields that satisfy the Zelmanov invariant criterion also satisfy the Zelmanov chr.inv.-criterion (5.3), where the Riemann-Christoffel curvature tensor's physically observable components X^{ij}, Y^{ijk}, Z^{iklj} play the part of wave functions.

The necessary condition for gravitational inertial waves is the fact that the chr.inv.-d'Alembert operator (5.2) is non-trivial, mathematically expressed as follows:

1. Chr.inv.-quantities f are non-stationary, i. e. $\frac{^*\partial f}{\partial t} \neq 0$;
2. The quantities f are inhomogeneous, i. e. $^*\nabla_i f_k \neq 0$.

The wave functions X_{ij} (4.16), Y_{ijk} (4.17) and Z_{iklj} (4.18) satisfy these requirements only if the mechanical chr.inv.-characteristics of the observer's reference space (the chr.inv.-quantities F_i, A_{ik}, D_{ik}) and the geometric chr.inv.-characteristic of the space (the chr.inv.-quantity C_{iklj}) also satisfy these requirements. Zelmanov himself in [42] formulated conditions of inhomogeneity inside a finite region located in the observer's space

$$\begin{gathered} ^*\nabla_i F_k \neq 0\,, \qquad ^*\nabla_j A_{ik} \neq 0\,, \\ ^*\nabla_j D_{ik} \neq 0\,, \quad ^*\nabla_j C_{ik} \neq 0\,. \end{gathered} \tag{5.5}$$

It is evident that under these conditions the wave functions X^{ij}, Y^{ijk}, Z^{iklj} shall be inhomogeneous.

The origin of non-stationary states of the gravitational inertial force vector F_i (4.1) is the non-stationarity of the gravitational potential w or the linear velocity of the space rotation v_i, consisting the force. Identities (4.6) and (4.7), linking quantities F_i and A_{ik}, lead us to conclude that the source of non-stationary states of v_i is the vortical nature of the vector F_i, i. e. $^*\nabla_k F_i - {}^*\nabla_i F_k \neq 0$. The origin of non-stationary states of the space deformation rate D_{ik} (4.3) and the space observable curvature C_{iklj} (4.14) is non-stationarity of the physical observable metric tensor h_{ik}, see [42],

$$h_{ik} = -g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}} = -g_{ik} + \frac{1}{c^2}\,v_i v_k\,. \tag{5.6}$$

Thus, the origin of non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} is the non-stationarity of components of the fundamental metric tensor $g_{\alpha\beta}$, namely:

(1) $g_{00} = \left(1 - \frac{\text{w}}{c^2}\right)^2$;

(2) $g_{0i} = -\frac{1}{c}\,v_i\left(1 - \frac{\text{w}}{c^2}\right)$;

(3) $g_{ik} = -h_{ik} + \frac{1}{c^2}\,v_i v_k$.

We consider each of the cases here,mindful of the need to find theoretical grounds for gravitational wave experiments:

1. Non-stationary states of g_{00} manifest as a result of time changes of the gravitational potential w. In experiments this non-stationarity is derived from very different geophysical sources, which, in a particular case, are due to changes in solar activity;
2. Non-stationary states of mixed components g_{0i} are derived from the non-stationarity of the space rotation linear velocity v_i and the gravitational potential w. The quantities g_{00} and g_{0i} are included in the formula for an interval of observable time $d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{\sqrt{g_{00}}}\,dx^i$ [42, 43]. Thus under non-stationary states of g_{00} and g_{0i} in the observer's laboratory (his reference frame) a standard clock located there should have some corrections (which change with time) with respect to a standard clock located in an region where the quantities g_{00} and g_{0i} are stationary.
3. Non-stationary states of g_{ik} are usually considered as deformations of the three-dimensional space. But the theory of physically observable quantities introduces substantial corrections to this thesis. The approach of Classical Mechanic looks at the spatial deformations as $\frac{1}{2}\frac{\partial g_{ik}}{\partial t}$, but the theory of physically observable quantities, taking properties of the observer into account, gives rise to a corrected formula for the spatial deformations which is $D_{ik} = \frac{1}{2\sqrt{g_{00}}}\frac{\partial}{\partial t}\left(-g_{ik} + \frac{1}{c^2}\,v_i v_k\right)$.*

*The presence of the minus sign here is a consequence of the fact that we use the signature $(+---)$, where plus is related to the time coordinate while minus is attributed to spatial coordinates. The minus sign has been chosen for the g_{ik} in the h_{ik} formula, because in this case the observable spatial interval $d\sigma = h_{ik}\,dx^i dx^k$ is positive, which is an important fact in the theory of physically observable quantities [42, 43].

The formulae coincide in that particular case where $g_{00}=1$ (w $=0$) and $g_{0i}=0$ ($v_i=0$). If $F_i=0$, according (to 4.6) the space rotation is stationary. If $v_i=0$, $A_{ik}=0$. Thus the necessary and sufficient condition to make w and v_i simultaneously zero is $F_i=0$ and $A_{ik}=0$ [42, 43]. In this case the observer's reference frame falls freely and is free of rotations. Such reference frames are known as *synchronous* [15], because there all clocks can be synchronized. Moreover, in this case time can be integrated: in calculations of the time interval $d\tau=dt$ between any two events, the integral of $d\tau$ is independent of the way we take this integral between the events (the path of integration). If $F_i\neq 0$ but $A_{ik}=0$, it is impossible to synchronize all the clocks simultaneously, but the synchronization itself can be realized because of the proportionality $d\tau=\sqrt{g_{00}}\,dt$ there. If $A_{ik}\neq 0$, the synchronization is impossible in principle, because the integral of $d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{\sqrt{g_{00}}}\,dx^i$ depends on the path of integration [42, 43].

Synchronous reference frames, because of their simplicity and associated simple calculations, are of broad utility in the General Theory of Relativity. In particular, they are used in relativistic cosmology and the gravitational wave problem. For instance, the well-known metric of weak plane gravitational waves takes the form [14, 15]

$$ds^2 = c^2dt^2 - (dx^1)^2 - (1-a)(dx^2)^2 + \\ + 2b\,dx^2dx^3 - (1+a)(dx^3)^2, \tag{5.7}$$

where $a=a(ct\pm x^1)$, $b=b(ct\pm x^1)$. So in this metric there is no gravitational potential (w $=0$) as soon as there is no space rotation ($v_i=0$). The condition w $=0$ prohibits the ultimate transit to Newton's theory of gravity. For this reason we arrive at an important conclusion:

> Weak plane gravitational waves are derived from sources other than gravitational fields of masses*.

An analogous situation arises in relativistic cosmology, where, until now, the main part is played by the theory of a homogeneous isotropic universe. Foundations of this theory are built on the metric of a homogeneous isotropic space [42]

$$\begin{gathered} ds^2 = c^2dt^2 - \\ - R^2\frac{(dx^1)^2+(dx^2)^2+(dx^3)^2}{\left[1+\frac{k}{4}\left[(dx^1)^2+(dx^2)^2+(dx^3)^2\right]\right]^2}, \\ R=R(t)\,, \qquad k=0,\pm 1\,. \end{gathered} \tag{5.8}$$

When one substitutes this metric into the Einstein equations taken with a specific value of the cosmological constant ($\lambda=0$, $\lambda<0$, $\lambda>0$), he obtains a spectra of solutions, which are known as *Friedmann's cosmological models* [42].

Taking our previous conclusion on the origin of weak plane gravitational waves into account, we come to a new and important conclusion:

> No gravitational fields derived from masses exist in any Friedmann universe. Moreover, any Friedmann universe is free of space rotations.

Currently there is no indubitable observational data supporting the absolute rotation of the Universe. This problem has been under considerable discussion between astronomers and physicists over last decade, and remains open. Rotations of bulk space bodies like planets, stars, and galaxies are beyond any doubt, but these rotations do not imply the absolute rotation of the whole Universe, including the absolute rotation of its gravitational field if one will describe it by the Friedmann models.

Looking back at the question of whether or not gravitational inertial waves exist, or whether or not non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} exist, we conclude that non-stationary states of the quantities are derived from:

1. A vortical nature of the field of the acting gravitational inertial force F_i;
2. Non-stationary states of the spatial components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$.

In the first case, the effect of gravitational inertial waves manifests as non-stationary corrections to the observer's time flow.

In the second case, the observer's time flow remains unchanged, but gravitational waves are waves of only the space deformation. Such pure deformation waves will deform a detector itself, so one simply waits for a gravitational wave to cause a resonance effect in a solid-body detector of the Weber kind [16]. Whether this conclusion is true or false will be considered in §7 and §8. Here we consider only the general theory of gravitational inertial waves and its relation to the invariant theory of gravitational waves.

As we showed above, those gravitational fields that satisfy the Zelmanov invariant criterion (3.9) also satisfy the Zelmanov chr.inv.-criterion (5.3), where the wave functions f are the Riemann-Christoffel tensor's observable components X^{ij}, Y^{ijk}, Z^{iklj}. As it was shown in the previous paragraph, "empty gravitational fields" (we mean gravitational fields permeating empty spaces, where no mass islands of matter exist) that satisfy the Zelmanov invariant criterion (3.9) are related to the 2nd kind (the sub-kind N) by Petrov's classification. Therefore it is appropriate to specify the algebraical kinds of the Riemann-Christoffel tensor in terms of physically observable quantities (chronometric invariants).

The whole problem of representing Petrov's classification in chronometrically invariant form has been solved in [66]. This solution, obtained Petrov in general covariant form [37],

*See §7 and §8 below for detailed calculations for the effect due to weak plane gravitational waves in solid-body detectors of the Weber kind (the Weber pigs) and also in antennae built on free masses.

was obtained for an ortho-frame, taken at an arbitrary fixed point of the space.

Chr.inv.-components of the Riemann-Christoffel curvature tensor have the properties

$$X_{ij} = X_{ji}\,, \quad X_k^k = -\kappa\,, \qquad Y_{[ijk]} = 0\,, \quad Y_{ijk} = -Y_{ikj}\,. \tag{5.9}$$

Equations (4.16), (4.17), (4.18) in an ortho-frame are

$$X_{ij} = -c^2 R_{0i0j}\,, \qquad Y_{ijk} = -cR_{0ijk}\,, \qquad Z_{iklj} = c^2 R_{iklj}\,. \tag{5.10}$$

When we write equations $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ in the orth-frame, we take the relationships (5.10) into account. Then, introducing three-dimensional matrices x and y such that

$$x \equiv \|x_{ik}\| = -\frac{1}{c^2}\|X_{ik}\|\,, \qquad y \equiv \|y_{ik}\| = -\frac{1}{2c}\|\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}\|\,, \tag{5.11}$$

where ε_{imn} is the three-dimensional discriminant tensor, we represent the six-dimensional matrix R_{ab} as follows

$$\|R_{ab}\| = \left\|\begin{matrix} x & y \\ y & -x \end{matrix}\right\|, \qquad a,b = 1,2,\dots 6\,, \tag{5.12}$$

satisfying the relations

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \qquad y_{11} + y_{22} + y_{33} = 0\,. \tag{5.13}$$

Now, let us compose a lambda-matrix

$$\|R_{ab} - \Lambda g_{ab}\| = \left\|\begin{matrix} x + \Lambda\varepsilon & y \\ y & -x - \Lambda\varepsilon \end{matrix}\right\|, \tag{5.14}$$

where ε is the three-dimensional unit matrix. Then, after transformations, we reduce this lambda-matrix to the form

$$\left\|\begin{matrix} x + iy + \Lambda\varepsilon & 0 \\ 0 & -x - iy - \Lambda\varepsilon \end{matrix}\right\| = \left\|\begin{matrix} \bar{Q}(\Lambda) & 0 \\ 0 & \bar{Q}(\Lambda) \end{matrix}\right\|. \tag{5.15}$$

The initial lambda-matrix can have one of the following characteristics:

$$(1)\ [111, \overline{111}]; \quad (2)\ [21, \overline{21}]; \quad (3)\ [3, 3]. \tag{5.16}$$

Then, using Petrov's had obtained the canonical form of the matrix $\|R_{ab}\|$ in the non-holonomic ortho-frame for each of the three kinds of the curvature tensor [37], we express the matrix $\|R_{ab}\|$ through components of the chr.inv.-tensors X_{ij} and Y_{ijk} [66]. We obtain

<u>The 1st Kind</u>

$$\|R_{ab}\| = \left\|\begin{matrix} x & y \\ y & -x \end{matrix}\right\|,$$

$$x = \left\|\begin{matrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{matrix}\right\|, \qquad y = \left\|\begin{matrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{matrix}\right\|, \tag{5.17}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \quad y_{11} + y_{22} + y_{33} = 0\,. \tag{5.18}$$

Using (5.11) we also express values of the stationary curvatures Λ_i $(i = 1,2,3)$ through the Riemann-Christoffel tensor's physically observable components

$$\Lambda_1 = -\frac{1}{c^2}X_{11} + \frac{i}{c}Y_{123}\,, \qquad \Lambda_2 = -\frac{1}{c^2}X_{22} + \frac{i}{c}Y_{231}\,, \qquad \Lambda_3 = -\frac{1}{c^2}X_{33} + \frac{i}{c}Y_{312}\,. \tag{5.19}$$

Thus, the components X_{ik} are included in the real parts of the stationary curvatures Λ_i, and components Y_{ijk} are included in the imaginary parts. In spaces of the sub-kind D ($\Lambda_2 = \Lambda_3$) we have: $X_{22} = X_{33}$, $Y_{231} = Y_{312}$. In spaces of the sub-kind O ($\Lambda_1 = \Lambda_2 = \Lambda_3$) we have: $X_{11} = X_{22} = X_{33} = -\frac{\kappa}{3}$, $Y_{123} = Y_{231} = Y_{312}$. Hence Einstein spaces of the sub-kind O have only real curvatures, while being empty they are flat.

For the 2nd kind we have

<u>The 2nd Kind</u>

$$\|R_{ab}\| = \left\|\begin{matrix} x & y \\ y & -x \end{matrix}\right\|,$$

$$x = \left\|\begin{matrix} x_{11} & 0 & 0 \\ 0 & x_{22}+1 & 0 \\ 0 & 0 & x_{33}-1 \end{matrix}\right\|, \qquad y = \left\|\begin{matrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 1 \\ 0 & 1 & y_{22} \end{matrix}\right\|, \tag{5.20}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \qquad x_{22} - x_{33} = 2\,, \qquad y_{11} + 2y_{22} = 0\,. \tag{5.21}$$

The stationary curvatures are

$$\Lambda_1 = -\frac{1}{c^2}X_{11} + \frac{i}{c}Y_{123}\,, \qquad \Lambda_2 = -\frac{1}{c^2}X_{22} - 1 + \frac{i}{c}Y_{231}\,, \qquad \Lambda_3 = -\frac{1}{c^2}X_{33} + 1 + \frac{i}{c}Y_{312}\,. \tag{5.22}$$

From this we conclude that values of the stationary curvatures Λ_2 and Λ_3 can never become zero, so Einstein spaces (gravitational fields) of the 2nd kind are curved in any case — they cannot approach Minkowski flat space.

In spaces of the sub-kind N ($\Lambda_1 = \Lambda_2$) in an ortho-frame the relations are true

$$X_{11} = X_{22} - c^2 = X_{33} + c^2, \qquad Y_{123} = Y_{231} = Y_{312} = 0\,, \tag{5.23}$$

so the stationary curvatures are real. In an empty space the matrices x and y become degenerate (its determinant becomes zero). For this reason spaces of the sub-kind N are *degenerate*, and, respectively, gravitational fields in spaces of the sub-kind N are known as *gravitational fields of the 2nd degenerate kind by Petrov's classification*. In emptiness ($\kappa = 0$) some elements of the matrices x and y take the numerical values $+1$ and -1 thereby making an ultimate transition to the Minkowski flat space impossible.

For the 3rd kind we have

The 3rd Kind

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|, \quad x = \left\| \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|, \quad y = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{matrix} \right\|. \tag{5.24}$$

Here the stationary curvatures are zero and both of the matrices x and y are degenerate. Einstein spaces of the 3rd kind can only be empty ($\kappa = 0$), but, at the same time, they can never be flat.

From the equations deduced for the canonical form of the matrix $\|R_{ab}\|$, we conclude: $Y_{ijk} = 0$ can be true only in gravitational fields of the 1st kind, which are derived from island masses of matter in emptiness or vacuum. Therefore we conclude that those gravitational fields where $Y_{ijk} = 0$ is true in the observer's accompanying reference frame can only be of the 1st kind, having stationary curvatures which are real.

Furthermore, in accordance with most of the criteria, the presence of gravitational waves is linked to spaces of the 2nd (N) kind and the 3rd kind, where the matrix y_{ik} has components equal to $+1$ or -1. Moreover, in fields of the 2nd (N) and 3rd kinds the values $+1$ or -1 are attributed also to components of the matrix x. This implies that:

> Those spaces which contain gravitational fields, satisfying the invariant criteria for gravitational waves, are curved independently of whether or not they are empty ($T_{\alpha\beta} = 0$) or filled with matter (in such spaces $T_{\alpha\beta} = g_{\alpha\beta}$). In any case, gravitational radiations are derived from interaction between two observable components X_{ij}, Y_{ijk} of the Rimeann-Christoffel curvature tensor.

The classification of gravitational fields built here applies only to Einstein spaces, because solving this problem for spaces of general kind, where $T_{\alpha\beta} \neq \kappa g_{\alpha\beta}$, would be very difficult, for mathematical reasons. Considering the details of these difficulties, we see that, having an arbitrary distribution of matter in a space, the matrix $\|R_{ab}\|$, taken in a non-holonomic ortho-frame, is not symmetrically doubled; on the contrary, the matrix takes the form

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y' & z \end{matrix} \right\|, \tag{5.25}$$

where the three-dimensional matrices x, y, z are built on the following elements, respectively*

$$x_{ik} = -\frac{1}{c^2}\, X_{ik}\,, \qquad z_{ik} = \frac{1}{c^2}\, \varepsilon_{imn}\varepsilon_{kpq} Z^{mnpq}, \qquad y_{ik} = \frac{1}{2c}\, \varepsilon_{imn} Y_{k\cdot\cdot}^{\cdot mn}\,, \tag{5.26}$$

and y' implies transposition. It is evident that reduction of this matrix to its canonical form is a very difficult problem.

Nevertheless Petrov's classification permits us to conclude:

> The physically observable components X^{ij} and Y^{ijk} of the Riemann-Christoffel curvature tensor are different in their physical origin†. Metrics can exist where $Y^{ijk} = 0$ but $X^{ij} \neq 0$ and $Z^{iklj} \neq 0$. Such spaces are of the 1st kind by Petrov's classification; they have real stationary curvatures. Such spaces do not satisfy the invariant criteria for gravitational waves. Thus no wave fields of gravity exist in spaces where $Y^{ijk} = 0$ but $X^{ij} \neq 0$ and $Z^{iklj} \neq 0$.

And further:

> In solutions of the Einstein equations there are no metrics where $Y^{ijk} \neq 0$ but $X^{ij} = 0$ and $Z^{iklj} = 0$. Thus in wave fields of gravity $Y^{ijk} \neq 0$ and $X^{ij} \neq 0$ (and as well $Z^{iklj} \neq 0$: see the footnote) everywhere and always.

*In ortho-frames there is no difference between upper and lower indices (see [37]). For this reason we can write $z_{ik} = \frac{1}{c^2}\, \varepsilon_{imn}\varepsilon_{kpq} Z_{mnpq}$ and $y_{ik} = \frac{1}{2c}\, \varepsilon_{imn} Y_{kmn}$ instead of $z_{ik} = \frac{1}{c^2}\, \varepsilon_{imn}\varepsilon_{kpq} Z^{mnpq}$ and $y_{ik} = \frac{1}{2c}\, \varepsilon_{imn} Y_{k\cdot\cdot}^{\cdot mn}$ in formula (3.26). This note relates to all formulae written in an ortho-frame. We met a similar case in formula (5.11), where we can also write $y \equiv \|y_{ik}\| = -\frac{1}{2c}\, \|\varepsilon_{imn} Y_{kmn}\|$ instead of $y \equiv \|y_{ik}\| = -\frac{1}{2c}\, \left\|\varepsilon_{imn} Y_{k\cdot\cdot}^{\cdot mn}\right\|$.

†We do not mention the third observable component Z^{iklj}, because in an ortho-frame the matrices x and z are connected by the equation $x = -z$.

We will show that in Einstein spaces filled with gravitational fields where the Riemann-Christoffel tensor's observable components X^{ij}, Y^{ijk}, Z^{iklj} play a part of the wave functions, the quantity X^{ij} is analogous to the electric component of an electromagnetic field, while Y^{ijk} is analogous to its magnetic component. All this will be discussed in §7.

6 Wave properties of Einstein's equations

In §2 we have showed that the gravitational field equations (the Einstein equations) do not contain a general covariant d'Alembert operator derived from the fundamental metric tensor $g_{\alpha\beta}$ (where $g_{\alpha\beta}$ is considered as a "four-dimensional gravitational potential"). Nevertheless this problem has been solved in linear approximation in the case where gravitational fields are occupy an empty space ($R_{\alpha\beta}=0$, "empty gravitational fields") [14, 15]. In this case a gravitational field is considered as a tiny addition to a flat space background described by the Minkowski metric. Thus

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \gamma_{\alpha\beta}\,, \tag{6.1}$$

where $\delta_{\alpha\beta}$ are components of the fundamental metric tensor in a Galilean reference frame $\delta_{\alpha\beta}=\{+1,-1,-1,-1\}$, and $\gamma_{\alpha\beta}$ describes weak corrections for the gravitational fields. The contravariant fundamental metric tensor $g^{\alpha\beta}$ to within the first order approximation of the $\gamma_{\alpha\beta}$ is

$$g^{\alpha\beta} = \delta^{\alpha\beta} - \gamma^{\alpha\beta}, \tag{6.2}$$

so the determinant of the tensor $g_{\alpha\beta}$ is

$$g = -\left(1+\gamma\right), \qquad \gamma = \det\|\gamma_{\alpha\beta}\|\,. \tag{6.3}$$

The requirement that components of the "additional" metric $\gamma_{\alpha\beta}$ must be infinitesimal fixes a prime reference frame. If this requirement is true in a reference frame, it will also be true after transformations

$$\tilde{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}, \tag{6.4}$$

where ξ^{α} are infinitesimal quantities $\xi^{\alpha}\ll 1$. Then we have

$$\tilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{\partial\xi_{\alpha}}{\partial x^{\beta}} - \frac{\partial\xi_{\beta}}{\partial x^{\alpha}}\,. \tag{6.5}$$

Because of (6.1), we impose an additional requirement on the tensor $\gamma_{\alpha\beta}$; this requirement is [15]

$$\frac{\partial\psi^{\alpha}}{\partial x^{\beta}} = 0\,, \qquad \psi^{\alpha}_{\beta} = \gamma^{\alpha}_{\beta} - \frac{1}{2}\,\delta^{\alpha}_{\beta}\,\gamma\,. \tag{6.6}$$

Taking (6.6) into account, the Ricci tensor takes the form

$$R_{\alpha\beta} = \frac{1}{2}\,\Box\gamma_{\alpha\beta}\,, \tag{6.7}$$

where

$$\Box \equiv g^{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,, \qquad \Delta = \frac{\partial^2}{\partial x^{1\,2}} + \frac{\partial^2}{\partial x^{2\,2}} + \frac{\partial^2}{\partial x^{3\,2}}\,. \tag{6.8}$$

Here $\Box$ is the d'Alembert operator, Δ is the Laplace operator. The calibrating requirements (6.6) are true in any metric $\gamma_{\alpha\beta}$ only if the quantities ξ^{α} are solutions of the equation

$$\Box\xi^{\alpha} = 0\,. \tag{6.9}$$

In [15] the requirement

$$\Box\gamma_{\alpha\beta} = 0 \tag{6.10}$$

was imposed on the quantities $\gamma_{\alpha\beta}$, which is interpreted as the *equation of weak gravitational waves in emptiness* — this formula (6.10) is a standard wave equation that describes a wave of the tensor field $\gamma_{\alpha\beta}$, traveling at the velocity c in emptiness.

One usually considers the equation (6.10) as the basis for the claim that the General Theory of Relativity predicts gravitational waves, which travel at the speed of light.

If we have a weak plane gravitational wave, so the field has changes along a single spatial direction (the x^1 axis, for instance), the formula (6.10) takes the form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^{1\,2}}\right)\gamma_{\alpha\beta} = 0\,, \tag{6.11}$$

and solutions of it can be any function of $ct\pm x^1$. After numerous transformations of the function $\gamma_{\alpha\beta}$ [14, 15] it obtains that in the field of a weak plane gravitational wave only the following components are non-zero: $\gamma_{22}=-\gamma_{33}\equiv a$, $\gamma_{23}\equiv b$. Thus, those weak plane gravitational waves that satisfy the Einstein equations in emptiness are transverse.

Thus if some additional requirements are imposed upon the Einstein equations in emptiness, the equations describe weak plane waves of the space deformation, the space metric of which is [15]

$$ds^2 = c^2dt^2 - (dx^1)^2 - (1+a)(dx^2)^2 + + 2b\,dx^2dx^3 - (1-a)(dx^3)^2, \tag{6.12}$$

where a and b are functions of $ct\pm x^1$. The field of gravitation, described by the metric (6.12), is of the sub-kind N by Petrov's classification, so it satisfies most of invariant criteria for gravitational waves.

The metric (6.12) has been written in a synchronous reference frame, so its space deforms, falls freely, and, at the same time, has no rotations. Hence, *under the given assumptions, weak plane gravitational waves are waves of "pure" deformation of the space*. This conclusion is the main reason why experimental physicists, and Weber in particular [16], expect that gravitational waves will cause a "pure" deformation effect in detectors.

Calculations for the interaction between a Weber solid-body detector and a weak plane gravitational wave field will be given in §7. Here we continue our argument for the wave nature of the Einstein equations in *strong gravitational fields* in the case where matter is arbitrarily distributed in the space. This research will be given in the terms of physically observable quantities for the reason that we will consider situations derived from different factors, generating gravitational wave fields, not only the space deformation.

The Einstein equations in the case where matter is arbitrarily distributed are [42]

$$\frac{{}^*\partial D}{\partial t} + D_{jl}D^{jl} + A_{jl}A^{lj} + \left({}^*\nabla_j - \frac{1}{c^2}F_j\right)F^j = -\frac{\kappa}{2}\left(\rho c^2 + U\right) + \lambda c^2, \tag{6.13}$$

$${}^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}F_j A^{ij} = \kappa J^i, \tag{6.14}$$

$$\begin{aligned}&\frac{{}^*\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D^j_k + A^{\cdot j}_{k\cdot}\right) + D D_{ik} + \\ &+ 3A_{ij}A^{\cdot j}_{k\cdot} + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}F_iF_k - \\ &- c^2 C_{ik} = \frac{\kappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right) + \lambda c^2 h_{ik}.\end{aligned} \tag{6.15}$$

Here ${}^*\nabla_j$ denotes the chr.inv.-derivative, while the quantities $\rho = \frac{T_{00}}{g_{00}}$, $J^i = \frac{cT^i_0}{\sqrt{g_{00}}}$, $U^{ik} = c^2 T^{ik}$ (from which we have $U = h^{ik}U_{ik}$) are the chr.inv.-components of the energy-momentum tensor $T_{\alpha\beta}$ of matter: the physically observable density ρ, the physically observable impulse density vector J^i, and the physically observable stress-tensor U^{ik}.

Zelmanov had deduced [42] that the chr.inv.-spatial curvature tensor C_{iklj} is linked to a chr.inv.-tensor H_{iklj}, which is like Schouten's tensor [67], by the equation

$$\begin{aligned}H_{lkij} = C_{lkij} + \frac{1}{c^2}\big(&2A_{kj}D_{jl} + A_{ij}A_{kl} + \\ &+ A_{jk}D_{il} + A_{kl}D_{ij} + A_{li}D_{jk}\big)\end{aligned} \tag{6.16}$$

and contracted tensors $H_{lk} = H_{lik\cdot}^{\cdots i}$ and $C_{lk} = C_{lik\cdot}^{\cdots i}$ are related as follows

$$H_{lk} = C_{lk} + \frac{1}{c^2}\left(A_{kj}D^j_l + A_{lj}D^j_k + A_{kl}D\right). \tag{6.17}$$

Taking the definition $D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}$ into account, and C_{lk} from (6.17), we reduce (6.15) to the form

$$\begin{aligned}&\frac{1}{2}\frac{{}^*\partial^2 h_{ik}}{\partial t^2} - D_{ij}D^j_k + D\left(D_{ik} - A_{ik}\right) + 2A_{ij}A^{\cdot j}_{k\cdot} + \\ &+ \frac{1}{2}\left({}^*\nabla_i F_k - {}^*\nabla_k F_i\right) - \frac{1}{c^2}F_iF_k - c^2 H_{ik} = \\ &= \kappa U_{ik} + \lambda c^2 h_{ik}.\end{aligned} \tag{6.18}$$

The quantity H_{ik}, by definition, is

$$H_{ik} = H_{ijk\cdot}^{\cdots j} = \frac{{}^*\partial \Delta^j_{ij}}{\partial x^k} - \frac{{}^*\partial \Delta^j_{ik}}{\partial x^j} + \Delta^m_{ij}\Delta^j_{km} - \Delta^m_{ik}\Delta^j_{jm}, \tag{6.19}$$

where $\Delta^m_{jm} = \frac{{}^*\partial \ln\sqrt{h}}{\partial x^j}$.

Taking into account (6.17), (6.19), and also Zelmanov's identities (4.6), (4.7) that link F_i and A_{ik}, we reduce (6.18) to the form

$$\begin{aligned}{}^*\Box h_{ik} &= 2\,\frac{{}^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^k} - \frac{2}{c^2}\left({}^*\nabla_i F_k + \frac{{}^*\partial A_{ik}}{\partial t}\right) - \\ &- \frac{4}{c^2}\left(A_{ij}A^{\cdot j}_{k\cdot} - D_{ij}D^j_k\right) - \frac{2D}{c^2}\left(D_{ik} + A_{ik}\right) + \\ &+ 2\left(h^{pq}\Delta^m_{pq}\Delta_{ik,m} + \Delta^m_{ij}\Delta^j_{km}\right) - \\ &- h^{pm}\frac{{}^*\partial}{\partial x^p}\left(\frac{{}^*\partial h_{im}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^i}\right) + \\ &+ \kappa\left(\rho h_{ik} + \frac{2}{c^2}U_{ik} - \frac{U}{c^2}h_{ik}\right) + 2\lambda h_{ik},\end{aligned} \tag{6.20}$$

where ${}^*\Box$ is the chr.inv.-d'Alembert operator, applied here to the chr.inv.-metric tensor h_{ik} (the observable metric tensor of the observer's three-dimensional space)*.

If we equate the right part of (6.20) in zero, the whole equation becomes a wave equation with respect to h_{ik}, namely

$${}^*\Box h_{ik} = \frac{1}{c^2}\frac{{}^*\partial^2 h_{ik}}{\partial t^2} - h^{jm}\frac{{}^*\partial^2 h_{ik}}{\partial x^j \partial x^m}. \tag{6.21}$$

In this case the spatial components of the Einstein equations describe gravitational inertial waves of the spatial metric h_{ik}, which travel at the velocity $u = c\left(1 - \frac{\mathrm{w}}{c^2}\right)$ which depends on the value of the gravitational potential w. This coincides with the results recently obtained by Rabounski [48]. If $\mathrm{w} = 0$, the waves travel at the velocity of light. The greater is w the smaller is u. The wave's velocity u becomes zero in the extreme case where $\mathrm{w} = c^2$ which occurs under collapse, hence under collapse gravitational waves stop — they become *standing gravitational waves*.

It is evident from the mathematical viewpoint, that reducing the right side of (6.20) to zero is a very difficult task, because the whole equation is a system of 6 nonlinear equations of the 2nd order, in which numerous variables are linked by relationships (6.13) and (6.14). Systems such as this cannot be solved analytically in general, but we can obtain solutions for various specific metrics.

Because experimental physicists. in their search for gravitational waves, propound experimental statements for detecting weak wave fields of gravitation, we are going to study a linearized form of the equation (6.20).

*Components of the chr.inv.-metric tensor h_{ik} satisfy the requirements ${}^*\nabla_j h_{ik} = {}^*\nabla_j h^k_i = {}^*\nabla_j h^{ik} = 0$. For this reason we can apply the chr.inv.-d'Alembert operator ${}^*\Box = \frac{1}{c^2}\frac{{}^*\partial^2}{\partial t^2} - h^{ik}\,{}^*\nabla^*_i\nabla_k$ to it.

For (6.20) in emptiness, the linear approximation is*

$$^*\Box h_{ik} = 2\,\frac{^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^k} - \frac{2}{c^2}\left(^*\nabla_i F_k + \frac{^*\partial A_{ik}}{\partial t}\right). \qquad (6.22)$$

As a matter of fact, equation (6.22) describes weak plane gravitational inertial waves without sources, if the wave field satisfies the obvious chr.inv.-condition

$$\frac{^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^k} = \frac{1}{c^2}\left(^*\nabla_i F_k + \frac{^*\partial A_{ik}}{\partial t}\right). \qquad (6.23)$$

In other words, the field of the observable metric tensor h_{ik} is a wave field if there are some relations between the inhomogeneity of the gravitational inertial force field, the non-stationary rotation of the space, and the volume transformations of the space element, taken in the field†. The condition (6.23) is true for the well-known metric of weak plane gravitational waves (6.12), because in the metric (6.12) we have $F_i = 0$, $A_{ik} = 0$, $\sqrt{h} = \sqrt{1-a^2-b^2} \approx 1$. Thus:

> Weak plane gravitational waves in emptiness are also weak plane gravitational inertial waves of the spatial observable metric h_{ik}.

As shown in [41], the metric (6.12) satisfies the Zelmanov chr.inv.-criterion for gravitational waves, where the wave functions are the Riemann-Christoffel tensor's physically observable components X^{ij}, Y^{ijk}, Z^{iklj}. Hence weak plane gravitational inertial waves (waves of the space curvature) can exist in emptiness, because of the Einstein equations. We have shown above that such wave gravitational fields can also exist in spaces of the sub-kind N by Petrov's classification (such spaces are curved themselves, and matter contributes only an additional component to the initial curvature). Hence such fields satisfy most of the known invariant criteria for gravitational waves.

As we showed above, on page 46, that fields of gravitational radiations cannot exist in spaces of the 1st kind by Petrov's classification. In spaces of the 1st kind $Y^{ijk} = 0$. Therefore it would be logical to express the Einstein equations in the physically observable components X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor, aiming to find relations between the ch.inv.-quantities X^{ij}, Y^{ijk}, Z^{iklj} and the physically observable components of the energy-momentum tensor $T_{\alpha\beta}$ of distributed matter (ρ, J^i, U_{ik}, see page 48).

In chr.inv.-components the Einstein equations become

$$\left.\begin{aligned} & Z_{mk\cdot\cdot}^{\cdot\cdot mk} = \kappa\left(\rho c^2 + U\right) - 2\lambda c^2, \\ & Y_{\cdot\cdot m}^{im\cdot} = \kappa J^i, \\ & X_{ik} - X h_{ik} + Z_{\cdot imk}^{m\cdot\cdot\cdot} = \\ & = \frac{\kappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right) + \lambda c^2 h_{ik}\,, \end{aligned}\right\} \qquad (6.24)$$

if matter is distributed arbitrarily. Here $X = h^{ik} X_{ik}$ is the trace (spur) of the tensor X_{ik}.

From here we see that the physical observable components of the Riemann-Christoffel tensor have different physical origins:

1. Quantities X^{ij} (and as well Z^{iklj}) are linked to the mass density ρ and the stress-tensor U_{ik};
2. Quantities Y^{ijk} are linked to the impulse density J^i of matter.

As we showed above, on page 46, in all the widely known metrics which satisfy both the invariant criteria and the chr.inv.-criterion for gravitational waves, we have $Y^{ijk} \neq 0$, although X^{ij} (and as well Z^{iklj}) can be zero. This fact leads us to a very important conclusion:

> Gravitational waves and gravitational inertial waves are mainly waves of the field of the Y^{ijk} physically observable component of the Riemann-Christoffel curvature tensor‡.

But this conclusion does not mean that only waves of the field Y^{ijk} can be discovered. As we will see in §7, relative accelerations of test-particles are derived from wave fields of all three observable components X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel tensor. Our conclusion means:

> If in a space, filled with a gravitational field, $Y^{ijk} = 0$ is true, the structure of the space itself prohibits the gravitational field from being a wave.

Contracting (6.26) and taking (6.24) into account, we obtain

$$X = \frac{\kappa}{2}\left(U - \rho c^2\right) - 2\lambda c^2. \qquad (6.25)$$

In an empty space where there are no λ-fields, the trace of X^{ij} and the contracted quantity $Z_{mk\cdot\cdot}^{\cdot\cdot mk}$ are zero, as well as the contracted quantity $Y_{\cdot\cdot m}^{im\cdot}$. Thus the chr.inv.-Einstein equations (6.24) in emptiness take the form§

$$\left.\begin{aligned} & Z_{mk\cdot\cdot}^{\cdot\cdot mk} = 0\,, \qquad X = 0\,, \\ & Y_{\cdot\cdot m}^{im\cdot} = 0\,, \\ & X_{ik} + Z_{\cdot imk}^{m\cdot\cdot\cdot} = 0\,, \end{aligned}\right\} \qquad (6.26)$$

so, while the quantities X^{ik} and Z^{iklj} are connected to one another, the quantity Y^{ijk} (which, being non-zero, $Y^{ijk} \neq 0$, permits gravitational fields to be a wave) is the independent observable component of the Riemann-Christoffel tensor.

*In obtaining this formula, in the initial equation (6.20), we neglect products of the chr.inv.-quantities and of their derivatives.

†The integral of $\sqrt{h}\,dx^1 dx^2 dx^3$ is the volume of an element of the space. Here the differentials dx^i themselves and an interval, where values of the x^i change where we take the integral, do not depend on x^0 [42].

‡Quadrupole mass-detectors, in particular, solid-body detectors (the Weber pigs) can only register waves of the X^{ij} component, not waves of Y^{ijk} if its particles are at rest in the initial moment of time (see §7 and §8 for details). Thus, the Weber experimental statement is false at its base.

§As a matter of fact, equality to zero of inflected forms of a tensor does not imply that the tensor quantity itself is zero. Thus, equalities $X = 0$, $Y_{\cdot\cdot m}^{im\cdot} = 0$, $Z_{mk\cdot\cdot}^{\cdot\cdot mk} = 0$ do not imply that the quantities X^{ik}, Y^{ijk}, Z^{iklj} themselves are zero. Therefore the chr.inv.-Einstein equations in emptiness (6.24) permit gravitational waves if, of course, $Y^{ijk} \neq 0$.

7 Expressing Synge-Weber equation (the world-lines deviation equation) in the terms of physical observable quantities, and its exact solutions

In the previous paragraphs we focused our attention on general criteria, which differentiate gravitational wave fields from other gravitational fields in the General Theory of Relativity. As a result, we have found the main properties of gravitational wave fields.

We are now going to introduce a substantial criticism of the contemporary theoretical foundations of current attempts to detect gravitational waves by solid-body detectors of the resonance kind (the Weber pigs) and quadrupole mass detectors in general.

As we showed in the previous paragraphs, only gravitational fields located in spaces where the Riemann-Christoffel curvature tensor has a specific structure, permit the presence of gravitational waves. Therefore it would be reasonable to design experiments by which a physical detector could register wave changes of the four-dimensional (space-time) curvature* — the waves of the Riemann-Christoffel curvature tensor field.

Such a physical detector could be a system of two test-particles: their relative world-trajectories will necessarily undergo changes through the action of a wave of the space curvature. These systems are described by the world-lines deviation equation — the Synge equation of geodesic deviation (2.8) if these are two free particles, and the Synge-Weber equation (2.12) if the particles are connected by a force of non-gravitational nature.

We propose gravitational wave detectors of two possible kinds. The system of two free particles is known as a *detector built on free masses*. In practice such a detector consists of two freely suspended massive bodies, separated by a suitable distance. The system of two particles connected by a spring is known as a *quadrupole mass-detector*— this is a detector of the resonance kind, a typical instance of which is the Weber cylindrical pig.

To understand how a graviational wave would affect the different types of detectors we need to make specific calculations for their behaviour in gravitational wave fields. But before making the calculations, it is required to describe the behaviour of two test-particles in regular gravitational fields (of non-wave nature) in the terms of physically observable quantities (chronometric invariants). This analysis will show how different kinds of gravitational inertial waves cause relative deviation (both spatial and time displacements) of two test-particles.

We will solve this problem first for a system two free particles as described by the Synge equation (2.8) where the right side is zero. The problem for spring-connected particles, described by the Synge-Weber equation (2.12), will be solved in the same way except that there will be a non-gravitational force acting, so that the right side of the equation will be non-zero.

Relative accelerations of free test-particles $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ as a whole and the quantity $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ are derived from components of the Riemann-Christoffel world-tensor, contracted with components of the particles' four-dimensional velocity vector U^β and their relative deviation vector η^γ, namely — from the quantity $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma$. To determine what effect is introduced by each observable component of the Riemann-Christoffel tensor into the spatial and time relative displacements, described by the relative displacement world-vector η^α, we consider the geodesic deviation equation (2.8), keeping the term $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ as a whole and the quantity $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ without expressing it in terms of the Christoffel symbols and their derivatives.

As well as any general covariant equation, the geodesic deviation equation (2.8) can be projected onto the observer's time line and spatial section (his three-dimensional space) as given in [42, 43] or on page 40 herein. Denoting

$$M^\alpha \equiv \frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = 0\,, \tag{7.1}$$

let us find equations which are its projection on the time line

$$\frac{M_0}{\sqrt{g_{00}}} = \frac{g_{0\alpha}}{\sqrt{g_{00}}}\,M^\alpha = \sqrt{g_{00}}\,M^0 - \frac{1}{c}\,v_i M^i = 0\,, \tag{7.2}$$

and its projection on the spatial section

$$M^i = 0\,. \tag{7.3}$$

To find the equations in expanded form we need first to find the chr.inv.-projections of them, consisting of the quantities η^α and U^α. Projections of the η^α onto the time line and spatial section are, respectively

$$\varphi \equiv \frac{\eta_0}{\sqrt{g_{00}}}\,, \qquad n^i \equiv \eta^i, \tag{7.4}$$

other components of the η^α are expressed through its physically observable components φ and n_i as follows

$$\eta^0 = \frac{\varphi + \frac{1}{c}\,v_k n^k}{\sqrt{g_{00}}}\,, \qquad \eta_i = -\frac{\varphi}{c}\,v_i - n_i\,. \tag{7.5}$$

The time and spatial components of the particles' world-velocity vector U^α are derived from the chr.inv.-definitions given by the theory of chronometric invariants for the space-time interval ds and the observable chr.inv.-velocity vector v^i

$$ds = c\,d\tau\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\,, \quad \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \quad \mathrm{v}^2 = h_{ik}\mathrm{v}^i\mathrm{v}^k, \tag{7.6}$$

*It is important to note that the expected gravitational waves are waves of the *space-time* curvature, not merely of the spatial curvature of the three-dimensional space. Consequently, waves of the four-dimensional curvature must produce changes not only in the distance between test-particles in a detector, but also in the time flow for the particles.

so the required quantities U^0 and U^i are

$$U^0 = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{dt}{d\tau}\,, \qquad U^i = \frac{\mathrm{v}^i}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,. \tag{7.7}$$

A formula for the time function $dt/d\tau$ is obtained from*

$$g_{\alpha\beta}\,U^\alpha U^\beta = g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 1\,, \tag{7.8}$$

which can be reduced to the quadratic equation

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{2v_i\mathrm{v}^i}{c^2\left(1-\frac{\mathrm{w}}{c^2}\right)}\frac{dt}{d\tau} + \frac{1}{\left(1-\frac{\mathrm{w}}{c^2}\right)^2}\left(\frac{1}{c^2}v_i v_k \mathrm{v}^i\mathrm{v}^k - 1\right) = 0\,, \tag{7.9}$$

which has two solutions

$$\left(\frac{dt}{d\tau}\right)_1 = \frac{\frac{1}{c^2}v_i\mathrm{v}^i+1}{1-\frac{\mathrm{w}}{c^2}}\,, \qquad \left(\frac{dt}{d\tau}\right)_2 = \frac{\frac{1}{c^2}v_i\mathrm{v}^i-1}{1-\frac{\mathrm{w}}{c^2}}\,. \tag{7.10}$$

The first solution is related to a space where time flows from past into future (a regular observer's space), the second solution is related to a space where time flows from future into past with respect to a regular observer's time flow (the mirror Universe [70, 71]). Taking only the first root, U^0 takes the form

$$U^0 = \frac{\frac{1}{c^2}v_i\mathrm{v}^i + 1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\left(1-\frac{\mathrm{w}}{c^2}\right)}\,. \tag{7.11}$$

Substituting formulae (7.5), (7.7), (7.11) into $\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = 0$ (7.1), and expressing the components of the Riemann-Christoffel tensor $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ in terms of its physically observable components X^{ij}, Y^{ijk}, Z^{iklj}, we obtain a formula for the relative spatial oscillations of two free test-particles

$$\frac{\mathrm{D}^2\eta^i}{ds^2} = \frac{1}{c^2-\mathrm{v}^2}\left(Y_{mk\cdot}^{\cdot\cdot i}\mathrm{v}^k - X^i_m - \frac{1}{c^2}Z_{mk\cdot n}^{\cdot\cdot i\cdot}\mathrm{v}^k\mathrm{v}^n\right)\eta^m. \tag{7.12}$$

From this formula we see that:

> The relative spatial deviations of two free particles can be caused by all three observable components of the Riemann-Christoffel curvature tensor. Moreover, each of the components acts on the particles in a different way: (1) the field of X^{ik} acts the particles only if they are at rest with respect to the observer's space references, so the field of X^{ik} can move particles only if they are at rest at the initial moment of time; (2) the fields of Y^{ijk} and Z^{iklj} can displace the particles with respect of each to other only if they are in motion ($\mathrm{v}^i \neq 0$) — the effect of Z^{iklj} is perceptible if the particles move at speeds close to the velocity of light.

*That is the evident equality.

Thus, with measurement taken by any observer, the physically observable components of the Riemann-Christoffel curvature tensor are of 3 different kinds:

1. The component X^{ik} —of "electric kind", because it can displace even resting particles;
2. The component Y^{ijk} — of "magnetic kind", because it can displace only moving particles;
3. The component Z^{iklj} of "magnetic relativistic kind", because it causes an effect only in particles moving at relativistic speeds.

Besides the observable spatial component η^i of the relative deviation vector η^α there is also its observable time component φ, which indicates the difference between time flows measured by clocks located at each of the particles.

We then obtain the relative time deviation equation for two free test-particles

$$\sqrt{g_{00}}\frac{\mathrm{D}^2\eta^0}{ds^2} - \frac{1}{c}v_i\frac{\mathrm{D}^2\eta^i}{ds^2} = -\sqrt{g_{00}}R^{0\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma + \frac{1}{c}v_i R^{i\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma. \tag{7.13}$$

Taking (7.10) into account and substituting the formulae for U^0, η^0, U^i, η^i into (7.11), then, expressing $R^{0\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ in terms of physically observable quantities, we reduce formula (7.13) to its final form

$$\sqrt{g_{00}}\frac{\mathrm{D}^2\eta^0}{ds^2} - \frac{1}{c}v_i\frac{\mathrm{D}^2\eta^i}{ds^2} = \frac{1}{c^2-\mathrm{v}^2}\left[\frac{1}{c}X_{ik}\left(n^i - \frac{\varphi}{c}\mathrm{v}^k\right)\mathrm{v}^k + \frac{1}{c}Y_{imk}\mathrm{v}^i\mathrm{v}^k\eta^m\right]. \tag{7.14}$$

Looking at this formula we note one simple thing about the effect of gravitational waves on the system of two free particles:

> The time observable component of the relative deviation vector for two free particles undergoes oscillations due only to the X^{ik} and Y^{ijk} observable components of the Riemann-Christoffel curvature tensor, not its Z^{iklj} component. Moreover, the fields of both the components X^{ik} and Y^{ijk} act on the particles only if they are in motion with respect to the space references. If the particles are at rest with respect to each other and the observer ($\mathrm{v}^i = 0$), the fact that the space has a Riemannian curvature makes no difference to the time flow measured in the particles.

It should be added that if the particles are in motion with respect to the space references and the observer, the effect of X^{ik} is both linearly and quadratically dependent on the speed, whilst the effect of Y^{ijk} is only quadratically dependent on the speed.

Thus, there is no complete analogy between the physically observable components of the Riemann-Christoffel curvature tensor and Maxwell's electromagnetic field tensor.

The components X^{ik} can be interpreted "electric" only in *relative spatial displacements* of two particles. In relative time deviations between the particles (the difference between the time flow measured in the them both) both X^{ik} and Y^{ijk} act on them depending on the particles' velocity with respect to the space references and the observer, so in this case both X^{ik} and Y^{ijk} are of the "magnetic" kind. Therefore the terms "electric" and "magnetic" are only applicable relative to observable components of the Riemann-Chrstoffel curvature tensor. This terminology is strictly true in that case where the particles have only relative spatial deviations, while the time flow is the same on the both world lines.

A formula for the observable relative time deviation $\varphi = \frac{\eta_0}{\sqrt{g_{00}}}$ between two free particles can be obtained from the requirement that the scalar product $U_\alpha \eta^\alpha$ remains unchanged along geodesic trajectories, so $U_\alpha \eta^\alpha = \text{const}$ must be true along trajectories of free particles. For this reason, if the vectors U^α and η^α are orthogonal, they are orthogonal on the entire world-trajectory [17]. Formulating the orthogonality condition $U_\alpha \eta^\alpha = \text{const}$ in terms of physically observable quantities, we introduce some corrections to the results obtained in [17].

In terms of physically observable quantities the orthogonality condition $U_\alpha \eta^\alpha = \text{const}$, because it is actually the same as $U_0 \eta^0 + U_k \eta^k = \text{const}$, reduces to

$$\varphi - \frac{1}{c}\, n_i \mathrm{v}^i = \text{const} \times \sqrt{1 - \frac{\mathrm{v}^2}{c^2}}\,. \tag{7.15}$$

From this we see that the vectors U^α and η^α are orthogonal only if $\mathrm{v}^2 = c^2$, i. e. U^α is isotropic: $g_{\alpha\beta} U^\alpha U^\beta = 0$. So if U^α and η^α are orthogonal, we have the deviation equation for *two isotropic geodesics* — world-lines of light-like particles moving at the velocity of light. We defer this case for the moment and consider only the case of *two neighbour non-isotropic geodesics*. In the particular case when two particles are moving on neighbouring geodesics, and are at rest with respect to the observer and his references (only the time flow is different in the particles), formula (7.15) leads to $\varphi = \text{const}$.

This formula verifies our previous conclusion that the particles have a time deviation only if they are in motion. The greater their velocity with respect to the space reference and the observer, the greater the deviation between the time flow on both world-lines. Thus measurement of time deviations between two particles in gravitational waves and gravitational inertial waves would be easier in experiments where the particles move at high speeds. In practice such an experimental statement could be realized using light-like particles (in particular, photons). A time deviation of two photons in gravitational wave fields can manifest as changes in the frequencies of two parallel light rays (laser beams, for instance), while a spatial deviation of the photons can manifest as changes in the phases of the light rays. Calculations of these effects will be presented in future article. Here we focus our attention on particles of non-zero rest-mass $m_0 \neq 0$ (so-called mass-bearing particles), which are at rest with respect to the space references and the observer or, alternatively, moving at sub-light speeds.

In equations (7.10) and (7.12), we kept the second absolute derivative $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ of the relative deviation vector η^α as a whole, because we were concerned only with the effects introduced by the Riemannian curvatureto the relative spatial acceleration $\frac{\mathrm{D}^2\eta^i}{ds^2}$ and relative time acceleration $\frac{\mathrm{D}^2\eta^0}{ds^2}$ of two free test-particles.

But if we wish to obtain solutions to the world-lines deviation equation, we need to express the quantity $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ and also $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ in terms of the Christoffel symbols and their derivatives.

We are now going to obtain solutions to the deviation equation for geodesic lines (the Synge equation).

Taking the definition

$$\frac{\mathrm{D}\eta^\alpha}{ds} = \frac{d\eta^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\eta^\mu U^\nu \tag{7.16}$$

into account, we obtain

$$\begin{gathered}\frac{\mathrm{D}^2\eta^\alpha}{ds^2} = \frac{d^2\eta^\alpha}{ds^2} + \frac{d\Gamma^\alpha_{\mu\nu}}{ds}\eta^\mu U^\nu + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \\ + \Gamma^\alpha_{\mu\nu}\eta^\mu\frac{dU^\nu}{ds} + \Gamma^\alpha_{\rho\sigma}\Gamma^\rho_{\mu\nu}\eta^\mu U^\nu U^\sigma = 0\,.\end{gathered} \tag{7.17}$$

We write $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ as

$$R^{\alpha\cdots}_{\cdot\beta\gamma\delta} = \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\sigma_{\beta\delta}\Gamma^\alpha_{\gamma\sigma} - \Gamma^\sigma_{\beta\gamma}\Gamma^\alpha_{\sigma\delta}\,, \tag{7.18}$$

express $\frac{dU^\alpha}{ds}$ via the geodesic equations

$$\frac{dU^\alpha}{ds} = -\Gamma^\alpha_{\mu\nu}U^\mu U^\nu, \tag{7.19}$$

and use the definition

$$\frac{d\Gamma^\alpha_{\mu\nu}}{ds} = \frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\sigma}U^\sigma. \tag{7.20}$$

Using the auxiliary formulae we obtain from (7.17) the Synge equation (the geodesic lines deviation equation) in its final form

$$\frac{d^2\eta^\alpha}{ds^2} + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma}U^\beta U^\delta\eta^\gamma = 0\,. \tag{7.21}$$

This is a differential equation of the 2nd order with respect to the quantity η^α: the equation is a system of 4 differential equations with respect to the quantities η^0 and η^i $(i = 1, 2, 3)$. The variable coefficients $\Gamma^\alpha_{\mu\nu}$ and their derivatives must be taken for that gravitational field, whose waves act on two free test-particles in our experiment. The world-quantities U^ν ($\nu = 0, 1, 2, 3$) can be found as solutions to the geodesic equations

$$\frac{dU^\nu}{ds} + \Gamma^\nu_{\mu\rho} U^\mu U^\rho = 0 \qquad (7.22)$$

only if the particles move with respect to the space references and the observer. If the particles are at rest with respect to the observer and his references, the components of their world-velocity vector U^ν are

$$U^0 = \frac{1}{\sqrt{g_{00}}}\,, \qquad U^i = 0\,, \qquad (7.23)$$

and, according to (7.13–7.15), their relative time deviation is zero, $\varphi = 0$ (the time flow measured on both geodesic lines is the same).

Current detectors used in the search for gravitational wave radiations are of such a construction that the particles therein, which detect the waves, are almost at rest with respect of each other and the observer. Experimental physicists, following Joseph Weber and his methods, think that gravitational waves can cause the rest-particles to undergo a relative displacement. With the current theory of the gravitational wave experiment, the experimental physicists limit themselves to the expected amplitude and energy of waves arriving from a proposed source of a gravitational wave field.

However, to set up the gravitational wave experiment correctly, we need to eliminate all extraneous assumptions and traditions. We merely need to obtain exact solutions to the world-lines deviation equation, applied to detectors of that kind which this experiment uses.

Detectors described by the geodesic lines deviation equation (the Synge equation), which we consider in this section, are known as "antennae built on free masses". We shall consider such detectors first.

The detectors consist of two freely suspended masses which are at rest with respect of each other and the observer, and separated by an appreciable distance. These could be two mirrors, located in a near-to-Earth orbit, for instance. Each of the mirrors is fitted with a laser range-finder, so we can measure the distance between them with high precision.

In order to interpret the possible results of such an experiment, we need to solve the Synge equation (7.21), expressing its solutions in the terms of physically observable quantities (chronometric invariants). Following "tradition", we solve the Synge equation for particles which are at rest with respect to each other and the observer's space references. So we consider that case where the particles' observable velocities are zero ($\mathrm{v}^i = 0$).

At first, because we are going to obtain solutions to the Synge equation in chr.inv.-from, we need to know the physically observable characteristics of the observer's reference space through which we express the solutions. We find the chr.inv.-characteristics from the geodesic equations taken in the main chr.inv.-from [42]

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = 0\,, \qquad (7.24)$$

$$\frac{d}{d\tau}(m\mathrm{v}^i) + 2m\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\mathrm{v}^k - mF^i + {} + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,, \qquad (7.25)$$

for each of the particles (because both particles are at rest with respect to one another, their geodesic equations are the same). Here m is the particle's relativistic mass, which, because in the case we are considering $\mathrm{v}^i = 0$, reduces to the rest-mass $m = m_0$. Then the geodesic equations take the very simple form

$$\frac{dm}{d\tau} = 0\,, \qquad (7.26)$$

$$mF^i = 0\,, \qquad (7.27)$$

so in this case the chr.inv.-vector of gravitational inertial force is $F^i = 0$: the particles are in free fall. In this case we can transform coordinates so that $g_{00} = 0$ and $\frac{\partial g_{0i}}{\partial t} = 0$ [42]. This implies that the Synge initial equation (7.19) can be solved correctly only for gravitational fields where the potential is weak $\mathrm{w} = 0$ (i. e. $g_{00} = 1$) and where the space rotation is stationary $\frac{\partial A_{ik}}{\partial t} = 0$. It should be noted that the metric of weak plane gravitational waves, the only metric used in the theory of gravitational wave experiments, satisfies these requirements.

Because $\varphi = \frac{1}{c} n_i \mathrm{v}^i$ (7.15), in the case we are considering the time observable component φ of the relative deviation vector η^α is zero $\varphi = 0$. For this reason we consider only the observable spatial component of the Synge equation (7.21).

In the accompanying reference frame (where the observer accompanies his references), according to the theory of chronometric invariants [42, 43], in the absence of gravitational fields $\mathrm{w} = 0$ and also gravitational inertial forces $F_i = 0$, we have: $\frac{d}{ds} = \frac{1}{c}\frac{d}{d\tau}$, $U^0 = \frac{1}{\sqrt{g_{00}}} = 1$, $U^i = \frac{1}{c}\mathrm{v}^i$, $\eta^0 = -g_{0i}\eta^i$, $\Gamma^i_{00} = -\frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 F^i = 0$, $\Gamma^i_{0k} = \frac{1}{c}\left(1 - \frac{\mathrm{w}}{c^2}\right)\left(D^i_k + A^{\cdot i}_{k\cdot} + \frac{1}{c^2} v_k F^k\right) = \frac{1}{c}\big(D^i_k + A^{\cdot i}_{k\cdot}\big)$. Employing now the formulae for the Synge equation (7.21) under $\mathrm{v}^i = 0$, we obtain the *Synge equation in chr.inv.-form**

$$\frac{d^2\eta^i}{d\tau^2} + 2\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\frac{d\eta^k}{d\tau} = 0\,. \qquad (7.28)$$

The quantity $\frac{d}{d\tau} = \frac{{}^*\partial}{\partial t} + \mathrm{v}^i \frac{{}^*\partial}{\partial x^i}$ [42, 43] here is

$$\frac{d}{d\tau} = \frac{\partial}{\partial t}\,, \qquad (7.29)$$

*As we mentioned, if the particles are at rest $\mathrm{v}^i = 0$, the chr.inv.-time component of the Synge equation becomes zero.

so the chr.inv.Synge -equation (7.28) takes its final form

$$\frac{\partial^2 \eta^i}{\partial t^2} + 2\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\frac{\partial \eta^k}{\partial t} = 0\,. \tag{7.30}$$

We find the exact solution to the Synge chr.inv.-equation (7.30) in the field of weak plane gravitational waves*. In the case we are considering ($\mathrm{v}^i = 0$) we have

$$\begin{aligned} &F_i = 0\,, \qquad A_{ik} = 0\,, \\ &D_{22} = -D_{33} = \frac{1}{2}\frac{\partial a}{\partial t}\,, \quad D_{23} = \frac{1}{2}\frac{\partial b}{\partial t}\,. \end{aligned} \tag{7.31}$$

Substituting the requirements into the initial equation (7.30) we obtain a system of three equations

$$\frac{\partial^2 \eta^1}{\partial t^2} = 0\,, \tag{7.32}$$

$$\frac{\partial^2 \eta^2}{\partial t^2} + \frac{\partial a}{\partial t}\frac{\partial \eta^2}{\partial t} - \frac{\partial b}{\partial t}\frac{\partial \eta^3}{\partial t} = 0\,, \tag{7.33}$$

$$\frac{\partial^2 \eta^3}{\partial t^2} - \frac{\partial a}{\partial t}\frac{\partial \eta^3}{\partial t} - \frac{\partial b}{\partial t}\frac{\partial \eta^2}{\partial t} = 0\,. \tag{7.34}$$

The solution of (7.32) is

$$\eta^1 = \eta^1_{(0)} + \dot{\eta}^1_{(0)} t\,, \tag{7.35}$$

where $\eta^1_{(0)}$ is the particle's initial deviation, $\dot{\eta}^1_{(0)}$ is its initial velocity.

This system can be easy solved in two particular cases of a linear polarized wave: (1) $b = 0$, and (2) $a = 0$.

In the first case ($b = 0$) we obtain

$$\frac{\partial \eta^2}{\partial t} = C_1\, e^{-a}, \qquad \frac{\partial \eta^3}{\partial t} = C_2\, e^{+a}, \tag{7.36}$$

where C_1 and C_2 are integration constants. Because values of a are weak, we can decompose e^{-a} into series. Then, assuming higher order terms infinitesimal, we obtain

$$\frac{\partial \eta^2}{\partial t} = C_1\,(1-a)\,, \qquad \frac{\partial \eta^3}{\partial t} = C_2\,(1+a)\,. \tag{7.37}$$

Assuming also that a falling gravitational wave is monochrome, bearing a constant amplitude A and a frequency ω,

$$a = A \sin\frac{\omega}{c}\left(ct \pm x^1\right), \tag{7.38}$$

we integrate the system (7.37). As a result we obtain

$$\eta^2 = C_1\left[t + \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + D_1\,, \tag{7.39}$$

$$\eta^3 = C_2\left[t - \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + D_2\,, \tag{7.40}$$

where D_1 and D_2 are integration constants. Assuming $x^1 = 0$ at the initial moment of time $t = 0$, we easily express the integration constants C_1, C_2, D_1, D_2 through the initial conditions. Finally, we obtain solutions

$$\eta^2 = \dot{\eta}^2_{(0)}\left[t + \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + \eta^2_{(0)} - \frac{A}{\omega}\,\dot{\eta}^2_{(0)}\,, \tag{7.41}$$

$$\eta^3 = \dot{\eta}^3_{(0)}\left[t - \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + \eta^3_{(0)} - \frac{A}{\omega}\,\dot{\eta}^3_{(0)}\,, \tag{7.42}$$

where $\eta^2_{(0)}$, $\eta^3_{(0)}$ and $\dot{\eta}^2_{(0)}$, $\dot{\eta}^3_{(0)}$ are the initial numerical values of the relative deviation η and relative velocity $\dot{\eta}$ of the particles along the x^2 and x^3 axes, respectively.

We have now obtained the exact solutions to the Synge equation (the geodesic lines deviation equation). From the solutions we see,

> If at the initial moment of time $t = 0$, two free particles are at rest with respect to each other and the observer $\dot{\eta}^2_{(0)} = \dot{\eta}^3_{(0)} = 0$, weak plane gravitational waves of the deformation kind (waves of the Riemannian curvature) cannot force the particles to go into relative motion. If at the initial moment of time the particles are in motion, the waves augment the particles' initial motion, accelerating them.

Thus our purely mathematical analysis of detectors built on free masses leads to the final conclusion:

> Weak plane gravitational waves of the deformation kind (the Riemannian curvature's waves) cannot be detected by any antenna composed on free masses, if the masses are at rest with respect to each other and the observer.

8 Criticism of Weber's conclusions on the possibility of detecting gravitational waves by solid-body detectors of the resonance kind

Historically, the first gravitational wave detector was the quadrupole mass-detector built in 1964 by Prof. Joseph Weber with his students David Zipoy and Robert Forward at Maryland University [70]. It was an aluminium cylindrical pig weighing 1.5 tons, suspended by a steel "thread" in a vacuum camera. At the point of connection between the pig and the thread, the pig was covered by a piezoelectric quartz film linked to a highly sensitive voltmeter. Weber expected that *a falling gravitational wave should make relative displacements of the butt-ends of the cylindrical pig — extension or compression of the pig*. In other words, they expected that falling gravitational waves will deform the pig, necessarily causing a piezoelectric effect in it. Modified by Sinsky [71], the first detector gave a possibility of registering a 10^{-16}cm relative displacement of its butt-ends.

Later, Weber built a system of two pigs. That system worked through the principle of coincident frequencies of

*Where the metric (5.7) is $ds^2 = c^2dt^2 - (dx^1)^2 - (1-a)(dx^2)^2 + 2b\,dx^2dx^3 - (1+a)(dx^3)^2$.

the signals registered in both pigs. The pigs had a relaxation time about 30 sec, were tuned for the frequency 10^4 rps, and were separated by 2 km. In 1967 Weber and his team registered coincident signals (to a precision within 0.2 sec) which appeared about once a month [1]. The registered relative displacements of the butt-ends in the pigs were $\sim 3\times 10^{-10}$ cm. Weber supposed that the origin of the observed signals were gravitational wave radiations.

Weber subsequently even used 6 pigs, one of which was located at Argonne National Laboratory (Illinois), the other 5 pigs located in his laboratory at Maryland University. The distance between the laboratories wss about 1000 km. The detectors were tuned for 1660 Hz — the frequency of supposed gravitational radiations excited from collapsing supernovae. During several months of observations, numerous coincident signals were registered [72]. A second cycle of the observations gave the same positive result [73]. Weber interpreted the registered signals as proof that strong gravitational radiations exist in the Galaxy. A peculiarity of those experiments was that the pigs located both in Illinois and Maryland were isolated as much as possible from external electromagnetic and seismic influences.

After Weber's pioneering experiments, experimental physicists built many similar detectors, much more sensitive than those of Weber. However, in contrast to those of Weber, not one of them registered any signals.

Therefore, using the world-lines deviation theory developed here in the terms of physically observable quantities, we are going to:

(1) investigate what in principle can be registered by a solid-body detector (a Weber pig) and

(2) compare our conclusion with that explanation given by Weber himself for his observed signals.

From the theoretical viewpoint we can conceive of a solid-body cylindrical detector as consisting of two test-particles, connected by a spring [16]. It is supposed that the system falls freely. It is also supposed that at the initial moment of time, when we start our measurements, the particles are at rest with respect to us (the observers) and each other. This is the standard problem statement, not only of Weber [16] or ourselves, but also of any other theoretical physicist.

The behaviour of two neighbouring particles in their motion along their neighbouring world-lines is described by the *world-lines deviation equation*. If the particles are not free, but connected by a non-gravitational force Φ^α (a spring, for instance), the Synge-Weber equation (2.12) applies, namely $\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = \frac{1}{m_0c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}\,dv$. This is an inhomogeneous differential equation of the 2nd order with respect to the relative deviation vector η^α of the particles. In order to solve the world-lines deviation equation we need to write $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ and $\frac{\mathrm{D}\Phi^\alpha}{dv}\,dv$ in expanded form. Because both terms contain the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, it would be reasonable to express the components of the Riemann-Christoffel tensor $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ in terms of the $\Gamma^\alpha_{\mu\nu}$ and their derivatives: in collecting similar terms some of them will cancel out (the same situation arose when we made the same calculations for the geodesic lines deviation equation).

Using formulae (7.17) and (7.18), and the quantity $\frac{dU^\alpha}{ds}$ from the world-line equation of a particle moved by a non-gravitational force Φ^α (2.11), we obtain

$$\frac{dU^\alpha}{ds} = -\Gamma^\alpha_{\rho\sigma}U^\rho U^\sigma + \frac{\Phi^\alpha}{m_0c^2}\,. \tag{8.1}$$

Expanding the formula for $\frac{\mathrm{D}\Phi^\alpha}{dv}\,dv$

$$\begin{aligned}\frac{\mathrm{D}\Phi^\alpha}{dv}\,dv &= \frac{\partial\Phi^\alpha}{\partial v}\,dv + \Gamma^\alpha_{\mu\nu}\Phi^\mu\frac{\partial x^\nu}{\partial v}\,dv = \\ &= \frac{\partial\Phi^\alpha}{\partial x^\sigma}\,\eta^\sigma + \Gamma^\alpha_{\mu\nu}\Phi^\mu\eta^\nu\end{aligned} \tag{8.2}$$

and substituting this into the world-lines deviation equation in its initial form (2.12), takeing into account that (8.1) and (8.2), we obtain

$$\begin{aligned}\frac{d^2\eta^\alpha}{ds^2} + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma}U^\beta U^\delta\eta^\gamma &= \\ = \frac{1}{m_0c^2}\frac{\partial\Phi^\alpha}{\partial x^\sigma}\,\eta^\sigma.\end{aligned} \tag{8.3}$$

This is the final form of the world-lines deviation equation for two test-particles connected by a spring. The quantities η^α and U^α are connected by (2.13): $\frac{\partial}{\partial s}(U_\alpha\eta^\alpha) = \frac{1}{m_0c^2}\,\Phi_\alpha\eta^\alpha$.

In a gravitational wave detector like Weber's, the cylindrical pig is isolated as much as possible from external influences of thermal, electromagnetic, seismic and another origins. To minimise external influences, experimental physicists place the detectors in mines located deep inside mountains or otherwise deep beneath the terrestrial surface, and cool the pigs to 2 K. Therefore particles of matter in the butt-ends and the pig in general, can be assumed at rest with respect to one another and to the observer.

Following Weber, experimental physicists expect that a falling gravitational wave will deform the pig, displacing its butt-ends with respect to each other. Relative displacements of the butt-ends of a pig are supposed to result in a piezo-electric effect which can be registered by a piezo-detector. In other words, experimental physicists expect that oscillations of the acting gravitational wave field give rise to a force in the world-lines deviation equation (the Synge-Weber equation), thereby displaceing the test-particles which were at rest at the initial moment of time. Oscillations of the acting gravitational wave field force the butt-ends of the pig to oscillate. As soon as the frequency of the pig's oscillations coincides with the falling wave's frequency, the amplitude of the pig's oscillations will increase significantly because of resonance, so the amplitude becomes measurable. Therefore the Weber detectors are said to be of the *resonance kind*.

Before ratifying the aforesaid conclusions it would be reasonable to study the world-lines deviation equation for two interacting test-particles that model a Weber pig, because this equation is the theoretical basis of all experimental attempts to register gravitational waves made by Weber and his followers during more than 30 years.

We will study this equation, proceeding from its form (8.3), because formula (8.3) gives a possibility of obtaining exact solutions to the relative deviation vector η^{α}; not the initial equation (2.12). Following analysis of the solutions we will come to a conclusion as to what effect a falling gravitational wave has on the detector*.

When we need to give a theoretical interpretation of experimental results, it is very important to analyse the results in the terms of physically observable quantities because such quantities can be registered in practice. For this reason we will study the behaviour of the Weber model (the system of two particles, connected by a non-gravitational force) in the terms of physically observable quantities (chronometric invariants) as we did in §7 when we solved a similar problem for the system of two free particles.

In detail, our task here is to consider commonly the world-lines equation (2.11) and the world-lines deviation equation (8.3), both written in chr.inv.-form. Note that the relationship (2.13), that is $\frac{\partial}{\partial s}(U_{\alpha}\eta^{\alpha}) = \frac{1}{m_0 c^2}\,\Phi_{\alpha}\eta^{\alpha}$, gives the exact solution for the quantity φ. The φ is the chr.inv.-time component of the relative deviation world-vector η^{α} with respect to which the world-lines deviation equation (8.3) is written. For this reason there is no need here to solve the chr.inv.-time projection of the world-lines deviation equation (8.3). We solve instead the relationship (2.13).

The world-lines equation (2.11) in chr.inv.-form is

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i \mathrm{v}^i + \frac{m}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k = \frac{\sigma}{c}\,, \tag{8.4}$$

$$\frac{d}{d\tau}(m\mathrm{v}^i) + 2m\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\mathrm{v}^k - mF^i + \Delta^i_{kn}\mathrm{v}^k\mathrm{v}^n = f^i, \tag{8.5}$$

where $\sigma \equiv \frac{\Phi_0}{\sqrt{g_{00}}}$ and $f^i \equiv \Phi^i$ are chr.inv.-components of the prevailing non-gravitational force Φ^{α}. In the case of the Weber model where the particles are at rest with respect to the observer ($\mathrm{v}^i = 0$), the chr.inv.-equations (8.4) and (8.5) become

$$\sigma = 0\,, \tag{8.6}$$

$$m_0 F^i = -f^i. \tag{8.7}$$

The condition $\sigma = 0$ comes from the fact that, when a particle is at rest its relativistic mass becomes the rest-mass $m = m_0$. Thus resting particles are under the action of only the spatial observable components f^i of the non-gravitational force Φ^{α}, so that the f^i are of the same value as the acting gravitational inertial force F^i, but acts in the opposite direction. Looking at definition (4.1), given by the theory of physical observable quantities for the gravitational inertial force F^i, we see that in this case the non-gravitational force f^i acts on a resting particle only if at least one of the following factor holds:

1. Inhomogeneity of the gravitational potential $\frac{\partial \mathrm{w}}{\partial x^i} \neq 0$;
2. Non-stationarity of the vector of the space rotation linear velocity $\frac{\partial v_i}{\partial t} \neq 0$.

If neither factor holds, $F^i = 0$ and hence $f^i = 0$, in which case both interacting particles, which are at rest with respect to each other and the observer, behaviour like free particles: their connecting force Φ^{α} does not manifest. Looking at the well-known metric (5.7) that describes weak plane gravitational waves, we see there that $F^i = 0$, $A_{ik} = 0$ and hence $v_i = 0$. Therefore:

> Weak plane gravitational waves described by the metric (5.7) **cannot be registered** by a solid-body detector of the resonance kind (a Weber detector).

Writing the relationship (2.13) in chr.inv.-form, we obtain

$$\frac{d}{d\tau}\left(\frac{\varphi - \frac{1}{c}\,n_i\mathrm{v}^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\right) = \frac{\sigma\varphi - f_i n^i}{mc}\,, \tag{8.8}$$

where again, $\varphi \equiv \frac{\eta_0}{\sqrt{g_{00}}}$ and $n^i \equiv \eta^i$ are chr.inv.-components of the relative deviation world-vector η^{α}.

From this we see that the angle between the vectors U_{α} and η^{α} is a variable depending on many factors, including the velocity v^i of the particles. At speeds close to that of light c, the angle increases. At $\mathrm{v} = c$ formula (8.8) becomes senseless: the denominator on the left side becomes zero.

If both particles are at rest, formula (8.8) becomes

$$\frac{d\varphi}{d\tau} = -\frac{f_i n^i}{m_0 c} = \frac{F_i n^i}{c}\,, \tag{8.9}$$

so that in the case of interacting rest-particles, in contrast to free ones, there is the time observable component φ of the relative deviation vector η^{α}. This implies that there are not only relative spatial displacements of the particles, but also a deviation between measurements of time made by the clocks of both particles. The "time deviation" φ can be found by integrating (8.9). We obtain

$$\varphi = \frac{1}{c}\int F_i n^i + \mathrm{const}\,, \tag{8.10}$$

so the value of the "time deviation" φ increases with time. It Note that $\frac{\varphi}{\tau} = 0$ only if the vector F^i (and hence, in this case, also f^i) is orthogonal to the vector n^i, so that $F_i n^i = -\frac{1}{m_0} f_i n^i = 0$.

*It is evident that equation (8.3) can be solved also for other forcing fields, which can be of a non-wave origin.

The integral (8.10) is the solution to the chr.inv.-time component of the world-lines deviation equation (8.3). This solution for φ itself, being a chronometric invariant, is a physically observable quantity.

We are now going to obtain solutions to the remaining three chr.inv.-equations with respect to η^α:

$$\frac{1}{c^2}\frac{d^2\eta^i}{d\tau^2}+2\Gamma^i_{00}\frac{1}{c}\frac{d\eta^0}{d\tau}U^0+2\Gamma^i_{k0}\frac{1}{c}\frac{d\eta^k}{d\tau}U^0+ \\ +\frac{\partial\Gamma^i_{00}}{\partial x^0}U^0U^0\eta^0+\frac{\partial\Gamma^i_{00}}{\partial x^k}U^0U^0\eta^k=\frac{1}{m_0c^2}\frac{\partial\Phi^i}{\partial x^\sigma}\eta^\sigma, \tag{8.11}$$

— the chr.inv.-spatial components of the world-lines deviation equation (8.3), in the case of two rest-particles $ds=cd\tau$.

In the left side of (8.11) we substitute the formulae for the quantities Γ^i_{00}, Γ^i_{00}, U^0, η^0, given on page 53, and also derivatives of the quantities. Then we transform the right part of (8.11) as follows

$$\frac{\partial\Phi^i}{\partial x^\sigma}\eta^\sigma=\frac{\partial f^i}{\partial x^0}\eta^0+\frac{\partial f^i}{\partial x^k}\eta^k=\frac{\varphi}{c}\frac{^*\partial f^i}{\partial t}+\frac{^*\partial f^i}{\partial x^k}n^k, \tag{8.12}$$

where we use the definitions of the chr.inv.-derivative operators (see page 40): $\frac{^*\partial}{\partial t}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}+\frac{1}{c^2}v_i\frac{^*\partial}{\partial t}$.

The initial equation (8.11) becomes

$$\frac{d^2n^i}{d\tau^2}+2\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\frac{dn^k}{d\tau}-\frac{2}{c}\frac{d\varphi}{d\tau}F^i+\frac{2}{c^2}F_kn^kF^i- \\ -\frac{\varphi}{c}\frac{^*\partial F^i}{\partial t}-\frac{^*\partial F^i}{\partial x^k}n^k=\frac{1}{m_0}\left(\frac{\varphi}{c}\frac{^*\partial f^i}{\partial t}+\frac{^*\partial F^i}{\partial x^k}n^k\right). \tag{8.13}$$

Owing to the particular conditions (8.7) and (8.9), derived from the requirement that the particles are at rest ($\mathrm{v}^i=0$), formula (8.13) becomes much more simple

$$\frac{d^2n^i}{d\tau^2}+2\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\frac{dn^k}{d\tau}=0\,, \tag{8.14}$$

which is the final form for the chr.inv.-spatial deviation equation for two rest-particles, connected by a non-gravitational force.

Equation (8.14) is like the chr.inv.-spatial deviation equation for two free rest-particles (7.30) — the chr.inv.-spatial part of the Synge general covariant equation. The difference is that (8.14) contains derivatives $\frac{d}{d\tau}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$, while (7.30) contains $\frac{\partial}{\partial t}$. This difference is derived from the fact that (7.30) is applicable to gravitational fields where $F_i=0$, the potential w is neglected and hence $\frac{\partial v_i}{\partial t}=0$, while (8.14) describes the relative deviation of two particles located in gravitational fields where $F_i\neq 0$.

The required condition $F_i\neq 0$ implies:

1. In this region the gravitational potential is $\mathrm{w}\neq 0$, hence, because the interval of physical observable time is $d\tau=\left(1-\frac{\mathrm{w}}{c^2}\right)dt-\frac{1}{c^2}v_idx^i$, the time flow differences at different points inside the region. In particular, if $v_i=0$, synchronization of clocks located at different points cannot be conserved. In the more general case where $v_i\neq 0$, clocks located at different points cannot be synchronized [42, 43];*
2. If the gravitational inertial force field F^i is vortical, the space rotation is non-stationary $\frac{\partial v_i}{\partial t}\neq 0$.

Let us get back to the chr.inv.-spatial equation for two particles connected by a non-gravitational force (8.14). There are quantities D_{ik} and A_{ik}, so relative accelerations of the particles can be derived from both the space deformations and rotation. In this problem statement, w implies that the gravitational potential of a distant source of gravitational radiations. So in a gravitational wave experiment we should specify the acting gravitational field as weak $\frac{\mathrm{w}}{c^2}\approx 0$, hence in the experiment the chr.inv.-gravitational inertial force vector F_i (4.1) becomes $F_i\approx\frac{\partial\mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}$. There are as well $\frac{d}{d\tau}=\frac{\partial}{\partial t}$.

We solve equation (8.14) in two cases, aiming to find what kind of gravitational field fluctuations were registered by Weber and his colleagues.

First case: $A_{ik}=0$, $D_{ik}\neq 0$.

In this case equation (8.14), with $\frac{\mathrm{w}}{c^2}\approx 0$, is the same as the chr.inv.-world-lines deviation equation for two free particles (7.30). As it was shown in §7, with solutions of equation (7.30) considered, a gravitational wave can affect the system of two free particles only if the particles are in motion at the initial moment of time. In that case a gravitational wave can only augment the initial motion of the particles. If they are at rest gravitational waves can have no effect on the particles.

*To realize the condition $\mathrm{w}\neq 0$ it is not necessary to have a wave gravitational field. In particular, $\mathrm{w}\neq 0$ is true even in stationary gravitational fields derived from island masses (like Schwarzschild's metric). Moreover, the phenomenon of different time flow in the Earth gravitational field is well-known from experimental tests of the General Theory of Relativity: a standard clock, located on the terrestrial surface, shows time which is $\sim 10^{-9}$sec different from time measured by the same standard clock, located in a balloon a few kilometers above the terrestrial surface (the difference increases with the duration of the experiment). But such corrections of time are not linked to the presence of gravitational waves.

There time corrections can also be registered, the origin of which are wave changes of the gravitational potential w. They can be interpreted as waves of the gravitational inertial force field F^i. In this case corrections to standard clocks, located at different points, should bear a relation to wave changes of w.

The presence of the space rotation $v_i\neq 0$ changes the time flow as well. Experiments, where a standard clock was moved by a jet plane around the world [49, 50, 51, 52], showed differential time flow with respect to the same standard clock located at rest at the air force base. Such difference of measured time, called the *desynchronization correction*, depends on the flight direction — with or opposite to Earth's rotation. Although such corrections are derived from the Earth rotation (the reference space rotation), in the "background" of such corrections there could also be registered additional tiny corrections derived from the rapid stationary rotation field of a massive space body, located far from the Earth.

When $A_{ik} = 0$ the chr.inv.-world-lines deviation equation (8.14), describing a Weber detector, coincides with equation (7.30), and we conclude:

> A Weber detector (a solid-body detector of the resonance kind) will have no response to a falling gravitational wave of the pure deformation kind, if the particles of which the detector is composed are at rest at the initial moment of measurements (the situation assumed in the Weber experiment).

Second case: $D_{ik} = 0$, $A_{ik} \neq 0$.

We assume that the space rotation has a constant angular velocity ω around the x^3 axis, while the linear velocity of this rotation is $v^i = \omega_{k\cdot}^{\cdot i} x^k \ll c$. For the background metric, following the classical approach [14, 15], we use the Minkowski line element, where the gravitational waves are superimposed as tiny corrections to it. Then the components of v^i are

$$v^1 = -\omega x^2, \quad v^2 = \omega x^1, \quad v^3 = 0\,, \tag{8.15}$$

and the space metric in its expanded form is

$$\begin{aligned} ds^2 = c^2 dt^2 + 2\omega\left(x^2 dx^1 - x^1 dx^2\right) dt - \\ - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \end{aligned} \tag{8.16}$$

This metric describes the four-dimensional space of a uniformly rotating reference frame, whose rotational linear velocity is negligible with respect to c.

Components of the tensor A_{ik} are

$$A_{2\cdot}^{\cdot 1} = \omega_{2\cdot}^{\cdot 1} = -\omega\,, \quad A_{1\cdot}^{\cdot 2} = \omega_{1\cdot}^{\cdot 2} = \omega\,, \quad A_{1\cdot}^{\cdot 3} = 0\,. \tag{8.17}$$

Substituting (8.17) into the chr.inv.-world-lines deviation equation (8.14) we obtain a system of deviation equations

$$\frac{\partial^2 \eta^1}{\partial t^2} - 2\omega \frac{\partial \eta^2}{\partial t} = 0\,, \tag{8.18}$$

$$\frac{\partial^2 \eta^2}{\partial t^2} + 2\omega \frac{\partial \eta^1}{\partial t} = 0\,, \tag{8.19}$$

$$\frac{\partial^2 \eta^3}{\partial t^2} = 0\,, \tag{8.20}$$

which commonly describe behaviour of two neighboring rest-particles in a uniformly rotating reference frame.

Equation (8.20) can be integrated immediately

$$\eta^3 = \eta^3_{(0)} + \dot{\eta}^3_{(0)} t\,, \tag{8.21}$$

where $\eta^3_{(0)}$ and $\dot{\eta}^3_{(0)}$ are the initial values of the relative displacement and velocity of the particles along the x^3 axis.

In integrating equations (8.19) and (8.20), we introduce the notation $\frac{\partial \eta^1}{\partial t} \equiv x$ and $\frac{\partial \eta^2}{\partial t} \equiv y$. Then we have

$$\left.\begin{aligned} \dot{x} - 2\omega y = 0 \\ \dot{y} + 2\omega x = 0 \end{aligned}\right\}. \tag{8.22}$$

We differentiate the first equation with respect to t

$$\ddot{x} = 2\omega \dot{y} \tag{8.23}$$

and substitute $\dot{y} = \ddot{x}/2\omega$ into the second one. We obtain a harmonic oscillation equation

$$\ddot{x} + 4\omega^2 x = 0\,, \tag{8.24}$$

with respect to the relative velocity $x = \frac{\partial \eta^1}{\partial t}$ of the particles. The solution to (8.24) is

$$x = \frac{\partial \eta^1}{\partial t} = C_1 \cos 2\omega t + C_2 \sin 2\omega t\,, \tag{8.25}$$

where C_1 and C_2 are integration constants, which can be obtained from the initial conditions. Thus we obtain

$$\frac{\partial \eta^1}{\partial t} = \left(\frac{\partial \eta^1}{\partial t}\right)_{(0)} \cos 2\omega t + \frac{1}{2\omega}\left(\frac{\partial^2 \eta^1}{\partial t^2}\right)_{(0)} \sin 2\omega t\,, \tag{8.26}$$

where terms marked with zero are the initial values of the relative velocity and acceleration of the particles. Integrating (8.26), we obtain

$$\eta^1 = \frac{\dot{\eta}^1_{(0)}}{2\omega} \sin 2\omega t - \frac{\ddot{\eta}^1_{(0)}}{4\omega^2} \cos 2\omega t + B_1\,, \tag{8.27}$$

where B_1 is an integration constant. Obtaining B_1 from the initial conditions, we obtain the final formula for η^1

$$\eta^1 = \frac{\dot{\eta}^1_{(0)}}{2\omega} \sin 2\omega t - \frac{\ddot{\eta}^1_{(0)}}{4\omega^2} \cos 2\omega t + \eta^1_{(0)} + \frac{\ddot{\eta}^1_{(0)}}{4\omega^2}\,. \tag{8.28}$$

In the same fashion we obtain a formula for η^2

$$\eta^2 = \frac{\dot{\eta}^2_{(0)}}{2\omega} \sin 2\omega t - \frac{\ddot{\eta}^2_{(0)}}{4\omega^2} \cos 2\omega t + \eta^2_{(0)} + \frac{\ddot{\eta}^2_{(0)}}{4\omega^2}\,. \tag{8.29}$$

By the exact solutions (8.21), (8.28), (8.29), obtained for the world-lines deviation equation taken in chr.inv.-form (8.14), it follows that:

> Stationary rotations of the space cannot force two neighbouring particles to initiate relative motion, if they are at rest at the initial moment of time.

In common with the result obtained in §7, where we discussed gravitational wave detectors built on free masses, we arrive at a final conclusion for the possibilities of gravitational wave detectors:

> Behaviour of both a gravitational wave detector built on free masses and a sold-body detector (a Weber pig) are **similar**. The only difference is that a solid-body detector can register both the time observable component and spatial observable components of the relative deviation vector, while a free-mass detector can register only spatial observable deviations. Deformations and stationary rotation of the space do not affect detectors of either kind.

Thus neither deformations nor stationary rotation of the space can not induce relative motion in the butt-ends of a Weber detector, if they are at rest. However Weber and his teamregistered signals. The question therefore arises:

> What signals did Weber register, and why, during the past 30 years, have his signals remained undetected by other researchers using superior detectors of the Weber kind?

We assume that the signals registered by Weber and his team, were much more than noise, and beyond doubt. Therefore, according to our theoretical analysis of the behaviour of solid-body detectors in weak gravitational waves, we make the following suppositions:

1. Weber registered signals which were an effect made in the pig by a vortex of the gravitational inertial force field. In other words, the origin of the signals could be rapid non-stationary rotation of a distant object in the depths of space;
2. The particles of the aluminium cylindrical pig, used by Weber, had substantial thermal motions. In this case parametric oscillations could appear as an effect of a falling gravitational wave. But in order to get such a real effect, the "background thermal oscillations" should be substantial;
3. The signals were registered only by Weber and his team. Not one signal has been registered by other experimental physicist during the subsequent 30 years, using superior detectors of the Weber kind. Either Weber registered gravitational waves derived from a non-stationary rotating object in the Universe, which occurred as a unique and short-lived phenomenon, or his original detector had a substantial peculiarity that made it differ in principle from the detectors used by other scientists.

We consider Weber's theory, aiming to ascertain what he registered with his solid-body detector.

9 Criticism of Weber's theory of detecting gravitational waves

In his book in 1960 [16], Weber propounded his theoretical arguments for the detection of gravitational waves by means of a solid-body detector of the resonance kind. He built his theory on the world-lines deviation equation for two particles, connected by a non-gravitational force (a spring, for instance). This is equation (2.12), being a modification of the well-known deviation equation for two free particles deduced by Synge (2.8), is known as the Synge-Weber equation. We considered both equations in detail above.

There is no doubt that the Synge-Weber equation is valid. Our main claim here is that Weber himself, in his analysis of the equation in order to build the theory for detecting gravitational waves, introduced a substantial assumption:

Weber's assumption 1 A falling gravitational wave should produce relative displacements of the butt-ends of a cylindrical pig.

So he obtained the same principle that he introduced, precluding himself from any possibility of obtaining anything else.

This line of reasoning constitutes a vicious circle. It would be been more reasonable and honest to have solved the world-lines deviation equation. Then he would have obtained exact solutions to the equation as was done in the previous sections herein.

In detail Weber's assumption 1 leads to the fact that, having a system of two test-particles connected by a spring, the resulting distance vector between them should be [16]

$$\eta^{\alpha} = r^{\alpha} + \xi^{\alpha}, \qquad r^{\alpha} \gg \xi^{\alpha}, \tag{9.1}$$

where the initial distance vector r^{α} is the such that

$$\frac{\mathrm{D}r^{\alpha}}{ds} = 0\,. \tag{9.2}$$

He supposed as well that $\eta^{\alpha} \to r^{\alpha}$ in the ultimate case where the friction rises infinitely or the Riemann-Christoffel curvature tensor becomes zero $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta} = 0$ [16].

Taking the main supposition (9.1) into account, Weber transforms the Synge-Weber equation (2.12) into

$$\frac{\mathrm{D}^2\xi^{\alpha}}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}\,U^{\beta}U^{\delta}\left(r^{\gamma} + \xi^{\gamma}\right) = \frac{1}{m_0c^2}\,f^{\alpha}, \tag{9.3}$$

where f^{α} is the difference between non-gravitational forces of the particles' interaction. Weber assumes f^{α} the sum of the elasticity force $f^{\alpha}_1 = -\mathcal{K}^{\alpha}_{\sigma}\,\xi^{\sigma}$ that restores the particles, and the oscillation relaxing force $f^{\alpha}_2 = -c\,\mathcal{D}^{\alpha}_{\sigma}\frac{\mathrm{D}\xi^{\sigma}}{ds}$, where $\mathcal{K}^{\alpha}_{\sigma}$ and $\mathcal{D}^{\alpha}_{\sigma}$ are the elasticity and friction coefficients, respectively. Then (9.3) takes the form

$$\frac{\mathrm{D}^2\xi^{\alpha}}{ds^2} + \frac{1}{m_0c}\,\mathcal{D}^{\alpha}_{\sigma}\frac{\mathrm{D}\xi^{\sigma}}{ds} + \frac{1}{m_0c^2}\,\mathcal{K}^{\alpha}_{\sigma}\,\xi^{\sigma} = -R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}\,U^{\beta}U^{\delta}\left(r^{\gamma} + \xi^{\gamma}\right). \tag{9.4}$$

Weber introduced additional substantial assumptions:

Weber's assumption 2 The whole detector is in the state of free falling;

Weber's assumption 3 The reference frame in his laboratory is such that the Christoffel symbols can be assumed zero.

Because of these assumptions, and the condition $|r| \gg |\xi|$, Weber writes equation (9.4) as follows

$$\frac{d^2\xi^{\alpha}}{dt^2} + \frac{1}{m_0}\,\mathcal{D}^{\alpha}_{\sigma}\frac{d\xi^{\sigma}}{dt} + \frac{1}{m_0c^2}\,\mathcal{K}^{\alpha}_{\sigma}\,\xi^{\sigma} = -c^2R^{\alpha\cdot\cdot\cdot}_{\cdot0\sigma0}\,r^{\sigma}. \tag{9.5}$$

Looking at the right side of Weber's equation (9.5) we see his fourth hidden assumption:

Weber's assumption 4 Particles located on two neighbouring world-lines in the Weber experimental statement (the butt-ends of his cylindrical pig) are at rest at the initial moment of time, so $U^i = 0$.

In §7, where we considered chr.inv.-equations of motion for two particles connected by a non-gravitational force (8.4) and (8.5), we came to the conclusion: a reference frame where interacting particles ($\Phi^\alpha \neq 0$) are at rest ($\mathrm{v}^i = 0$) cannot be in astate of free fall. Really, the free fall condition is $F^i = 0$. Equation $m_0 F^i = -f^i$ (8.7), which is the chr.inv.-form of spatial equations of motion of the interacting particles, implies that when $F^i = 0$, $f^i = 0$. Therefore:

The Weber assumption 2 is **inapplicable** to his experimental statement.

Moreover, a reference frame where the Christoffel symbols are zero can be applicable only at a point, it is unapplicable to a finite region. At the same time, in the Weber experimental statement, the detector itself is a system of two particles located at the distance η from each other. In a Riemannian space the Riemann-Christoffel curvature tensor is different from zero, so the Riemannian coherence objects (the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$) cannot be reduced to zero by coordinate transformations. We can merely choose a reference system where, at a given point P, the coherence objects are zero $(\Gamma^\alpha_{\beta\gamma})_P = 0$. Such a reference frame is known as a *geodesic reference frame* [37]. Therefore:

The Weber assumption 3 is **inapplicable** to his experimental statement.

Thus if we retain the rest-condition $U^i = 0$ and the free fall condition in the Weber equation (9.4), there must still be the non-gravitational force $\Phi^\alpha = 0$. So the Weber equation becomes the free particles deviation equation, which in chr.inv.-form is (7.30).

If we reject free fall in the Weber equation (9.4), but retain $U^i = 0$, it takes the same form as (8.14), which is not a free oscillation equation, in which case weak plane gravitational waves can act on the particles only if they are in motion at the initial moment of time.

Collecting these results we conclude that:

The Weber equation (9.4) is **incorrect**, because the free fall condition in common with the rest-condition for two neighbouring particles, connected by a non-gravitational force, lead to the requirement that this force should be zero, thus contradicting the initial conditions of the Weber experimental statement.

It is evident that in aiming to determine the sort of effects a falling gravitational wave has on a free-mass detector or a Weber detector, it would be reasonable to consider a case where the particles are in motion $U^i \neq 0$. In this case, before solving the deviation equation (2.8) for two free particles or (2.12) for two interacting particles (depending on the type of detector used), we should solve the equations of motion for free particles (2.6) or forced particles (2.11), respectively.

It should be noted that the main structure of motion is determined by the left (geometrical) side of equations of motion, while the right side introduces only an additional effect into the motion.

In my previous articles [74, 75, 76] common exact solutions to the geodesic equations and the deviation equation had been obtained in the field of weak plane gravitational waves, described by the metric (6.12). The exact solutions had been obtained in general covariant form.

The solutions to the equations of motion for a free particle, equations (2.6), in a linear polarized harmonic wave $a = A \sin \frac{\omega}{c}(ct \pm x^1)$, $b = 0$ are as follows

$$U^0 + U^1 = \varepsilon = \text{const}\,, \tag{9.6}$$

$$U^1 = -\frac{1}{4\varepsilon}\Big[\big(U^2_{(0)}\big)^2 e^{2A\sin\frac{2\omega}{c}(ct\pm x^1)} + \big(U^3_{(0)}\big)^2 e^{-2A\sin\frac{\omega}{c}(ct\pm x^1)}\Big]_a + U^1_{(0)}\,, \tag{9.7}$$

$$U^2 = U^2_{(0)}\, e^{A\sin\frac{\omega}{c}(ct\pm x^1)}, \tag{9.8}$$

$$U^3 = U^3_{(0)}\, e^{A\sin\frac{\omega}{c}(ct\pm x^1)}, \tag{9.9}$$

where $U^1_{(0)}$, $U^2_{(0)}$, $U^3_{(0)}$ are the initial values of the particle's velocity along each of the spatial axes.

From the solutions two important conclusions follow:

1. A weak plane gravitational wave, falling in the x^1 direction, acts on free particles only if they have non-zero velocities in directions x^2 and x^3 orthogonal to the wave motion.
2. The presence of transverse oscillations in the plane (x^2, x^3) leads also to longitudinal oscillations in the direction x^1.

The solutions to the free-particles deviation equation (2.8) in the field of a weak plane gravitational wave are

$$\begin{aligned}\eta^1 &= \frac{A\Big[\big(U^3_{(0)}\big)^2 - \big(U^2_{(0)}\big)^2\Big]}{2\varepsilon^2}\Big(\eta^1_{(0)} + \dot\eta^1_{(0)}t\Big)\times \\ &\quad\times \sin\frac{\omega}{c}(ct\pm x^1) + \frac{AL}{2\varepsilon\omega}\cos\frac{\omega}{c}(ct\pm x^1) + \\ &\quad+\left\{\dot\eta^1_{(0)} - \frac{A\Big[\big(U^3_{(0)}\big)^2 - \big(U^2_{(0)}\big)^2\Big]}{2\varepsilon^2}\,\omega\eta^1_{(0)}\right\}t + \\ &\quad+\eta^1_{(0)} - \frac{AL}{2\varepsilon}\,,\end{aligned} \tag{9.10}$$

$$\begin{aligned}\eta^2 &= \dot\eta^2_{(0)}\Big[t + \frac{A}{\omega}\cos\frac{\omega}{c}(ct\pm x^1)\Big] + \eta^2_{(0)} - \frac{A}{\omega}\dot\eta^2_{(0)} - \\ &\quad- \frac{AU^2_{(0)}}{\varepsilon}\Big\{\dot\eta^1_{(0)}\Big[t - \frac{1}{\omega}\cos\frac{\omega}{c}(ct\pm x^1)\Big] - \\ &\quad-\Big(\eta^1_{(0)} + \dot\eta^1_{(0)}t\Big)\cos\frac{\omega}{c}(ct\pm x^1) + \frac{\dot\eta^1_{(0)}}{\omega} + \eta^1_{(0)}\Big\},\end{aligned} \tag{9.11}$$

$$\eta^3 = \dot{\eta}^3_{(0)}\left[t - \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + \eta^3_{(0)} + \frac{A}{\omega}\dot{\eta}^3_{(0)} - \\ + \frac{AU^3_{(0)}}{\varepsilon}\left\{\dot{\eta}^1_{(0)}\left[t - \frac{1}{\omega}\cos\frac{\omega}{c}(ct \pm x^1)\right] - \right. \\ \left. - \left(\eta^1_{(0)} + \dot{\eta}^1_{(0)}\,t\right)\cos\frac{\omega}{c}(ct \pm x^1) - \frac{\dot{\eta}^1_{(0)}}{\omega} - \eta^1_{(0)}\right\}, \tag{9.12}$$

where

$$L = U^2_{(0)}\dot{\eta}^2_{(0)} - U^3_{(0)}\dot{\eta}^3_{(0)} = \frac{\eta^1_{(0)}}{\varepsilon}\left[\left(U^3_{(0)}\right)^2 - \left(U^2_{(0)}\right)^2\right]. \tag{9.13}$$

The solutions η^1, η^2, η^3 are the relative deviations of two free particles in directions orthogonal to the direction of the wave's motion. The deviations are actually generalizations of the solutions (7.41) and (7.42), where the particles were at rest. The only difference is that here (9.10–9.12) there are additional parts, where the particles' initial velocities $U^2_{(0)}$ and $U^3_{(0)}$ are added.

Here we see that, besides regular harmonic oscillations, the term $t\cos\frac{\omega}{c}\left(ct \pm x^1\right)$ describes oscillations with an amplitude that increases without bound with time. Another substantial difference is that, in contrast to solutions (7.35), (7.42), (7.43) given for rest-particles, the solutions (9.10), (9.11), (9.12) contain longitudinal oscillations — they are described by solution (9.10). Both harmonic oscillations and unbounded-rising oscillations exist there only if, at the initial moment of time, the particles are in motion along x^2 and x^3 (orthogonal to the x^1 direction of the wave's motion).

So, we come to our final conclusions on both free-mass detectors and solid-body detectors of gravitational waves:

> The greater the velocities of particles (atoms and molecules) in a gravitational wave detector (built on either free masses or of the Weber kind), the more sensitive is the detector to a falling weak plane gravitational wave. In current experiments researchers cool the Weber pigs to super low temperatures, about 2 K, aiming to minimize the inherent oscillations of the particles of which they consist. This is a counterproductive procedure by which experimental physicists actually reduce the sensitivity of the Weber detectors to practically zero. We see the same vicious drawback in current experiments with free-mass detectors, where such a detector consists of two satellites located in the same orbit near the Earth. Because the observer (a laser range-finder) is located in one of the satellites, both satellites are at rest with respect to each other and the observer. All the current experiments cannot register gravitational waves in principle. In a valid experiment for discovering gravitational waves, the particles of which the detector consists must be in as rapid motion as possible. It would be better to design a detector using two laser beams directed parallel to each other, because of the light velocity of the moving particles (photons). The indicative quantities to be observed are the light frequency and phase.

Acknowledgements

I am very grateful to Dmitri Rabounski for his help in this research, over many years, and his substantial additions and corrections to it. Many thanks to Stephen J. Crothers for his corrections and help in editing the manuscript.

References

1. Weber J. *Phys. Rev. Lett.*, 1968, v. 20, 1367.
2. Braginsky V. B., Zeldovich Ya. B., Rudenko V. N. *JETP-Letters*, 1969, v. 10, 441.
3. Kafka P. *Nature*, 1970, v. 226, 436.
4. Braginsky V. B., Zeldovich Ya. B., Rudenko V. N. Measuring the gravitational constant and numerous weak gravitational effects. Moscow, Nauka, 1973 (cited page 8).
5. Braginsky V. B., Manukin A. B., Popov E. J., Rudenko V. N. *Phys. Rev. Ser. A*, 1973, v. 45, 271.
6. Douglas D. N., Gram R. Q., Tyson J. A., Lee R. W. *Phys. Rev. Lett.*, 1975, v. 35, 480.
7. Levine J. L., Garwin R. L. *Phys. Rev. Lett.*, 1973, v. 31, 173.
8. Bramanti D., Maischberger K., Parkinson D. *Nuovo Cimento*, 1973, v. 7, 665.
9. Drever R., Hough J., Bland R., Lessnoff G. *Nature*, 1973, v. 246, 340.
10. Partridge R. B. *Phys. Rev. Lett.*, 1971, v. 26, 912.
11. Bahcall J. N., Davis R. Jr. *Phys. Rev. Lett.*, 1971, v. 26, 662.
12. Adamiantz R. A., Alexeev R. A., Kolosnitzin N. I. *JETP-Letters*, v. 15, 418.
13. Tyson J. A., Maclennan C. G., Lanzorotti L. J. *Phys. Rev. Letters*, 1973, v. 30, 1006.
14. Eddington A. S. The mathematical theory of relativity. Cambridge University Press, Cambridge, 1924 (referred with the 3rd expanded edition, GTTI, Moscow, 1934, 508 pages).
15. Landau L. D. and Lifshitz E. M. The classical theory of fields. GITTL, Moscow, 1939 (ref. with the 4th final exp. edition, Butterworth–Heinemann, 1980).
16. Weber J. General Relativity and gravitational waves. R. Marshak, New York, 1961 (referred with the 2nd edition, Foreign Literature, Moscow, 1962, 271 pages).
17. Synge J. L. Relativity: the General Theory. North Holland, Amsterdam, 1960 (referred with the 2nd expanded edition, Foreign Literature, Moscow, 1963, 432 pages).
18. Pirani F. *Phys. Rev.*, 1957, v. 105, 1089.
19. Bondi H., Pirani F., Robinson J. *Proc. Roy. Soc. A*, 1959, v. 251, 519.
20. Lichnerowicz A. *Compt. Rend. Acad. Sci.*, 1958, v. 246, 893; 1959, v. 248, 2728.
21. Lichnerowicz A. *Ann. mat. pura ed appl.*, 1960, v. 50, 1–95.
22. Bel L. *Compt. Rend. Acad. Sci.*, 1958, v. 247, 1094; 1958, v. 246, 3015.
23. Bel L. *Cahiers Phys.*, 1962, v. 16, No. 138, 59.

24. Bel L. *Colloq. intern. Centre nat. rech. scient.*, Paris, 1962, 119.
25. Debever R. *Compt. Rend. Acad. Sci.*, 1959, v. 249, 1324.
26. Debever R. *Bull. Soc. Math. Belgique*, 1958, v. 10, No. 2, 112.
27. Debever R. *Cahiers Phys.*, 1964, v. 18, 303.
28. Hély J. *Compt. Rend. Acad. Sci.*, 1960, v. 251, 1981; 1961, v. 252, 3754.
29. Trautman A. *Compt. Rend. Acad. Sci.*, 1958, v. 246, 1500.
30. Bondi H. *Rend. School Intern. Phys. "Enrico Fermi"*, N.Y.–London, 1962, 202.
31. Bondi H. *Nature*, 1957, v. 179, 1072.
32. Einstein A., Rosen N. J. *Franklin Inst.*, 1937, v. 223, 43.
33. Rosen N. *Res. Council Israel*, 1954, v. 3, No. 4, 328.
34. Peres A. *Phys. Rev.*, 1960, v. 118, 1105; *Phys. Rev. Lett.*, 1959, v. 3, No. 12.
35. Takeno H. *Tensor*, 1956, v. 6, 15; 1958, v. 8, 59.
36. Takeno H. *Tensor*, 1962, v. 12, 197.
37. Petrov A. Z. Einstein spaces. Pergamon, London, 1969.
38. Kompaneetz A. S. *JETP-USSR*, 1958, v. 34, 953.
39. Robinson J., Trautman A. *Phys. Rev. Lett.*, 1960, v. 4, 431; *Proc. Roy. Soc. A*, 1962, v. 265, 463.
40. Zakharov V. D. To the gravitational wave problem. *Problems in the Theory of Grav. and Elem. Particles*, Moscow, Atomizdat, 1966, v. 1, 114.
41. Borissova L. B. and Zakharov V. D. Gravitational inertial waves in vacuum. The linear approximation. *Izvestia VUZov*, 1974, v. 12, 106.
42. Zelmanov A. L. Chronometric invariants. Dissertation, 1944. First published: CERN, EXT-2004-117, 236 pages.
43. Zelmanov A. L. Chronometric invariants and co-moving coordinates in the general relativity theory. *Doklady Acad. Nauk USSR*, 1956, v. 107 (6), 815–818.
44. Schwarzschild K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsber Akad. Wiss. Berlin*, 1916.
45. Trautman A. *Phys. Rev. Lett.*, 1960, v. 4, 431.
46. Gertzenstein E. M., Pustovoyt D. D. *JETP-USSR*, 1962, v. 22, 222.
47. Pirani F. L. *Gravitation*, Ed. by L. Witten, New York, 1962.
48. Rabounski D. A new method to measure the speed of gravitation. *Progress in Physics*, 2005, v. 1, 3–6.
49. Hafele J. C. Relativistic behaviour of moving terrestrial clocks. *Nature*, July 18, 1970, v. 227, 270–271.
50. Hafele J. C. Relativistic time for terrestrial circumnavigations. *American Journal of Physics*, 1972, v. 40, 81–85.
51. Hafele J. C. and Keating R. E. Around-the-world atomic clocks: predicted relativistic time gains. *Science*, July 14, 1972, v. 177, 166–168.
52. Hafele J. C. and Keating R. E. Around-the-world atomic clocks: observed relativistic time gains. *Science*, July 14, 1972, v. 177, 168–170.
53. Abrams L. S. Black holes: the legacy of Hilbert's error. *Can. J. Phys.*, 1989, v. 67, 919 (see also in arXiv: gr-qc/0102055).
54. Crothers S. J. On the general solution to Einstein's vacuum field and its implications for relativistic degeneracy. *Progress in Physics*, 2005, v. 1, 68–73.
55. Crothers S. J. On the ramifications of the Schwarzschild spacetime metric. *Progress in Physics*, 2005, v. 1, 74–80.
56. Rudenko V. N. Relativistic experiments in gravitational field. *Uspekhi Fizicheskikh Nauk*, 1978, v. 126 (3), 361–401.
57. Zelmanov A. L. Orthometric form of monad formalism and its relations to chronometric and kinemetric invariants. *Doklady Acad. Nauk USSR*, 1976, v. 227 (1), 78–81.
58. Zakharov V. D. Gravitational waves in Einstein's theory of gravitation. Nauka, Moscow, 1972.
59. Schouten J. A. und Struik D. J. Einführung in die neuren Methoden der Differentialgeometrie. *Zentralblatt für Mathematik*, 1935, Bd. 11 und Bd. 19.
60. Eisenhart L. P. *Ann. Math.*, 1938, v. 39, 316.
61. Peres A. *Ann. Phys.*, 1961, v. 14, 419.
62. Nordtvedt K., Pagels H. *Ann. Phys.*, 1962, v. 17, 426.
63. Gigoreva (Borissova) L. B., Zakharov V. D. Spaces of a recursion curvature in the General Theory of Relativity. *Doklady Acad. Sci. USSR*, 1972, v. 207, No. 4, 814.
64. Walker A. G. *Proc. Lond. Math. Soc.*, 1950, v. 52, 36.
65. Collinson C. D., Dodd R. K. *Nuovo Cimento*, 1969, v. 62B, No. 2, 229.
66. Gigoreva (Borissova) L. B. Chronometrically invariant representation of Petrov classification of gravitational fields. *Doklady Acad. Sci. USSR*, 1970, v. 192, No. 6, 1251.
67. Schouten J. A. und Struik D. J. Einführung in die neuren Methoden der Differentialgeometrie. *Zentralblatt für Mathematik*, 1935, Bd. 11 und Bd. 19.
68. Rabounski D. D. and Borissova L. B. Particles here and beyond the Mirror. Editorial URSS, Moscow, 2001, 84 pages.
69. Borissova L. B. and Rabounski D. D. Fields, vacuum, and the mirror Universe. Editorial URSS, Moscow, 2001, 272 pages (the 2nd revised ed.: CERN, EXT-2003-025).
70. Weber J. Gravitation and Relativity. New York and Amsterdam, 1964, 90.
71. Sinsky J. and Weber J. *Phys. Rev. Lett.*, 1967, v. 18, 795.
72. Weber J. *Phys. Rev. Lett.*, 1969, v. 22, 1320.
73. Weber J. *Phys. Rev. Lett.*, 1970, v. 24, 276.
74. Borissova L. B. Relative oscillations of test-particles in accompanying reference frames. *Doklady Acad. Nauk USSR*, 1975, v. 225 (4), 786–789.
75. Borissova L. B. Quadrupole mass-detector in a field of weak flat gravitational waves. *Izvestia VUZov*, *Physica*, 1978, v. 10, 109–114.
76. Borissova L. B. and Zakharov V. D. The system of test-particles in the field of plane gravitational waves. *Izvestia VUZov*, *Physica*, 1976, v. 12, 111–117.

On Geometric Probability, Holography, Shilov Boundaries and the Four Physical Coupling Constants of Nature

Carlos Castro

Center for Theor. Studies of Physical Systems, Clark Atlanta University, Atlanta, Georgia, 30314, USA

E-mail: czarlosromanov@yahoo.com; castro@ctsps.cau.edu

By recurring to Geometric Probability methods, it is shown that the coupling constants, $\alpha_{EM}; \alpha_W; \alpha_C$ associated with Electromagnetism, Weak and the Strong (color) force are given by the *ratios of the ratios* of the measures of the Shilov *boundaries* $Q_2 = S^1 \times RP^1$; $Q_3 = S^2 \times RP^1$; S^5, respectively, with respect to the ratios of the measures $\mu[Q_5]/\mu_N[Q_5]$ associated with the $5D$ conformally compactified real Minkowski spacetime $\bar{M}_5$ that has the same topology as the Shilov boundary Q_5 of the 5 complex-dimensional poly-disc D_5. The homogeneous symmetric complex domain $D_5 = SO(5,2)/SO(5) \times SO(2)$ corresponds to the conformal relativistic *curved* 10 real-dimensional phase space $\mathcal{H}^{10}$ associated with a particle moving in the $5D$ Anti de Sitter space AdS_5. The geometric coupling constant associated to the gravitational force can also be obtained from the ratios of the measures involving Shilov boundaries. We also review our derivation of the observed vacuum energy density based on the geometry of de Sitter (Anti de Sitter) spaces.

1 The fine structure constant and Geometric Probability

Geometric Probability [21] is the study of the probabilities involved in geometric problems, e. g., the distributions of length, area, volume, etc. for geometric objects under stated conditions. One of the most famous problem is the Buffon's Needle Problem of finding the probability that a needle of length l will land on a line, given a floor with equally spaced parallel lines a distance d apart. The problem was first posed by the French naturalist Buffon in 1733. For $l < d$ the probability is

$$P = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{l|\cos(\theta)|}{d} = \frac{4l}{2\pi d} \int_0^{\pi/2} \cos(\theta) = \frac{2l}{\pi d} = \frac{2ld}{\pi d^2}\,. \tag{1}$$

Hence, the Geometric Probability is essentially the *ratio* of the areas of a rectangle of length $2d$, and width l and the area of a circle of radius d. For $l > d$, the solution is slightly more complicated [21]. The Buffon needle problem provides with a numerical experiment that determines the value of π empirically. Geometric Probability is a vast field with profound connections to Stochastic Geometry.

Feynman long ago speculated that the fine structure constant may be related to π. This is the case as Wyler found long ago [1]. We will based our derivation of the fine structure constant based on Feynman's physical interpretation of the electron's charge as the probability amplitude that an electron emits (or absorbs) a photon. The clue to evaluate this probability within the context of Geometric Probability theory is provided by the electron self-energy diagram. Using Feynman's rules, the self-energy $\Sigma(p)$ as a function of the electron's incoming (outgoing) energy-momentum p_μ is given by the integral involving the photon and electron propagator along the internal lines

$$-i\Sigma(p) = (-ie)^2 \times \times \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu \, \frac{i}{\gamma^\rho(p_\rho - k_\rho) - m} \, \frac{-ig_{\mu\nu}}{k^2}\, \gamma^\nu. \tag{2}$$

The integral is taken with respect to the values of the photon's energy-momentum k^μ. By inspection one can see that the electron self-energy is proportional to the fine structure constant $\alpha_{EM} = e^2$, the square of the probability amplitude (in natural units of $\hbar = c = 1$) and physically represents the electron's emission of a virtual photon (off-shell, $k^2 \neq 0$) of energy-momentum k_ρ at a given moment, followed by an absorption of this virtual photon at a later moment.

Based on this physical picture of the electron self-energy graph, we will evaluate the Geometric Probability that an electron emits a photon at $t = -\infty$ (infinite past) and reabsorbs it at a much later time $t = +\infty$ (infinite future). The off-shell (virtual) photon associated with the electron self-energy diagram *asymptotically* behaves on-shell at the very moment of emission ($t = -\infty$) and absorption ($t = +\infty$). However, the photon can remain off-shell in the intermediate region between the moments of emission and absorption by the electron.

The topology of the boundaries (at conformal infinity) of the past and future light-cones are spheres S^2 (the celestial sphere). This explains why the (Shilov) boundaries are essential mathematical features to understand the geometric derivation of all the coupling constants. In order to describe the physics at infinity we will recur to Penrose's ideas [10] of conformal compactifications of Minkowski spacetime by attaching the light-cones at conformal infinity. Not unlike the one-point compactification of the complex plane by adding the points at infinity leading to the Gauss-Riemann sphere. The conformal group leaves the light-cone fixed and it does not alter the causal properties of spacetime despite the rescalings of the metric. The topology of the conformal compactification of real Minkowski spacetime $\bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$ is precisely the same as the topology of the Shilov boundary Q_4 of the 4 complex-dimensional poly-disc D_4. The action of the discrete group Z_2 amounts to an antipodal identification of the future null infinity $\mathcal{I}^+$ with the past null infinity $\mathcal{I}^-$; and the antipodal identification of the past timelike infinity i^- with the future timelike infinity, i^+, where the electron emits, and absorbs the photon, respectively.

Shilov boundaries of homogeneous (symmetric spaces) complex domains, G/K [7, 8, 9] are not the same as the ordinary topological boundaries (except in some special cases). The reason being that the action of the isotropy group K of the origin is not necesarily *transitive* on the ordinary topological boundary. Shilov boundaries are the minimal subspaces of the ordinary topological boundaries which implement the Maldacena-'T Hooft-Susskind *holographic* principle [13] in the sense that the holomorphic data in the interior (bulk) of the domain is fully determined by the holomorphic data on the Shilov boundary. The latter has the property that the maximum modulus of any holomorphic function defined on a domain is attained at the Shilov boundary.

For example, the poly-disc D_4 of 4 complex dimensions is an 8 real-dim Hyperboloid of constant negative scalar curvature that can be identified with the conformal relativistic *curved* phase space associated with the electron (a particle) moving in a $4D$ Anti de Sitter space AdS_4. The poly-disc is a Hermitian symmetric homogeneous coset space associated with the $4D$ conformal group $SO(4,2)$ since $D_4 = SO(4,2)/SO(4) \times SO(2)$. Its Shilov boundary $Shilov$ $(D_4) = Q_4$ has precisely the *same* topology as the $4D$ conformally compactified real Minkowski spacetime $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$. For more details about Shilov boundaries, the conformal group, future tubes and holography we refer to the article by Gibbons [12] and [7, 16].

In order to define the Geometric Probability associated with this process of the electron's emission of a photon at i^- ($t = -\infty$), followed by an absorption at i^+ ($t = +\infty$), we must take into account the important fact that the photon is on-shell $k^2 = 0$ *asymptotically* (at $t = \pm\infty$), but it can move off-shell $k^2 \neq 0$ in the intermediate region which is represented by the *interior* of the conformally compactified real Minkowski spacetime $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$.

Denoting by $\hat{\mu}[Q_4]$ the measure-density (the measure-current) whose *flux* through the future and past celestial spheres S^2 (associated with the future/past light-cones) at timelike infinity i^+, i^-, respectively, is $V(S^2)\hat{\mu}[Q_4]$. The *net* flux through the two celestial spheres S^2 at timelike infinity $i^\pm$ requires an overall factor of 2 giving then the value of $2V(S^2)\hat{\mu}[Q_4]$. The Geometric Probability is defined by the ratio of the measures associated with the celestial spheres S^2 at i^+, i^- timelike infinity, where the photon moves on-shell, relative to the measure of the full *interior* region of $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$, where the photon can move off-shell, as it propagates from i^- to i^+:

$$\alpha = \frac{2V(S^2)\, \hat{\mu}[Q_4]}{\mu[Q_4]}. \tag{3}$$

The ratio $(\hat{\mu}[Q_4]/\mu[Q_4]\ )$ can be re-written in terms of the ratios of the normalized measures of

$$\bar{M}_5 = Q_5 = Shilov\ [D_5] = S^4 \times S^1/Z_2 = S^4 \times RP^1, \tag{4}$$

namely, in terms of the normalized measures of the conformally compactified $5D$ Minkowski spacetime. This is achieved as follows [4]

$$\frac{\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{1}{V(S^4)}\, \frac{\mu_N[Q_5]}{\mu[Q_5]}, \tag{5}$$

resulting from the embeddings (inmersions) of $D_4 \to D_5$.

The origin of the factor $V(S^4)$ in the r. h. s of (5), as one goes from the ratio of measures in Q_4 to the ratio of the measures in Q_5, is due to the reduction from the action of the isotropy group of the origin $SO(5) \times SO(2)$ on Q_5, to the action of the isotropy group of the origin $SO(4) \times SO(2)$ on Q_4, furnishing an overall reduction factor of $V[SO(5)/SO(4)] = V(S^4)$. The 5 complex-dimensional poly-disc $D_5 = SO(5,2)/SO(5) \times SO(2)$ is the 10 real-dim Hyperboloid $\mathcal{H}^{10}$ corresponding to the conformal relativistic curved phase space of a particle moving in $5D$ Anti de Sitter Space AdS_5. This picture is also consistent with the Kaluza-Klein compactification procedure of obtaining $4D$ EM from pure Gravity in $5D$. The $\mathcal{H}^{10}$ can be embedded in the 11-dim pseudo-Euclidean $R^{9,2}$ space, with two-time like directions. This is where 11-dim lurks into our construction.

Next we turn to the Hermitian metric on D_5 constructed by Hua [8] which is $SO(5,2)$-invariant and is based on the Bergmann kernel [15] involving a crucial normalization factor of $1/V(D_5)$. However, the standard normalized measure $\mu_N[Q_5]$ based on the Poisson kernel and involving a normalization factor of $1/V(Q_5)$ is *not* invariant under the full group $SO(5,2)$. It is only invariant under the isotropy group of the origin $SO(5) \times SO(2)$. In order to construct an invariant measure on Q_5 under the full group $SO(5,2)$ one requires to introduce a crucial factor related to the Jacobian measure involving the action of the conformal group $SO(5,2)$ on the full bulk domain D_5. As explained by [4] one has:

$$\frac{\mu_N[Q_5]}{\mu[Q_5]} = \frac{1}{V(Q_5)}\, ||\mathcal{J}_C^{-1}|| =$$

$$= \frac{1}{V(Q_5)}\, \sqrt{||\mathcal{J}_C^{-1}(\mathcal{J}_C^*)^{-1}||} = \frac{1}{V(Q_5)}\, \sqrt{||\mathcal{J}_R^{-1}||} =$$

$$= \frac{1}{V(Q_5)} \sqrt{\sqrt{|\det g|^{-1}}} = \frac{1}{V(Q_5)} [|\det(g)|]^{-\frac{1}{4}} = = \frac{1}{V(Q_5)} [V(D_5)]^{\frac{1}{4}}, \tag{6}$$

the z dependence of the complex Jacobian is no longer explicit because the determinant of the $SO(5,2)$ matrices is unity.

This explains very clearly the origins of the factor $[V(D_5)]^{\frac{1}{4}}$ in Wyler's formula for the fine structure constant [1]. This reduction factor of $V(Q_5)$ is in this case given by $V(D_5)^{\frac{1}{4}}$. As we shall see below, the power of $\frac{1}{4}$ is related to the inverse of the $\dim(S^4) = 4$. This summarizes, briefly, the role of Bergmann kernel [15] in the construction by Hua [8], and adopted by Wyler [1], of the Hermitian metric of a bounded homogenous (symmetric) complex domain. To sum up, we must perform the reduction from $V(Q_5) \to V(Q_5)/V(D_5)^{\frac{1}{4}}$ in the construction of the normalized measure $\mu_N[Q_5]$. This approach is very different than the interpretation given by Smith [3] and later adopted by Smilga [5].

Hence, the Geometric Probability ratio becomes

$$\frac{\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{1}{V(S^4)} \frac{\mu_N[Q_5]}{\mu[Q_5]} = = \frac{1}{V(S^4)} \frac{1}{V(Q_5)} [\, V(D_5)\,]^{\frac{1}{4}} \equiv \frac{1}{\alpha_G}. \tag{7a}$$

This last ratio, for reasons to be explained below, is nothing but the inverse of the geometric coupling strength of gravity, $1/\alpha_G$. The relationship to the gravitational constant is based on the definition of the coupling appearing in the Einstein-Hilbert Lagrangian $(R/16\pi G)$, as follows

$$(16\pi G)(m^2_{Planck}) \equiv \alpha_{EM}\ \alpha_G = 8\pi \ \Rightarrow$$
$$G = \frac{1}{16\pi} \frac{8\pi}{m^2_{Planck}} = \frac{1}{2m^2_{Planck}} \ \Rightarrow \tag{7b}$$
$$Gm^2_{proton} = \frac{1}{2} \left(\frac{m_{proton}}{m_{Planck}} \right)^2 \sim 5.9{\times}10^{-39},$$

and in natural units $\hbar = c = 1$ yields the physical force strength of Gravity at the Planck Energy scale $1.22{\times}10^{19}$ GeV. The Planck mass is obtained by equating the Schwarzschild radius $2\,G\,m_{Planck}$ to the Compton wavelength $1/m_{Planck}$ associated with the mass; where $m_{Planck}\sqrt{2} = 1.22{\times}10^{19}$ GeV and the proton mass is 0.938 GeV. Some authors define the Planck mass by absorbing the factor of $\sqrt{2}$ inside the definition of $m_{Planck} = 1.22{\times}10^{19}$ GeV.

The role of the conformal group in Gravity in these expressions (besides the holographic bulk/boundary AdS/CFT duality correspondence [13]) stems from the MacDowell-Mansouri-Chamseddine-West formulation of Gravity based on the conformal group $SO(3,2)$ which has the same number of 10 generators as the $4D$ Poincare group. The $4D$ vielbein e^a_μ which gauges the spacetime translations is identified with the $SO(3,2)$ generator $A^{[a5]}_\mu$, up to a crucial scale factor R, given by the size of the Anti de Sitter space (de Sitter space) throat. It is known that the Poincare group is the Wigner-Inonu group contraction of the de Sitter Group $SO(4,1)$ after taking the throat size $R = \infty$. The spin-connection ω^{ab}_μ that gauges the Lorentz transformations is identified with the $SO(3,2)$ generator $A^{[ab]}_\mu$. In this fashion, the e^a_μ, ω^{ab}_μ are encoded into the $A^{[mn]}_\mu$ $SO(3,2)$ gauge fields, where m,n run over the group indices $1,2,3,4,5$. A word of caution, Gravity is a gauge theory of the full diffeomorphisms group which is infinite-dimensional and which includes the translations. Therefore, strictly speaking gravity is not a gauge theory of the Poincare group. The Ogiovetsky theorem shows that the diffeomorphisms algebra in $4D$ can be generated by an infinity of *nested* commutators involving the $GL(4,R)$ and the $4D$ Conformal Group $SO(4,2)$ generators.

In [17] we have shown why the MacDowell-Mansouri-Chamseddine-West formulation of Gravity, with a cosmological constant and a topological Gauss-Bonnet invariant term, can be obtained from an action inspired from a BF-Chern-Simons-Higgs theory based on the conformal $SO(3,2)$ group. The AdS_4 space is a natural vacuum of the theory. The vacuum energy density was derived to be the geometric-mean between the UV Planck scale and the IR throat size of de Sitter (Anti de Sitter) space. Setting the throat size to coincide with the future horizon scale (of an accelerated de Sitter Universe) given by the Hubble scale (today) R_H, the geometric mean relationship yields the observed value of the vacuum energy density $\rho \sim (L_P)^{-2}(R_H)^{-2} = (L_P)^{-4} \times (L^2_P/R^2_H) \sim 10^{-122} M^4_{Planck}$. Nottale [23] gave a different argument to explain the small value of ρ based on Scale Relativistic arguments. It was also shown in [17] why the Euclideanized AdS_{2n} spaces are $SO(2n-1,2)$ instantons solutions of a non-linear sigma model obeying a double self duality condition.

Therefore, the Geometric Probability α_{EM} for an electron to emit a photon at $t = -\infty$ and to absorb it at $t = +\infty$ agrees with the Wyler's celebrated expression for the fine structure constant

$$\alpha_{EM} = \frac{2V(S^2)\hat{\mu}[Q_4]}{\mu[Q_4]} = (8\pi) \frac{1}{V(S^4)} \frac{1}{V(Q_5)} \times \times [V(D_5)]^{\frac{1}{4}} = \frac{9}{8\pi^4} \left(\frac{\pi^5}{2^4{\times}5!} \right)^{\frac{1}{4}} = \frac{1}{137.03608}\,, \tag{8}$$

after one inserts the values of the volumes:

$$V(D_5) = \frac{\pi^5}{2^4{\times}5!}\,, \quad V(Q_5) = \frac{8\pi^3}{3}\,, \quad V(S^4) = \frac{8\pi^2}{3}\,. \tag{9}$$

In general

$$V(D_n) = \frac{\pi^n}{2^{n-1}n!}\,, \quad V(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}\,, \tag{10a}$$

$$V(Q_n)=V(S^{n-1}\times RP^1)=V(S^{n-1})\times V(RP^1) = \\ = \frac{2\pi^{n/2}}{\Gamma(n/2)} \times \pi = \frac{2\pi^{(n+2)/2}}{\Gamma(n/2)}\,. \tag{10b}$$

Objections were raised to Wyler's original expression by Robertson [2]. One of them was that the hyperboloids (discs) are not compact and whose volumes diverge since the Lobachevsky metric diverges on the boundaries of the poly-discs. Gilmore explained [2] why one requires to use the Euclideanized regularized volumes as Wyler did. Furthermore, in order to resolve the scaling problems of Wyler's expression, Gilmore showed why it is essential to use dimensionless volumes by setting the throat sizes of the Anti de Sitter hyperboloids to $r = 1$, because this is the only choice for r where all elements in the bounded domains are also coset representatives, and therefore, amount to honest group operations. Hence the so-called scaling objections against Wyler raised by Robertson were satisfactory solved by Gilmore [2].

The question as to *why* the value of α_{EM} obtained in Wyler's formula is precisely the value of α_{EM} observed at the *scale* of the Bohr radius a_B, has not been solved, to my knowledge. The Bohr radius is associated with the ground (most stable) state of the Hydrogen atom [3]. The spectrum generating group of the Hydrogen atom is well known to be the conformal group $SO(4,2)$ due to the fact that there are two conserved vectors, the angular momentum and the Runge-Lentz vector. After quantization, one has two commuting $SU(2)$ copies $SO(4) = SU(2) \times SU(2)$. Thus, it makes physical sense why the Bohr-scale should appear in this construction. Bars [14] has studied the many physical applications and relationships of many seemingly distinct models of particles, strings, branes and twistors, based on the (super) conformal groups in diverse dimensions. In particular, the relevance of two-time physics in the formulation of M, F, S theory has been advanced by Bars for some time. The Bohr radius corresponds to an energy of $137.036\times 2\times 13.6$ eV $\sim 3.72\times 10^3$ eV. It is well known that the Rydberg scale, the Bohr radius, the Compton wavelength of electron, and the classical electron radius are all related to each other by a successive scaling in products of α_{EM}.

2 The fiber bundle interpretation of the Wyler formula

Having found Wyler's expression from Geometric Probability, we shall present a Fiber Bundle interpretation of the Wyler expression by starting with a Fiber bundle E over the base curved-space $D_5 = SO(5,2)/SO(5) \times SO(2)$. The subgroup $H = SO(5)$ of the isotropy group $K = SO(5)\times SO(2)$ acts on the Fibers $F = S^4$ (the internal symmetry space). Locally, and only locally, the Fiber bundle E is the product $D_5 \times S^4$. However, this is *not* true globally. On the Shilov boundary Q_5, the restriction of the Fiber bundle E to the Shilov boundary Q_5 is written by $E|_{Q_5}$ and *locally* is the product of $Q_5 \times S^4$, but this is *not* true globally. For this reason one has that the volume $V(E|_{Q_5}) \neq V(Q_5 \times S^4) = V(Q_5) \times V(S^4)$. But instead, $V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$.

This is the reasoning behind the construction of the quantity $\hat{\mu}[Q_4]/\mu[Q_4]$ that has the units of a density. Its inverse $\mu[Q_4]/\hat{\mu}[Q_4]$ is the volume associated with the restriction of the Fiber Bundle E to the Shilov boundary Q_5: $V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$.

The reason why one embeds $D_4 \to D_5$ and $Q_4 \to Q_5$ is because the space $Q_4 = S^3 \times RP^1$ is *not* large enough to implement the action of the $SO(5)$ group, the compact version of the Anti de Sitter Group $SO(3,2)$ that is required in the MacDowell-Mansouri-Chamseddine-West formulation of Gravity. However, the space $Q_5 = S^4 \times RP^1$ is large enough to implement the action of $SO(5)$ via the internal symmetry space $S^4 = SO(5)/SO(4)$. This justifies the embedding procedure of $D_4 \to D_5$. This Fiber Bundle interpretation is not very different from Smith's interpretation [3]. Following the Fiber Bundle interpretation of the volume $V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$, we will now prove why

$$2\,V(S^2) = \frac{\mu(S^1)}{\hat{\mu}(S^1)} = 8\pi\,. \tag{11}$$

The space S^1 is associated with the $U(1)$ group action and naturally encodes the $U(1)$ gauge invariance linked to Electromagnetism (EM). The result of eq-(11) is what will allow us to *define* α_{EM} as the *ratio of the ratios* of suitable measures in S^1 and Q_4, respectively,

$$\alpha_{EM} = \frac{2V(S^2)\;\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{(\mu(S^1)/\hat{\mu}(S^1))}{(\mu[Q_4]/\hat{\mu}[Q_4])}\,. \tag{12}$$

We may notice that $S^1 \equiv Q_1$ (very special case) since the circle is both the Shilov and ordinary topological boundary of the disc D_1. However, $Q_2 \equiv S^1 \times S^1/Z_2 = S^1 \times RP^1$. Once again, we will write the ratio of the measures in $Q^1 = S^1$ in terms of the ratio of the normalized measures in Q^2 via the reduction from $S^1 \times S^1/Z_2$ to S^1. This requires the embedding (inmersion) of $D_1 \to D_2$ in order to construct the measures on D_1, Q_1 as induced from the measures in D_2, Q_2 resulting from the embedding (inmersion):

$$\frac{\hat{\mu}(S^1)}{\mu(S^1)} = \frac{\hat{\mu}(Q_1)}{\mu(Q_1)} = \frac{1}{V(S^1/Z_2)}\frac{\mu_N[Q_2]}{\mu[Q_2]} = \\ = \frac{1}{V(S^1/Z_2)}\;\frac{1}{(V(Q_2)/V(D_2)}\,. \tag{13}$$

Notice that $\hat{\mu}(S^1)$ as explained before is a measure-density on S^1. Likewise, $\hat{\mu}(Q_4)$ was a measure-density on Q_4. We should not confuse these measure-densities with the normalized measures in one-higher dimension.

By inserting the values of the measures and using

$$V(S^1/Z_2) = V(RP^1) = \pi\,, \quad V(D_2) = \frac{\pi^2}{2\times 2!}\,, \qquad V(Q^2) = \frac{2\pi^2}{\Gamma(1)} = 2\pi^2, \tag{14}$$

it yields then

$$\frac{\mu(S^1)}{\hat\mu(S^1)} = (2\pi^2)\,(\pi)\,\frac{1}{(\pi^2/2\times 2!)} = 8\pi = 2\,V(S^2) \tag{15}$$

as claimed. Therefore, $2V(S^2) = \mu(S^1)/\hat\mu(S^1) = 8\pi$ is the crucial factor appearing in Wyler's formula which admits a natural Geometric probability explanation which is very different from the different interpretations provided in [3, 4, 5].

The Fiber Bundle interpretation associated with the $U(1) \sim SO(2)$ group is the following. The Fiber bundle E is defined over the curved space $D_2 = SO(2,2)/SO(2)\times SO(2)$. The subgroup $H = SO(2) \sim U(1)$ of the isotropy group $K = SO(2)\times SO(2)$ acts on the fibers identified with the symmetry space S^1 (where the $U(1)$ group acts). The Fiber bundle E locally can be written as $D_2 \times S^1$ but not globally. The restriction of the Fiber bundle E to the Shilov boundary $Q_2 = S^1\times S^1/Z_2 = S^1\times RP^1$ is $E|_{Q_2}$ and locally can be written as $Q_2 \times S^1$, but *not* globally. This is why the volume $V(E|_{Q_2}) \neq V(Q_2)\times V(S_1)$ but instead it equals $(V(Q_2)/V(D_2))\times V(S^1/Z_2) = 2V(S^2) = 8\pi$.

Concluding, the Geometric Probability that an electron emits a photon at $t = -\infty$ and absorbs it at $t = +\infty$ is given by the *ratio* of the *ratios* of measures, and it agrees with Wheeler's ideas that one must normalize the couplings with respect to the geometric coupling strength of Gravity:

$$\alpha_{EM} = \frac{2V(S^2)\hat\mu[Q_4]}{\mu[Q_4]} = \frac{(\mu(S^1)/\hat\mu(S^1))}{(\mu[Q_4]/\hat\mu[Q_4])} = = (8\pi)\frac{1}{V(S^4)}\frac{1}{V(Q_5)}[V(D_5)]^{\frac14} = \frac{1}{137.03608}\,. \tag{16}$$

The second important conclusion that can be *derived* from Geometric Probability theory is the general numerical values of the exponents s_n appearing in the factors $V(D_n)^{s_n}$. The normalization factor $V(Q_5)/V(D_5)^{1/4}$ in the construction of the ratio of measures $\mu_N[Q_5]/\mu[Q_5]$ involves in this case powers of the type $V(D_5)^{1/4}$. The power of $\frac14$ is related to the inverse of the $\dim(S^4) = 4$ (the internal symmetry space $SO(5)/SO(4)$). From eq-(13) we learnt that the reduction factor of $V(Q^2)/V(D_2)$ was $V(D_2)$; i. e. the exponent is unity. The power of *unity* is related to the inverse of the $\dim(S^1/Z_2) = 1$. Thus, the arguments based on Geometric Probability leads to normalized measures by factors of $V(Q_n)/V(D_n)^{s_n}$ and whose exponents s_n are given by the *inverse* of the dimensions of the internal symmetry spaces $s_n = (\dim(S^{n-1}))^{-1}$. There is a different interpretation of these factors $V(D_n)^{s_n}$ given by Smith [3]. In general, for other homogeneous complex domains, this power is given by the inverse of the dimension of the internal symmetry space.

3 The weak and strong coupling constants from Geometric Probability

We turn now to the derivation of the other coupling constants. The Fiber Bundle picture of the previous section is essential in our construction. The Weak and the Strong geometric coupling constant strength, defined as the probability for a particle to emit and later absorb a $SU(2), SU(3)$ gauge boson, respectively, can both be obtained by using the main formula derived from Geometric Probability after one identifies the suitable homogeneous domains and their Shilov boundaries to work with. We will show why the weak and strong couplings are given by

$$\alpha_{Weak} = \frac{(\mu[Q_2]/\hat\mu[Q_2])}{(\mu[Q_4]/\hat\mu[Q_4])} = \frac{(\mu[Q_2]/\hat\mu[Q_2])}{\alpha_G} = = \frac{(\mu[Q_2]/\hat\mu[Q_2])}{(8\pi/\alpha_{EM})}\,, \tag{17}$$

and

$$\alpha_{Color} = \frac{(\mu[S^4]/\hat\mu[S^4])}{(\mu[Q_4]/\hat\mu[Q_4]} = \frac{(\mu[S^4]/\hat\mu[S^4])}{\alpha_G} = = \frac{(\mu[S^4]/\hat\mu[S^4])}{(8\pi/\alpha_{EM})}\,. \tag{18}$$

At this point we must emphasize that we define α_{weak}, α_{color} as g_w^2, g_c^2 instead of the conventional $(g_w^2/4\pi)$, $(g_c^2/4\pi)$ definitions used in the Renormalization Group program. The Shilov boundary of (D_2) is $Q_2 = S^1\times RP^1$ but is not large enough to accommodate the action of the isospin group $SU(2)$. One needs a Fiber Bundle over $D_3 = = SO(3,2)/SO(3)\times SO(2)$ whose subgroup $H = SO(3)$ of the isotropy group $K = SO(3)\times SO(2)$ acts on the internal symmetry space S^2 (the fibers). Since the coset space $SU(2)/U(1)$ is a double-cover of the S^2 as one goes from the $SO(3)$ action to the $SU(2)$ action one must take into account an extra factor of 2. This is the reason why one jumps to one-dimension higher from Q_2 to $Q_3 = S^2\times RP^1$, because the coset $SU(2)/U(1)$ is a double-cover of the sphere $S^2 = SO(3)/SO(2)$ and can accommodate the action of the $SU(2)$ group.

By following the same procedure as above, i. e. by re-writing the ratio of the measures $(\hat\mu[Q_2]/\mu[Q_2])$ in terms of the ratio of the measures $(\mu_N[Q_3]/\mu[Q_3])$ via the embeddings of $D_2 \to D_3$, one has

$$(\hat\mu[Q_2]/\mu[Q_2]) = \frac{1}{V(SU(2)/U(1))}\,\frac{\mu_N[Q_3]}{\mu[Q_3]}. \tag{19}$$

Notice that because $SU(2)$ is a $2-1$ covering of the $SO(3)$, this implies that the measure

$$V(SU(2)/U(1)) = 2V(SO(3)/U(1)) = 2V(S^2) = 8\pi\,. \tag{20}$$

As indicated above, because the dimension of the internal symmetry space is $\dim(S^2) = 2$, the construction of the normalized measure $\mu_N[Q_3]$ will require a reduction of $V(Q_3)$ by a factor of $V(D_3)$ raised to the power of $(\dim(S^2))^{-1} = \frac{1}{2}$:

$$\frac{\mu_N[Q_3]}{\mu[Q_3]} = \frac{1}{V(Q_3)/V(D_3)^{1/2}} = \frac{1}{V(Q_3)}\, V(D_3)^{1/2}. \quad (21)$$

Therefore, the ratio of the measures is

$$\frac{\hat{\mu}[Q_2]}{\mu[Q_2]} = \frac{1}{2V(S^2)}\,\frac{1}{V(Q_3)}\, V(D_3)^{1/2}, \quad (22)$$

whose Fiber Bundle interpretation is that the volume of the Fiber Bundle over D_3, but restricted to the Shilov boundary Q_3, and whose structure group is $SU(2)$ (the double cover of $SO(3)$), is $V(E|_{Q_3}) = 2V(S^2) \times (V(Q_3)/V(D_3)^{1/2})$. Thus, that the Geometric probability expression is

$$\alpha_{Weak} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(\mu[Q_4]/\hat{\mu}[Q_4])} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(8\pi/\alpha_{EM})} = = 2V(S^2)V(Q_3)\frac{1}{V(D_3)^{1/2}}\frac{\alpha_{EM}}{8\pi} = 0.2536, \quad (23)$$

that corresponds to the weak geometric coupling constant α_W at an energy of the order of

$$E = M = 146 \text{ GeV} \sim \sqrt{M^2_{W_+} + M^2_{W_-} + M^2_Z}\,, \quad (24)$$

after we have inserted the expressions

$$V(S^2) = 4\pi\,, \quad V(Q_3) = 4\pi^2\,, \quad V(D_3) = \frac{\pi^3}{24}\,, \quad (25a)$$

into the formula (23). The relationship to the Fermi coupling G_{Fermi} goes as follows (after indentifying the energy scale $E = M = 146$ GeV):

$$G_F \equiv \frac{\alpha_W}{M^2} \Rightarrow G_F\, m^2_{proton} = \left(\frac{\alpha_W}{M^2}\right) m^2_{proton} = = 0.2536 \times \left(\frac{m_{proton}}{146 \text{ GeV}}\right)^2 \sim 1.04{\times}10^{-5} \quad (25b)$$

in very good agreement with experimental observations.

Once more, it is unknown why the value of α_{Weak} obtained from Geometric Probability corresponds to the energy scale related to the W_+, W_-, Z_0 boson mass, after spontaneous symmetry breaking.

Finally, we shall derive the value of α_{Color} from eq-(18). Since S^4 is not large enough to accommodate the action of the color group $SU(3)$ one needs to work with one-dimension higher S^5 , that can be interpreted as the boundary of the $6D$ Ball $B_6 = SU(4)/U(3) = SU(4)/SU(3) \times U(1)$. Thus, the $SU(3)$ group is part of the isotropy group $K = = SU(3) \times U(1)$ that defines the coset space B_6. In this special case the Shilov and ordinary topological boundaries of B_6 coincide with S^5 [3]. Hence, following the same procedures as above, the ratio of the measures in S^4 (boundary of B_5) can be re-written in terms of the ratio of the measures in S^5 (boundary of B_6) via the embeddings of $B_5 \to B_6$ as follows:

$$\frac{\hat{\mu}[S^4]}{\mu[S^4]} = \frac{1}{V(S^4)}\,\frac{\mu_N[S^5]}{\mu[S^5]} = \frac{1}{V(S^4)}\,\frac{1}{V(S^5)/V(B_6)^{1/4}} = = \frac{1}{V(S^4)}\,\frac{1}{V(S^5)}\, V(B_6)^{1/4}, \quad (26)$$

since the exponent of the reduction factor $V(B_6)^{1/4}$ is given by $(\dim(S^4))^{-1} = \frac{1}{4}$. Notice, again, that $\hat{\mu}[S^4]$ is the measure-density in S^4 and must not be confused with the normalized measures.

Therefore, one arrives at

$$\alpha_{Color} = V(S^4)\, V(S^5)\, \frac{1}{V(B_6)^{1/4}}\, \frac{\alpha_{EM}}{8\pi} = 0.6286, \quad (27)$$

that corresponds to the strong coupling constant at an energy related to the pion masses [3]:

$$E = 241 \text{ MeV} \sim \sqrt{m^2_{\pi^+} + m^2_{\pi^-} + m^2_{\pi^0}} \quad (28)$$

and where we have used the expressions:

$$V(S^4) = \frac{8\pi^2}{3}\,, \quad V(S^5) = 4\pi^3\,, \quad V(B_6) = \frac{\pi^3}{6}\,. \quad (29)$$

The pions are the known lightest quark/antiquark pairs that feel the strong interaction [3]. For a detailed analysis of volumes of compact manifolds (coset spaces) see [24].

Once again, it is unknown why the value of α_{Color} obtained from Geometric Probability (28) corresponds to the energy scale related to the masses of the three pions [3]. Masses of the fundamental particles were derived in [3] based on the definitions that mass is the probability amplitude for a particle to change direction.

To conclude, by defining the geometric coupling constants $\alpha = g^2$ as the Geometric Probability to emit (and later absorb) a gauge boson, all the three geometric coupling constants, $\alpha_{EM}; \alpha_{Weak}; \alpha_{Color}$ are given by the *ratios of the ratios* of the measures of the Shilov *boundaries* $Q_2 = S^1 \times RP^1$; $Q_3 = S^2 \times RP^1$; S^5, respectively, with respect to the ratios of the measures $\mu[Q_5]/\mu_N[Q_5]$ associated with the $5D$ conformally compactified real Minkowski spacetime $\bar{M}_5$ that has the same topology as the Shilov boundary Q_5 of the 5 complex-dimensional poly-disc D_5. The latter corresponds to the conformal relativistic *curved* 10 real-dimensional phase space $\mathcal{H}^{10}$ associated with a particle moving in the $5D$ Anti de Sitter space AdS_5. The ratios of particle masses, like the proton to electron mass ratio $m_p/m_e \sim 6\pi^5$ has also been calculated using the volumes of homogeneous bounded domains [3, 4].

It is not known whether this procedure would work for Grand Unified Theories based on the groups

$$SU(5), SO(10), E_6, E_7, E_8\,. \qquad (30)$$

Beck [6] has obtained all the Standard Model parameters by studying the numerical minima (and zeros) of certain potentials associated with the Kaneko coupled two-dim lattices based on Stochastic Quantization methods. The results above and by Smith [3] are analytical rather than being numerical [6] and it is not clear if there is any relationship between these two approaches. Noyes has proposed an iterated numerical hierarchy based on Mersenne primes $M_p = 2^p - 1$ for certain values of $p =$ primes [18] and obtained many numerical values for the physical parameters. Pitkanen has developed methods to calculate the physical masses recurring to a p-adic hierarchy of scales based on Mersenne primes [19].

An important connection between anomaly cancellation in string theory and perfect even numbers was found in [22]. These are numbers which can be written in terms of sums of its divisors, including unity, like $6 = 1 + 2 + 3$, and are of the form $P(p) = \frac{1}{2}\, 2^p(2^p - 1)$ if, and only if, $2^p - 1$ is a Mersenne prime. Not all values of $p =$ prime yields primes. The number $2^{11} - 1$ is not a Mersenne prime, for example. The number of generators of the anomaly free groups $SO(32)$, $E_8 \times E_8$ of the 10-dim superstring is 496 which is an even perfect number. Another important group related to the unique tadpole-free bosonic string theory is the $SO(2^{13}) = SO(8192)$ group related to the bosonic string compactified on the $E_8 \times SO(16)$ lattice. The number of generators of $SO(8192)$ is an even perfect number since $2^{13} - 1$ is a Mersenne prime. For an introduction to p-adic numbers in Physics and String theory see [20]. A lot more work needs to be done to be able to answer the question: Is all this just a mere numerical coincidence or is it design?

4 Acknowledgements

We are indebted to Frank (Tony) Smith and C. Beck for numerous discussions about their work. To M. Bowers and C. Handy (CTSPS) for their kind support.

References

1. Wyler A. *Comptes Rendus Acad. Sci. Paris*, 1969, v. A269, 743. *Comptes Rendus Acad. Sci. Paris*, 1971, v. A272, 186.
2. Gilmore R. *Phys. Rev. Lett.*, 1972, v. 28, No. 7, 462. Robertson B. *Phys. Rev. Lett.*, 1972, v. 27, 1845.
3. Smith Jr F. D., *Int. J. Theor. Phys.*, 1985, v. 24, 155; *Int. J. Theor. Phys.*, 1985, v. 25, 355. From sets to quarks. arXiv: hep-ph/9708379, CERN CDS EXT-2003-087.
4. Gonzalez-Martin G. Physical Geometry. Univ. of Simon Bolivar Publ., Caracas, 2000. The proton/electron geometric mass ratio. arXiv: physics/0009052. The fine structure constant from relativistic groups. arXiv: physics/ 0009051.
5. Smilga W. Higher order terms in the contraction of $S0(3, 2)$. arXiv: hep-th/0304137.
6. Beck C. Spatio-temporal vacuum fluctuations of quantized fields. World Scientific, Singapore 2002.
7. Coquereaux R., Jadcyk A. *Reviews in Mathematical Physics*, 1990, No. 1, v. 2, 1–44.
8. Hua L. K. Harmonic analysis of functions of several complex variables in the classical domains. Birkhauser, Boston-Basel-Berlin, 2000.
9. Faraut J., Kaneyuki S., Koranyi A., Qi-Keng Lu and Roos G. Analysis and geometry on complex homogeneous domains. *Progress in Math.*, v. 185, Birkhauser, Boston-Basel-Berlin.
10. Penrose R., Rindler W. Spinors and space-time. Cambridge University Press, 1986.
11. Hugget S.A., Todd K. P. An introduction to twistor theory. *London Math. Soc. Stud. Texts*, v. 4, Cambridge Univ. Press, 1985.
12. Gibbons G. Holography and the future tube. arXiv: hep-th/ 9911027.
13. Maldacena J. The large *N* limit of superconformal theories and supergravity. arXiv: hep-th/9711200.
14. Bars I. Twistors and two-time physics. arXiv: hep-th/0502065.
15. Bergman S. The kernel function and conformal mapping. *Math. Surveys*, 1970, v. 5, AMS, Providence.
16. Odzijewicz A. *Int. Jour. of Theor. Phys.*, 1986, v. 107, 561–575.
17. Castro C. *Mod. Phys. Letts.*, 2002, v. A17, No. 32, 2095–2103. *Class. Quan. Grav.*, 2003, v. 20, 3577–3592.
18. Noyes P. Bit-strings physics: a discrete and finite approach to natural philosophy. *Series in Knots in Physics*, v. 27, Singapore, World Scientific, 2001.
19. Pitkanen M. Topological Geometrodynamics I, II. *Chaos, Solitons and Fractals*, 2002, v. 13, No. 6, 1205, 1217.
20. Vladimorov V., Volovich I. and Zelenov I. p-Adic numbers in mathematical physics. Singapore, World Scientific, 1994. Brekke L., Freund P. *Physics Reports*, 1993, v. 1, 231.
21. Ambartzumian R. V. (Ed.) Stochastic and integral geometry. Dordrecht (Netherlands), Reidel, 1987. Kendall M. G. and Moran P. A. P. Geometric probability. New York, Hafner, 1963. Kendall W.S, Barndorff-Nielson O. and van Lieshout M. C. Current trends in stochastic geometry: likelihood and computation. Boca Raton (FL), CRC Press, 1998. Klain D. A. and Rota G.-C. Introduction to geometric probability. New York, Cambridge Univ. Press, 1997. Santalo L. A. Integral geometry and geometric probability. Reading (MA) Addison-Wesley, 1976. Solomon H. Geometric probability. Philadelphia (PA), SIAM, 1978. Stoyan D., Kendall W. S. and Mecke J. Stochastic geometry and its applications. New York, Wiley, 1987. http://mathworld.wolfram.com/GeometricProbability.html
22. Frampton P., Kephart T. *Phys. Rev.*, 1999, v. D60, 08790.
23. Nottale L. Fractal spacetime and microphysics, towards the theory of scale relativity. World Scientific, Singapore 1992. *Chaos, Solitons and Fractals*, 2003, v. 16, 539.
24. Boya L., Sudarshan E. C. G. and Tilma T. Volumes of compact manifolds. arXiv: math-ph/0210033.

On the Generalisation of Kepler's 3rd Law for the Vacuum Field of the Point-Mass

Stephen J. Crothers

Sydney, Australia

E-mail: thenarmis@yahoo.com

I derive herein a general form of Kepler's 3rd Law for the general solution to Einstein's vacuum field. I also obtain stable orbits for photons in all the configurations of the point-mass. Contrary to the accepted theory, Kepler's 3rd Law is modified by General Relativity and leads to a finite angular velocity as the proper radius of the orbit goes down to zero, without the formation of a black hole. Finally, I generalise the expression for the potential function of the general solution for the point-mass in the weak field.

1 Introduction

In previous papers [1, 2] I derived the general solution for Einstein's vacuum field and showed that black holes do not exist in Einstein's universe. In those papers I also obtained expressions for Kepler's 3rd Law for the simple (i. e. non-rotating) point-mass and the simple point-charge. In this paper I obtain expressions for Kepler's 3rd Law for the rotating point-mass and the rotating point-charge. Owing to the rotation of the source of the field, Kepler's 3rd Law for the polar orbit is not the same as that for the equatorial orbit, so that stable photon orbits are also different in the polar and equatorial orbits, showing that in the rotating configurations spacetime is no longer isotropic.

The expressions I obtain readily reduce to those I have previously derived for the non-rotating configurations.

2 Definitions

I have already shown [3] that the most general static metric for the point-mass is,

$$ds^2 = A(D)dt^2 - B(D)dD^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$D = |r - r_0|,$$

$$A, B, C > 0,$$

where r_0 is an arbitrary real number. With respect to this metric I identify the coordinate radius, the r-parameter, the radius of curvature, and the proper radius thus:

1. *The coordinate radius is* $D = |r - r_0|$.
1. *The* r*-parameter is the variable* r.
2. *The radius of curvature is* $R = \sqrt{C(D)}$.
3. *The proper radius is* $R_p = \int \sqrt{B(D)}\, dD$.

3 The equatorial orbit

The general Kerr-Newman form in Boyer-Lindquist coordinates is,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(R^2+a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\theta^2.$$

This can be written as,

$$ds^2 = \left(\frac{\Delta - a^2\sin^2\theta}{\xi}\right)dt^2 - \frac{\xi}{\Delta}dR^2 - \xi d\theta^2 + \left[\frac{a^2\Delta\sin^4\theta - \left(R^2+a^2\right)^2\sin^2\theta}{\xi}\right]d\varphi^2 - \left[\frac{2a\Delta\sin^2\theta - 2a\left(R^2+a^2\right)\sin^2\theta}{\xi}\right]dt d\varphi\,, \quad (1)$$

where I have previously shown [2, 3] in the case of the rotating point-charge,

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re,\ r \in \Re,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,$$

$$a^2 + q^2 < m^2, \quad n \in \Re^+, \quad \xi = \rho^2 = R^2 + a^2\cos^2\theta\,,$$

$$a = \frac{L}{m}, \quad \Delta = R^2 - \alpha R + a^2 + q^2\,,$$

$$0 < |r - r_0| < \infty\,,$$

where L is the angular momentum, and n and r_0 are arbitrary.

I have also shown previously that Kepler's 3rd Law for the simple point-mass is,

$$\omega^2 = \frac{\alpha}{2R^3}\,, \quad (2)$$

where

$$\lim_{r \to r_0^{\pm}} \sqrt{C_n(r)} = R_0 = \alpha = 2m\ \forall\ r_0\,,$$

is a scalar invariant; and for the simple point-charge is,

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}\,, \quad (3)$$

where, $\forall\, r_0$,

$$\lim_{r \to r_0^{\pm}} \sqrt{C_n(r)} = R_0 = \beta = m + \sqrt{m^2 - q^2}, \;\; q^2 < m^2\,,$$

is a scalar invariant.

In the case of the equatorial orbit, $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$, so (1) becomes,

$$\begin{aligned} ds^2 = & \left(\frac{\Delta - a^2}{\xi}\right) dt^2 - \frac{\xi}{\Delta} dR^2 + \\ & + \left[\frac{a^2\Delta - \left(R^2 + a^2\right)^2}{\xi}\right] d\varphi^2 - \\ & - \left[\frac{2a\Delta - 2a\left(R^2 + a^2\right)}{\xi}\right] dt d\varphi\,. \end{aligned} \tag{4}$$

$$R^2 = C_n(r) = \Big(\left|r - r_0\right|^n + \beta^n\Big)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,,\;\; q^2 < m^2\,,$$

$$\xi = R^2\,,\;\; \Delta = R^2 - \alpha R + a^2 + q^2\,,$$

$$0 < |r - r_0| < \infty\,.$$

Consider the associated Lagrangian, where the dot indicates $\partial/\partial\tau$,

$$\begin{aligned} L = & \frac{1}{2}\left[\frac{\Delta - a^2}{\xi}\dot{t}^2 - \frac{\xi}{\Delta}\dot{R}^2\right] + \\ & + \frac{1}{2}\left[\frac{a^2\Delta - \left(R^2 + a^2\right)^2}{\xi}\right]\dot{\varphi}^2 - \\ & - \frac{1}{2}\left[\frac{2a\Delta - 2a\left(R^2 + a^2\right)}{\xi}\right]\dot{t}\,\dot{\varphi}\,. \end{aligned} \tag{5}$$

Then,

$$\begin{aligned} & \frac{\partial L}{\partial R} - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{R}}\right) = 0 \Rightarrow \frac{\xi\Delta' - \xi'\left(\Delta - a^2\right)}{2\xi^2}\dot{t}^2 + \\ & + \frac{\xi\left[a^2\Delta' - 4R\left(R^2 + a^2\right)\right]}{2\xi^2}\dot{\varphi}^2 - \\ & - \frac{\xi'\left[a^2\Delta - \left(R^2 + a^2\right)^2\right]}{2\xi^2}\dot{\varphi}^2 - \\ & - \frac{\xi\left(2a\Delta' - 4aR\right) - \xi'\left[2a\Delta - 2a\left(R^2 + a^2\right)\right]}{2\xi^2}\dot{t}\dot{\varphi} + \\ & + \frac{\Delta\xi' - \xi\Delta'}{2\Delta^2}\dot{R}^2 + \frac{\xi}{\Delta}\ddot{R} = 0\,. \end{aligned} \tag{6}$$

Taking $R = \text{const.}$ reduces (6) to,

$$\begin{aligned} & \Big\{\xi\left[a^2\Delta' - 4R\left(R^2 + a^2\right)\right] - \\ & - \xi'\left[a^2\Delta - \left(R^2 + a^2\right)^2\right]\Big\}\omega^2 - \\ & - \left\{\xi\left(2a\Delta' - 4aR\right) - \xi'\left[2a\Delta - 2a\left(R^2 + a^2\right)\right]\right\}\omega + \\ & + \xi\Delta' - \xi'\left(\Delta - a^2\right) = 0\,, \end{aligned} \tag{7}$$

where $\omega = \frac{\dot{\varphi}}{\dot{t}}$. The solutions for ω are,

$$\omega = \frac{a\alpha R - 2aq^2 \pm R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4}\,.$$

In order for this to reduce to the non-rotating configurations, the plus sign must be taken so,

$$\omega = \frac{a\alpha R - 2aq^2 + R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4}\,, \tag{8}$$

$$R^2 = C_n(r) = \Big(\left|r - r_0\right|^n + \beta^n\Big)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,,\;\; q^2 < m^2\,,$$

$$\alpha = 2m\,,$$

$$0 < |r - r_0| < \infty\,.$$

Equation (8) is Kepler's 3rd Law for the equatorial plane of the rotating point-charge. I remark that the radius of curvature in the equatorial orbit is precisely that for the simple point-charge. The expression for Kepler's 3rd Law for the equatorial plane of the rotating point-mass is obtained from (8) by setting $q = 0$,

$$\omega = \frac{a\alpha R + R^2\sqrt{2\alpha R}}{a^2\alpha R - 2R^4}\,,$$

$$R^2 = C_n(r) = \Big(\left|r - r_0\right|^n + \alpha^n\Big)^{\frac{2}{n}}\,,$$

$$\alpha = 2m\,,$$

$$0 < |r - r_0| < \infty\,,$$

in which case the radius of curvature in the equatorial orbit is precisely that for the simple point-mass.

Taking the near-field limit on (8) gives,

$$\lim_{r \to r_0^{\pm}} \omega = \frac{a\alpha\beta - 2aq^2 + \beta^2\sqrt{2\alpha\beta - 4q^2}}{a^2\alpha\beta - 2a^2q^2 - 2\beta^4}\,, \tag{9}$$

which is a scalar invariant.

When $a = 0$ and $q \neq 0$, equation (8) reduces to,

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}\,,$$

which recovers Kepler's 3rd Law (3) for the simple point-charge. If $a = q = 0$, equation (8) reduces to,

$$\omega^2 = \frac{\alpha}{2R^3}\,,$$

$$\beta = \alpha = 2m\,,$$

which recovers Kepler's 3rd Law (2) for the simple point-mass.

When $a = 0$ and $q \neq 0$, (9) reduces in the near-field limit, to

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{\alpha}{2\beta^3} - \frac{q^2}{\beta^4}\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,,$$

the scalar invariant of Kepler's 3rd Law for the simple point-charge; and when $a = q = 0$, (9) reduces to the near-field limit,

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{1}{2\alpha^2}\,,$$

$$\alpha = 2m\,,$$

the scalar invariant for Kepler's 3rd Law for the simple point-mass, as originally obtained by Karl Schwarzschild [4] for his particular solution.

4 Photons in equatorial orbit

Setting $\theta = \frac{\pi}{2}$ in (1) and setting (1) equal to zero gives,

$$\begin{aligned} &\left[a^2\Delta - \left(R^2 + a^2\right)^2\right]\omega^2 - \\ &- \left[2a\Delta - 2a\left(R^2 + a^2\right)\right]\omega + \left(\Delta - a^2\right) = 0\,, \end{aligned} \tag{10}$$

from which it follows,

$$\omega = \frac{\dot\varphi}{\dot t} = \frac{a\left(q^2 - \alpha R\right) + R^2\sqrt{R^2 - \alpha R + a^2 + q^2}}{a^2q^2 - \alpha a^2 R - a^2R^2 - R^4}\,. \tag{11}$$

Equating (8) to (11) gives,

$$\begin{aligned} &\frac{a\alpha R - 2aq^2 + R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4} = \\ &= \frac{a\left(q^2 - \alpha R\right) + R^2\sqrt{R^2 - \alpha R + a^2 + q^2}}{a^2q^2 - \alpha a^2R - a^2R^2 - R^4}\,, \end{aligned} \tag{12}$$

$$\alpha = 2m\,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2},\ q^2 < m^2\,,$$

$$r_0 \in \Re,\ n \in \Re^+\,,$$

for the radius of curvature $R_{ph-e} = R = \sqrt{C_n(r_{ph-e})}$ of the equatorial orbit of a photon for the rotating point-charge. When $a = 0$ equation (12) reduces to,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = \frac{3\alpha + \sqrt{9\alpha^2 - 32q^2}}{4}\,,$$

recovering the stable radius of curvature for the photon orbit about the simple point-charge [2]. When $a = q = 0$, equation (12) reduces to,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = \frac{3\alpha}{2} = 3m\,, \tag{13}$$

which recovers the stable radius of curvature for the photon around the simple point-mass [1].

When $n = 1$ and $r_0 = \alpha$, equation (13) gives,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = r_{ph-e} = 3m\,,$$

This radius is taken incorrectly by the orthodox relativists as a measurable proper radius in the gravitational field of the simple point-mass. The actual proper radius associated with (13) is,

$$R_p = \frac{\alpha\sqrt{3}}{2} + \alpha\ln\left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right),$$

which is a scalar invariant for the photon orbit about the point-mass.

The expression for the radius of curvature of the stable photon equatorial orbit for the rotating point-mass is obtained from (12) by setting $q = 0$, thus

$$\frac{a\alpha R + R^2\sqrt{2\alpha R}}{a^2\alpha R - 2R^4} = \frac{a\alpha R - R^2\sqrt{R^2 - \alpha R + a^2}}{\alpha a^2R + a^2R^2 + R^4}\,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}\,,$$

$$\alpha = 2m\,,$$

$$r_0 \in \Re,\ n \in \Re^+\,.$$

5 The polar orbit

According to (1), if $R = \sqrt{C_n(r)}$ is a function of t,

$$R = R(t, \theta) = = \sqrt{C_n(r(t))} = \left(\left|r(t) - r_0\right|^n + \beta^n\right)^{\frac{1}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,,$$

so if $\dot r = 0$, $\dot R = 0$.

In the polar orbit there is no loss of generality in taking $\varphi = \text{const.}$, $\dot\varphi = 0$. Then (1) becomes,

$$ds^2 = \frac{\Delta - a^2\sin^2\theta}{\xi}dt^2 - \frac{\xi}{\Delta}dR^2 - \xi d\theta^2\,, \tag{14}$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}},\quad r_0 \in \Re,\ r \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,$$

$$a^2 + q^2 < m^2\,,\quad n \in \Re^+,\quad \xi = \rho^2 = R^2 + a^2\cos^2\theta\,,$$

$$a = \frac{L}{m}\,,\quad \Delta = R^2 - \alpha R + a^2 + q^2\,,$$

$$0 < \left|r - r_0\right| < \infty\,.$$

Consider the associated Lagrangian,

$$L = \frac{1}{2}\left[\frac{\Delta - a^2\sin^2\theta}{\xi}\dot t^2 - \frac{\xi}{\Delta}\dot R^2 - \xi\dot\theta^2\right].$$

Then,

$$\frac{\partial L}{\partial R} - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{R}}\right) = \frac{1}{2}\left[\frac{\xi\Delta' - \xi'(\Delta - a^2\sin^2\theta)}{\xi^2}\dot{t}^2\right] + - \frac{1}{2}\left[\frac{(\Delta\xi' - \xi\Delta')}{\Delta^2}\dot{R}^2 + \xi'\dot{\theta}^2\right] + \frac{\xi}{\Delta}\ddot{R} = 0\,. \tag{15}$$

If $\dot{R} = 0$, then (15) yields,

$$\omega^2 = \frac{\dot{\theta}^2}{\dot{t}^2} = \frac{\xi\Delta' - \xi'(\Delta - a^2\sin^2\theta)}{\xi'\xi^2} = = \frac{\alpha R^2 - \alpha a^2\cos^2\theta - 2q^2 R}{2R\left(R^2 + a^2\cos^2\theta\right)^2} = = \frac{\alpha C_n - \alpha a^2\cos^2\theta - 2q^2\sqrt{C_n}}{2\sqrt{C_n}\left(C_n + a^2\cos^2\theta\right)^2}\,, \tag{16}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,\ a^2 + q^2 < m^2\,,$$
$$n \in \Re^+\ r_0 \in \Re\,,$$
$$0 < |r - r_0| < \infty\,.$$

Equation (16) is Kepler's 3rd Law for the polar orbit of the rotating point-charge. I remark that the angular velocity depends upon azimuth.

Let $a = 0,\ q \neq 0$, then (16) reduces to,

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}\,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}},\ \beta = m + \sqrt{m^2 - q^2},$$
$$q^2 < m^2,\ r_0 \in \Re,\ n \in \Re^+\,,$$
$$0 < |r - r_0| < \infty,$$

which recovers Kepler's 3rd Law (3) for the simple point-charge. Setting $a = q = 0$ reduces (16) to,

$$\omega^2 = \frac{\alpha}{2R^3}\,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}\,,$$
$$n \in \Re^+,\ r_(0) \in \Re\,,$$
$$0 < |r - r_0| < \infty,$$

which recovers Kepler's 3rd Law (2) for the simple point-mass.

Taking the near-field limit on (16),

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{\alpha\beta^2 - \alpha a^2\cos^2\theta - 2q^2\beta}{2\beta\left(\beta^2 + a^2\cos^2\theta\right)^2}\,, \tag{17}$$

which is a scalar invariant, subject to azimuth, for the polar orbit of the rotating point-charge.

When $q = 0,\ a \neq 0$, equation (16) reduces to,

$$\omega^2 = \frac{\alpha R^2 - \alpha a^2\cos^2\theta}{2R\left(R^2 + a^2\cos^2\theta\right)^2} = = \frac{\alpha C_n - \alpha a^2\cos^2\theta}{2\sqrt{C_n}\left(C_n + a^2\cos^2\theta\right)^2}\,, \tag{18}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,,\ a^2 < m^2\,,$$
$$n \in \Re^+\ r_0 \in \Re\,,$$
$$0 < |r - r_0| < \infty\,.$$

This is Kepler's 3rd Law for the polar orbit of the rotating point-mass.

Taking the near-field limit on (18),

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{\alpha\beta^2 - \alpha a^2\cos^2\theta}{2\beta\left(\beta^2 + a^2\cos^2\theta\right)^2}\,, \tag{19}$$

which is a scalar invariant, subject to azimuth, for the polar orbit of the rotating point-mass.

Thus, ω varies with azimuth as does $R = \sqrt{C_n(r)}$. At the poles of the rotating point-charge,

$$R^2 = C_n(r) = \left(|r - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,,$$
$$\beta = m + \sqrt{m^2 - a^2 - q^2}\,, \tag{20}$$
$$\omega^2 = \frac{\alpha R^2 - \alpha a^2 - 2q^2 R}{2R\left(R^2 + a^2\right)^2}\,,$$

and at the equator,

$$R^2 = C_n(r) = \left(|r - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,,$$
$$\beta = m + \sqrt{m^2 - q^2}\,, \tag{21}$$
$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}\,.$$

It is noted that at the momentary equator in a polar orbit, the radius of curvature and Kepler's 3rd Law are precisely those for the simple point-charge.

In the case of the rotating point-mass, at the poles,

$$R^2 = C_n(r) = \left(|r - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,,$$
$$\beta = m + \sqrt{m^2 - a^2}\,, \tag{22}$$
$$\omega^2 = \frac{\alpha R^2 - \alpha a^2}{2R\left(R^2 + a^2\right)^2}\,,$$

and at the equator,

$$R^2 = C_n(r) = \left(|r - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,,$$
$$\beta = 2m = \alpha\,, \tag{23}$$
$$\omega^2 = \frac{\alpha}{2R^3}\,.$$

At the momentary equator in a polar orbit the radius of curvature and Kepler's 3rd Law are precisely those for the simple point-mass.

6 Photons in the polar orbit

Setting (14) equal to zero, with $\dot{R} = 0$, gives

$$\omega^2 = \frac{\Delta - a^2 \sin^2\theta}{\xi^2} = \frac{R^2 - \alpha R + a^2\cos^2\theta + q^2}{\left(R^2 + a^2\cos^2\theta\right)^2}\,. \qquad (24)$$

Denote the stable photon radius of curvature for a photon in polar orbit by $R_{ph-p} = \sqrt{C_n(r_{ph-p})}$. Then equating (24) to (16) gives,

$$2R^3_{ph-p} - 3\alpha R^2_{ph-p} +$$

$$+ \left(2a^2\cos^2\theta + 4q^2\right) R_{ph-p} + \alpha a^2\cos^2\theta = 0\,,$$

$$R^2_{ph-p} = C_n(r_{ph-p}) = \left(|r_{ph} - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,, \qquad (25)$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2},\ a^2 + q^2 < m^2\,,$$

$$r_0 \in \Re, \quad n \in \Re^+\,.$$

Equation (25) gives the stable photon radius of curvature in the polar orbit. The orbit has a variable radius of curvature with azimuth.

When $a = 0$, $q \neq 0$, equation (25) reduces to

$$R_{ph-p} = \sqrt{C_n(r_{ph-p})} = \frac{3\alpha + \sqrt{9\alpha^2 - 32q^2}}{4}\,, \qquad (26)$$

$$C_n(r_{ph-p}) = \left(\left|r_{ph-p} - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2},\ q^2 < m^2\,,$$

$$r_0 \in \Re,\ n \in \Re^+\,,$$

which recovers the radius of curvature for the stable orbit of a photon about the simple point-charge. When $a = q = 0$, (25) reduces to,

$$R_{ph-p} = \sqrt{C_n(r_{ph-p})} = \frac{3\alpha}{2}\,, \qquad (27)$$

$$C_n(r_{ph-p}) = \left(\left|r_{ph-p} - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}\,,$$

$$\alpha = 2m,\ r_0 \in \Re,\ n \in \Re^+\,,$$

which recovers the curvature radius for the stable orbit of a photon about the simple point-mass. When $n = 1$ and $r_0 = \alpha$, equation (27) gives,

$$R_{ph-p} = \sqrt{C_n(r_{ph-p})} = r_{ph-p} = 3m\,,$$

which is the stable radius of curvature for the photon about the simple point-mass, but which is misinterpreted by the orthodox relativists as a measurable proper radius.

To obtain the stable photon radius of curvature of the polar orbit for the rotating point-mass, set $q = 0$ in (25),

$$2R^3_{ph-p} - 3\alpha R^2_{ph-p} + 2a^2\cos^2\theta R_{ph-p} +$$

$$+ \alpha a^2\cos^2\theta = 0\,,$$

$$R^2_{ph-p} = C_n(r_{ph-p}) = \left(|r_{ph} - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,, \qquad (28)$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta},\ a^2 < m^2\,,$$

$$r_0 \in \Re, \quad n \in \Re^+\,.$$

7 Potential functions in the weak field

In the case of the rotating point-charge,

$$g_{00} = \frac{\Delta - a^2\sin^2\theta}{\rho^2}\,, \qquad (29)$$

$$\Delta = C_n(r) - \alpha\sqrt{C_n(r)} + a^2 + q^2\,,$$

$$\rho^2 = C_n(r) + a^2\cos^2\theta\,.$$

The potential Φ for a general metric is given by,

$$g_{00} = (1 - \Phi)^2 = 1 - 2\Phi + \Phi^2\,.$$

In the weak field,

$$g_{00} \approx 1 - 2\Phi\,.$$

Now (29) gives,

$$g_{00} = \frac{C_n(r) - \alpha\sqrt{C_n(r)} + a^2\cos^2\theta + q^2}{C_n(r) + a^2\cos^2\theta} =$$

$$= 1 - \frac{\alpha\sqrt{C_n(r)} - q^2}{C_n(r) + a^2\cos^2\theta}\,,$$

so the potential is,

$$\Phi = \frac{\alpha\sqrt{C_n(r)} - q^2}{2\left(C_n(r) + a^2\cos^2\theta\right)}\,, \qquad (30)$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,$$

$$a^2 + q^2 < m^2\,, \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,.$$

The potential therefore depends upon azimuth.

The potential for the rotating point-mass is obtained from (30) by setting $q=0$,

$$\Phi = \frac{\alpha\sqrt{C_n(r)}}{2\left(C_n(r) + a^2\cos^2\theta\right)}\,, \tag{31}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,,$$

$$a^2 < m^2\,, \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,.$$

If $a=0$ the potential for the simple point-charge is recovered from (30),

$$\Phi = \frac{\alpha}{2\sqrt{C_n(r)}} - \frac{q^2}{2C_n(r)}\,, \tag{32}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,,$$

$$q^2 < m^2\,, \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,,$$

and if $a=q=0$ the potential for the simple point-mass is recovered,

$$\Phi = \frac{\alpha}{2\sqrt{C_n(r)}}\,, \tag{33}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,.$$

According to (30), orbit in the equatorial gives equations (32) for the simple point-charge. According to (31), orbit in the equatorial gives equations (33) for the simple point-mass. For orbits in the polar, equations (32) and (33) are momentarily realised at the equator for a test particle orbiting the rotating point-charge and the rotating point-mass respectively. Thus, the effects of rotation of the source of the field do not manifest for a test particle in an equatorial orbit.

Taking the near-field limit on (30) gives,

$$\lim_{r \to r_0^{\pm}} \Phi = \frac{\alpha\beta - q^2}{2\left(\beta^2 + a^2\cos^2\theta\right)}\,, \tag{34}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,$$

$$a^2 + q^2 < m^2\,.$$

The potential approaches a finite limit with azimuth. The limiting values for the simpler configurations are easily obtained from (34) in the obvious way.

References

1. Crothers S. J. On the general solution to Einstein's vacuum field and it implications for relativistic degeneracy. *Progress in Physics*, 2005, v. 1, 68–73.
2. Crothers S. J. On the ramifications of the Schwarzschild spacetime metric. *Progress in Physics*, 2005, v. 1, 74–80.
3. Crothers S. J. On the geometry of the general solution for the vacuum field of the point-mass. *Progress in Physics*, 2005, v. 2, 3–14.
4. Schwarzschild K. On the gravitational field of a mass point according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 189 (arXiv: physics/9905030).

On the Vacuum Field of a Sphere of Incompressible Fluid

Stephen J. Crothers

Sydney, Australia

E-mail: thenarmis@yahoo.com

The vacuum field of the point-mass is an unrealistic idealization which does not occur in Nature — Nature does not make material points. A more realistic model must therefore encompass the extended nature of a real object. This problem has also been solved for a particular case by K. Schwarzschild in his neglected paper on the gravitational field of a sphere of incompressible fluid. I revive Schwarzschild's solution and generalise it. The black hole is necessarily precluded. A body cannot undergo gravitational collapse to a material point.

1 Introduction

In my previous papers [1, 2] concerning the general solution for the point-mass I showed that the black hole is not consistent with General Relativity and owes its existence to a faulty analysis of the Hilbert [3] solution. In this paper I shall show that, along with the black hole, gravitational collapse to a point-mass is also untenable. This was evident to Karl Schwarzschild who, immediately following his derivation of his exact solution for the mass-point [4], derived a particular solution for an extended body in the form of a sphere of incompressible, homogeneous fluid [5]. This is also an idealization, and so too has its shortcomings, but represents a somewhat more plausable end result of gravitational collapse.

The notion that Nature makes material points, i. e. masses without extension, I view as an oxymoron. It is evident that there has been a confounding of a mathematical point with a material object which just cannot be rationally sustained. Einstein [6, 7] objected to the introduction of singularities in the field but could offer no viable alternative, even though Schwarzschild's extended body solution was readily at his hand.

The point-mass and the singularity are equivalent. Abrams [8] has remarked that singularities associated with a spacetime manifold are not uniquely determined until a boundary is correctly attached to it. In the case of the point-mass the source of the gravitational field is identified with a singularity in the manifold. The fact that the vacuum field for the point-mass is singular at a boundary on the manifold indicates that the point-mass does not occur in Nature. Oddly, the conventional view is that it embodies the material point. However, there exists no observational or experimental data supporting the idea of a point-mass or point-charge. I can see no way an electron, for instance, could be compressed into a material point-charge, which must occur if the point-mass is to be admitted. The idea of electron compression is meaningless, and therefore so is the point-mass. Eddington [9] has remarked in similar fashion concerning the electron, and relativistic degeneracy in general.

I regard the point-mass as a mathematical artifice and consider it in the fashion of a centre-of-mass, and therefore not as a physical object. In Newton's theory of gravitation, $r = 0$ is singular, and equivalently in Einstein's theory, the proper radius $R_p(r_0) \equiv 0$ is singular, as I have previously shown. Both theories therefore share the non-physical nature of the idealized case of the point-mass.

To obtain a model for a star and for the gravitational collapse thereof, it follows that the solution to Einstein's field equations must be built upon some manifold without boundary. In more recent years Stavroulakis [10, 11, 12] has argued the inappropriateness of the solutions on a manifold with boundary on both physical and mathematical grounds, and has derived a stationary solution from which he has concluded that gravitational collapse to a material point is impossible.

Utilizing Schwarzschild's particular solution I shall extend his result to a general solution for a sphere of incompressible fluid.

2 The general solution for Schwarzschild's incompressible sphere of fluid

At the surface of the sphere the required solution must maintain a smooth transition from the field outside the sphere to the field inside the sphere. Therefore, the metric for the interior and the metric for the exterior must attain the same value for the radius of curvature at the surface of the sphere. Furthermore, owing to the extended nature of the sphere, the exterior metric must take the form of the metric for the point-mass, but with a modified invariant containing the factors giving rise to the field, reflecting the non-pointlike nature of the source, thereby treating the source as a mass concentrated at the centre-of-mass of the sphere, just as in Newton's theory. Schwarzschild has achieved this in his particular case. He obtained the following metric for the field inside his sphere,

$$ds^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 dt^2 - \\ - \frac{3}{\kappa\rho_0} d\chi^2 - \frac{3\sin^2\chi}{\kappa\rho_0}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{1}$$

$$\sin\chi = \sqrt{\frac{\kappa\rho_0}{3}}\,\eta^{\frac{1}{3}}, \quad \eta = r^3 + \rho,$$

$$\rho = \left(\frac{\kappa\rho_0}{3}\right)^{\frac{-3}{2}}\left[\frac{3}{2}\sin^3\chi_a - \frac{9}{4}\cos\chi_a\left(\chi_a - \frac{1}{2}\sin 2\chi_a\right)\right],$$

$$\kappa = 8\pi k^2,$$

$$0 \leqslant \chi \leqslant \chi_a < \frac{\pi}{2},$$

where ρ_0 is the constant density of the fluid, k^2 Gauss' gravitational constant, and the subscript a denotes values at the surface of the sphere. Metric (1) is non-singular.

Schwarzschild's particular metric outside the sphere is,

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - \\ - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2}$$

$$R^3 = r^3 + \rho, \quad \alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3\chi_a,$$

$$0 \leqslant \chi_a < \frac{\pi}{2},$$

$$r_a \leqslant r < \infty.$$

Metric (2) is non-singular for an extended body.

In the case of the simple point-mass (i. e. non-rotating, no charge) I have shown elsewhere [13] that the general solution is,

$$ds^2 = \left[\frac{(\sqrt{C_n} - \alpha)}{\sqrt{C_n}}\right]dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n} - \alpha)}\right]\frac{C_n'^{\,2}}{4C_n}dr^2 - \\ - C_n(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}, \quad \alpha = 2m,$$

$$n \in \Re^+, \; r \in \Re, \; r_0 \in \Re,$$

$$0 < |r - r_0| < \infty,$$

where n and r_0 are arbitrary.

Now Schwarzschild fixed his solution for $r_0 = 0$. I note that his equations, rendered herein as equations (1) and (2), can be easily generalised to an arbitrary $r_0 \in \Re$ and arbitrary $\chi_0 \in \Re$ by replacing his r and χ by $|r - r_0|$ and $|\chi - \chi_0|$ respectively. Furthermore, equation (3) must be modified to account for the extended configuration of the gravitating mass. Consequently, equation (1) becomes,

$$ds^2 = \left[\frac{3\cos|\chi_a - \chi_0| - \cos|\chi - \chi_0|}{2}\right]^2 dt^2 - \\ - \frac{3}{\kappa\rho_0}d\chi^2 - \frac{3\sin^2|\chi - \chi_0|}{\kappa\rho_0}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{4}$$

$$\sin|\chi - \chi_0| = \sqrt{\frac{\kappa\rho_0}{3}}\eta^{\frac{1}{3}}, \quad \eta = |r - r_0|^3 + \rho,$$

$$\rho = \left(\frac{\kappa\rho_0}{3}\right)^{\frac{-3}{2}}\left\{\frac{3}{2}\sin^3|\chi_a - \chi_0| - \right. \\ \left. - \frac{9}{4}\cos|\chi_a - \chi_0|\left[|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right]\right\},$$

$$\kappa = 8\pi k^2, \; r_0 \in \Re, \; r \in \Re, \; \chi_a \in \Re, \; \chi_0 \in \Re,$$

$$0 \leqslant |\chi - \chi_0| \leqslant |\chi_a - \chi_0| < \frac{\pi}{2},$$

and outside the sphere, equation (2) becomes,

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1}dR^2 - \\ - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{5}$$

$$R^3 = |r - r_0|^3 + \rho, \quad \alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0|,$$

$$n \in \Re^+, \; r_0 \in \Re, \; r \in \Re, \; \chi_0 \in \Re, \; \chi_a \in \Re,$$

$$0 \leqslant |\chi_a - \chi_0| < \frac{\pi}{2},$$

$$|r_a - r_0| \leqslant |r - r_0| < \infty,$$

and outside the sphere, equation (3) becomes,

$$ds^2 = \left[\frac{(\sqrt{C_n} - \alpha)}{\sqrt{C_n}}\right]dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n} - \alpha)}\right]\frac{C_n'^{\,2}}{4C_n}dr^2 - \\ - C_n(d\theta^2 + \sin^2\theta d\varphi^2), \tag{6}$$

$$C_n(r) = \left(|r - r_0|^n + \epsilon^n\right)^{\frac{2}{n}},$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0|,$$

$$\epsilon = \sqrt{\frac{3}{\kappa\rho_0}}\left\{\frac{3}{2}\sin^3|\chi_a - \chi_0| - \right. \\ \left. - \frac{9}{4}\cos|\chi_a - \chi_0|\left[|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right]\right\}^{\frac{1}{3}},$$

$$r_0 \in \Re, \; r \in \Re, \; n \in \Re^+, \; \chi_0 \in \Re, \; \chi_a \in \Re,$$

$$|r_a - r_0| \leqslant |r - r_0| < \infty\,.$$

The general solution for the interior of the incompressible Schwarzschild sphere is given by (4), and (6) gives the general solution on the exterior of the sphere.

Consider the general form for a static metric for the gravitational field [13],

$$ds^2 = A(D)dt^2 - B(D)dD^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,,$$

$$D = |r - r_0|\,,$$

$$A, B, C > 0\ \forall\ r \neq r_0\,.$$

With respect to this metric I identify the real r-parameter, the radius of curvature, and the proper radius thus:

1. *The real r-parameter is the variable r.*
2. *The radius of curvature is $R_c = \sqrt{C(D)}$.*
3. *The proper radius is $R_p = \int \sqrt{B(D)}\, dD$.*

According to the foregoing, the proper radius of the sphere of incompressible fluid determined from *inside* the sphere is, from (4),

$$R_p = \int\limits_{\chi_0}^{\chi_a} \sqrt{\frac{3}{\kappa\rho_0}}\frac{(\chi - \chi_0)}{|\chi - \chi_0|}d\chi = \sqrt{\frac{3}{\kappa\rho_0}}\,|\chi_a - \chi_0|\,. \qquad (7)$$

The proper radius of the sphere cannot be determined from *outside* the sphere. According to (6) the proper radius to a spacetime event outside the sphere is,

$$\begin{aligned} R_p &= \int \sqrt{\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}}\,\frac{C_n'}{2\sqrt{C_n}}\,dr = \\ &= K + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} + \\ &+ \alpha \ln\left|\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}\right|\,, \end{aligned} \qquad (8)$$

$$K = \text{const.}$$

At the surface of the sphere the proper radius from outside has some value R_{p_a}, for some value r_a of the parameter r. Therefore, at the surface of the sphere,

$$\begin{aligned} R_{p_a} &= K + \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} + \\ &+ \alpha \ln\left|\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}\right|\,. \end{aligned}$$

Solving for K,

$$\begin{aligned} K &= R_{p_a} - \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} - \\ &- \alpha \ln\left|\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}\right|\,. \end{aligned}$$

Substituting into (8) for K gives for the proper radius from outside the sphere,

$$\begin{aligned} R_p(r) &= R_{p_a} + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} - \\ &- \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} + \\ &+ \alpha \ln\left|\frac{\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}}{\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}}\right|\,. \end{aligned} \qquad (9)$$

Then by (9), for $|r - r_0| \geqslant |r_a - r_0|$

$$|r - r_0| \to |r_a - r_0| \Rightarrow R_p \to R_{p_a}^{+}\,,$$

but R_{p_a} cannot be determined.

According to (4) the radius of curvature $R_c = P_a$ at the surface of the sphere is,

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0|\,. \qquad (10)$$

Furthermore, inside the sphere,

$$\frac{G}{R_p} \leqslant 2\pi\,,$$

and

$$\lim_{\chi \to \chi_0^{\pm}} \frac{G}{R_p} = 2\pi\,,$$

where $G = 2\pi R_c$ is the circumference of a great circle.

But outside the sphere,

$$\frac{G}{R_p} \geqslant 2\pi\,,$$

with the equality only when $R_p \to \infty$.

The radius of curvature of (6) at the surface of the sphere is $\sqrt{C_n(r_a)}$ so,

$$\sqrt{C_n(r_a)} = \left(|r_a - r_0|^n + \epsilon^n\right)^{\frac{1}{n}}\,. \qquad (11a)$$

At the surface of the sphere the measured circumference G_a of a great circle is,

$$G_a = 2\pi P_a = 2\pi\sqrt{C_n(r_a)}\,.$$

Therefore, at the surface of the sphere equations (10) and (11a) are equal,

$$\left(|r_a - r_0|^n + \epsilon^n\right)^{\frac{1}{n}} = \sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0|\,, \qquad (11b)$$

and so,

$$|r_a - r_0| = \left[\left(\frac{3}{\kappa\rho_0}\right)^{\frac{n}{2}}\sin^n|\chi_a - \chi_0| - \epsilon^n\right]^{\frac{1}{n}}\,. \qquad (11c)$$

The variable r is just a *parameter* for the radial quantities R_p and R_c associated with (6). Similarly, χ is also a *parameter* for the radial quantities R_p and R_c associated with (4). I remark that r_0 and χ_0 are both *arbitrary*, and *independent* of one another, and that $|r - r_0|$ and $|\chi - \chi_0|$ do not of themselves denote radii in any direct way. The arbitrary values of the parameter "origins", r_0 and χ_0, are simply boundary points on r and χ respectively. Indeed, by (7), $R_p(\chi_0) \equiv 0$, and by (9), $R_p(r_a) \equiv R_{p_a}$, irrespective of the values of r_0, r_a, and χ_0. The centre-of-mass of the sphere of fluid is always located precisely at $R_p(\chi_0) \equiv 0$. Furthermore, $R_p(r)$ for $|r - r_0| < |r_a - r_0|$ has no meaning since inside the sphere (4) describes the geometry, not (6).

According to (11b), equation (9) can be written as,

$$R_p(r) = R_{p_a} + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} -$$
$$-\sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0|\left(\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0| - \alpha\right)} + \tag{12}$$
$$+\alpha\ln\left|\frac{\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}}{\sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0|} + \sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0| - \alpha}}\right|,$$
$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0| .$$

Note that in (4), $|\chi - \chi_0|$ cannot grow up to $\frac{\pi}{2}$, so that Schwarzschild's sphere does not constitute the whole spherical space, which has a radius of curvature of $\sqrt{\frac{3}{\kappa\rho_0}}$. From (4) and (6),

$$\frac{\alpha}{P_a} = \sin^2|\chi_a - \chi_0|, \quad \alpha = \frac{\kappa\rho_0}{3}P_a^3 . \tag{13}$$

The volume of the sphere is,

$$V = \left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\int_{\chi_0}^{\chi_a}\sin^2|\chi - \chi_0|\frac{(\chi - \chi_0)}{|\chi - \chi_0|}d\chi \times$$
$$\times\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi =$$
$$= 2\pi\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\left(|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right),$$

so the mass of the sphere is,

$$M = \rho_0 V = \frac{3}{4k^2}\sqrt{\frac{3}{\kappa\rho_0}}\left(|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right).$$

Schwarzschild [5] has also shown that the velocity of light inside his sphere of incompressible fluid is given by,

$$v_c = \frac{2}{3\cos\chi_a - \cos\chi},$$

which generalises to,

$$v_c = \frac{2}{3\cos|\chi_a - \chi_0| - \cos|\chi - \chi_0|}. \tag{14}$$

At the centre $\chi = \chi_0$, so v_c reaches a maximum value there of,

$$v_c = \frac{2}{3\cos|\chi_a - \chi_0| - 1},$$

Equation (14) is singular when $\cos|\chi_a - \chi_0| = \frac{1}{3}$, which means that there is a lower bound on the possible radii of curvature for spheres of incompressible, homogeneous fluid, which is, by (13) and (6),

$$P_{a\ (min)} = \frac{9}{8}\alpha = \sqrt{\frac{8}{3\kappa\rho_0}}, \tag{15a}$$

and consequently, by equation (11a),

$$|r_a - r_0|_{(min)} = \left[\left(\frac{9\alpha}{8}\right)^n - \epsilon^n\right]^{\frac{1}{n}} = \tag{15b}$$
$$= \left[\left(\frac{8}{3\kappa\rho_0}\right)^{\frac{n}{2}} - \epsilon^n\right]^{\frac{1}{n}},$$

from which it is clear that a body cannot collapse to a material point.

From (13), a sphere of given gravitational mass $\frac{\alpha}{k^2}$, cannot have a radius of curvature, determined from outside, smaller than,

$$P_{a\ (min)} = \alpha,$$

so

$$|r_a - r_0|_{(min)} = [\alpha^n - \epsilon^n]^{\frac{1}{n}},$$
$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0| .$$

3 Kepler's 3rd Law for the sphere of incompressible fluid

There is no loss of generality in considering only the equatorial plane, $\theta = \frac{\pi}{2}$. Equation (6) then leads to the Lagrangian,

$$L = \frac{1}{2}\left[\left(\frac{\sqrt{C} - \alpha}{\sqrt{C}}\right)\dot{t}^2 - \left(\frac{\sqrt{C}}{\sqrt{C} - \alpha}\right)\left(\dot{\sqrt{C}}\right)^2 - C\dot{\varphi}^2\right],$$

where the dot indicates $\partial/\partial\tau$.

Let $R = \sqrt{C_n(r)}$. Then,

$$\frac{\partial}{\partial\tau}\frac{\partial L}{\partial\dot{R}} - \frac{\partial L}{\partial R} = \frac{R}{R-\alpha}\,\ddot{R} + \frac{\alpha}{2R^2}\,\dot{t}^2 - \frac{\alpha}{2\,(R-\alpha)}\,\dot{R}^2 - R\,\dot{\varphi}^2 = 0\,.$$

Now let $R = \text{const}$. Then,

$$\frac{\alpha}{2R^2}\,\dot{t}^2 = R\,\dot{\varphi}^2\,,$$

so

$$\omega^2 = \frac{\alpha}{2R^3} = \frac{\alpha}{2C^{\frac{3}{2}}} = \frac{\alpha}{2\left(\left|r-r_0\right|^n + \epsilon^n\right)^{\frac{3}{n}}}\,. \tag{16}$$

Equation (16) is Kepler's 3rd Law for the sphere of incompressible fluid.

Taking the near-field limit gives,

$$\omega_a^2 = \lim_{|r-r_0|\to|r_a-r_0|^+} \omega^2 = \frac{\alpha}{2\left(\left|r_a-r_0\right|^n + \epsilon^n\right)^{\frac{3}{n}}}\,.$$

According to (11b) and (10) this becomes,

$$\omega_a^2 = \frac{\alpha}{2\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\sin^3\left|\chi_a-\chi_0\right|} = \frac{\alpha}{2P_a^3}\,.$$

Finally, using (13),

$$\omega_a = \frac{\sin^3\left|\chi_a-\chi_0\right|}{\alpha\sqrt{2}}\,, \tag{17}$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3\left|\chi_a-\chi_0\right|\,.$$

In contrast, the limiting value of ω for the simple point-mass [4] is,

$$\omega_0 = \frac{1}{\alpha\sqrt{2}}\,,$$

$$\alpha = 2m\,.$$

When P_a is minimum, (17) becomes,

$$\omega_a^2 = \frac{16}{27\alpha}\,, \tag{18}$$

$$\alpha = \frac{16}{27}\sqrt{\frac{6}{\kappa\rho_0}}\,.$$

Clearly, equation (17) is an invariant,

$$\omega_a = \sqrt{\frac{\kappa\rho_0}{6}}\,.$$

4 Passive and active mass

The relationship between passive and active mass manifests, owing to the difference established by Schwazschild, between what he called "substantial mass" (passive mass) and the gravitational (i.e. active) mass. He showed that the former is larger than the latter.

Schwarzschild has shown that the substantial mass M is given by,

$$M = 2\pi\rho_0\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\left(\chi_a - \frac{1}{2}\sin 2\,\chi_a\right),$$

$$0 \leqslant \chi_a < \frac{\pi}{2}\,,$$

and the gravitational mass is,

$$m = \frac{\alpha c^2}{2G} = \frac{1}{2}\sqrt{\frac{3}{\kappa\rho_0}}\sin^3\chi_a = \frac{\kappa\rho_0}{6}P_a^3 = \frac{4\pi}{3}P_a^3\rho_0\,,$$

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}}\sin\chi_a\,,$$

$$0 \leqslant \chi_a < \frac{\pi}{2}\,.$$

I have generalised Schwarzschild's result to,

$$M = 2\pi\rho_0\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\left(\left|\chi_a-\chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a-\chi_0\right|\right),$$

$$\begin{aligned} m = \frac{\alpha c^2}{2G} = \frac{1}{2}\sqrt{\frac{3}{\kappa\rho_0}}\sin^3\left|\chi_a-\chi\rho_0\right| &= \\ = \frac{\kappa\rho_0}{6}P_a^3 = \frac{4\pi}{3}P_a^3\rho_0\,, \end{aligned} \tag{19}$$

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}}\sin\left|\chi_a-\chi_0\right|\,,$$

$$0 \leqslant \left|\chi_a-\chi_0\right| < \frac{\pi}{2}\,,$$

where G is Newton's gravitational constant. Equation (19) is only formally the same as that for the Euclidean sphere, because the radius of curvature P_a is not a Euclidean quantity, and cannot be measured in the gravitational field.

The ratio between the gravitational mass and the substantial mass is,

$$\frac{m}{M} = \frac{2\sin^3\left|\chi_a-\chi_0\right|}{3\left(\left|\chi_a-\chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a-\chi_0\right|\right)}\,.$$

Schwarzschild has shown that the naturally measured fall velocity of a test particle, falling from rest at infinity down to the surface of the sphere of incompressible fluid is,

$$v_a = \sin\chi_a\,,$$

which I generalise to,

$$v_a = \sin\left|\chi_a - \chi_0\right| .$$

The quantity v_a is the escape velocity.

Therefore, as the escape velocity increases, the ratio $\frac{m}{M}$ decreases, owing to the increase in the mass concentration.

In the case of the fictitious point-mass,

$$\lim_{|\chi_a - \chi_0| \to 0} \left(\frac{m}{M}\right) = 1 .$$

However, according to equation (14), for an incompressible sphere of fluid,

$$\cos\left|\chi_a - \chi_0\right|_{min} = \frac{1}{3} ,$$

so

$$\left(\frac{m}{M}\right)_{max} \approx 0.609 .$$

Finally,

$$\text{as } \left|\chi_a - \chi_0\right| \to \frac{\pi}{2},\ \frac{m}{M} \to \frac{4}{3\pi} .$$

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

References

1. Crothers S. J. On the general solution to Einstein's vacuum field and it implications for relativistic degeneracy. *Progress in Physics*, 2005, v. 1, 68–73.
2. Crothers S. J. On the ramifications of the Schwarzschild spacetime metric. *Progress in Physics*, 2005, v. 1, 74–80.
3. Hilbert D. *Nachr. Ges. Wiss. Gottingen, Math. Phys. Kl.*, 1917, v. 53, (see also in arXiv: physics/0310104).
4. Schwarzschild K. On the gravitational field of a mass point according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 189 (see also in arXiv: physics/9905030).
5. Schwarzschild K. On the gravitational field of a sphere of incompressible fluid according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 424 (see also in arXiv: physics/9912033).
6. Einstein A., Rosen N. The particle problem in the general theory of relativity. *Phys. Rev.,* 1935, v. 48, 73.
7. Einstein A. The Meaning of Relativity. 5th Edition, Science Paperbacks and Methuen and Co. Ltd., London, 1967.
8. Abrams L. S. Black holes: the legacy of Hilbert's error. *Can. J. Phys.*, v. 67, 1989, 919 (see also in arXiv: gr-qc/0102055).
9. Eddington A. S. The mathematical theory of relativity, Cambridge University Press, Cambridge, 2nd edition, 1960.
10. Stavroulakis N. A statical smooth extension of Schwarzschild's metric. *Lettere al Nuovo Cimento*, 1974, v. 11, 8 (www.geocities.com/theometria/Stavroulakis-3.pdf).
11. Stavroulakis N. On the Principles of General Relativity and the $S\Theta(4)$-invariant metrics. *Proc. 3rd Panhellenic Congr. Geometry*, Athens, 1997, 169 (www.geocities.com/theometria/Stavroulakis-2.pdf).
12. Stavroulakis N. On a paper by J. Smoller and B. Temple. *Annales de la Fondation Louis de Broglie*, 2002, v. 27, 3 (www.geocities.com/theometria/Stavroulakis-1.pdf).
13. Crothers S. J. On the geometry of the general solution for the vacuum field of the point-mass. *Progress in Physics*, 2005, v. 2, 3–14.

Power as the Cause of Motion and a New Foundation of Classical Mechanics

Efthimios Harokopos

P.O. Box 77034, Paleo Faliro, 17510 Athens, Greece

E-mail: makischarokopos@yahoo.com

Laws of motion are derived based on power rather than on force. I show how power extends the law of inertia to include curvilinear motion and I also show that the law of action-reaction can be expressed in terms of the mutual time rate of change of kinetic energies instead of mutual forces. I then compare the laws of motion based on power to Newton's Laws of Motion and I investigate the relation of power to Leibniz's notion of vis viva. I also discuss briefly how the metaphysics of power as the cause of motion can be grounded in a modern version of occasionalism for the purpose of establishing an alternative foundation of mechanics. The laws of motion derived in this paper along with the metaphysical foundation proposed come in defense of the hypotheses that time emerges as an ordered progression of now and that gravitation is the effect of energy transfer between an unobservable substance and all matter in the Universe.

1 Introduction

This paper's central aim is the derivation of laws of motion based on the notion of power rather than on the classical notion of force. Although the derivation of laws of motion is traditionally a subject of mechanics, several references are made herein to the history and philosophy of science. This is necessary because this paper deals primarily with the foundations of mechanics. Specifically, the hypothesis that power is the cause of motion, as contrasted to the Newtonian hypothesis according to which force is the cause of motion, leads to a major revision of the foundations of Classical Mechanics.

Most contemporary philosophers of science focus on the foundational problems of General Relativity and Quantum Mechanics and, unlike their seventeenth-century counterparts, think of Classical Mechanics as unproblematic. Butterfield mentions two errors found in this view that correspond to what he calls the matter-in-motion picture and the particle-in-motion picture [1]. According to the matter-in-motion picture, for example, bodies are collections of particles separated by voids, can move in vacuum and interact with each other, whilst their motion is completely determined by Newton's Second Law. This view has become a part of an "educated layperson's" common sense nowadays but according to Butterfield it is problematic: it does not offer, amongst other things, any explanation of the mechanism(s) of the assumed interactions but resorts to concepts such as forces acting across an intervening void ("action-at-a-distance").

The failure of modern theories to provide solutions to the foundational problems of Classical Mechanics is partly due to the fact that alternative rigid foundations have not been proposed but issues seem to have been further perplexed. Quantum uncertainty and the four-dimensional space-time of relativity have taken the place of the determinism and of the unobservable absolute space and universal time of Classical Mechanics. Mysterious action-at-a-distance still prevails in the quantum world and attempts to quantize gravity and unite Quantum Mechanics with General Relativity have failed to this date. In presenting an alternative system of laws of motion based on power, I aim primarily in the investigation of a new foundation, which offers an alternative approach for solutions to some of the unsolved problems of Classical Mechanics.

In a similar way to the matter-in-motion picture, the notion of force has also become part of an "educated layperson's" common sense, thanks to the empirical support the laws of mechanics have enjoyed over the past 300 years. It is well known, however, that Newton was heavily criticized for his use of the notion of force in an effort to ground his physics on his metaphysics and there is still considerable interest in the metaphysics of his *Principia*. In *Science and Hypothesis*, Poincaré writes [2]:

> When are two forces equal? We are told that it is when they give the same acceleration to the same mass, or when acting in opposite directions they are in equilibrium. This definition is a sham.

In *Principles of Dynamics*, Donald T. Greenwood offers an introduction to the issues raised by Newton's concept of force [3]:

> The concept of force as a fundamental quantity in the study of mechanics has been criticized by various scientists and philosophers of science from shortly after Newton's enunciation of the laws of motion until the present time. Briefly, the idea of a force, and a field force in particular, was considered to be an

> intellectual construction, which has no real existence. It is merely another name for the product of mass and acceleration, which occurs in the mathematics of solving a problem. *Furthermore, the idea of force as a cause of motion should be discarded since the assumed cause and effect relationship cannot be proven*. (Italics added)

The questions raised from Newton's definition of force and postulation of absolute space are well known to the philosophers of science and will be further discussed in sections 4, 5 and 6. In the following two sections, 2 and 3, I will show that using the notion of power as *a priori* principle, laws of motion can be derived with remarkably different definitions of inertia and action-reaction. I will then argue in section 4, where I discuss the relation of this alternative system of laws to Newton's, that the existence of a more general principle of motion is even acknowledged by Newton, in his own writings. In section 5, the relation of the notion of power to Leibniz's notion of vis viva is examined. Then, in section 6, I discuss how the metaphysics of power can be grounded in a modern version of occasionalism for the purpose of establishing an alternative foundation of Classical Mechanics. I argue that the alternative foundation proposed, along with an appropriate space-time structure, support a new hypothesis about time and about the nature of gravitation.

2 The axiom of motion

I begin the derivation of the laws of motion by stating the axiom of motion, an expression relating the velocity and the time rate of change of momentum of a particle, to a scalar quantity called the time rate of change of kinetic energy, also known as (instantaneous) power. The status of this axiom is assumed here to be that of *a priori* truth as opposed to a self-evident or empirical principle.

Axiom of Motion: The time rate of change of the kinetic energy of a particle is equal to the scalar product of its velocity and time rate of change of its momentum.

Denoting the kinetic energy by E_k and the momentum by $\mathbf{p}$, the axiom of motion can be expressed as follows:

$$\frac{dE_k}{dt} = \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{r}}{dt}\,, \tag{1}$$

where $\mathbf{r}$ is the position vector of the particle. The momentum $\mathbf{p}$ is defined as

$$\mathbf{p} = m\frac{d\mathbf{r}}{dt}\,. \tag{2}$$

If the mass m of the particle is independent of time t and position $\mathbf{r}$, then by combining equations (1) and (2), the time rate of change of the kinetic energy E_k can be written as follows:

$$\frac{dE_k}{dt} = m\frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt}\,. \tag{3}$$

Corollary I: The kinetic energy of a particle with a constant mass m is given by

$$E_k = \frac{1}{2}\, m\mathbf{v} \cdot \mathbf{v}\,, \tag{4}$$

where $\mathbf{v}$ is defined as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}\,. \tag{5}$$

Proof: From equation (3) we obtain

$$\frac{dE_k}{dt} = m\frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt} = m\frac{d\mathbf{r}}{dt} \cdot \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = m\frac{d}{dt}\left(\frac{1}{2}\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}\right),$$

which yields

$$E_k = \frac{1}{2}\, m\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{2}\, m\mathbf{v} \cdot \mathbf{v}\,. \tag{6}$$

The axiom of motion is the only principle required for deriving the laws of motion, as it will be shown in the next section.

3 The laws of motion

Law of Inertia: If the time rate of change of the kinetic energy of a particle is zero, the particle will continue in its state of motion.

Proof: If the time rate of change of the kinetic energy of a particle is zero, then from equation (3) we obtain

$$m\frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt} = 0\,. \tag{7}$$

Assuming m remains constant, the following satisfy equation (7)

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_0\,, \tag{8}$$

$$\frac{d\mathbf{r}}{dt} = 0\,, \tag{9}$$

$$\frac{d^2\mathbf{r}}{dt^2} \cdot \mathbf{v} = 0\,, \tag{10}$$

where $\mathbf{v}_0$ is a constant. Thus, solutions to equation (7) include motion with a constant velocity $\mathbf{v}_0$, given by equation (8), or a state of rest, given by equation (9) and in both these cases the time rate of change of kinetic energy is zero. These are trivial solutions to equation (7) arising when either the velocity or the acceleration of the particle, are null vectors. Yet, these two trivial solutions result in the simplest kinematic states possible and the only two states allowed when there are no forces acting on a particle according to Newton's First Law. However, if power is postulated as the cause of motion there is another trivial solution, that of uniform circular motion, as it will be shown below.

General solutions to equation (10) include all curvilinear paths with a constant kinetic energy E_k. The requirement of a constant kinetic energy could have been included in the statement of the law of inertia but this is obviously redundant since, if the time rate of change of the kinetic energy is zero then kinetic energy is constant. Clearly, the states of motion resulting from (8) and (9) are trivial solutions to (10) with zero velocity and zero acceleration, respectively. From equations (5), (6) and (10) we obtain:

$$\frac{d^2\mathbf{r}}{dt^2}\cdot\frac{d\mathbf{r}}{dt} = 0 \Leftrightarrow \frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt} = \mathbf{v}\cdot\mathbf{v} = \frac{2E_k}{m} = k\,, \tag{11}$$

where k is a constant equal to twice the kinetic energy per unit mass. Thus, all motion paths that satisfy equation (10) also satisfy the following equation

$$\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt} = k\,, \tag{12}$$

which is equivalent to the statement that the magnitude of velocity, or the speed, must be constant. In the case of motion in a plane, **v** can be expressed in polar coordinates as follows:

$$\mathbf{v} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\theta}\,. \tag{13}$$

From equations (12) and (13) we obtain:

$$\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = k^2. \tag{14}$$

A trivial solution to equation (14) is uniform circular motion given by

$$\mathbf{r}(t) = r\,\hat{\mathbf{r}}(t)\,, \tag{15}$$

where r is a constant radius and the unit radial vector $\hat{\mathbf{r}}$ rotates at a constant rate $d\theta/dt$. In the context of this law of inertia, if a particle is in uniform circular motion and the time rate of change of its kinetic energy remains zero, the state of uniform circular motion will be maintained. Notice that no claim of any sort is made herein that zero power is the cause of uniform circular motion. Obviously, a zero of something cannot be the real cause of anything. The only claim made is that if a particle is in uniform circular motion -or in any other curvilinear path that satisfies equation (12) — and power, the postulated cause of motion, remains zero then the particle will continue in its state of motion. I would like to stretch this point because, as it will be discussed further in chapter 4, the laws of motion presented in this paper can be considered as an alternative to Newton's Laws of Motion. Thus, one should refrain from evaluating these laws in the context of Newtonian mechanics, since the two systems of laws are grounded in different metaphysics. The question then of how a particle is set on a uniform circular motion in the first place is a metaphysical one and it will be placed in its proper context in chapter 6.

Non-trivial solutions to equation (14) include motion in a plane where the magnitude of the velocity **v** remains constant up to sign changes. Such motion possibilities are virtually unlimited, including for instance motion in eight-shaped figures and cycloid paths. However, some of these paths may represent physical possibilities and others may not. Uniform circular motion is a physical possibility in both micro and macro scales and this has been confirmed empirically. The choice of specific curvilinear motions over others as an effect of inertia, if power is postulated to be the cause of motion, is the subject of metaphysics discussed in section 6. The law of inertia presented in this section is a statement that the state of such motions is maintained in the absence of a cause, if power is postulated to be the cause of motion. However, the law does not provide a justification for the existence or preference of certain states of motions over others in the absence of a cause of motion.

General solutions to equation (12) in three-dimensional Euclidean space include motion along any curve. It is known from differential geometry that if a curve is regular, then there exists a reparametrization such that the curve has unit speed [4]. Thus, a particle can be made to move with constant speed along any curve in space using proper arc-length reparametrization resulting in constant kinetic energy and as a consequence, zero power.

The law of inertia is a statement about the tendency of particles to maintain their state of motion when the time rate of change in their kinetic energy is zero and this tendency is called *inertia*. Again, the law of inertia was derived based on the metaphysical hypothesis that power is the cause of motion. A consequence from such hypothesis is that the set of "cause-free" paths now includes all paths where the kinetic energy remains constant, instead of just uniform rectilinear motion and the state of rest defined in Newtonian mechanics. As it will be discussed in section 4.1, from an empirical viewpoint it is irrelevant whether one considers just rectilinear or curvilinear motion as an effect of inertia, since no experiment can be devised to prove that in the case of a freely moving particle. This is because, there is always a cause present affecting the motion of all particles. In the case of Newtonian mechanics, this cause is a gravity force and in the case of the laws of motion discussed in this paper there is always a power cause acting and giving rise to gravitational effects as it will be discussed in chapter 6.

Corollary II: If the time rate of change of the kinetic energy of a particle is zero, linear momentum is conserved.

Proof: As a direct consequence of the law of inertia, if the time rate of change of kinetic energy is zero and the velocity is denoted by **v**, then from equations (1) and (5) we obtain

$$\frac{d\mathbf{p}}{dt}\cdot\mathbf{v} = 0\,. \tag{16}$$

By using equation (2) and since **v** is not the null vector in general, we obtain from equation (16) the result:

$$\frac{d(m\mathbf{v})}{dt} = 0 \Rightarrow (m\mathbf{v})_2 - (m\mathbf{v})_1 = 0 \Rightarrow \\ \Rightarrow (m\mathbf{v})_2 = (m\mathbf{v})_1 = m\mathbf{v} = \text{const}\,. \tag{17}$$

Equation (17) is the mathematical statement of the theorem of the conservation of linear momentum [5].

Law of Interaction: To every action there is an equal and opposite reaction; that is, in an isolated system of two particles acting upon each other, the mutual time rate of change of kinetic energies are equal in magnitude and opposite in sign.

Proof: We denote the two interacting particles as m_1 and m_2. Furthermore, we denote m_1 as the agent causing the action in the system. The total kinetic energy of the interacting system of particles is the sum of the kinetic energies of the two particles:

$$E_k = E_{k_1} + E_{k_2}\,. \tag{18}$$

From equations (1), (5) and (18) we obtain

$$\frac{dE_k}{dt} = \frac{d\mathbf{p_1}}{dt}\cdot\mathbf{v_1} + \frac{d\mathbf{p_2}}{dt}\cdot\mathbf{v_2}\,, \tag{19}$$

where $\mathbf{v_1}$ and $\mathbf{v_2}$ are the velocities of the two particles with momentum $\mathbf{p_1}$ and $\mathbf{p_2}$, respectively.

Next, we consider the mutual time rate of change of kinetic energy imposed by the particles upon each other. The time rate of change of kinetic energy of particle m_2, denoted as E_{k_2}, is equal to the action imposed on it by particle m_1, denoted as $E_{k_{12}}$ and given by

$$\frac{dE_{k_2}}{dt} = \frac{d\mathbf{p_2}}{dt}\cdot\mathbf{v_2} = \frac{dE_{k_{12}}}{dt}\,. \tag{20}$$

The time rate of change of the kinetic energy of particle m_1 is equal to the sum of the time rate of change of the kinetic energy of the system due to its action as an agent and that imposed on it by particle m_2 in the form of a reaction and denoted as $E_{k_{21}}$

$$\frac{dE_{k_1}}{dt} = \frac{dE_k}{dt} + \frac{dE_{k_{21}}}{dt} = \frac{d\mathbf{p_1}}{dt}\cdot\mathbf{v_1}\,. \tag{21}$$

By combining equations (19), (20) and (21), we obtain the result:

$$\frac{dE_{k_{12}}}{dt} = -\frac{dE_{k_{21}}}{dt}\,. \tag{22}$$

Equation (22) is the mathematical statement of the law of interaction. According to the law, the reaction on a horse pulling on a cart, — to use Newton's example in the Principia — is equal to the action applied by the horse on the cart. In general, part of the action produced by the horse is used to change its own state of motion and the remaining to change that of the cart. In the case where the total action of the horse is reacted by the cart, from equation (21) it may be seen that dE_k/dt is equal to zero and the state of motion does not change. Then, in this special case, action is equal to reaction *by definition*. This can serve the purpose of clearing any confusion that may arise when the action by the horse on the cart is thought to be equal to the total action produced by the horse, a statement that is not true in the most general case.

The philosophical issues arising from the law of interaction will be discussed in more detail in section 4.

Corollary III: In an isolated system of two particles acting upon each other and both having velocity $\mathbf{v}$, the mutual time rate of change of momentum vectors are equal in magnitude and opposite in direction.

Proof: By denoting the mutual momentum vectors by $\mathbf{p_{12}}$ and $\mathbf{p_{21}}$, from equations (1), (5) and (22) we obtain

$$\frac{d\mathbf{p_{12}}}{dt}\cdot\mathbf{v} = -\frac{d\mathbf{p_{21}}}{dt}\cdot\mathbf{v} \Leftrightarrow \left(\frac{d\mathbf{p_{12}}}{dt} + \frac{d\mathbf{p_{21}}}{dt}\right)\cdot\mathbf{v} = 0\,. \tag{23}$$

Since $\mathbf{v}$ is not in general a null vector, we obtain the result:

$$\frac{d\mathbf{p_{12}}}{dt} = -\frac{d\mathbf{p_{21}}}{dt}\,. \tag{24}$$

In the case where $\mathbf{v}$ is orthogonal to the sum of the mutual time rate of change of the momentum vectors of the two particles, then equation (23) will still hold. However, in this case, the mutual time rate of change of momentum vectors will not in general be equal in magnitude and opposite in direction.

The axiom of motion of section 2, together with the law of inertia and the law of interaction, combined further with the axiom of conservation of energy of isolated systems, provide a framework for deriving the differential equations of motion of particles and by extension of rigid bodies in dynamical motion. Next, I will examine the relation of the laws of motion presented in this section to Newton's Laws of Motion.

4 Power versus force

Newton stated his laws of motion in *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), first published in 1686 [6]. The Principia was revised by Newton in 1713 and 1726. Using modern terminology, the laws can be stated as follows [3]:

First Law: Every body continuous in its state of rest, or of uniform motion in a straight line, unless compelled to change that state by forces acting upon it.

Second Law: The time rate of change of linear momentum of a body is proportional to the force acting upon it and occurs in the direction in which the force acts.

Third Law: To every action there is an equal and opposite reaction and thus, the mutual forces of two bodies acting upon each other are equal in magnitude and opposite in direction.

4.1 Newton's First Law: A priori truth or an experimental fact?

Newton's First Law can be deduced from the law of inertia stated in section 3 and specifically from equations (8) and (9), or from corollary II. According to the law of inertia, when the time rate of change of the momentum of a particle is zero, then that particle will either remain at rest or move in a straight line with constant velocity $\mathbf{v}_0$.

It is interesting to recall Newton's comments in Principia that follow the First Law [6]:

> Projectiles continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continuously drawn aside from rectilinear motion, do not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

The first part of Newton's comments regarding the projectile motion is problematic from an empirical perspective. No experiment can be devised where a projectile will move in the absence of gravity. Thus, there can be no cause free motion experiments in the context of Newtonian mechanics in order to observe what the resulting motion would be if the cause were to be removed. Therefore, it seems that Newton was referring to a thought experiment than to a well-established empirical fact. Furthermore, in the remaining part of Newton's comment regarding the First Law, things become even more interesting as he attempts to draw conclusions regarding the validity of the First Law from the motion of rotating bodies, such as spinning disks and planets. This is obviously a peculiar attempt for a connection between the rectilinear motion the First Law deals exclusively with, and rotational motion in the absence of a resisting medium. It appears that Newton's attempt to provide conclusive empirical support of the First Law is fraught with difficulties simply because no experiments can be devised from which the First Law can be inferred from the phenomena and rendered general by induction. This fact turns out to conflict with Newton's statement in the general scholium in book III of the Principia [6]:

> In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive forces of bodies, and the laws of motion and gravitation, were discovered.

The First Law and specifically the statement that bodies remain at rest or move uniformly in a straight line unless a force acts upon them, does not comply with the rules of the (experimental) philosophy Newton claims to abide with. The First Law does not deal with circular orbits, even if such orbits were employed by Newton as an example in his attempt to justify it. The First Law is actually an axiom, which must be accepted without proof, and not a statement derived via the use of inductive methodology. This is again due to the fact that no experiment can be devised on our planet for the purpose of observing what the motion of a projectile would be when there is no force acting upon it. According to Newton's Law of Universal Gravitation, gravity forces act upon a body unless it is set in motion in a region of space sufficiently far away from the influence of other bodies. Is then Newton alluding to the possibility of the existence of a more general First Law similar to the law of inertia of section 3? Let us recall what Poincaré said [2]:

> *The Principle of Inertia*. — A body under the action of no force can only move uniformly in a straight line. Is this a truth imposed on the mind *à priori*? If this be so, how is it that the Greeks have ignored it? How could they have believed that motion ceases with the cause of motion? Or, again, that every body, if there is nothing to prevent it, will move in a circle, the noblest of all forms of motion? If it be said that the velocity of a body cannot change, or there is no reason for it to change, may we not just as legitimately maintain that the position of a body cannot change, or that the curvature of its path cannot change, without the agency of an external cause? Is, then, the principle of inertia, which is not an *à priori* truth, an experimental fact? Have there ever been experiments on bodies acted on by no forces? And, if so, how did we know that no forces were acting?

Poincaré continues with his discussion of the principle of inertia by stating that

> Newton's First Law could be the consequence of a more general principle, of which the principle of inertia is only a particular case.

In turn, I argue that the axiom of motion, equation (1), can serve the role of this more general principle and Newton's First Law is indeed a special case of a more general law of inertia, such as the one derived in section 3.

Thus, I essentially argue that Newton's First Law makes reference to phenomena that are just two possibilities within a broader range of possibilities mandated by a more general principle of inertia, such as the law of inertia of section 3. As I will demonstrate in the proceedings, the same holds true with Newton's Third Law. There, matters are even clearer regarding my argument that Newton's laws are just a special case of the laws presented in section 3.

4.2 Newton's Second Law: The metaphysical cause of motion

The mathematical expression of Newton's Second Law, after a suitable choice of units is made is the following [3]:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v})\,. \tag{25}$$

With the Second Law, Newton defines force as the cause of motion and equates it to the time rate of change of momentum. The laws of motion presented in section 3, based on the axiom of motion, challenge the notion that the Newtonian force is the cause of motion and the metaphysical foundation of mechanics. However, in these laws of motion, the metaphysics of force are replaced by those of the time rate of change of kinetic energy, also known as *power*. In a way analogous to Newton's Second Law, the axiom of motion stated in section 2 can be expressed as follows

$$P = \frac{d(E_k)}{dt}\,, \tag{26}$$

where P is the (instantaneous) power and E_k the kinetic energy of a particle.

> When we say force is the cause of motion, we are talking metaphysics...

writes Poincaré in *Science and Hypothesis* [2]. This statement made by Poincaré also applies when the time rate of change of kinetic energy, or power, is defined as the cause of motion. Whether using force or power, the physics of the associated laws of motion must be grounded in some metaphysics and this is done in section 6. It is important to understand that the particular choice of a quantity to assume the role of the cause of motion becomes the link between the empirical world of physics and the metaphysics of what exists and is real. Thus, although one can choose either force or power as the basis of stating laws of motion, the metaphysical foundations of such laws will turn out to be profoundly different. Newton used his notion of force to ground his physics in the metaphysics of absolute space and time. In section 6, I will discuss how the notion of power grounds the physics of the laws of motion of section 3 in the metaphysics of a modern version of Cartesian occasionalism and a dual space-time account. It turns out that the view of the world implied by such metaphysics is very different from the Newtonian or Leibnizian ones.

Besides the difference in metaphysics, the alternative to Newton's second law given by equation (26) offers an advantage in resolving some philosophical issues regarding the foundations of Classical Mechanics and in particular the need to consider fictitious forces when applying Newton's Second Law in non-inertial reference frames. In the case of observers at rest in accelerated reference frames in either rectilinear or uniform circular motion, the time rate of change of kinetic energy is zero and thus no additional fictitious power cause is needed to explain the state of motion. Again, this is only true if power is defined as the cause of motion. If force is defined as the cause of motion then in both non-inertial reference frames mentioned fictitious causes must be considered. Specifically, in the case of rectilinear motion, observers at rest in an accelerated frame must assume inertial fictitious forces acting and in the case of observers at rest in a uniformly rotating reference frame, centrifugal forces acting must be assumed.

The same conclusion holds in the case of fictitious Coriolis forces acting on freely moving particles in rotating reference frames. Since such fictitious forces are always orthogonal to the velocity of a particle in motion, for rotating observers it turns out that the time rate of change of kinetic energy of the particle is equal to zero, as obtained by equation (1). The same result is true for observers at rest since in that case the time rate of change of momentum of a freely moving particle is zero. Fictitious forces need to be considered regardless of whether force or power is defined as the cause of motion when a force analysis is carried out. However, when power is defined as the cause of motion, there are no philosophical issues arising from the need to consider fictitious causes of motion in non-inertial reference frames and this is the point just made. Thus, the transition from force to power as the cause of motion leads to a compatibility with the epistemological principle which states that every phenomenon is to receive the same interpretation from any given moving coordinate system. This epistemological principle also plays an important role in the axiomatic foundation of the theory of relativity [7].

4.3 Newton's Third Law: a special case of a more general action-reaction law?

Newton's Third Law may be deduced from the law of interaction of section 3 and in particular from equation (24) of corollary III. In the scholium following the Laws of Motion, Newton attempts to provide additional support for the Third Law through a host of observations related to various modes of mechanical interaction between bodies. From the closing comments in the scholium, some interesting conclusions can be drawn [6]:

> ...But to treat of mechanics is not my present business. I was aiming to show by those examples the greater extent and certainty of the third Law of Motion. *For if we estimate the action of the agent from the product of its force and velocity* and likewise the reaction of the impediment from the product of the velocities of its several parts, and the forces of resistance arising from friction, cohesion, weight, and acceleration of those parts, the action and reaction in the use of all sorts of machines will be found always equal to one another. And so far as the action is propagated by the intervening instruments, and at last impressed upon the resisting body, the ultimate action will be always contrary to the reaction. (Italics added)

It is clear that Newton was well aware of the product of velocity and force being a measure of action and of reaction, as defined in the law of interaction of section 3. Newton actually made use of the law of interaction in his scholium above to justify some particular situations where his Third

Law of action-reaction does not apply directly. But why is it the case that Newton stated his Third Law in terms of forces and not in terms of the product of force and velocity he mentions in his scholium quoted above? Why does it appear that a more general law was used to justify some particular situations Newton's Third Law does not directly apply to, but the latter was stated as a law of mechanics? The answer can be found in the attempt to model gravity in Newtonian mechanics as the effect of mutual attraction caused by central forces acting at a distance. The Third Law had to be stated in terms of the mutual action-reaction forces being equal in magnitude and opposite in direction to justify the particular form of Newton's Law of Universal Gravitation. But again, the Third Law fails the requirement set forth by the rules of the experimental philosophy of Newton, for it being deduced from the phenomena; it is just another axiom that must be accepted without proof. Forces acting on different bodies, and especially celestial ones, cannot be experimentally determined to be equal. Only forces acting on the same body can be determined to be equal by experiment.

I have shown that even Newton himself made both indirect and direct use of the notion of power in an attempt to provide a general justification of his Third Law. Can we simply assume that Newton was unaware that there is a single principle that could serve as the basis of a system of laws of mechanics that are in a certain way more general than his laws? I suspect that he was aware of it. But the consequences from stating laws based on this principle of motion would be devastating on the metaphysics of force. If force were to be just an intellectual construction and not the cause of motion, then Newton's whole system of the world was at stake. Motion then would have to be explained based on some other metaphysics, such as Cartesian occasionalism for example and the notion that all causes are due to God, or Spinoza's doctrine that everything is a mode of God [8], or even Leibniz's notion of a living force.

5 Power versus vis viva

Leibniz rejected the doctrine of Cartesian occasionalism and Newtonian substantivalism but his efforts to ground his relationism on the metaphysics of a living force were also met with difficulties. Leibniz realized that for motion to be real, it must be grounded on something that is not mere relation, something absolute and unobservable that serves as its cause [8]. Leibniz stated his laws of motion in his unpublished during his lifetime work *Dynamica de Potentia et Legibus Naturae Corporeae* in which he attempted to explain the world in terms of the conservation laws of vis viva and momentum of colliding bodies.

The laws of inertia and interaction of section 3 were derived from the axiom of motion of section 2. The latter is related to the living force, or vis viva, defined by Leibniz as being a real metaphysical property of a substance. Leibniz measure of vis viva is the quantity $m\mathrm{v}^2$, in contrast to the Cartesian definition of the *quantity of motion* being equal to size multiplied by speed, and later redefined by Newton as being equal to the product of mass and velocity. In turn, the axiom of motion stated in section 2 is related to the time rate of change of vis viva, the quantity Leibniz argued is conserved and a real metaphysical property of a substance, in an effort to support his relational account of space-time.

Leibniz's definition of vis viva as a real metaphysical property of a substance is fraught with difficulties. Roberts has argued that, in his later communications with Samuel Clarke, who was a defender of Newton's substantivalism, Leibniz seems to commit to a richer space-time structure that can support absolute velocities [9]. Roberts' work has cast light into a little known, or maybe misinterpreted, aspect of Leibniz's metaphysics. Specifically, into Leibniz's efforts to come up with laws of motion based on vis viva being a measure of force, while at the same time his relationism implies a space-time structure that is a well-founded phenomenon. This might be an indication of Leibniz's later realization that relationism fails unless absolute velocities are supported by a richer space-time structure than what is commonly referred to as Leibnizian space-time. In section 6, I define an account of space-time that can support relationism and absolute velocities in an attempt to ground the physics of the axiom and laws of motion in the metaphysics of power.

Along these lines, in a similar way to the link between the Newtonian force and momentum, the former being the time rate of change of the latter, I argue that vis viva is actually a quantity of motion and power, its time rate of change, is the cause of motion. In this way the similarities between the laws of conservation of momentum and vis viva become evident, because they are both defined as quantities of motion. In essence, I argue, the time rate of change of vis viva is the real metaphysical cause of motion. Of course, such a switch in the definitions is not compatible with Leibniz's metaphysics. This is because the time rate of change of the kinetic energy of a body moving with constant linear velocity, or even in uniform circular motion, is zero. A zero of something cannot assume the role of a real metaphysical property of a substance and the cause of motion in a Leibnizian world. Despite these metaphysical difficulties I will deal with in more detail in the next section, on the physics side it is clear that the laws of motion of section 3 were derived from a quantity that is proportional to the time derivative of vis viva. Thus, they have a direct link to Leibniz's Laws of Motion [8]. Specifically, Leibniz's laws of conservation of vis viva and momentum can be derived from the laws of inertia and interaction of section 3, respectively, but the details are left out.

6 The metaphysics of power

Before I discuss the metaphysics of power and specifically the notion that power is the cause of motion, I will briefly review the philosophical debate about the ontology of space-time. I argue that the space-time debate and the debate about the cause of motion are closely related in the sense that an answer to the former provides an answer to the latter. Thus, I essentially argue that the space-time debate is not a mere philosophical one and its resolution will have a decisive impact on which laws of motion and gravitation are assigned the status of "laws of nature" as opposed to that of mere heuristics.

6.1 The space-time debate

The publication of Newton's Principia in 1686 was the cause of the start of one of the most interesting debates in the history of the philosophy of science, dealing mainly with the ontology of space-time. Leibniz ignited the debate by arguing that Newton's substantival space-time, the notion that space and time exist independently of material things and their spatiotemporal relations, was not a well-founded phenomenon. Leibniz confronted Newtonian substantivalists with his relationism, based on which space is defined as the set of (possible) relations among material things and the only well-defined quantities of motion are relative ones [10]. Newton just grounded his physics in the metaphysics of force and absolute space and time. For Newton, the only well-defined quantities of motion are the absolute ones, like absolute position, velocity and acceleration. Substantivalism and relationism then appear in modern literature as two completely different accounts of space-time.

The key issue regarding the space-time debate, which is still alive by the way, is whether it does really make sense to speak of *either* a substantival *or* a relational account of space-time. Since diametrically opposite views of this kind have only led to sharp conflict and irreconcilable differences, maybe it would make sense to investigate whether both a substantival and relational space-time is a possibility. This two-level approach seems not to have been considered seriously because it implies a superfluous world. However, both Newtonian substantivalism and Leibnizian relationism are fraught with difficulties. On one hand, the metaphysics of Newtonian force require the postulation of unobservables, like absolute space. On the other hand, in Leibniz's relationism, for motion to be real, it must be grounded in something that is not mere relation, something absolute and unobservable that serves as its cause, what Leibniz called a vis viva [9]. The differences seem to reconcile when a two-level, or if I may call it a dual, space-time account is postulated and I will throw in here the term *substantival relationism*.

6.2 From cause-free motion to gravitation

The hypothesis about the duality of space-time just put forward is next examined in the context of gravitation and its observable effects, i. e. the motion of celestial bodies and free-falling particles. This step is of great importance since any laws of motion must account for all observable phenomena including those that are attributed to gravitation. Newton accomplished the step of grounding the physics of the Laws of Motion to his metaphysics of substantival space and universal time, by assuming that the cause of gravitation was also some type of force. Next, in what was a remarkable achievement in the history of science, he derived the famous Law of Universal Gravitation (LUG). In a similar way, I argue that power is the cause of gravitation in order to maintain a compatibility with the axiom and laws of motion of sections 2 and 3, respectively. Thus, the time rate of change of a potential energy function $E_p(r)$ is the cause of gravitation and equation (1), the axiom of motion, becomes

$$\frac{dE_k}{dt} = \frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{r}}{dt} = -\frac{dE_p}{dt}\,. \tag{27}$$

The law of conservation of mechanical energy can be derived from equation (27) as follows:

$$\frac{dE_k}{dt} = -\frac{dE_p}{dt} \;\Leftrightarrow\; \frac{d}{dt}(E_k + E_p) = 0 \;\Leftrightarrow\; \Leftrightarrow\; E_k + E_p = \text{const}\,. \tag{28}$$

The Law of Universal Gravitation may be restated as follows:

Law of Universal Gravitation: All particles move in such a way as for the time rate of change of their kinetic energy to be equal the time rate of change of their potential energy.

In fact, I argue that Newton's Law of Universal Gravitation is a statement about the form of the potential function $E_p(r)$ in equation (27) and thus it can assume a variety of interpretations regarding mechanisms giving rise to it. If we postulate that energy transfer affects all particles in motion, in accordance with equation (27), this can support the hypothesis that gravitation is the result of energy transfer between all bodies in motion with some substance. Substantival space-time can serve the role of this substance and can facilitate the energy transfer to and from all bodies in motion and in such a way that all spatiotemporal quantities evolve according to certain rules giving rise to the well-known potential function $E_p(r)$ first discovered by Newton.

Since the above metaphysics are compatible with the concept of a mechanical universe, one could then postulate the existence of some type of mechanism that facilitates the transfer of energy between all bodies in motion and substantival space-time. This mechanism must be part of the substance level, whereas at the phenomenal level its effect is the observed motions. According to this dual scheme, at the phenomenal level the only well-founded quantities of motion are relative ones and space-time is relational, whereas, at the substance level, the only well-defined quantities of motion are the absolute ones and the space-time is substantival.

6.3 A new foundation of mechanics

The hypothesis just made, attributing gravitation to energy transfer between all bodies in motion and substantival space-time requires that at every instance something must accomplish this task and bring about the perceived effects. I will relate this to occasionalism in the following way: according to Nicolas Malebranche and other seventeenth-century Cartesian occasionalists, what we actually call causes are really no more than *occasions* on which, in accordance with his own laws, God acts to bring about the effect [11]. If one were to replace the notion of God by the notion of a mechanism, then a modern (or mechanical) occasionalist could assert that what we actually call causes are no more than occasions on which a mechanism acts to bring about the effect. In this sense we immediately resolve two more issues: first, time emerges as an ordered progression of instances, or nows, on which the mechanism acts to bring about the effect. Then, the matter-in-motion picture [1] is better illuminated by asserting that all motion and interactions of material bodies are facilitated by a mechanism that operates based on its own rules rather than taking place due to forces or based on rules inherent in the bodies themselves.

The concept of time as a collection of nows is in fact similar to that found in Barbour [12]. The main difference with the view I express here is that time emerges due to the actions of a mechanism hidden in substantival space-time in an orderly fashion and has a direction, i. e. there is an arrow of time. More importantly, the universal clock of Newton is now part of the mechanism that resides in substantival space-time but at the phenomenal level time and motion cannot be separated because there is no motion without time and no time without motion, i. e. time and motion are inextricably related.

What I argue essentially is that gravitation has an external cause to the phenomenal level and space-time is a substance of some kind that facilitates the energy transfer required for the manifestation of gravitational effects. These ideas may not be completely new. What is new here is the derivation of a system of laws of motion based on the notion of power. Power allows grounding the physics that all phenomena are caused by energy transfer, including those attributed to gravitation, to the metaphysics of substantival space-time being a giant mechanism and a substance. Since the times of the Greeks, Anaximander of Miletus (c. 650 BCE) expressed the view that

> The apeiron, from which the elements are formed, is something that is different (from the elements).

Then, Newton argued that all motion must be referenced to an absolute, unobservable space. Even in general relativity space-time retains its substantival account and it exists independently of the events occurring in it [10]. Baker has argued that the space-time of general relativity must be a substance and attempts to support this claim of his based on the observed expansion of the Universe [13]. Baker's argument about the requirement of a carrier of gravitational energy from its source to a detector, if it is to be compelling, must apply to all forms of energy transfer traditionally assumed to take place in vacuum. But such generalization can be further coupled with the hypothesis that some causes are external to the world of observable phenomena. In Wiithrich there are references made to the hypothesis that gravity forces have an external cause in an attempt to explain the failure in quantizing the field equations of general relativity [14]. Thus, arguments have already been made in favor of the hypothesis that space-time is some kind of a substance and that any causal connections attributed to gravitation are apparent. Usually, arguments leading to such provocative hypotheses are treated at the level of epistemological skepticism but as McCabe argues the hypothesis, for instance, that our universe is part of a computer simulation implementation generates empirical predictions and it is therefore a falsifiable hypothesis [15]. One question that arises from this discussion is the following: does the existence of external causes imply that our world is some type of virtual reality? My own answer to this important question is both yes and no. Yes, because according to the hypothesis there are external causes to the world of perceived phenomena and thus part of another world. No, because a cause being external and unobservable does not preclude it being part of an all-encompassing entity, which we can call Universe. Therefore, the answer to the question seems to depend on how one defines *Universe*. But the presence of external causes to the world of observable phenomena must not be rejected *a priori* on the basis that it leads to the provocative virtual reality hypothesis and experimental physics must pursue seriously its falsification or corroboration. Although such task is highly challenging, the state-of-the-art in precision instrumentation has reached levels that allow the initiation of a program of this nature.

7 Summary

The axiom and laws of motion presented in sections 2 and 3, respectively, are:

Axiom of Motion: The time rate of change of the kinetic energy of a particle is the scalar product of its velocity and time rate of change of its momentum.

Law of Inertia: If the time rate of change of the kinetic energy of a particle is zero, the particle will continue in its state of motion.

Law of Interaction: To every action there is an equal and opposite reaction; that is, in an isolated system of two particles acting upon each other, the mutual time rate of change of kinetic energies are equal in magnitude and opposite in sign.

A restatement of the Law of Universal Gravitation was presented in section 6 as follows:

Law of Universal Gravitation: All particles move in such a way as for the time rate of change of their kinetic energy to be equal to the time rate of change of their potential energy.

In section 4, I argued that the above laws of motion are, in a certain sense, more general than Newton's, and that this claim is even supported by Newton's own writings, especially in the case of the Third Law. Furthermore, in section 5, I discussed the relation of the axiom and laws of motion to Leibniz's laws of the conservation of vis viva and momentum. I argued that kinetic energy can be defined as a quantity of motion and its time derivative as the cause of motion, in a similar way to the Newtonian force being the time derivative of momentum and a postulated cause of motion.

In section 6, I discussed how the axiom and laws of motion of sections 2 and 3, combined further with a modified version of Cartesian occasionalism and a dual space-time account form an alternative foundation of classical mechanics in the context of a mechanical Universe. Specifically, I proposed a substantival-relational account of space-time and a mechanism residing in the substance level whose actions coordinate all motion and interactions. I argued that the proposed foundation supports the hypothesis about gravitation being the effect of energy transfer between all bodies in motion and substantival space-time and I stated a version of the Law of Universal Gravitation which is compatible with the hypothesis that power is the cause of motion. These metaphysics also provide solutions to some foundational problems of Classical Mechanics, such as the matter-in-motion picture and the emergence and direction of time. Finally, I briefly referred to the ramifications on the nature of our physical reality when the cause of gravitation is considered part of an unobservable substance. I argued that the soundness of the virtual reality or computer simulation hypothesis depends on how Universe is defined. The fact that such hypothesis about the nature of our reality is provocative should not be an excuse for rejecting *a priori* external causes of motion and gravitation. Theoretical physicists ought to seriously investigate new models incorporating such assumptions about the nature of our physical reality and experimental physicists should pursue their falsification.

References

1. Butterfield J. Between laws and models: some philosophical morals of Lagrangian mechanics, 2004. e-print, Pittsburgh Archives/00001937.
2. Henri P. Science and hypothesis. Dover, New York, 1952.
3. Greenwood D. T. Principles of dynamics. Prentice-Hall, New Jersey, 1965.
4. O'Neill B. Elementary differential geometry. Academic Press, San Diego, 1997.
5. Meirovitch S. L. Methods of analytical dynamics. McGraw-Hill, New York, 1970.
6. Newton I. Mathematical principles of natural philosophy. Trans. by Andrew Motte and rev. by Florian Cajori, R. M. Hutchins, ed. in Great Books of the Western World: 34. Newton Huygens. Encyclopaedia Britannica, Chicago, 1952, 1–372.
7. Reichenbach H. The philosophy of space and time. Dover, New York, 1958.
8. Garber S. D. Leibniz: physics and philosophy. N. Jolley, ed. in The Cambridge Companion to Leibniz, Cambridge University Press, Cambridge, 1995, 270–352.
9. Roberts J. T. Leibniz on force and absolute motion. *J. Phil. Sci.*, 2003, v. 70, 553–571.
10. Sklar L. Space, time and space-time. University of California Press, Berkeley, 1997.
11. Brown S. The seventeenth-century intellectual background. N. Jolley, ed. in The Cambridge Companion to Leibniz, Cambridge University Press, Cambridge, 1995, 43–66.
12. Barbour J. The end of time. Oxford University Press, New York, 1999.
13. Baker D. Spacetime substantivalism and Einstein's cosmological constant. *Proc. Phil. Sci. Assoc.*, 19th Biennial Meeting, PSA2004, 2004.
14. Wiithrich, C. To quantize or not to quantize: fact and folklore in quantum gravity. *Proc. Phil. of Sci. Assoc.*, 19th Biennial Meeting, PSA2004 2004.
15. McCabe G. Universe creation on a computer, 2004. e-print, Pittsburgh Archives/00001891.

Hydrodynamic Covariant Symplectic Structure from Bilinear Hamiltonian Functions

Salvatore Capozziello*, Salvatore De Martino†, Stephan Tzenov†

*Dipartimento di Scienze Fisiche and INFN sez. di Napoli, Universitá "Federico II" di Napoli, Complesso Universitario di Monte S. Angelo Edificio N, via Cinthia 45, I-80126 Napoli, Italy

E-mail: capozziello@na.infn.it

†Dipartimento di Fisica "E.R. Caianiello" and INFN sez. di Napoli, Universitá di Salerno, Via S. Allende, I-84081 Baronissi (SA), Italy

E-mail: demartino@sa.infn.it, tzenov@sa.infn.it

Starting from generic bilinear Hamiltonians, constructed by covariant vector, bivector or tensor fields, it is possible to derive a general symplectic structure which leads to holonomic and anholonomic formulations of Hamilton equations of motion directly related to a hydrodynamic picture. This feature is gauge free and it seems a deep link common to all interactions, electromagnetism and gravity included. This scheme could lead toward a full canonical quantization.

1 Introduction

It is well known that a self-consistent quantum field theory of space-time (quantum gravity) has not been achieved, up to now, using standard quantization approaches. Specifically, the request of general coordinate invariance (one of the main features of General Relativity) gives rise to unescapable troubles in understanding the dynamics of gravitational field. In fact, for a physical (non-gravitational) field, one has to assign initially the field amplitudes and their first time derivatives, in order to determine the time development of such a field considered as a dynamical entity. In General Relativity, these quantities are not useful for dynamical determination since the metric field $g_{\alpha\beta}$ can evolve at any time simply by a general coordinate transformation. No change of physical observables is the consequence of such an operation since it is nothing else but a relabelling under which the theory is invariant. This apparent "shortcoming" (from the quantum field theory point of view) means that it is necessary a separation of metric degrees of freedom into a part related to the true dynamical information and a part related only to the coordinate system. From this viewpoint, General Relativity is similar to classical Electromagnetism: the coordinate invariance plays a role analogous to the electromagnetic gauge invariance and in both cases (Lorentz and gauge invariance) introduces redundant variables in order to insure the maintenance of transformation properties. However, difficulties come out as soon as one try to disentangle dynamical from gauge variables. This operation is extremely clear in Electromagnetism while it is not in General Relativity due to its intrinsic non-linearity. A determination of independent dynamical modes of gravitational field can be achieved when the theory is cast into a canonical form involving the minimal number of degrees of freedom which specify the state of the system. The canonical formalism is essential in quantization program since it leads directly to Poisson bracket relations among conjugate variables. In order to realize it in any fundamental theory, one needs first order field equations in time derivatives (Hamilton-like equations) and a $(3+1)$-form of dynamics where time has been unambiguously singled out. In General Relativity, the program has been pursued using the first order Palatini approach [6], where metric $g_{\alpha\beta}$ is taken into account independently of affinity connections $\Gamma^{\gamma}_{\alpha\beta}$ (this fact gives rise to first order field equations) and the so called ADM formalism [7] where $(3+1)$-dimensional notation has led to the definition of gravitational Hamiltonian and time as a conjugate pair of variables. However, the genuine fundament of General Relativity, the covariance of all coordinates without the distinction among space and time, is impaired and, despite of innumerable efforts, the full quantization of gravity has not been achieved up to now. The main problems are related to the lack of a well-definite Hilbert space and a quantum concept of measure for $g_{\alpha\beta}$. An extreme consequence of this lack of full quantization for gravity could be related to the dynamical variables: very likely, the true variables could not be directly related to metric but to something else as, for example, the connection $\Gamma^{\gamma}_{\alpha\beta}$. Despite of this lack, a covariant symplectic structure can be identified also in the framework of General Relativity and then also this theory could be equipped with the same features of other fundamental theories. This statement does not still mean that the identification of a symplectic structure immediately leads to a full quantization but it could be a useful hint toward it.

The aim of this paper is to show that a prominent role in the identification of a covariant symplectic structure is played by bilinear Hamiltonians which have to be conserved. In fact, taking into account generic Hamiltonian invariants, constructed by covariant vectors, bivectors or tensors, it is possible to show that a symplectic structure can be achieved in any case. By specifying the nature of such vector fields (or, in general, tensor invariants), it gives rise to intrinsically symplectic structure which is always related to Hamilton-like equations (and a Hamilton-Jacobi-like approach is always found). This works for curvature invariants, Maxwell theory and so on. In any case, the only basic assumption is that conservation laws (in Hamiltonian sense) have to be identified in the framework of the theory.

The layout of the paper is the following. In Sec.II, we give the generalities on the symplectic structure and the canonical description of mechanics. Sec.III is devoted to the discussion of symplectic structures which are also generally covariant. We show that a covariant analogue of Hamilton equations can be derived from covariant vector (or tensor) fields in holonomic and anholonomic coordinates. In Sec. IV, the covariant symplectic structure is casted into the hydrodynamic picture leading to the recovery of the covariant Hamilton equations. Sec.V is devoted to applications, discussion and conclusions.

2 Generalities on the Symplectic Structure and the Canonical description

In order to build every fundamental theory of physics, it is worth selecting the symplectic structure of the manifold on which such a theory is formulated. This goal is achieved if suitable symplectic conjugate variables and even-dimensional vector spaces are chosen. Furthermore, we need an antisymmetric, covariant tensor which is non-degenerate.

We are dealing with a symplectic structure if the couple

$$\{\mathbf{E_{2n}}, \mathbf{w}\}\,, \tag{1}$$

is defined, where $\mathbf{E_{2n}}$ is a vector space and the tensor $\mathbf{w}$ on $\mathbf{E_{2n}}$ associates scalar functions to pairs of vectors, that is

$$[\mathbf{x},\mathbf{y}] = \mathbf{w}(\mathbf{x},\mathbf{y}), \tag{2}$$

which is the *antiscalar* product. Such an operation satisfies the following properties

$$[\mathbf{x},\mathbf{y}] = -[\mathbf{y},\mathbf{x}] \qquad \forall \mathbf{x},\mathbf{y}\in\mathbf{E_{2n}} \tag{3}$$

$$[\mathbf{x},\mathbf{y}+\mathbf{z}] = [\mathbf{x},\mathbf{y}]+[\mathbf{x}\mathbf{z}] \quad \forall \mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{E_{2n}}, \tag{4}$$

$$\mathrm{a}[\mathbf{x},\mathbf{y}] = [\mathrm{a}\mathbf{x},\mathbf{y}] \qquad \forall \mathrm{a}\in R,\ \ \mathbf{x},\mathbf{y}\in\mathbf{E_{2n}} \tag{5}$$

$$[\mathbf{x},\mathbf{y}] = 0 \qquad \forall y\in\mathbf{E_{2n}} \;\Rightarrow\; \mathbf{x}=0 \tag{6}$$

$$[\mathbf{x},[\mathbf{y},\mathbf{z}]]+[\mathbf{y},[\mathbf{z},\mathbf{x}]]+[\mathbf{z},[\mathbf{x},\mathbf{y}]] = 0\,. \tag{7}$$

The last one is the Jacobi cyclic identity.

If $\{\mathbf{e}_i\}$ is a vector basis in $\mathbf{E_{2n}}$, the antiscalar product is completely singled out by the matrix elements

$$w_{ij} = [\mathbf{e}_i,\mathbf{e}_j]\,, \tag{8}$$

where $\mathbf{w}$ is an antisymmetric matrix with determinant different from zero. Every antiscalar product between two vectors can be expressed as

$$[\mathbf{x},\mathbf{y}] = w_{ij}\,x^i y^j, \tag{9}$$

where x^i and y^j are the vector components in the given basis.

The form of the matrix $\mathbf{w}$ and the relation (9) become considerably simpler if a canonical basis is taken into account for $\mathbf{w}$. Since $\mathbf{w}$ is an antisymmetric non-degenerate tensor, it is always possible to represent it through the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{10}$$

where I is a $(n\times n)$ unit matrix. Every basis where $\mathbf{w}$ can be represented through the form (10) is a *symplectic basis*. In other words, the symplectic bases are the canonical bases for any antisymmetric non-degenerate tensor $\mathbf{w}$ and can be characterized by the following conditions:

$$[\mathbf{e}_i,\mathbf{e}_j]=0\,,\quad [\mathbf{e}_{n+i},\mathbf{e}_{n+j}]=0\,,\quad [\mathbf{e}_i,\mathbf{e}_{n+j}]=\delta_{ij}\,, \tag{11}$$

which have to be verified for every pair of values i and j ranging from 1 to n.

Finally, the expression of the antiscalar product between two vectors, in a symplectic basis, is

$$[\mathbf{x},\mathbf{y}] = \sum_{i=1}^{n}\left(x^{n+i}y^i - x^i y^{n+i}\right), \tag{12}$$

and a symplectic transformation in $\mathbf{E_{2n}}$ leaves invariant the antiscalar product

$$\mathbf{S}[\mathbf{x},\mathbf{y}] = [\mathbf{S}(\mathbf{x}),\mathbf{S}(\mathbf{y})] = [x,y]. \tag{13}$$

It is easy to see that standard Quantum Mechanics satisfies such properties and so it is endowed with a symplectic structure.

On the other hand a standard canonical description can be sketched as follows. For example, the relativistic Lagrangian of a charged particle interacting with a vector field $A(q;s)$ is

$$\mathcal{L}(q,u;s) = \frac{mu^2}{2} - eu\cdot A(q;s), \tag{14}$$

where the scalar product is defined as

$$z\cdot w = z_\mu w^\mu = \eta_{\mu\nu}z^\mu w^\nu, \tag{15}$$

and the signature of the Minkowski spacetime is the usual one with

$$z_\mu = \eta_{\mu\nu}z^\nu, \qquad \widehat{\eta} = \mathrm{diag}(1,-1,-1,-1). \tag{16}$$

Furthermore, the contravariant vector u^μ with components $u = \left(u^0, u^1, u^2, u^3\right)$ is the four-velocity

$$u^\mu = \frac{\mathrm{d}q^\mu}{\mathrm{d}s} \,. \tag{17}$$

The canonical conjugate momentum π^μ is defined as

$$\pi^\mu = \eta^{\mu\nu} \frac{\partial \mathcal{L}}{\partial u^\nu} = m u^\mu - e A^\mu \,, \tag{18}$$

so that the relativistic Hamiltonian can be written in the form

$$\mathcal{H}(q, \pi; s) = \pi \cdot u - \mathcal{L}(q, u; s). \tag{19}$$

Suppose now that we wish to use any other coordinate system x^α as Cartesian, curvilinear, accelerated or rotating one. Then the coordinates q^μ are functions of the x^α, which can be written explicitly as

$$q^\mu = q^\mu(x^\alpha). \tag{20}$$

The four-vector of particle velocity u^μ is transformed according to the expression

$$u^\mu = \frac{\partial q^\mu}{\partial x^\alpha} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \frac{\partial q^\mu}{\partial x^\alpha} v^\alpha, \tag{21}$$

where

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \,. \tag{22}$$

is the transformed four-velocity expressed in terms of the new coordinates. The vector field A^μ is also transformed as a vector

$$\mathcal{A}^\mu = \frac{\partial x^\mu}{\partial q^\alpha} A^\alpha. \tag{23}$$

In the new coordinate system x^α the Lagrangian (14) becomes

$$\mathcal{L}(x, v; s) = g_{\mu\nu} \left[\frac{m}{2} v^\mu v^\nu - e v^\mu \mathcal{A}^\nu(x; s) \right], \tag{24}$$

where

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial q^\mu}{\partial x^\alpha} \frac{\partial q^\nu}{\partial x^\beta}. \tag{25}$$

The Lagrange equations can be written in the usual form

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\partial \mathcal{L}}{\partial v^\lambda} \right) - \frac{\partial \mathcal{L}}{\partial x^\lambda} = 0 \,. \tag{26}$$

In the case of a free particle (no interaction with an external vector field), we have

$$\frac{\mathrm{d}}{\mathrm{d}s} (g_{\lambda\mu} v^\mu) - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} v^\mu v^\nu = 0 \,. \tag{27}$$

Specifying the covariant velocity v_λ as

$$v_\lambda = g_{\lambda\mu} v^\mu, \tag{28}$$

and using the well-known identity for connections $\Gamma^\alpha_{\mu\nu}$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} + \Gamma^\alpha_{\lambda\nu} g_{\alpha\mu} \,, \tag{29}$$

we obtain

$$\frac{\mathrm{D}v_\lambda}{\mathrm{D}s} = \frac{\mathrm{d}v_\lambda}{\mathrm{d}s} - \Gamma^\mu_{\lambda\nu} v^\nu v_\mu = 0 \,. \tag{30}$$

Here $\mathrm{D}v_\lambda / \mathrm{D}s$ denotes the covariant derivative of the covariant velocity v_λ along the curve $x^\nu(s)$. Using Eqs. (28) and (29) and the fact that the affine connection $\Gamma^\lambda_{\mu\nu}$ is symmetric in the indices μ and ν, we obtain the equation of motion for the contravariant vector v^λ

$$\frac{\mathrm{D}v^\lambda}{\mathrm{D}s} = \frac{\mathrm{d}v^\lambda}{\mathrm{d}s} + \Gamma^\lambda_{\mu\nu} v^\mu v^\nu = 0 \,. \tag{31}$$

Before we pass over to the Hamiltonian description, let us note that the generalized momentum p_μ is defined as

$$p_\mu = \frac{\partial \mathcal{L}}{\partial v^\mu} = m g_{\mu\nu} v^\nu, \tag{32}$$

while, from Lagrange equations of motion, we obtain

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}s} = \frac{\partial \mathcal{L}}{\partial x^\mu}. \tag{33}$$

The transformation from $(x^\mu, v^\mu; s)$ to $(x^\mu, p_\mu; s)$ can be accomplished by means of a Legendre transformation, and instead of the Lagrangian (24), we consider the Hamilton function

$$\mathcal{H}(x, p; s) = p_\mu v^\mu - \mathcal{L}(x, v; s). \tag{34}$$

The differential of the Hamiltonian in terms of x, p and s is given by

$$\mathrm{d}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial x^\mu} \mathrm{d}x^\mu + \frac{\partial \mathcal{H}}{\partial p_\mu} \mathrm{d}p_\mu + \frac{\partial \mathcal{H}}{\partial s} \mathrm{d}s. \tag{35}$$

On the other hand, from Eq.(34), we have

$$\mathrm{d}\mathcal{H} = v^\mu \mathrm{d}p_\mu + p_\mu \mathrm{d}v^\mu - \frac{\partial \mathcal{L}}{\partial v^\mu} \mathrm{d}v^\mu - \frac{\partial \mathcal{L}}{\partial x^\mu} \mathrm{d}x^\mu - \frac{\partial \mathcal{L}}{\partial s} \mathrm{d}s. \tag{36}$$

Taking into account the defining Eq.(32), the second and the third term on the right-hand-side of Eq.(36) cancel out. Eq.(33) can be further used to cast Eq.(36) into the form

$$\mathrm{d}\mathcal{H} = v^\mu \mathrm{d}p_\mu - \frac{\mathrm{d}p_\mu}{\mathrm{d}s} \mathrm{d}x^\mu - \frac{\partial \mathcal{L}}{\partial s} \mathrm{d}s \,, \tag{37}$$

Comparison between Eqs.(35) and (37) yields the Hamilton equations of motion

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{\partial \mathcal{H}}{\partial p_\mu} \,, \qquad \frac{\mathrm{d}p_\mu}{\mathrm{d}s} = -\frac{\partial \mathcal{H}}{\partial x^\mu} \,, \tag{38}$$

where the Hamiltonian is given by

$$\mathcal{H}(x, p; s) = \frac{g^{\mu\nu}}{2m} p_\mu p_\nu + \frac{e}{m} p_\mu \mathcal{A}^\mu. \tag{39}$$

In the case of a free particle, the Hamilton equations can be written explicitly as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{g^{\mu\nu}}{m} p_\nu\,, \qquad \frac{\mathrm{d}p_\lambda}{\mathrm{d}s} = -\frac{1}{2m}\frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu p_\nu\,. \tag{40}$$

To obtain the equations of motion we need the expression

$$\frac{\partial g^{\mu\nu}}{\partial x^\lambda} = -\Gamma^\mu_{\lambda\alpha} g^{\alpha\nu} - \Gamma^\nu_{\lambda\alpha} g^{\alpha\mu}, \tag{41}$$

which can be derived from the obvious identity

$$\frac{\partial}{\partial x^\lambda}\left(g^{\mu\alpha} g_{\alpha\nu}\right) = 0\,, \tag{42}$$

and Eq.(29). From the second of Eqs. (40), we obtain

$$\frac{\mathrm{D}p_\lambda}{\mathrm{D}s} = \frac{\mathrm{d}p_\lambda}{\mathrm{d}s} - \Gamma^\mu_{\lambda\nu} v^\nu p_\mu = 0\,, \tag{43}$$

similar to equation (30). Differentiating the first of the Hamilton equations (40) with respect to s and taking into account equations (41) and (43), we again arrive to the equation for the geodesics (31).

Let us now show that on a generic curved (torsion-free) manifolds the Poisson brackets are conserved. To achieve this result, we need the following identities

$$g^{\mu\nu} = g^{\nu\mu} = \eta^{\alpha\beta}\frac{\partial x^\mu}{\partial q^\alpha}\frac{\partial x^\nu}{\partial q^\beta}\,, \tag{44}$$

$$\frac{\partial^2 x^\lambda}{\partial q^\alpha \partial q^\beta} = -\Gamma^\lambda_{\mu\nu}\frac{\partial x^\mu}{\partial q^\alpha}\frac{\partial x^\nu}{\partial q^\beta}\,, \tag{45}$$

■ To prove (45), we differentiate the obvious identity

$$\frac{\partial x^\lambda}{\partial q^\rho}\frac{\partial q^\rho}{\partial x^\nu} = \delta^\lambda_\nu\,. \tag{46}$$

As a result, we find

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial q^\rho}\frac{\partial^2 q^\rho}{\partial x^\mu \partial x^\nu} = -\frac{\partial q^\rho}{\partial x^\nu}\frac{\partial q^\sigma}{\partial x^\mu}\frac{\partial^2 x^\lambda}{\partial q^\rho \partial q^\sigma}.\,\blacksquare \tag{47}$$

The next step is to calculate the fundamental Poisson brackets in terms of the variables (x^μ, p_ν), initially defined using the canonical variables (q^μ, π_ν) according to the relation

$$[U, V] = \frac{\partial U}{\partial q^\mu}\frac{\partial V}{\partial \pi_\mu} - \frac{\partial V}{\partial q^\mu}\frac{\partial U}{\partial \pi_\mu}, \tag{48}$$

where $U(q^\mu, \pi_\nu)$ and $V(q^\mu, \pi_\nu)$ are arbitrary functions. Making use of Eqs.(18) and (21), we know that the variables

$$q^\mu \quad \Leftrightarrow \quad \pi_\mu = m u_\mu = m\eta_{\mu\nu} u^\nu = m\eta_{\mu\nu}\frac{\partial q^\nu}{\partial x^\alpha} v^\alpha, \tag{49}$$

form a canonical conjugate pair. Using Eq.(32), we would like to check whether the variables

$$x^\mu \quad \Leftrightarrow \quad p_\mu = m g_{\mu\nu} v^\nu = g_{\mu\nu}\eta^{\alpha\lambda}\pi_\lambda\frac{\partial x^\nu}{\partial q^\alpha}, \tag{50}$$

form a canonical conjugate pair. We have

$$\begin{aligned}[U, V] = &\left[\frac{\partial U}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial q^\mu} + \frac{\partial U}{\partial p_\sigma}\eta^{\beta\lambda}\pi_\lambda\frac{\partial}{\partial q^\mu}\left(g_{\sigma\nu}\frac{\partial x^\nu}{\partial q^\beta}\right)\right]\times \\ &\times \frac{\partial V}{\partial p_\alpha} g_{\alpha\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho} - \\ &-\left[\frac{\partial V}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial q^\mu} + \frac{\partial V}{\partial p_\sigma}\eta^{\beta\lambda}\pi_\lambda\frac{\partial}{\partial q^\mu}\left(g_{\sigma\nu}\frac{\partial x^\nu}{\partial q^\beta}\right)\right]\times \\ &\times \frac{\partial U}{\partial p_\alpha} g_{\alpha\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho}.\end{aligned} \tag{51}$$

The first and the third term on the right-hand-side of Eq.(51) can be similarly manipulated as follows

$$\begin{aligned}\textbf{I}\text{-st term} &= \frac{\partial U}{\partial x^\alpha}\frac{\partial V}{\partial p_\beta} g_{\beta\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho}\frac{\partial x^\alpha}{\partial q^\mu} = \\ &= g_{\beta\chi} g^{\chi\alpha}\frac{\partial U}{\partial x^\alpha}\frac{\partial V}{\partial p_\beta} = \frac{\partial U}{\partial x^\alpha}\frac{\partial V}{\partial p_\alpha}\,,\end{aligned} \tag{52}$$

$$\textbf{III}\text{-rd term} = -\frac{\partial V}{\partial x^\alpha}\frac{\partial U}{\partial p_\alpha}\,. \tag{53}$$

Next, we manipulate the second term on the right-hand-side of Eq.(51). We obtain

$$\begin{aligned}\textbf{II}\text{-nd term} &= \frac{\partial U}{\partial p_\sigma}\frac{\partial V}{\partial p_\alpha} g_{\alpha\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho}\eta^{\beta\lambda}\pi_\lambda \times \\ &\times\left[g_{\sigma\nu}\frac{\partial^2 x^\nu}{\partial q^\mu \partial q^\beta} + \frac{\partial x^\nu}{\partial q^\beta}\frac{\partial g_{\sigma\nu}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial q^\mu}\right] = \\ &= \frac{\partial U}{\partial p_\sigma}\frac{\partial V}{\partial p_\alpha} g_{\alpha\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho}\eta^{\beta\lambda}\pi_\lambda \times \\ \times&\left[-g_{\sigma\nu}\Gamma^\nu_{\gamma\delta}\frac{\partial x^\gamma}{\partial q^\mu}\frac{\partial x^\delta}{\partial q^\beta} + \frac{\partial x^\delta}{\partial q^\beta}\frac{\partial x^\gamma}{\partial q^\mu}\left(\Gamma^\nu_{\gamma\sigma} g_{\nu\delta} + \Gamma^\nu_{\gamma\delta} g_{\nu\sigma}\right)\right] = \\ &= \frac{\partial U}{\partial p_\sigma}\frac{\partial V}{\partial p_\alpha} g_{\alpha\chi} g^{\chi\gamma}\eta^{\beta\lambda}\pi_\lambda\frac{\partial x^\delta}{\partial q^\beta} g_{\nu\delta}\Gamma^\nu_{\gamma\sigma} = \\ &= \frac{\partial U}{\partial p_\sigma}\frac{\partial V}{\partial p_\beta} g_{\mu\nu}\eta^{\alpha\lambda}\pi_\lambda\frac{\partial x^\nu}{\partial q^\alpha}\Gamma^\mu_{\beta\sigma} = \Gamma^\lambda_{\mu\nu} p_\lambda\frac{\partial U}{\partial p_\nu}\frac{\partial V}{\partial p_\mu}.\end{aligned} \tag{54}$$

The fourth term is similar to the second one but with U and V interchanged

$$\textbf{IV}\text{-th term} = -\Gamma^\lambda_{\mu\nu} p_\lambda\frac{\partial U}{\partial p_\mu}\frac{\partial V}{\partial p_\nu}\,. \tag{55}$$

In the absence of torsion, the affine connection $\Gamma^\lambda_{\mu\nu}$ is symmetric with respect to the lower indices, so that the second and the fourth term on the right-hand-side of Eq.(51) cancel each other. Therefore,

$$[U, V] = \frac{\partial U}{\partial x^\mu}\frac{\partial V}{\partial p_\mu} - \frac{\partial V}{\partial x^\mu}\frac{\partial U}{\partial p_\mu}\,, \tag{56}$$

which means that the fundamental Poisson brackets are conserved. On the other hand, this implies that the variables $\{x^\mu, p_\nu\}$ are a canonical conjugate pair.

As a final remark, we have to say that considering a generic metric $g_{\alpha\beta}$ and a connection $\Gamma^{\alpha}_{\mu\nu}$ is related to the fact that we are passing from a Minkowski-flat spacetime (local inertial reference frame) to an accelerated reference frame (curved spacetime). In what follows, we want to show that a generic bilinear Hamiltonian invariant, which is conformally conserved, gives always rise to a canonical symplectic structure. The specific theory is assigned by the vector (or tensor) fields which define the Hamiltonian invariant.

3 A symplectic structure compatible with general covariance

The above considerations can be linked together leading to a more general scheme where a covariant symplectic structure is achieved. Summarizing, the main points which we need are: (i) an even-dimensional vector space $\mathbf{E_{2n}}$ equipped with an antiscalar product satisfying the algebra (3)-(7); (ii) generic vector fields defined on such a space which have to satisfy the Poisson brackets; (iii) first-order equations of motion which can be read as Hamilton-like equations; (iv) general covariance which has to be preserved.

Such a program can be pursued by taking into account covariant and contravariant vector fields. In fact, it is possible to construct the Hamiltonian invariant

$$\mathcal{H} = V^{\alpha}V_{\alpha}\,, \tag{57}$$

which is a scalar quantity satisfying the relation

$$\delta\mathcal{H} = \delta(V^{\alpha}V_{\alpha}) = 0\,, \tag{58}$$

being δ a spurious variation due to the transport. It is worth stressing that the vectors V^{α} and V_{α} are not specified and the following considerations are completely general. Eq.(57) is a so called "*already parameterized*" invariant which can constitute the "density" of a parameterized action principle where the time coordinate is not distinguished *a priori* from the other coordinates [8, 9].

Let us now take into account the intrinsic variation of V^{α}. On a generic curved manifold, we have

$$DV^{\alpha} = dV^{\alpha} - \delta V^{\alpha} = \partial_{\beta}V^{\alpha}dx^{\beta} - \delta V^{\alpha}, \tag{59}$$

where D is the intrinsic variation, d the total variation and δ the spurious variation due to the transport on the curved manifold. The spurious variation has a very important meaning since, in General Relativity, if such a variation for a given quantity is equal to zero, this means that the quantity is conserved. From the definition of covariant derivative, applied to the contravariant vector, we have

$$DV^{\alpha} = \partial_{\beta}V^{\alpha}dx^{\beta} + \Gamma^{\alpha}_{\sigma\beta}V^{\sigma}dx^{\beta}, \tag{60}$$

and

$$\nabla_{\beta}V^{\alpha} = \partial_{\beta}V^{\alpha} + \Gamma^{\alpha}_{\sigma\beta}V^{\sigma}, \tag{61}$$

and then

$$\delta V^{\alpha} = -\Gamma^{\alpha}_{\sigma\beta}V^{\sigma}dx^{\beta}. \tag{62}$$

Analogously, for the covariant derivative applied to the covariant vector,

$$DV_{\alpha} = dV_{\alpha} - \delta V_{\alpha} = \partial_{\beta}V_{\alpha}dx^{\beta} - \delta V_{\alpha}\,, \tag{63}$$

and then

$$DV_{\alpha} = \partial_{\beta}V_{\alpha}dx^{\beta} - \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}dx^{\beta}, \tag{64}$$

and

$$\nabla_{\beta}V_{\alpha} = \partial_{\beta}V_{\alpha} - \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}\,. \tag{65}$$

The spurious variation is now

$$\delta V_{\alpha} = \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}dx^{\beta}. \tag{66}$$

Developing the variation (58), we have

$$\delta\mathcal{H} = V_{\alpha}\delta V^{\alpha} + V^{\alpha}\delta V_{\alpha}\,, \tag{67}$$

and

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = V_{\alpha}\frac{\delta V^{\alpha}}{dx^{\beta}} + V^{\alpha}\frac{\delta V_{\alpha}}{dx^{\beta}}, \tag{68}$$

which becomes

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = \frac{\delta V^{\alpha}}{dx^{\beta}}\frac{\partial\mathcal{H}}{\partial V^{\alpha}} + \frac{\delta V_{\alpha}}{dx^{\beta}}\frac{\partial\mathcal{H}}{\partial V_{\alpha}}, \tag{69}$$

being

$$\frac{\partial\mathcal{H}}{\partial V^{\alpha}} = V_{\alpha}, \qquad \frac{\partial\mathcal{H}}{\partial V_{\alpha}} = V^{\alpha}. \tag{70}$$

From Eqs.(62) and (66), it is

$$\frac{\delta V^{\alpha}}{dx^{\beta}} = -\Gamma^{\alpha}_{\sigma\beta}V^{\sigma} = -\Gamma^{\alpha}_{\sigma\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right), \tag{71}$$

$$\frac{\delta V_{\alpha}}{dx^{\beta}} = \Gamma^{\sigma}_{\alpha\beta}V_{\sigma} = \Gamma^{\sigma}_{\alpha\beta}\left(\frac{\partial\mathcal{H}}{\partial V^{\sigma}}\right), \tag{72}$$

and substituting into Eq.(69), we have

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = -\Gamma^{\alpha}_{\sigma\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\alpha}}\right) + \Gamma^{\sigma}_{\alpha\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\alpha}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\sigma}}\right), \tag{73}$$

and then, since α and σ are mute indexes, the expression

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = \left(\Gamma^{\alpha}_{\sigma\beta} - \Gamma^{\alpha}_{\sigma\beta}\right)\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\alpha}}\right) \equiv 0\,, \tag{74}$$

is identically equal to zero. In other words, $\mathcal{H}$ is absolutely conserved, and this is very important since the analogy with a canonical Hamiltonian structure
is straightforward. In fact, if, as above,

$$\mathcal{H} = \mathcal{H}(p, q) \tag{75}$$

is a classical generic Hamiltonian function, expressed in the canonical phase-space variables $\{p, q\}$, the total variation (in a vector space $\mathbf{E_{2n}}$ whose dimensions are generically given by p_i and q_j with $i, j = 1, ..., n$) is

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q} dq + \frac{\partial \mathcal{H}}{\partial p} dp \,, \tag{76}$$

and

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial q}\dot{q} + \frac{\partial \mathcal{H}}{\partial p}\dot{p} = = \frac{\partial \mathcal{H}}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial \mathcal{H}}{\partial p}\frac{\partial \mathcal{H}}{\partial q} \equiv 0 \,, \tag{77}$$

thanks to the Hamilton canonical equations

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \,, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \,. \tag{78}$$

Such a canonical approach holds also in our covariant case if we operate the substitutions

$$V^\alpha \longleftrightarrow p \qquad\qquad V_\alpha \longleftrightarrow q \tag{79}$$

and the canonical equations are

$$\frac{\delta V^\alpha}{dx^\beta} = -\Gamma^\alpha_{\sigma\beta}\left(\frac{\partial \mathcal{H}}{\partial V_\sigma}\right) \quad \longleftrightarrow \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \,, \tag{80}$$

$$\frac{\delta V_\alpha}{dx^\beta} = \Gamma^\sigma_{\alpha\beta}\left(\frac{\partial \mathcal{H}}{\partial V^\sigma}\right) \quad \longleftrightarrow \quad \frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p} \,. \tag{81}$$

In other words, starting from the (Hamiltonian) invariant (57), we have recovered a covariant canonical symplectic structure. The variation (67) may be seen as the generating function $\mathcal{G}$ of canonical transformations where the generators of $q-$, $p-$ and $t-$changes are dealt under the same standard.

At this point, some important remarks have to be done. The covariant and contravariant vector fields can be also of different nature so that the above fundamental Hamiltonian invariant can be generalized as

$$\mathcal{H} = W^\alpha V_\alpha \,, \tag{82}$$

or, considering scalar smooth and regular functions, as

$$\mathcal{H} = f(W^\alpha V_\alpha), \tag{83}$$

or, in general

$$\mathcal{H} = f\big(W^\alpha V_\alpha, B^{\alpha\beta}C_{\alpha\beta}, B^{\alpha\beta}V_\alpha V'_\beta, \ldots\big), \tag{84}$$

where the invariant can be constructed by covariant vectors, bivectors and tensors. Clearly, as above, the identifications

$$W^\alpha \longleftrightarrow p \qquad\qquad V_\alpha \longleftrightarrow q \tag{85}$$

hold and the canonical equations are

$$\frac{\delta W^\alpha}{dx^\beta} = -\Gamma^\alpha_{\sigma\beta}\left(\frac{\partial \mathcal{H}}{\partial V_\sigma}\right), \quad \frac{\delta V_\alpha}{dx^\beta} = \Gamma^\sigma_{\alpha\beta}\left(\frac{\partial \mathcal{H}}{\partial W^\sigma}\right). \tag{86}$$

Finally, conservation laws are given by

$$\frac{\delta \mathcal{H}}{dx^\beta} = \left(\Gamma^\alpha_{\sigma\beta} - \Gamma^\alpha_{\sigma\beta}\right)\left(\frac{\partial \mathcal{H}}{\partial V_\sigma}\right)\left(\frac{\partial \mathcal{H}}{\partial W^\alpha}\right) \equiv 0 \,. \tag{87}$$

In our picture, this means that the canonical symplectic structure is assigned in the way in which covariant and contravariant vector fields are related. However, if the Hamiltonian invariant is constructed by bivectors and tensors, equations (86) and (87) have to be generalized but the structure is the same. It is worth noticing that we never used the metric field but only connections in our derivations.

These considerations can be made independent of the reference frame if we define a suitable system of unitary vectors by which we can pass from holonomic to anholonomic description and viceversa. We can define the reference frame on the event manifold $\mathcal{M}$ as vector fields $e_{(k)}$ in event space and dual forms $e^{(k)}$ such that vector fields $e_{(k)}$ define an orthogonal frame at each point and

$$e^{(k)}\big(e_{(l)}\big) = \delta^{(k)}_{(l)} \,. \tag{88}$$

If these vectors are unitary, in a Riemannian 4-spacetime are the standard *vierbiens* [5].

If we do not limit this definition of reference frame by orthogonality, we can introduce a *coordinate reference frame* $(\partial_\alpha, ds^\alpha)$ based on vector fields tangent to line $x^\alpha = \text{const}$. Both reference frames are linked by the relations

$$e_{(k)} = e^\alpha_{(k)}\partial_\alpha; \qquad e^{(k)} = e^{(k)}_\alpha dx^\alpha. \tag{89}$$

From now on, Greek indices will indicate holonomic coordinates while Latin indices between brackets, the anholonomic coordinates (*vierbien* indices in 4-spacetimes). We can prove the existence of a reference frame using the orthogonalization procedure at every point of spacetime. From the same procedure, we get that coordinates of frame smoothly depend on the point. The statement about the existence of a global reference frame follows from this. A smooth field on time-like vectors of each frame defines congruence of lines that are tangent to this field. We say that each line is a world line of an observer or a *local reference frame*. Therefore a reference frame is a set of local reference frames. The *Lorentz transformation* can be defined as a transformation of a reference frame

$$x'^\alpha = f\big(x^0, x^1, x^2, x^3, \ldots, x^n\big), \tag{90}$$

$$e'^\alpha_{(k)} = A^\alpha_\beta B^{(l)}_{(k)} e^\beta_{(l)} \,, \tag{91}$$

where

$$A^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x'^\beta}, \qquad \delta_{(i)(l)} B^{(i)}_{(j)} B^{(l)}_{(k)} = \delta_{(j)(k)} \,. \tag{92}$$

We call the transformation A^α_β the holonomic part and transformation $B^{(l)}_{(k)}$ the anholonomic part.

A vector field V has two types of coordinates: *holonomic coordinates* V^α relative to a coordinate reference frame and *anholonomic coordinates* $V^{(k)}$ relative to a reference frame. For these two kinds of coordinates, the relation

$$V^{(k)} = e^{(k)}_\alpha V^\alpha \,, \tag{93}$$

holds. We can study parallel transport of vector fields using any form of coordinates. Because equations (90) and (91) are linear transformations, we expect that parallel transport in anholonomic coordinates has the same form as in holonomic coordinates. Hence we write

$$DV^\alpha = dV^\alpha + \Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma , \tag{94}$$

$$DV^{(k)} = dV^{(k)} + \Gamma^{(k)}_{(l)(p)} V^{(l)} dx^{(p)} . \tag{95}$$

Because DV^α is also a tensor, we get

$$\Gamma^{(k)}_{(l)(p)} = e^\alpha_{(l)} e^\beta_{(p)} e^{(k)}_\gamma \Gamma^\gamma_{\alpha\beta} + e^\alpha_{(l)} e^\beta_{(p)} \frac{\partial e^{(k)}_\alpha}{\partial x^\beta} . \tag{96}$$

Eq.(96) shows the similarity between holonomic and anholonomic coordinates. Let us introduce the symbol $\partial_{(k)}$ for the derivative along the vector field $e_{(k)}$

$$\partial_{(k)} = e^\alpha_{(k)} \partial_\alpha \,. \tag{97}$$

Then Eq.(96) takes the form

$$\Gamma^{(k)}_{(l)(p)} = e^\alpha_{(l)} e^\beta_{(p)} e^{(k)}_\gamma \Gamma^\gamma_{\alpha\beta} + e^\alpha_{(l)} \partial_{(p)} e^{(k)}_\alpha \,. \tag{98}$$

Therefore, when we move from holonomic coordinates to anholonomic ones, the connection also transforms the way similarly to when we move from one coordinate system to another. This leads us to the model of anholonomic coordinates. The vector field $e_{(k)}$ generates lines defined by the differential equations

$$e^\alpha_{(l)} \frac{\partial \tau}{\partial x^\alpha} = \delta^{(k)}_{(l)} \,, \tag{99}$$

or the symbolic system

$$\frac{\partial \tau}{\partial x^{(l)}} = \delta^{(k)}_{(l)} \,. \tag{100}$$

Keeping in mind the symbolic system (100), we denote the functional τ as $x^{(k)}$ and call it the anholonomic coordinate. We call the regular coordinate holonomic. Then we can find derivatives and get

$$\frac{\partial x^{(k)}}{\partial x^\alpha} = \delta^{(k)}_\alpha \,. \tag{101}$$

The necessary and sufficient conditions to complete the integrability of system (101) are

$$\omega^{(i)}_{(k)(l)} = e^\alpha_{(k)} e^\beta_{(l)} \left(\frac{\partial e^{(i)}_\alpha}{\partial x^\beta} - \frac{\partial e^{(i)}_\beta}{\partial x^\alpha} \right) = 0 \,, \tag{102}$$

where we introduced the anholonomic object $\omega^{(i)}_{(k)(l)}$.

Therefore each reference frame has n vector fields

$$\partial_{(k)} = \frac{\partial}{\partial x^{(k)}} = e^\alpha_{(k)} \partial_\alpha \,, \tag{103}$$

which have the commutators

$$\begin{aligned} \left[\partial_{(i)}, \partial_{(j)}\right] &= \left(e^\alpha_{(i)} \partial_\alpha e^\beta_{(j)} - e^\alpha_{(j)} \partial_\alpha e^\beta_{(i)} \right) e^{(m)}_\beta \partial_{(m)} = \\ &= e^\alpha_{(i)} e^\beta_{(j)} \left(-\partial_\alpha e^{(m)}_\beta + \partial_\beta e^{(m)}_\beta \right) \partial_{(m)} = \omega^{(m)}_{(i)(j)} \partial_{(m)} . \end{aligned} \tag{104}$$

For the same reason, we introduce the forms

$$dx^{(k)} = e^{(k)} = e^{(k)}_\beta dx^\beta , \tag{105}$$

and a differential of this form is

$$\begin{aligned} d^2 x^{(k)} &= d\left(e^{(k)}_\alpha dx^\alpha \right) = \left(\partial_\beta e^{(k)}_\alpha - \partial_\alpha e^{(k)}_\beta \right) dx^\alpha \wedge dx^\beta = \\ &= -\omega^{(m)}_{(k)(l)} dx^{(k)} \wedge dx^{(l)} . \end{aligned} \tag{106}$$

Therefore when $\omega^{(i)}_{(k)(l)} \neq 0$, the differential $dx^{(k)}$ is not an exact differential and the system (101), in general, cannot be integrated. However, we can consider meaningful objects which model the solution. We can study how the functions $x^{(i)}$ changes along different lines. The functions $x^{(i)}$ is a natural parameter along a flow line of vector fields $e_{(i)}$. It is defined along any line.

All the above results can be immediately achieved in holonomic and anholonomic formalism considering the equation

$$\mathcal{H} = W^\alpha V_\alpha = W^{(k)} V_{(k)} \,, \tag{107}$$

and the analogous ones. This means that the results are independent of the reference frame and the symplectic covariant structure always holds.

4 The hydrodynamic picture

In order to further check the validity of the above approach, we can prove that it is always consistent with the hydrodynamic picture (see also [10] for details on hydrodynamic covariant formalism).

Let us define a phase space density $f(x, p; s)$ which evolves according to the Liouville equation

$$\frac{\partial f}{\partial s} + \frac{1}{m} \frac{\partial}{\partial x^\mu} \left(g^{\mu\nu} p_\nu f \right) - \frac{1}{2m} \frac{\partial}{\partial p_\lambda} \left(\frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu p_\nu f \right) = 0 \,. \tag{108}$$

Next we define the density $\varrho(x; s)$, the covariant current velocity $v_\mu(x; s)$ and the covariant stress tensor $\mathcal{P}_{\mu\nu}(x; s)$ according to the relations

$$\varrho(x; s) = mn \int \mathrm{d}^4 p f(x, p; s) \,, \tag{109}$$

$$\varrho(x;s)v_\mu(x;s) = n\int \mathrm{d}^4 p p_\mu f(x,p;s)\,, \tag{110}$$

$$\mathcal{P}_{\mu\nu}(x;s) = \frac{n}{m}\int \mathrm{d}^4 p p_\mu p_\nu f(x,p;s)\,. \tag{111}$$

It can be verified, by direct substitution, that a solution to the Liouville Eq.(108) of the form

$$f(x,p;s) = \frac{1}{mn}\varrho(x;s)\delta^4[p_\mu - m v_\mu(x;s)]\,, \tag{112}$$

leads to the equation of continuity

$$\frac{\partial\varrho}{\partial s} + \frac{\partial}{\partial x^\mu}(g^{\mu\nu}v_\nu\varrho) = 0\,, \tag{113}$$

and to the equation for balance of momentum

$$\frac{\partial}{\partial s}(\varrho v_\mu) + \frac{\partial}{\partial x^\lambda}\left(g^{\lambda\alpha}\mathcal{P}_{\alpha\mu}\right) + \frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}\mathcal{P}_{\alpha\beta} = 0\,. \tag{114}$$

Taking into account the fact that for the particular solution (112), the stress tensor, as defined by Eq.(111), is given by the expression

$$\mathcal{P}_{\mu\nu}(x;s) = \varrho v_\mu v_\nu\,, \tag{115}$$

we obtain the final form of the hydrodynamic equations

$$\frac{\partial\varrho}{\partial s} + \frac{\partial}{\partial x^\mu}(\varrho v^\mu) = 0\,, \tag{116}$$

$$\frac{\partial v_\mu}{\partial s} + v^\lambda\left(\frac{\partial v_\mu}{\partial x^\lambda} - \Gamma^\nu_{\mu\lambda}v_\nu\right) = \frac{\partial v_\mu}{\partial s} + v^\lambda\nabla_\lambda v_\mu = 0\,. \tag{117}$$

It is straightforward to see that, through the substitution $v_\mu \to V_\mu$, Eq.(72) is immediately recovered along a geodesic, that is our covariant symplectic structure is consistent with a hydrodynamic picture. It is worth noting that if $\frac{\partial v_\mu}{\partial s}$ in Eq.(117), the motion is not geodesic. The meaning of this term different from zero is that an extra force is acting on the system.

5 Applications, Discussion and Conclusions

Many applications of the previous results can be achieved specifying the nature of vector (or tensor) fields which define the Hamiltonian conserved invariant $\mathcal{H}$. Considerations in General Relativity and Electromagnetism are particularly interesting at this point. Let us take into account the Riemann tensor $R^\rho_{\sigma\mu\nu}$. It comes out when a given vector V^ρ is transported along a closed path on a generic curved manifold. It is

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma, \tag{118}$$

where ∇_μ is the covariant derivative. We are assuming a Riemannian $\mathbf{V}_n$ manifold as standard in General Relativity. If connection is not symmetric, an additive torsion field comes out from the parallel transport.

Clearly, the Riemann tensor results from the commutation of covariant derivatives and it can be expressed as the sum of two commutators

$$R^\rho_{\sigma\mu\nu} = \partial_{[\mu}, \Gamma^\rho_{\nu]\sigma} + \Gamma^\rho_{\lambda[\mu}, \Gamma^\lambda_{\nu]\sigma}\,. \tag{119}$$

Furthermore, (anti) commutation relations and cyclic identities (in particular Bianchi's identities) hold for the Riemann tensor [5].

All these straightforward considerations suggest the presence of a symplectic structure whose elements are covariant and contravariant vector fields, V^α and V_α, satisfying the properties (3)-(7). In this case, the dimensions of vector space $\mathbf{E}_{2n}$ are assigned by V^α and V_α. It is important to notice that such properties imply the connections (Christoffel symbols) and not the metric tensor.

As we said, the invariant (57) is a generic conserved quantity specified by the choice of V^α and V_α. If

$$V^\alpha = \frac{dx^\alpha}{ds}\,, \tag{120}$$

is a 4-velocity, with $\alpha = 0, 1, 2, 3$, immediately, from Eq.(80), we obtain the equation of geodesics of General Relativity,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,. \tag{121}$$

On the other hand, being

$$\delta V^\alpha = R^\alpha_{\beta\mu\nu}V^\beta dx_1^\mu dx_2^\nu\,, \tag{122}$$

the result of the transport along a closed path, it is easy to recover the geodesic deviation considering the geodesic (121) and the infinitesimal variation ξ^α with respect to it, i. e.

$$\frac{d^2(x^\alpha+\xi^\alpha)}{ds^2} + \Gamma^\alpha_{\mu\nu}(x+\xi)\frac{d(x^\mu+\xi^\mu)}{ds}\frac{d(x^\nu+\xi^\nu)}{ds} = 0, \tag{123}$$

which gives, through Eq.(119),

$$\frac{d^2\xi^\alpha}{ds^2} = R^\alpha_{\mu\lambda\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\lambda. \tag{124}$$

Clearly the symplectic structure is due to the fact that the Riemann tensor is derived from covariant derivatives either as

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma, \tag{125}$$

or

$$[\nabla_\mu, \nabla_\nu]V_\rho = R^\sigma_{\mu\nu\rho}V_\sigma\,. \tag{126}$$

In other words, fundamental equations of General Relativity are recovered from our covariant symplectic formalism.

Another interesting choice allows to recover the standard Electromagnetism. If $V^\alpha = A^\alpha$, where A^α is the vector potential and the Hamiltonian invariant is

$$\mathcal{H} = A^\alpha A_\alpha\,, \tag{127}$$

it is straightforward, following the above procedure, to obtain, from the covariant Hamilton equations, the electromagnetic tensor field

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \nabla_{[\alpha} A_{\beta]}\,, \tag{128}$$

and the Maxwell equations (in a generic empty curved space-time)

$$\nabla^\alpha F_{\alpha\beta} = 0\,, \qquad \nabla_{[\alpha} F_{\lambda\beta]} = 0\,. \tag{129}$$

The standard Lorentz gauge is

$$\nabla^\alpha A_\alpha = 0\,, \tag{130}$$

and electromagnetic wave equation is easily recovered.

In summary, a covariant, symplectic structure can be found for every Hamiltonian invariant which can be constructed by covariant vectors, bivectors and tensor fields. In fact, any theory of physics has to be endowed with a symplectic structure in order to be formulated at a fundamental level.

We pointed out that curvature invariants of General Relativity can show such a feature and, furthermore, they can be recovered from Hamiltonian invariants opportunely defined. Another interesting remark deserves the fact that, starting from such invariants, covariant and contravariant vector fields can be read as the configurations q^i and the momenta p_i of classical Hamiltonian dynamics so then the Hamilton-like equations of motion are recovered from the application of covariant derivative to both these vector fields. Besides, the approach can be formulated in a holonomic and anholonomic representations, once vector fields (or tensors in general) are represented in *vierbien* or coordinate–frames. This feature is essential to be sure that general covariance and symplectic structure are conserved in any case.

Specifying the nature of vector fields, we select the particular theory. For example, if the vector field is the 4-velocity, we obtain geodesic motion and geodesic deviation. If the vector is the vector potential of Electromagnetism, Maxwell equations and Lorentz gauge are recovered. The scheme is independent of the nature of vector field and, in our opinion, it is a strong hint toward a unifying view of basic interactions, gravity included.

Acknowledgements

We wish to acknowledge Francesco Guerra for the useful discussions and suggestions on the topic.

References

1. Sakurai J. J. Modern Quantum Mechanics. Revised Edition, Addison-Wesley Publ. Co., New York, 1994.
2. Itzykson C., Zuber J. B. Quantum field theory. McGraw-Hill, Singapore, 1980.
3. Kaku M. Quantum field theory. Oxford Univ. Press, Oxford, 1993.
4. Birrell N., Davies P. C. Quantum fields in curved space. Cambridge Univ., 1984.
5. Landau L. D., Lifshitz E. M. Theorie du Champs. Mir, Moscow, 1960.
6. Palatini A. *Rend. Circ. Mat. Palermo*, 1919, v. 43, 203.
7. Misner C. W., Thorne K. S., and Wheeler J. W. Gravitation. W. H. Freeman and Company, New York, 1970.
8. Lanczos C. The variational principles of mechanics. Toronto Univ. Press, 1949.
9. Schwinger E. *Phys. Rev.*, 1951, v. 82, 914; *Phys. Rev.*, 1953, v. 91, 713.
10. Ferapontov E. V., Pavlov M. V. nlin.SI/0212026, 2002.

Entangled States and Quantum Causality Threshold in the General Theory of Relativity

Dmitri Rabounski, Larissa Borissova, Florentin Smarandache

Dept. of Mathematics, University of New Mexico, 200 College Road, Gallup, NM 87301, USA

E-mail: rabounski@yahoo.com; lborissova@yahoo.com; fsmarandache@yahoo.com

This article shows, Synge-Weber's classical problem statement about two particles interacting by a signal can be reduced to the case where the same particle is located in two different points A and B of the basic space-time in the same moment of time, so the states A and B are entangled. This particle, being actual two particles in the entangled states A and B, can interact with itself radiating a photon (signal) in the point A and absorbing it in the point B. That is our goal, to introduce entangled states into General Relativity. Under specific physical conditions the entangled particles in General Relativity can reach a state where neither particle A nor particle B can be the cause of future events. We call this specific state Quantum Causality Threshold.

1 Disentangled and entangled particles in General Relativity. Problem statement

In his article of 2000, dedicated to the 100th anniversary of the discovery of quanta, Belavkin [1] generalizes definitions assumed *de facto* in Quantum Mechanics for entangled and disentangled particles. He writes:

> "The only distinction of the classical theory from quantum is that the prior mixed states cannot be dynamically achieved from pure initial states without a procedure of either statistical or chaotic mixing. In quantum theory, however, the mixed, or decoherent states can be dynamically induced on a subsystem from the initial pure disentangled states of a composed system simply by a unitary transformation.
>
> Motivated by Eintein-Podolsky-Rosen paper, in 1935 Schrödinger published a three part essay* on *The Present Situation in Quantum Mechanics*. He turns to EPR paradox and analyses completeness of the description by the wave function for the entangled parts of the system. (The word *entangled* was introduced by Schrödinger for the description of nonseparable states.) He notes that if one has pure states $\psi(\sigma)$ and $\chi(\upsilon)$ for each of two completely separated bodies, one has maximal knowledge, $\psi_1(\sigma,\upsilon) = \psi(\sigma)\chi(\upsilon)$, for two taken together. But the converse is not true for the entangled bodies, described by a non-separable wave function $\psi_1(\sigma,\upsilon) \neq \psi(\sigma)\chi(\upsilon)$: Maximal knowledge of a total system does not necessary imply maximal knowledge of all its parts, not even when these are completely separated one from another, and at the time can not influence one another at all."

In other word, because Quantum Mechanics considers particles as stochastic clouds, there can be entangled particles — particles whose states are entangled, they build a whole system so that if the state of one particle changes the state of the other particles changes immediately as they are far located one from the other.

In particular, because of the permission for entangled states, Quantum Mechanics permits quantum teleportation — the experimentally discovered phenomenon. The term "quantum teleportation" had been introduced into theory in 1993 [2]. First experiment teleporting massless particles (quantum teleportation of photons) was done five years later, in 1998 [3]. Experiments teleporting mass-bearing particles (atoms as a whole) were done in 2004 by two independent groups of scientists: quantum teleportation of the ion of Calcium atom [4] and of the ion of Beryllium atom [5].

There are many followers who continue experiments with quantum teleportation, see [6–16] for instance.

It should be noted, the experimental statement on quantum teleportation has two channels in which information (the quantum state) transfers between two entangled particles: "teleportation channel" where information is transferred instantly, and "synchronization channel" — classical channel where information is transferred in regular way at the light speed or lower of it (the classical channel is targeted to inform the receiving particle about the initial state of the first one). After teleportation the state of the first particle destroys, so there is data transfer (not data copying).

General Relativity draws another picture of data transfer: the particles are considered as point-masses or waves, not stochastic clouds. This statement is true for both mass-bearing particles and massless ones (photons). Data transfer between any two particles is realized as well by point-mass particles, so in General Relativity this process is not of stochastic origin.

In the classical problem statement accepted in General Relativity [17, 18, 19], two mass-bearing particles are con-

*Schrödinger E. *Naturwissenschaften*, 1935, Band 23, 807–812, 823–828, 844–849.

sidered which are moved along neighbour world-lines, a signal is transferred between them by a photon. One of the particles radiates the photon at the other, where the photon is absorbed realizing data transfer between the particles. Of course, the signal can as well be carried by a mass-bearing particle.

If there are two free mass-bering particles, they fall freely along neighbour geodesic lines in a gravitational field. This classical problem has been developed in Synge's book [20] where he has deduced the geodesic lines deviation equation (Synge's equation, 1950's). If these are two particles connected by a non-gravitational force (for instance, by a spring), they are moved along neighbour non-geodesic world-lines. This classical statement has been developed a few years later by Weber [21], who has obtained the world-lines deviation equation (Synge-Weber's equation).

Anyway in this classical problem of General Relativity two interacting particles moved along both neighbour geodesic and non-geodesic world-lines are *disentangled*. This happens, because of two reasons:

1. In this problem statement a signal moves between two interacting particles at the velocity no faster than light, so their states are absolutely separated — these are *disentangled states*;
2. Any particle, being considered in General Relativity's space-time, has its own four-dimensional trajectory (world-line) which is the set of the particle's states from its birth to decay. Two different particles can not occupy the same world-line, so they are in absolutely separated states — they are *disentangled* particles.

The second reason is much stronger than the first one. In particular, the second reason leads to the fact that, in General Relativity, *entangled* are only neighbour states of the same particle along its own world-line — its own states separated in time, not in the three-dimensional space. No two different particles could be entangled. Any two different particles, both mass-bearing and massless ones, are *disentangled* in General Relativity.

On the other hand, experiments on teleportation evident that *entanglement* is really an existing state that happens with particles if they reach specific physical conditions. This is the fact, that should be taken into account by General Relativity.

Therefore our task in this research is to introduce entangled states into General Relativity. Of course, because of the above reasons, two particles can not be in entangled state if they are located in the basic space-time of General Relativity — the four-dimensional pseudo-Riemannian space with sign-alternating label $(+---)$ or $(-+++)$. Its metric is strictly non-degenerated as of any space of Riemannian space family, namely — there the determinant $g = \det\|g_{\alpha\beta}\|$ of the fundamental metric tensor $g_{\alpha\beta}$ is strictly negative $g<0$. We expand the Synge-Weber problem statement, considering it in a *generalized space-time* whose metric can become degenerated $g=0$ under specific physical conditions. This space is one of Smarandache geometry spaces [22–28], because its geometry is partially Riemannian, partially not.

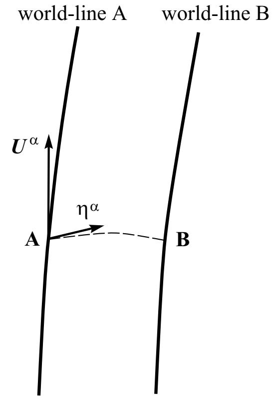


Fig. 1: Synge-Weber's statement.

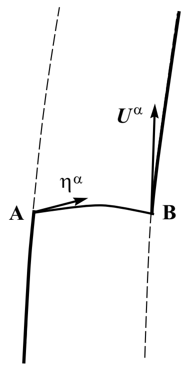


Fig. 2: The advanced statement.

As it was shown in [29, 30] (Borissova and Rabounski, 2001), when General Relativity's basic space-time degenerates physical conditions can imply *observable teleportation* of both a mass-bearing and massless particle — its instant displacement from one point of the space to another, although it moves no faster than light in the degenerated space-time area, outside the basic space-time. In the generalized space-time the Synge-Weber problem statement about two particles interacting by a signal (see Fig. 1) can be reduced to the case where the same particle is located in two different points A and B of the basic space-time in the same moment of time, so the states A and B are entangled (see Fig. 2). This particle, being actual two particles in the entangled states A and B, can interact with itself radiating a photon (signal) in the point A and absorbing it in the point B. That is our goal, to introduce entangled states into General Relativity.

Moreover, as we will see, under specific physical conditions the entangled particles in General Relativity can reach a state where neither particle A nor particle B can be the cause of future events. We call this specific state *Quantum Causality Threshold*.

2 Introducing entangled states into General Relativity

In the classical problem statement, Synge [20] considered two free-particles (Fig. 1) moving along neighbour geodesic world-lines $\Gamma(v)$ and $\Gamma(v+dv)$, where v is a parameter along the direction orthogonal to the geodesics (it is taken in the plane normal to the geodesics). There is $v=\text{const}$ along each the geodesic line.

Motion of the particles is determined by the well-known geodesic equation

$$\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\, U^\mu \frac{dx^\nu}{ds} = 0\,, \tag{1}$$

which is the actual fact that the absolute differential $\mathrm{D}U^\alpha = dU^\alpha + \Gamma^\alpha_{\mu\nu} U^\mu dx^\nu$ of a tangential vector U^α (the velocity world-vector $U^\alpha = \frac{dx^\alpha}{ds}$, in this case), transferred along that geodesic line to where it is tangential, is zero. Here s is an invariant parameter along the geodesic (we assume it the space-time interval), and $\Gamma^\alpha_{\mu\nu}$ are Christoffel's symbols of the 2nd kind. Greek $\alpha = 0, 1, 2, 3$ sign for four-dimensional (space-time) indices.

The parameter v is different for the neighbour geodesics, the difference is dv. Therefore, in order to study relative displacements of two geodesics $\Gamma(v)$ and $\Gamma(v+dv)$, we shall study the vector of their infinitesimal relative displacement

$$\eta^\alpha = \frac{\partial x^\alpha}{\partial v}\, dv\,, \tag{2}$$

As Synge had deduced, a deviation of the geodesic line $\Gamma(v+dv)$ from the geodesic line $\Gamma(v)$ can be found as the solution of his obtained equation

$$\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \eta^\gamma = 0\,, \tag{3}$$

that describes relative accelerations of two neighbour free-particles ($R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ is Riemann-Chrostoffel's curvature tensor). This formula is known as the geodesic lines deviation equation or the *Synge equation*.

In Weber's statement [21] the difference is that he considers two particles connected by a non-gravitational force Φ^α, a spring for instance. So their world-trajectories are non-geodesic, they are determined by the equation

$$\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\, U^\mu \frac{dx^\nu}{ds} = \frac{\Phi^\alpha}{m_0 c^2}\,, \tag{4}$$

which is different from the geodesic equation in that the right part in not zero here. His deduced improved equation of the world lines deviation

$$\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \eta^\gamma = \frac{1}{m_0 c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}\, dv\,, \tag{5}$$

describes relative accelerations of two particles (of the same rest-mass m_0), connected by a spring. His deviation equation is that of Synge, except of that non-gravitational force Φ^α in the right part. This formula is known as the *Synge-Weber equation*. In this case the angle between the vectors U^α and η^α does not remain unchanged along the trajectories

$$\frac{\partial}{\partial s}(U_\alpha \eta^\alpha) = \frac{1}{m_0 c^2}\,\Phi_\alpha \eta^\alpha. \tag{6}$$

Now, proceeding from this problem statement, we are going to introduce entangled states into General Relativity. At first we determine such states in the space-time of General Relativity, then we find specific physical conditions under which two particles reach a state to be entangled.

Definition Two particles A and B, located in the same spatial section* at the distance $dx^i \neq 0$ from each other, are filled in non-separable states if the observable time interval $d\tau$ between linked events in the particles† is zero $d\tau = 0$. If only $d\tau = 0$, the states become non-separated one from the other, so the particles A and B become entangled.

So we will refer to $d\tau = 0$ as the *entanglement condition* in General Relativity.

Let us consider the entanglement condition $d\tau = 0$ in connection with the world-lines deviation equations.

In General Relativity, the interval of physical observable time $d\tau$ between two events distant at dx^i one from the other is determined through components of the fundamental metric tensor as

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i, \tag{7}$$

see §84 in the well-known *The Classical Theory of Fields* by Landau and Lifshitz [19]. The mathematical apparatus of physical observable quantities (Zelmanov's theory of chronometric invariants [31, 32], see also the brief account in [30, 29]) transforms this formula to

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i, \tag{8}$$

where $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ is the gravitational potential of an acting gravitational field, and $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation.

So, following the theory of physical observable quantities, in real observations where the observer accompanies his references the space-time interval $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ is

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \tag{9}$$

where $d\sigma^2 = \left(-g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}\right) dx^i dx^k$ is a three-dimensional (spatial) invariant, built on the metric three-dimensional observable tensor $h_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}$. This metric observable tensor, in real observations where the observer accompanies his references, is the same that the analogous built general covariant tensor $h_{\alpha\beta}$. So, $d\sigma^2 = h_{ik}\, dx^i dx^k$ is the spatial observable interval for any observer who accompanies his references.

As it is easy to see from (9), there are two possible cases where the entanglement condition $d\tau = 0$ occurs:

(1) $ds = 0$ and $d\sigma = 0$,

(2) $ds^2 = -d\sigma^2 \neq 0$, so $d\sigma$ becomes imaginary,

*A three-dimensional section of the four-dimensional space-time, placed in a given point in the time line. In the space-time there are infinitely many spatial sections, one of which is our three-dimensional space.

†Such linked events in the particles A and B can be radiation of a signal in one and its absorbtion in the other, for instance.

we will refer to them as the *1st kind and 2nd kind entanglement auxiliary conditions*.

Let us get back to the Synge equation and the Synge-Weber equation.

According to Zelmanov's theory of physical observable quantities [31, 32], if an observer accompanies his references the projection of a general covariant quantity on the observer's spatial section is its spatial observable projection.

Following this way, Borissova has deduced (see eqs. 7.16–7.28 in [33]) that the spatial observable projection of the Synge equation is*

$$\frac{d^2\eta^i}{d\tau^2} + 2\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\frac{d\eta^k}{d\tau} = 0\,, \qquad (10)$$

she called it the *Synge equation in chronometrically invariant form*. The Weber equation is different in its right part containing the non-gravitational force that connects the particles (of course, the force should be filled in the spatially projected form). For this reason, conclusions obtained for the Synge equation will be the same that for the Weber one.

In order to make the results of General Relativity applicable to practice, we should consider tensor quantities and equations designed in chronometrically invariant form, because in such way they contain only chronometrically invariant quantities — physical quantities and geometrical properties of space, measurable in real experiment [31, 32].

Let us look at our problem under consideration from this viewpoint.

As it easy to see, the Synge equation in its chronometrically invariant form (10) under the entanglement condition $d\tau = 0$ becomes nonsense. The Weber equation becomes nonsense as well. So, the classical problem statement becomes senseless as soon as particles reach entangled states.

At the same time, in the recent theoretical research [29] two authors of the paper (Borissova and Rabounski, 2005) have found two groups of physical conditions under which particles can be teleported in non-quantum way. They have been called the *teleportation conditions*:

(1) $d\tau = 0$ $\{ds = 0\,,\ d\sigma = 0\}$, the conditions of photon teleportation;

(2) $d\tau = 0$ $\{ds^2 = -d\sigma^2 \neq 0\}$, the conditions of substantial (mass-bearing) particles teleportation.

There also were theoretically deduced physical conditions†, which should be reached in a laboratory in order to teleport particles in the non-quantum way [29].

As it is easy to see the non-quantum teleportation condition is identical to introduce here the entanglement main condition $d\tau = 0$ in couple with the 1st kind and 2nd kind auxiliary entanglement conditions!

Taking this one into account, we transform the classical Synge and Weber problem statement into another. In our statement the world-line of a particle, being entangled to itself by definition, splits into two different world-lines under teleportation conditions. In other word, as soon as the teleportation conditions occur in a research laboratory, the world-line of a teleported particle breaks in one world-point A and immediately starts in the other world-point B (Fig. 2). Both particles A and B, being actually two different states of the same teleported particle at a remote distance one from the other, are in *entangled states*. So, in this statement, the particles A and B themselves are *entangled*.

Of course, this entanglement exists in only the moment of the teleportation when the particle exists in two different states simultaneously. As soon as the teleportation process has been finished, only one particle of them remains so the entanglement disappears.

It should be noted, it follows from the entanglement conditions, that only substantial particles can reach entangled states in the basic space-time of General Relativity — the four-dimensional pseudo-Riemannian space. Not photons. Here is why.

As it is known, the interval $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$ can not be fully degenerated in a Riemannian space‡: the condition is that the determinant of the metric fundamental tensor $g_{\alpha\beta}$ must be strictly negative $g = \det\|\,g_{\alpha\beta}\| < 0$ by definition of Riemannian spaces. In other word, in the basic space-time of General Relativity the fundamental metric tensor must be strictly non-degenerated as $g < 0$.

The observable three-dimensional (spatial) interval $d\sigma^2 = h_{ik}\,dx^i dx^k$ is positive determined [31, 32], proceeding from physical sense. It fully degenerates $d\sigma^2 = 0$ if only the space compresses into point (the senseless case) or the determinant of the metric observable tensor becomes zero $h = \det\|\,h_{ik}\| = 0$.

As it was shown by Zelmanov [31, 32], in real observations where an observer accompanies his references, the determinant of the metric observable tensor is connected with the determinant of the fundamental one by the relationship $h = -\frac{g}{g_{00}}$. From here we see, if the three-dimensional observable metric fully degenerates $h = 0$, the four-dimensional metric degenerates as well $g = 0$.

We have obtained that states of two substantial particles can be entangled, if $d\tau = 0$ $\{ds^2 = -d\sigma^2 \neq 0\}$ in the space neighbourhood. So $h > 0$ and $g < 0$ in the neighbourhood,

*In this formula, according to Zelmanov's mathematical apparatus of physical observable quantities [31, 32], $D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} = \frac{1}{2\sqrt{g_{00}}}\frac{\partial h_{ik}}{\partial t}$ is the three-dimensional symmetric tensor of the space deformation observable rate while $A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right)$ is the three-dimensional antisymmetric tensor of the space rotation observable angular velocities, which indices can be lifted/lowered by the metric observable tensor so that $D^i_k = h^{im} D_{km}$ and $A^{\cdot i}_{k\cdot} = h^{im} A_{km}$. See brief account of the Zelmanov mathematical apparatus in also [30, 33, 34, 35].

†A specific correlation between the gravitational potential w, the space rotation linear velocity v_i and the teleported particle's velocity u^i.

‡It can only be partially degenerated. For instance, a four-dimensional Riemannian space can be degenerated into a three-dimensional one.

hence the four-dimensional pseudo-Riemannian space is not degenerated.

Conclusion Substantial particles can reach entangled states in the basic space-time of General Relativity (the four-dimensional pseudo-Riemannian space) under specific conditions in the neighbourhood.

Although $ds^2 = -d\sigma^2$ in the neighbourhood ($d\sigma$ should be imaginary), the substantial particles remain in regular sub-light area, they do not become super-light tachyons. It is easy to see, from the definition of physical observable time (8), the entanglement condition $d\tau = 0$ occurs only if the specific relationship holds

$$\mathrm{w} + v_i u^i = c^2 \tag{11}$$

between the gravitational potential w, the space rotation linear velocity v_i and the particles' true velocity $u^i = dx^i/dt$ in the observer's laboratory. For this reason, in the neighbourhood the space-time metric is

$$ds^2 = -d\sigma^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k, \tag{12}$$

so the substantial particles can become entangled if the space initial signature $(+---)$ becomes inverted $(-+++)$ in the neighbourhood, while the particles' velocities u^i remain no faster than light.

Another case — massless particles (photons). States of two phonos can be entangled, only if there is in the space neighbourhood $d\tau = 0$ $\{ds = 0\,,\, d\sigma = 0\}$. In this case the determinant of the metric observable tensor becomes $h = 0$, so the space-time metric as well degenerates $g = -g_{00}\, h = 0$. This is not the four-dimensional pseudo-Riemannian space.

Where is that area? In the previous works (Borissova and Rabounski, 2001 [30, 29]) a generalization to the basic space-time of General Relativity was introduced — the four-dimensional space which, having General Relativity's sign-alternating label $(+---)$, permits the space-time metric to be fully degenerated so that there is $g \leqslant 0$.

As it was shown in those works, as soon as the specific condition $\mathrm{w} + v_i u^i = c^2$ occurs, the space-time metric becomes fully degenerated: there are $ds = 0$, $d\sigma = 0$, $d\tau = 0$ (it can be easy derived from the above definition for the quantities) and, hence $h = 0$ and $g = 0$. Therefore, in a space-time where the *degeneration condition* $\mathrm{w} + v_i u^i = c^2$ is permitted the determinant of the fundamental metric tensor is $g \leqslant 0$. This case includes both Riemannian geometry case $g < 0$ and non-Riemannian, fully degenerated one $g = 0$. For this reason a such space is one of Smarandache geometry spaces [22–28], because its geometry is partially Riemannian, partially not*. In the such generalized space-time the 1st kind entanglement conditions $d\tau = 0$ $\{ds = 0\,,\, d\sigma = 0\}$ (the entanglement conditions for photons) are permitted in that area where the space metric fully degenerates (there $h = 0$ and, hence $g = 0$).

Conclusion Massless particles (photons) can reach entangled states, only if the basic space-time fully degenerates $g = \det \| g_{\alpha\beta} \| = 0$ in the neighbourhood. It is permitted in the generalized four-dimensional space-time which metric can be fully degenerated $g \leqslant 0$ in that area where the degeneration conditions occur. The generalized space-time is attributed to Smarandache geometry spaces, because its geometry is partially Riemannian, partially not.

So, entangled states have been introduced into General Relativity for both substantial particles and photons.

3 Quantum Causality Threshold in General Relativity

This term was introduced by one of the authors two years ago (Smarandache, 2003) in our common correspondence [36] on the theme:

Definition Considering two particles A and B located in the same spatial section, *Quantum Causality Threshold* was introduced as a special state in which neither A nor B can be the cause of events located "over" the spatial section on the Minkowski diagram.

The term *Quantum* has been added to the *Causality Threshold*, because in this problem statement an interaction is considered between two infinitely far away particles (in infinitesimal vicinities of each particle) so this statement is applicable to only quantum scale interactions that occur in the scale of elementary particles.

Now, we are going to find physical conditions under which particles can reach the threshold in the space-time of General Relativity.

Because in this problem statement we look at causal relations in General Relativity's space-time from "outside", it is required to use an "outer viewpoint" — a point of view located outside the space-time.

We introduce a such point of outlook in an Euclidean flat space, which is tangential to our's in that world-point, where the observer is located. In this problem statement we have a possibility to compare the absolute cause relations in that tangential flat space with those in ours. As a matter, a tangential Euclidean flat space can be introduced at any point of the pseudo-Riemannian space.

*In foundations of geometry it is known the *S-denying* of an axiom [22–25], i. e. in the same space an "axiom is false in at least two different ways, or is false and also true. Such axiom is said to be Smarandachely denied, or S-denied for short" [26]. As a result, it is possible to introduce geometries, which have common points bearing mixed properties of Euclidean, Lobachevsky-Bolyai-Gauss, and Riemann geometry in the same time. Such geometries has been called paradoxist geometries or Smarandache geometries. For instance, Iseri in his book *Smarandache Manifolds* [26] and articles [27, 28] introduced manifolds that support particular cases of such geometries.

At the same time, according to Zelmanov [31, 32], within infinitesimal vicinities of any point located in the pseudo-Riemannian space a *locally geodesic reference frame* can be introduced. In a such reference frame, within infinitesimal vicinities of the point, components of the metric fundamental tensor (marked by tilde)

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{\alpha\beta}}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \ldots \qquad (13)$$

are different from those $g_{\alpha\beta}$ at the point of reflection to within only the higher order terms, which can be neglected. So, in a locally geodesic reference frame the fundamental metric tensor can be accepted constant, while its first derivatives (Christoffel's symbols) are zeroes. The fundamental metric tensor of an Euclidean space is as well a constant, so values of $\tilde{g}_{\mu\nu}$, taken in the vicinities of a point of the pseudo-Riemannian space, converge to values of $g_{\mu\nu}$ in the flat space tangential at this point. Actually, we have a system of the flat space's basic vectors $\vec{e}_{(\alpha)}$ tangential to curved coordinate lines of the pseudo-Riemannian space. Coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other (the latest is true if the space rotates). Therefore the lengths of the basic vectors may be very different from the unit.

Writing the world-vector of an infinitesimal displacement as $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$, we obtain $d\vec{r} = \vec{e}_{(\alpha)} dx^\alpha$, where the components of the basic vectors $\vec{e}_{(\alpha)}$ tangential to the coordinate lines are $\vec{e}_{(0)} = \{e^0_{(0)}, 0, 0, 0\}$, $\vec{e}_{(1)} = \{0, e^1_{(1)}, 0, 0\}$, $\vec{e}_{(2)} = \{0, 0, e^2_{(2)}, 0\}$, $\vec{e}_{(3)} = \{0, 0, 0, e^2_{(3)}\}$. Scalar product of $d\vec{r}$ with itself is $d\vec{r}d\vec{r} = ds^2$ or, in another $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$, so $g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)}e_{(\beta)}\cos(x^\alpha; x^\beta)$. We obtain

$$g_{00} = e^2_{(0)}\,, \qquad g_{0i} = e_{(0)}e_{(i)}\cos(x^0; x^i)\,, \qquad (14)$$

$$g_{ik} = e_{(i)}e_{(k)}\cos(x^i; x^k)\,, \qquad i, k = 1, 2, 3\,. \qquad (15)$$

Then, substituting g_{00} and g_{0i} from formulas that determine the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ and the space rotation linear velocity $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$, we obtain

$$v_i = -c\,e_{(i)}\cos(x^0; x^i)\,, \qquad (16)$$

$$h_{ik} = e_{(i)}e_{(k)}\Big[\cos(x^0; x^i)\cos(x^0; x^k) - \cos(x^i; x^k)\Big]. \qquad (17)$$

From here we see: if the pseudo-Riemannian space is free of rotation, $\cos(x^0; x^i) = 0$ so the observer's spatial section is strictly orthogonal to time lines. As soon as the space starts to do rotation, the cosine becomes different from zero so the spatial section becomes non-orthogonal to time lines (Fig. 3). Having this process, the light hypercone inclines with the time line to the spatial section. In this inclination the light hypercone does not remain unchanged, it "compresses" because of hyperbolic transformations in pseudo-Riemannian space. The more the light hypercone inclines, the more it symmetrically "compresses" because the space-time's geometrical structure changes according to the inclination.

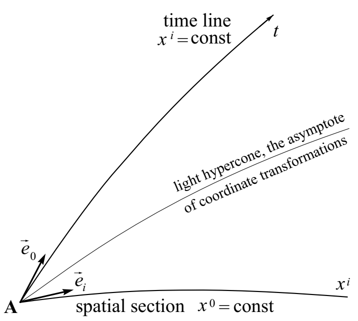


Fig. 3: The spatial section becomes non-orthogonal to time lines, as soon as the space starts rotation.

In the ultimate case, where the cosine reach the ultimate value $\cos(x^0; x^i) = 1$, time lines coincide the spatial section: time "has fallen" into the three-dimensional space. Of course, in this case the light hypercone overflows time lines and the spatial section: the light hypercone "has as well fallen" into the three-dimensional space.

As it is easy to see from formula (16), this ultimate case occurs as soon as the space rotation velocity v_i reaches the light velocity. If particles A and B are located in the space filled into this ultimate state, neither A nor B can be the cause of events located "over" the spatial section in the Minkowski diagrams we use in the pictures. So, in this ultimate case the space-time is filled into a special state called Quantum Causality Threshold.

Conclusion Particles, located in General Relativity's space-time, reach Quantum Causality Threshold as soon as the space rotation reaches the light velocity. Quantum Causality Threshold is impossible if the space does not rotate (holonomic space), or if it rotates at a sub-light speed.

So, Quantum Causality Threshold has been introduced into General Relativity.

References

1. Belavkin V. P. Quantum causality, decoherence, trajectories and information. arXiv: quant-ph/0208087, 76 pages.
2. Bennett C. H., Brassard G., Crepeau C., Jozsa R., Peres A., and Wootters W. K. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett.*, 1993, v. 70, 1895–1899.
3. Boschi D., Branca S., De Martini F., Hardy L., and Popescu S. Experimental realization of teleporting an unknown pure quantum state via dual classical and Einstein-Podolsky-Rosen Channels. *Phys. Rev. Lett.*, 1998, v. 80, 1121–1125.
4. Riebe M., Häffner H., Roos C. F., Hänsel W., Benhelm J., Lancaster G. P. T., Korber T. W., Becher C., Schmidt-Kaler F., James D. F. V., and Blatt R. Deterministic quantum teleportation with atoms. *Nature*, 2004, v. 429 (June, 17), 734–736.

5. Barrett M. D., Chiaverini J., Schaetz T., Britton J., Itano W. M., Jost J. D., Knill E., Langer C., Leibfried D., Ozeri R., Wineland D. J. Deterministic quantum teleportation of atomic qubits. *Nature*, 2004, v. 429 (June, 17), 737–739.
6. Pan J.-W., Bouwmeester D., Daniell M., Weinfurter H., Zeilinger A. Experimental test of quantum nonlocality in three-photon Greenberger-Horne-Zeilinger entanglement. *Nature*, 2000, v. 403 (03 Feb 2000), 515–519.
7. Mair A., Vaziri A., Weihs G., Zeilinger A. Entanglement of the orbital angular momentum states of photons. *Nature*, v. 412 (19 July 2001), 313–316.
8. Lukin M. D., Imamoglu A. Controlling photons using electromagnetically induced transparency *Nature*, v. 413 (20 Sep 2001), 273–276.
9. Julsgaard B., Kozhekin A., Polzik E. S. Experimental long-lived entanglement of two macroscopic objects. *Nature*, v. 413 (27 Sep 2001), 400–403.
10. Duan L.-M., Lukin M. D., Cirac J. I., Zoller P. Long-distance quantum communication with atomic ensembles and linear optics. *Nature*, v. 414 (22 Nov 2001), 413–418.
11. Yamamoto T., Koashi M., Özdemir Ş.K., Imoto N. Experimental extraction of an entangled photon pair from two identically decohered pairs. *Nature*, v. 421 (23 Jan 2003), 343–346.
12. Pan J.-W., Gasparoni S., Aspelmeyer M., Jennewein T., Zeilinger A. Experimental realization of freely propagating teleported qubits. *Nature*, v. 421 (13 Feb 2003), 721–725.
13. Pan J.-W., Gasparoni S., Ursin R., Weihs G., Zeilinger A. Experimental entanglement purification of arbitrary unknown states. *Nature*, v. 423 (22 May 2003), 417–422.
14. Zhao Zhi, Chen Yu-Ao, Zhang An-Ning, Yang T., Briegel H. J., Pan J.-W. Experimental demonstration of five-photon entanglement and open-destination teleportation. *Nature*, v. 430 (01 July 2004), 54–58.
15. Blinov B. B., Moehring D. L., Duan L.-M., Monroe C. Observation of entanglement between a single trapped atom and a single photon. *Nature*, v. 428 (11 Mar 2004), 153–157.
16. Ursin R., Jennewein T., Aspelmeyer M., Kaltenbaek R., Lindenthal M., Walther P., Zeilinger A. Communications: Quantum teleportation across the Danube. *Nature*, v. 430 (19 Aug 2004), 849–849.
17. Pauli W. Relativitätstheorie. *Encyclopäedie der mathematischen Wissenschaften*, Band V, Heft IV, Art. 19, 1921 (Pauli W. Theory of Relativity. Pergamon Press, 1958).
18. Eddington A. S. The mathematical theory of relativity. Cambridge University Press, Cambridge, 1924 (referred with the 3rd expanded edition, GTTI, Moscow, 1934, 508 pages).
19. Landau L. D. and Lifshitz E. M. The classical theory of fields. GITTL, Moscow, 1939 (ref. with the 4th final exp. edition, Butterworth–Heinemann, 1980).
20. Synge J. L. Relativity: the General Theory. North Holland, Amsterdam, 1960 (referred with the 2nd expanded edition, Foreign Literature, Moscow, 1963, 432 pages).
21. Weber J. General Relativity and gravitational waves. R. Marshak, New York, 1961 (referred with the 2nd edition, Foreign Literature, Moscow, 1962, 271 pages).
22. Smarandache F. Paradoxist mathematics. *Collected papers*, v. II, Kishinev University Press, Kishinev, 1997, 5–29.
23. Ashbacher C. Smarandache geometries. *Smarandache Notions*, book series, v. 8, ed. by C. Dumitrescu and V. Seleacu, American Research Press, Rehoboth, 1997, 212–215.
24. Chimienti S. P., Bencze M. Smarandache paradoxist geometry. *Bulletin of Pure and Applied Sciences*, 1998, v. 17E, No. 1, 123–124. See also *Smarandache Notions*, book series, v. 9, ed. by C. Dumitrescu and V. Seleacu, American Research Press, Rehoboth, 1998, 42–43.
25. Kuciuk L. and Antholy M. An introduction to Smarandache geometries. *New Zealand Math. Coll.*, Massey Univ., Palmerston North, New Zealand, Dec 3–6, 2001 (on-line http://atlas-conferences.com/c/a/h/f/09.htm).
26. Iseri H. Smarandache manifolds. American Research Press, Rehoboth, 2002.
27. Iseri H. Partially paradoxist Smarandache geometry. *Smarandache Notions*, book series, v. 13, ed. by J. Allen, F. Liu, D. Costantinescu, Am. Res. Press, Rehoboth, 2002, 5–12.
28. Iseri H. A finitely hyperbolic point in a smooth manifold. *JP Journal on Geometry and Topology*, 2002, v. 2 (3), 245–257.
29. Borissova L. B. and Rabounski D. D. On the possibility of instant displacements in the space-time of General Relativity. *Progress in Physics*, 2005, v. 1, 17–19.
30. Borissova L. B. and Rabounski D. D. Fields, vacuum, and the mirror Universe. Editorial URSS, Moscow, 2001, 272 pages (the 2nd revised ed.: CERN, EXT-2003-025).
31. Zelmanov A. L. Chronometric invariants. Dissertation, 1944. First published: CERN, EXT-2004-117, 236 pages.
32. Zelmanov A. L. Chronometric invariants and co-moving coordinates in the general relativity theory. *Doklady Acad. Nauk USSR*, 1956, v. 107 (6), 815–818.
33. Borissova L. Gravitational waves and gravitational inertial waves in the General Theory of Relativity: A theory and experiments. *Progress in Physics*, 2005, v. 2, 30–62.
34. Rabounski D. A new method to measure the speed of gravitation. *Progress in Physics*, 2005, v. 1, 3–6.
35. Rabounski D. A theory of gravity like electrodynamics. *Progress in Physics*, 2005, v. 2, 15–29.
36. Smarandache F. Private communications with D. Rabounski and L. Borissova, May 2005.

Bootstrap Universe from Self-Referential Noise

Reginald T. Cahill and Christopher M. Klinger

School of Chemistry, Physics and Earth Sciences, Flinders University,
GPO Box 2100, Adelaide 5001, Australia

E-mail: Reg.Cahill@flinders.edu.au

We further deconstruct Heraclitean Quantum Systems giving a model for a universe using pregeometric notions in which the end-game problem is overcome by means of self-referential noise. The model displays self-organisation with the emergence of 3-space and time. The time phenomenon is richer than the present geometric modelling.

1 Heraclitean Quantum Systems

From the beginning of theoretical physics in the 6th and 5th centuries BC there has been competition between two classes of modelling of reality: one class has reality explained in terms of things, and the other has reality explained purely in terms of relationships (information).* While in conventional physics a mix of these which strongly favours the "things" approach is currently and very efficaciously used, here we address the problem of the "ultimate" modelling of reality. This we term the *end-game* problem: at higher levels in the phenomenology of reality one chooses economical and effective models — which usually have to be accompanied by meta-rules for interpretation, but at the lower levels we are confronted by the problem of the source of "things" and their rules or "laws". At one extreme we could have an infinite regress of ever different "things", another is the notion of a Platonic world where mathematical things and their rules reside [1]. In both instances we still have the fundamental problem of why the universe "ticks" — that is, why it is more than a mathematical construct; why is it experienced?

This "end-game" problem is often thought of as the unification of our most successful and deepest, but incompatible, phenomenologies: General Relativity and Quantum Theory. We believe that the failure to find a common underpinning of these models is that it is apparently often thought it would be some amalgamation of the two, and not something vastly different. Another difficulty is that the lesson from these models is often confused; for instance from the success of the geometrical modelling of space and time it is often argued that the universe "*is* a 4-dimensional manifold". However the geometrical modelling of time is actually deficient: it lacks much of the experienced nature of time — for it fails to model both the directionality of time and the phenomenon of the (local) "present moment". Indeed the geometrical model might better be thought of as a "historical model" of time, because in histories the notion of direction and present moment are absent — they must be provided by external meta-rules. General relativity then is about possible histories of the universe, and in this it is both useful and successful. Similarly quantum field theories have fields built upon a possible (historical) spacetime, and subjected to quantisation. But such quantum theories have difficulties with classicalisation and the individuality of events — as in the "measurement problem". At best the theory invokes ensemble measurement postulates as external meta-rules. So our present-day quantum theories are also historical models.

The problem of unifying general relativity and quantum theories then comes down to going beyond *historical* modelling, which in simple terms means finding a better model of time. The historical or *being* model of reality has been with us since Parmenides and Zeno, and is known as the Eleatic model. The *becoming* or *processing* model of reality dates back further to Heraclitus of Ephesus (540–480 BC) who argued that common sense is mistaken in thinking that the world consists of stable things; rather the world is in a state of flux. The appearances of "things" depend upon this flux for their continuity and identity. What needs to be explained, Heraclitus argued, is not change, but the appearance of stability.

Although "process" modelling can be traced through to the present time it has always been a speculative notion because it has never been implemented in a mathematical form and subjected to comparison with reality. Various proposals of a *pregeometric* nature have been considered [2, 3, 4]. Here we propose a mathematical *pregeometric process* model of reality — which in [5] was called a *Heraclitean Quantum System* (HQS). There we arrived at a HQS by deconstruction of the functional integral formulation of quantum field theories retaining only those structures which we felt would not be emergent. In this we still started with "things", namely a Grassmann algebra, and ended with the need to de-

*This is the original 1997 version of the paper which introduced the notion that reality has an *information-theoretic* intrinsic randomness. Since this pioneering paper the model of reality known as *Process Physics* has advanced enormously, and has been confirmed in numerous experiments. The book Cahill, R. T. *Process Physics: From Information Theory to Quantum Space and Matter*, Nova Science Pub. NY 2005, reviews subsequent developments. Numerous papers are available at http://www.mountainman.com.au/process_physics/ and http://www.scieng.flinders.edu.au/cpes/people/cahill_r/processphysics.html compose the mathematical structures into possible histories — each corresponding to a different possible decoherent classical sequencing. However at that level of the HQS we cannot expect anything other than the usual historical modelling of time along with its deficiencies. The problem there was that the deconstruction began with ensembled quantum field theory, and we can never recover individuality and actuality from ensembles — that has been the problem with quantum theory since its inception.

Here we carry the deconstruction one step further by exploiting the fact that functional integrals can be thought of as arising as ensemble averages of Wiener processes. These are normally associated with Brownian-type motions in which random processes are used in modelling many-body dynamical systems. We argue that random processes are a fundamental and necessary aspect of reality — that they arise in the resolution presented here to the end-game problem of modelling reality. In sect. 2 we argue that this "noise" arises as a necessary feature of the self-referential nature of the universe. In sect. 3 we discuss the nature of the self-organised space and time phenomena that arise, and argue that the time modelling is richer and more "realistic" than the geometrical model. In sect. 4 we show how the ensemble averaging of possible universe behaviour is expressible as a functional integral.

2 Self-Referential Noise

Our proposed solution to the end-game problem is to avoid the notion of things and their rules; rather to use a bootstrapped self-referential system. Put simply, this models the universe as a self-organising and self-referential information system — "information" denoting relationships as distinct from "things". In such a system there is no bottom level and we must consider the system as having an iterative character and attempt to pick up the structure by some mathematical modelling.

Chaitin [6] developed some insights into the nature of complex self-referential information systems: combining Shannon's information theory and Turing's computability theory resulted in the development of Algorithmic Information Theory (AIT). This shows that number systems contain randomness and unpredictability, and extends Gödel's discovery, which resulted from self-referenc-ing problems, of the incompleteness of such systems (see [7] for various discussions of the *physics of information*; here we are considering *information as physics*).

Hence if we are to model the universe as a closed system, and thus self-referential, then the mathematical model must necessarily contain randomness. Here we consider one very simple such model and proceed to show that it produces a dynamical 3-space and a theory for time that is richer than the historical/geometrical model.

We model the self-referencing by means of an iterative map

$$B_{ij} \to B_{ij} - (B + B^{-1})_{ij}\, \eta + w_{ij}, \qquad i, j = 1, 2, \dots, M \to \infty. \tag{1}$$

We think of B_{ij} as relational information shared by two monads i and j. The monads concept was introduced by Leibniz, who espoused the *relational* mode of thinking in response to and in contrast to Newton's *absolute* space and time. Leibniz's ideas were very much in the *process* mould of thinking: in this the monad's *view* of available information and the commonality of this information is intended to lead to the emergence of space. The monad i acquires its meaning entirely by means of the information $B_{i1}, B_{i2}, \dots$, where $B_{ij} = -B_{ji}$ to avoid self-information, and real number valued. The map in (1) has the form of a Wiener process, and the $w_{ij} = -w_{ji}$ are independent random variables for each ij and for each iteration, and with variance 2η for later convenience. The w_{ij} model the self-referential noise. The beginning of a universe is modelled by starting the iterative map with $B_{ij} \approx 0$, representing the absence of information or order. Clearly due to the B^{-1} term iterations will rapidly move the B_{ij} away from such starting conditions.

The non-noise part of the map involves B and B^{-1}. Without the non-linear inverse term the map would produce independent and trivial random walks for each B_{ij} — the inverse introduces a linking of all information. We have chosen B^{-1} because of its indirect connection with quantum field theory (see sec. 4) and because of its self-organising property. It is the conjunction of the noise and non-noise terms which leads to the emergence of self-organisation: without the noise the map converges (and this determines the signs in formula 1), in a deterministic manner to a degenerate condensate type structure, discussed in [5], corresponding to a pairing of linear combinations of monads. Hence the map models a non-local and noisy information system from which we extract spatial and time-like behaviour, but we expect residual non-local and random processes characteristic of quantum phenomena including EPR/Aspect type effects. While the map already models some time-like behaviour, it is in the nature of a bootstrap system that we start with *process*. In this system the noise corresponds to the Heraclitean flux which he also called the "cosmic fire", and from which the emergence of stable structures should be understood. To Heraclitus the flame represented one of the earliest examples of the interplay of order and disorder. The contingency and self-ordering of the process clearly suggested a model for reality.

3 Emergent Space and Time

Here we show that the HQS iterative map naturally results in dynamical 3-dimensional spatial structures. Under the mapping the noise term will produce rare large value B_{ij}.

Because the order term is generally much smaller, for small η, than the disorder term these values will persist under the mapping through more iterations than smaller valued B_{ij}. Hence the larger B_{ij} correspond to some temporary background structure which we now identify.

Consider this relational information from the point of view of one monad, call it monad i. Monad i is connected via these large B_{ij} to a number of other monads, and the whole set forms a tree-graph relationship. This is because the large links are very improbable, and a tree-graph relationship is much more probable than a similar graph with additional links. The simplest distance measure for any two nodes within a graph is the smallest number of links connecting them. Let $D_1, D_2, \ldots, D_L$ be the number of nodes of distance $1, 2, \ldots, L$ from node i (define $D_0 = 1$ for convenience), where L is the largest distance from i in a particular tree-graph, and let N be the total number of nodes in the tree. Then $\sum_{k=0}^{L} D_k = N$. See Fig.1 for an example.

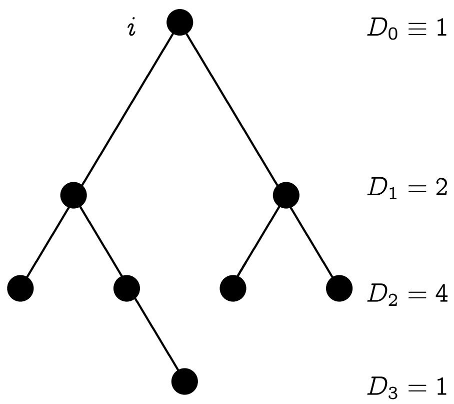


Fig. 1: An $N = 8$, $L = 3$ tree, with indicated distance distributions from monad i.

Now consider the number of different N-node trees, with the same distance distribution $\{D_k\}$, to which i can belong. By counting the different linkage patterns, together with permutations of the monads we obtain

$$\mathcal{N}(D, N) = \frac{(M-1)! D_1^{D_2} D_2^{D_3} \ldots D_{L-1}^{D_L}}{(M-N-2)! D_1! D_2! \ldots D_L!}, \qquad (2)$$

here $D_k^{D_{k+1}}$ is the number of different possible linkage patterns between levels k and $k+1$, and $(M-1)!/(M-N-2)!$ is the number of different possible choices for the monads, with i fixed. The denominator accounts for those permutations which have already been accounted for by the $D_k^{D_{k+1}}$ factors. Nagels [8] analysed $\mathcal{N}(D, N)$, and the results imply that the most likely tree-graph structure to which a monad can belong has the distance distribution

$$D_k \approx \frac{L^2 \ln L}{2\pi^2} \sin^2\left(\frac{\pi k}{L}\right) \quad k = 1, 2, \ldots, L. \qquad (3)$$

for a given arbitrary L value. The remarkable property of this most probable distribution is that the $\sin^2$ indicates that the tree-graph is embeddable in a 3-dimensional hypersphere, S^3. Most importantly, monad i "sees" its surroundings as being 3-dimensional, since $D_k \sim k^2$ for small $\pi k/L$. We call these 3-spaces *gebits* (geometrical bits). We note that the $\ln L$ factor indicates that larger gebits have a larger number density of points.

Now the monads for which the B_{ij} are large thus form disconnected gebits. These gebits however are in turn linked by smaller and more transient B_{kl}, and so on, until at some low level the remaining B_{mn} are noise only; that is they will not survive an iteration. Under iterations of the map this spatial network undergoes growth and decay at all levels, but with the higher levels (larger $\{B_{ij}\}$ gebits) showing most persistence. By a similarity transformation we can arrange the gebits into block diagonal matrices $b_1, b_2, \ldots$, within B, and embedded amongst the smaller and more common noise entries. Now each gebit matrix has $\det(b) = 0$, since a tree-graph connectivity matrix is degenerate. Hence under the mapping the B^{-1} order term has an interesting dynamical effect upon the gebits since, in the absence of the noise, B^{-1} would be singular. The outcome from the iterations is that the gebits are seen to compete and to undergo mutations, for example by adding extra monads to the gebit. Numerical studies reveal gebits competing and "consuming" noise, in a Darwinian process.

Hence in combination the order and disorder terms synthesise an evolving dynamical 3-space with hierarchical structures, possibly even being fractal. This emergent 3-space is entirely relational; it does not arise within any *a priori* geometrical background structure. By construction it is the most robust structure, — however other softer emergent modes of behaviour will be seen as attached to or embedded in this flickering 3-space. The possible fractal character could be exploited by taking a higher level view: identifying each gebit $\rightarrow I$ as a higher level monad, with appropriate informational connections $\mathcal{B}_{IJ}$, we could obtain a higher level iterative map of the form (1), with new order/disorder terms. This would serve to emphasise the notion that in self-referential systems there are no "things", but rather a complex network of iterative relations.

In the model the iterations of the map have the appearance of a cosmic time. However the analysis to reveal the internal experiential time phenomenon is non-trivial, and one would certainly hope to recover the local nature of experiential time as confirmed by special and general relativity experiments. However it is important to notice that the modelling of the time phenomenon here is much richer than that of the historical/geometric model. First the map is clearly uni-directional (there is an "arrow of time") as there is no way to even define an inverse mapping because of the role of the noise term, and this is very unlike the conventional differential equations of traditional physics. In the analysis of the gebits we noted that they show strong persistence, and in that sense the mapping shows a natural partial-memory phenomenon, but the far "future" detailed structure of even this spatial network is completely unknowable without performing the iterations. Furthermore the sequencing of the spatial and other structures is individualistic in that a re-run of the model will always produce a different outcome. Most important of all is that we also obtain a modelling of the "present moment" effect, for the outcome of the next iteration is contingent on the noise. So the system shows overall a sense of a recordable past, an unknowable future and a contingent present moment.

The HQS process model is expected to be capable of a better modelling of our experienced reality, and the key to this is the noisy processing the model requires. As well we need the "internal view", rather than the "external view" of conventional modelling in physics. Nevertheless we would expect that the internally recordable history could be indexed by the usual real-number/geometrical time coordinate.

This new self-referential process modelling requires a new mode of analysis since one cannot use externally imposed meta-rules or interpretations, rather, the internal experiential phenomena and the characterisation of the simpler ones by emergent "laws" of physics must be carefully determined. There has indeed been an ongoing study of how (unspecified) closed self-referential noisy information systems acquire self-knowledge and how the emergent hierarchical structures can "recognise" the same "individuals" [9]. These *Combinatoric Hierarchy* (CH) studies use the fact that only recursive constructions are possible in Heraclitean/Leibnizian systems. We believe that our HQS process model may provide an explicit representation for the CH studies.

4 Possible-Histories Ensemble

While the actual history of the noisy map can only be found in a particular "run", we can nevertheless show that averages over an ensemble of possible histories can be determined, and these have the form of functional integrals. The notion of an ensemble average for any function f of the B, at iteration $c = 1, 2, 3, \ldots$, is expressed by

$$< f[B] >_c = \int \mathcal{D}B f[B]\, \Phi_c[B]\,, \tag{4}$$

where $\Phi_c[B]$ is the ensemble distribution. By the usual construction for Wiener processes we obtain the Fokker-Planck equation

$$\Phi_{c+1}[B] = \Phi_c[B] - \\ - \sum_{ij} \eta \left\{ \frac{\partial}{\partial B_{ij}} \left[(B + B^{-1})_{ij}\, \Phi_c[B] \right] - \frac{\partial^2}{\partial B_{ij}^2}\, \Phi_c[B] \right\}. \tag{5}$$

For simplicity, in the quasi-stationary regime, we find

$$\Phi[B] \sim \exp\left(-S[B]\right), \tag{6}$$

where the action is

$$S[B] = \sum_{i>j} B_{ij}^2 - \mathrm{TrLn}(B)\,. \tag{7}$$

Then the ensemble average is

$$\frac{1}{Z} \int \mathcal{D}B f[B] \exp\left(-S[B]\right), \tag{8}$$

where Z ensures the correct normalisation for the averages. The connection between (1) and (7) is given by

$$(B^{-1})_{ij} = \frac{\partial}{\partial B_{ji}} \mathrm{TrLn}(B) = \frac{\partial}{\partial B_{ji}} \ln \prod_\alpha \lambda_\alpha[B]\,. \tag{9}$$

which probes the sensitivity of the invariant ensemble information to changes in B_{ji}, where the information is in the eigenvalues $\lambda_\alpha[B]$ of B. A further transformation is possible [5]:

$$< f[B] > = \frac{1}{Z} \int \mathcal{D}\overline{m}\, \mathcal{D}m \mathcal{D}B\, f[B] \times \\ \times \exp\left[-\sum_{i>j} B_{ij}^2 + \sum_{i,j} B_{ij} (\overline{m}_i m_j - \overline{m}_j m_i) \right] = \\ = \frac{1}{Z} f\left[\frac{\partial}{\partial J}\right] \int \mathcal{D}\overline{m}\, \mathcal{D}m \exp\left[-\sum_{i>j} \overline{m}_i m_j \overline{m}_j m_i + \\ + \textstyle\sum_{ij} J_{ij} (\overline{m}_i m_j - \overline{m}_j m_i) \right]. \tag{10}$$

This expresses the ensemble average in terms of an anti-commuting Grassmannian algebraic computation [5]. This suggests how the noisy information map may lead to fermionic modes. While functional integrals of the above forms are common in quantum field theory, it is significant that in forming the ensemble average we have lost the contingency or present-moment effect. This always happens — ensemble averages do not tell us about individuals — and then the meta-rules and "interpretations" must be supplied in order to generate some notion of what an individual might have been doing.

The Wiener iterative map can be thought of as a resolution of the functional integrals into different possible histories. However this does not imply the notion that in some sense *all* these histories must be realised, rather only *one* is required. Indeed the basic idea of the process modelling is that of individuality. Not unexpectedly we note that the modelling in (1) must be done from within that *one* closed system.

In conventional quantum theory it has been discovered that the individuality of the measurement process — the "click" of the detector — can be modelled by adding a noise term to the Schrödinger equation [10]. Then by performing an ensemble average over many individual runs of this modified Schrödinger equation one can derive the ensemble measurement postulate — namely $<A> = (\psi, A\psi)$ for the "expectation value of the operator A". This individualising of the ensemble average has been shown to also relate to the decoherence functional formalism [11]. There are a number of other proposals considering noise in spacetime modelling [12, 13].

5 Conclusion

We have addressed here the unique end-game problem which arises when we attempt to model and comprehend the universe as a closed system. The outcome is the suggestion that the peculiarities of this end-game problem are directly relevant to our everyday experience of time and space; particularly the phenomena of the contingent present moment and the three-dimensionality of space. This analysis is based upon the basic insight that a closed self-referential system is necessarily noisy. This follows from Algorithmic Information Theory. To explore the implications we have considered a simple *pregeometric non-linear noisy iterative map*. In this way we construct a process bootstrap system with minimal structure. The analysis shows that the first self-organised structure to arise is a dynamical 3-space formed from competing pieces of 3-geometry — the gebits. The analysis of experiential time is more difficult, but it will clearly be a contingent and process phenomenon which is more complex than the current geometric/historic modelling of time. To extract emergent properties of self-referential systems requires that an internal view be considered, and this itself must be a recursive process. We suggest that the non-local self-referential noise has been a major missing component of our modelling of reality. Two particular applications are an understanding of why quantum detectors "click" and of the physics of consciousness [1], since both clearly have an essential involvement with the modelling of the present-moment effect, and cannot be understood using the geometric/historic modelling of time.

We thank Susan Gunner and Khristos Nizamis for useful comments. Research supported by an ARC Small Grant from Flinders University.

References

1. Penrose R. The large, the small and the human mind. Cambridge Univ. Press, 1997.
2. Wheeler J. A. Pregeometry: motivations and prospects. *Quantum Theory and Gravitation*, ed. by Marlow A. R. Academic Press, New York, 1980.
3. Gibbs P. The small scale structure of space-time: a bibliographical review. arXiv: hep-th/9506171.
4. Finkelstein D. R. Quantum Relativity. Springer, 1996.
5. Cahill R. T. and Klinger C. M. *Phys. Lett. A*, 1996, v. 223, 313.
6. Chaitin G. J. Information, randomness and incompleteness. 2nd ed., World Scientific, 1990.
7. Zurek W. H. (ed.) Complexity, entropy and the physics of information. Addison-Wesley, 1990.
8. Nagels G. *Gen. Rel. and Grav.*, 1985, v. 17, 545.
9. Bastin T. and Kilmister C. Combinatorial physics. World Scientific, 1995.
10. Gisin N. and Percival I. C. Quantum state diffusion: from foundations to applications. arXiv: quant-ph/9701024.
11. Diosi L., Gisin N., Halliwell J., Percival I. C. *Phys. Rev. Lett.*, 1995, v. 74, 203.
12. Percival I. C. *Proc. R. Soc. Lond. A*, 1997, v. 453, 431.
13. Calogero F. *Phys. Lett. A*, 1997, v. 228, 335.

Verifying Unmatter by Experiments, More Types of Unmatter, and a Quantum Chromodynamics Formula

Florentin Smarandache

Dept. of Mathematics, University of New Mexico, 200 College Road, Gallup, NM 87301, USA

E-mails: fsmarandache@yahoo.com; smarand@unm.edu

As shown, experiments registered unmatter: a new kind of matter whose atoms include both nucleons and anti-nucleons, while their life span was very short, no more than 10^{-20}sec. Stable states of unmatter can be built on quarks and anti-quarks: applying the unmatter principle here it is obtained a quantum chromodynamics formula that gives many combinations of unmatter built on quarks and anti-quarks.

In the last time, before the apparition of my articles defining "matter, antimatter, and unmatter" [1, 2], and Dr. S. Chubb's pertinent comment [3] on unmatter, new development has been made to the unmatter topic.

1 Definition of Unmatter

In short, unmatter is formed by matter and antimatter that bind together [1, 2]. The building blocks (most elementary particles known today) are 6 quarks and 6 leptons; their 12 antiparticles also exist. Then *unmatter* will be formed by at least a building block and at least an antibuilding block which can bind together.

2 Exotic atom

If in an atom we substitute one or more particles by other particles of the same charge (constituents) we obtain an exotic atom whose particles are held together due to the electric charge. For example, we can substitute in an ordinary atom one or more electrons by other negative particles (say π^-, anti-ρ-meson, D^-, D_s^-- muon, τ, Ω^-, Δ^-, etc., generally clusters of quarks and antiquarks whose total charge is negative), or the positively charged nucleus replaced by other positive particle (say clusters of quarks and antiquarks whose total charge is positive, etc).

3 Unmatter atom

It is possible to define the unmatter in a more general way, using the exotic atom. The classical unmatter atoms were formed by particles like:

(a) electrons, protons, and antineutrons, or

(b) antielectrons, antiprotons, and neutrons.

In a more general definition, an unmatter atom is a system of particles as above, or such that one or more particles are replaces by other particles of the same charge. Other categories would be:

(c) a matter atom with where one or more (but not all) of the electrons and/or protons are replaced by antimatter particles of the same corresponding charges, and

(d) an antimatter atom such that one or more (but not all) of the antielectrons and/or antiprotons are replaced by matter particles of the same corresponding charges.

In a more composed system we can substitute a particle by an unmatter particle and form an unmatter atom.

Of course, not all of these combinations are stable, semi-stable, or quasi-stable, especially when their time to bind together might be longer than their lifespan.

4 Examples of unmatter

During 1970-1975 numerous pure experimental verifications were obtained proving that "atom-like" systems built on nucleons (protons and neutrons) and anti-nucleons (anti-protons and anti-neutrons) are real. Such "atoms", where nucleon and anti-nucleon are moving at the opposite sides of the same orbit around the common centre of mass, are very unstable, their life span is no more than 10^{-20} sec. Then nucleon and anti-nucleon annihilate into gamma-quanta and more light particles (pions) which can not be connected with one another, see [6, 7, 8]. The experiments were done in mainly Brookhaven National Laboratory (USA) and, partially, CERN (Switzerland), where "proton — anti-proton" and "anti-proton — neutron" atoms were observed, called them $\bar{p}p$ and $\bar{p}n$ respectively, see Fig. 1 and Fig. 2.

After the experiments were done, the life span of such "atoms" was calculated in theoretical way in Chapiro's works [9, 10, 11]. His main idea was that nuclear forces, acting between nucleon and anti-nucleon, can keep them far way from each other, hindering their annihilation. For instance, a proton and anti-proton are located at the opposite sides in the same orbit and they are moved around the orbit centre. If the diameter of their orbit is much more than the diameter of "annihilation area", they can be kept out of annihilation (see Fig. 3). But because the orbit, according to Quantum Mechanics, is an actual cloud spreading far around the average radius, at any radius between the proton and the anti-proton there is a probability that they can meet one another at the annihilation distance. Therefore "nucleon — anti-nucleon" system annihilates in any case, this system

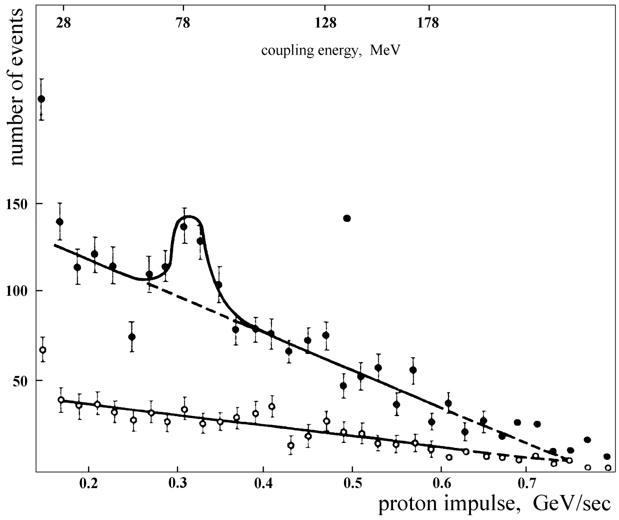


Fig. 1: Spectra of proton impulses in the reaction $\bar{p}+d \rightarrow (\bar{p}n)+p$. The upper arc — annihilation of $\bar{p}n$ into even number of pions, the lower arc — its annihilation into odd number of pions. The observed maximum points out that there is a connected system$\bar{p}n$. Abscissa axis represents the proton impulse in GeV/sec (and the connection energy of the system $\bar{p}n$). Ordinate axis — the number of events. Cited from [6].

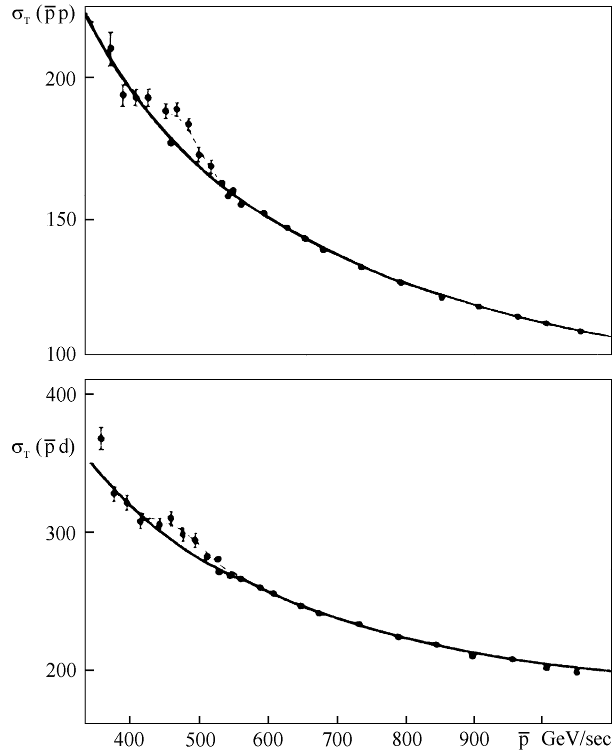


Fig. 2: Probability σ of interaction between $\bar{p}$, p and deutrons d (cited from [7]). The presence of maximum stands out the existence of the resonance state of "nucleon — anti-nucleon".

is unstable by definition having life span no more than 10^{-20} sec.

Unfortunately, the researchers limited the research to the consideration of $\bar{p}p$ and $\bar{p}n$ "atoms" only. The reason was that they, in the absence of a theory, considered $\bar{p}p$ and $\bar{p}n$ "atoms" as only a rare exception, which gives no classes of matter.

Despite Benn Tannenbaum's and Randall J. Scalise's rejections of unmatter and Scalise's personal attack on me in a true Ancient Inquisitionist style under MadSci moderator John Link's tolerance (MadSci web site, June-July 2005), the unmatter does exists, for example some messons and antimessons, through for a trifling of a second lifetime, so the pions are unmatter*, the kaon K^+ (us^), K^- (u^s), Phi (ss^), D^+ (cd^), D^0 (cu^), D_s^+ (cs^), J/Psi (cc^), B^- (bu^), B^0 (db^), B_s^0 (sb^), Upsilon (bb^), etc. are unmatter too†.

Also, the pentaquark theta-plus Θ^+, of charge $+1$, uudds^ (i. e. two quarks up, two quarks down, and one anti-strange quark), at a mass of 1.54 GeV and a narrow width of 22 MeV, is unmatter, observed in 2003 at the Jefferson Lab in Newport News, Virginia, in the experiments that involved multi-GeV photons impacting a deuterium target. Similar pentaquark evidence was obtained by Takashi Nakano of Osaka University in 2002, by researchers at the ELSA accelerator in Bonn in 1997-1998, and by researchers at ITEP in Moscow in 1986. Besides theta-plus, evidence has been found in one experiment [4] for other pentaquarks, Ξ_s^- (ddssu^) and Ξ_s^+ (uussd^).

In order for the paper to be self-contained let's recall that the *pionium* is formed by a π^+ and π^- mesons, the *positronium* is formed by an antielectron (positron) and an electron in a semi-stable arrangement, the *protonium* is formed by a proton and an antiproton also semi-stable, the *antiprotonic helium* is formed by an antiproton and electron together with the helium nucleus (semi-stable), and *muonium* is formed by a positive muon and an electron. Also, the *mesonic atom* is an ordinary atom with one or more of its electrons replaced by negative mesons. The *strange matter* is a ultra-dense matter formed by a big number of strange quarks bounded together with an electron atmosphere (this strange matter is hypothetical).

From the exotic atom, the pionium, positronium, protonium, antiprotonic helium, and muonium are unmatter. The mesonic atom is unmatter if the electron(s) are replaced by negatively-charged antimessons. Also we can define a mesonic antiatom as an ordinary antiatomic nucleous with one or more of its antielectrons replaced by positively-charged mesons. Hence, this mesonic antiatom is unmatter if the antielectron(s) are replaced by positively-charged messons. The strange matter can be unmatter if these exists at least an antiquark together with so many quarks in the nucleous. Also, we can define the strange antimatter as formed by

*Which have the composition u^d and ud^, where by u^ we mean anti-up quark, d = down quark, and analogously u = up quark and d^ = anti-down quark, while by ^ we mean "anti".

†Here c = charm quark, s = strange quark, b = bottom quark.

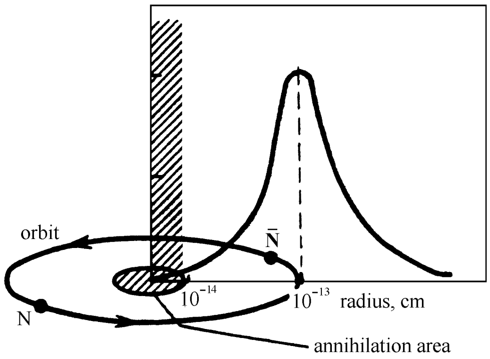


Fig. 3: Annihilation area and the probability arc in "nucleon — anti-nucleon" system (cited from [11]).

a large number of antiquarks bound together with an anti-electron around them. Similarly, the strange antimatter can be unmatter if there exists at least one quark together with so many antiquarks in its nucleous.

The bosons and antibosons help in the decay of unmatter. There are 13 + 1 (Higgs boson) known bosons and 14 antibosons in present.

5 Quantum Chromodynamics formula

In order to save the colorless combinations prevailed in the Theory of Quantum Chromodynamics (QCD) of quarks and antiquarks in their combinations when binding, we devise the following formula:

$$Q - A \in \pm M3 \,, \qquad (1)$$

where M3 means multiple of three, i. e. $\pm M3 = \{3k \mid k \in Z\} = \{\dots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \dots\}$, and Q = number of quarks, A = number of antiquarks. But (1) is equivalent to

$$Q \equiv A \pmod 3 \qquad (2)$$

(Q is congruent to A modulo 3).

To justify this formula we mention that 3 quarks form a colorless combination, and any multiple of three (M3) combination of quarks too, i. e. 6, 9, 12, etc. quarks. In a similar way, 3 antiquarks form a colorless combination, and any multiple of three (M3) combination of antiquarks too, i. e. 6, 9, 12, etc. antiquarks. Hence, when we have hybrid combinations of quarks and antiquarks, a quark and an antiquark will annihilate their colors and, therefore, what's left should be a multiple of three number of quarks (in the case when the number of quarks is bigger, and the difference in the formula is positive), or a multiple of three number of antiquarks (in the case when the number of antiquarks is bigger, and the difference in the formula is negative).

6 Quark-antiquark combinations

Let's note by q = quark ∈ {Up, Down, Top, Bottom, Strange, Charm}, and by a = antiquark ∈ {Upˆ, Downˆ, Topˆ, Bottomˆ, Strangeˆ, Charmˆ}. Hence, for combinations of n quarks and antiquarks, $n \geqslant 2$, prevailing the colorless, we have the following possibilities:

- if n = 2, we have: qa (biquark — for example the mesons and antimessons);
- if n = 3, we have qqq, aaa (triquark — for example the baryons and antibaryons);
- if n = 4, we have qqaa (tetraquark);
- if n = 5, we have qqqqa, aaaaq (pentaquark);
- if n = 6, we have qqqaaa, qqqqqq, aaaaaa (hexaquark);
- if n = 7, we have qqqqqaa, qqaaaaa (septiquark);
- if n = 8, we have qqqqaaaa, qqqqqqaa, qqaaaaaa (octoquark);
- if n = 9, we have qqqqqqqqq, qqqqqqaaa, qqqaaaaaa, aaaaaaaaa (nonaquark);
- if n = 10, we have qqqqqaaaaa, qqqqqqqqaa, qqaaaaaaaa (decaquark); etc.

7 Unmatter combinations

From the above general case we extract the unmatter combinations:

- For combinations of 2 we have: qa (unmatter biquark), mesons and antimesons; the number of all possible unmatter combinations will be 6×6 = 36, but not all of them will bind together.

It is possible to combine an entity with its mirror opposite and still bound them, such as: uuˆ, ddˆ, ssˆ, ccˆ, bbˆ which form mesons. It is possible to combine, unmatter + unmatter = unmatter, as in udˆ + usˆ = uudˆsˆ (of course if they bind together).

- For combinations of 3 (unmatter triquark) we can not form unmatter since the colorless can not hold.
- For combinations of 4 we have: qqaa (unmatter tetraquark); the number of all possible unmatter combinations will be $6^2 \times 6^2 = 1{,}296$, but not all of them will bind together.
- For combinations of 5 we have: qqqqa, or aaaaq (unmatter pentaquarks); the number of all possible unmatter combinations will be $6^4 \times 6 + 6^4 \times 6 = 15{,}552$, but not all of them will bind together.
- For combinations of 6 we have: qqqaaa (unmatter hexaquarks); the number of all possible unmatter combinations will be $6^3 \times 6^3 = 46{,}656$, but not all of them will bind together.
- For combinations of 7 we have: qqqqqaa, qqaaaaa (unmatter septiquarks); the number of all possible unmatter combinations will be $6^5 \times 6^2 + 6^2 \times 6^5 = 559{,}872$, but not all of them will bind together.

- For combinations of 8 we have: qqqqaaaa, qqqqqqqa, qaaaaaaa (unmatter octoquarks); the number of all the unmatter combinations will be $6^4 \times 6^4 + 6^7 \times 6^1 + 6^1 \times 6^7 = 5{,}038{,}848$, but not all of them will bind together.
- For combinations of 9 we have types: qqqqqqaaa, qqqaaaaaa (unmatter nonaquarks); the number of all the unmatter combinations will be $6^6 \times 6^3 + 6^3 \times 6^6 = 2 \times 6^9 = 20{,}155{,}392$, but not all of them will bind together.
- For combinations of 10 we have types: qqqqqqqqaa, qqqqqaaaaa, qqaaaaaaaa (unmatter decaquarks); the number of all the unmatter combinations will be $3 \times 6^{10} = 181{,}398{,}528$, but not all of them will bind together. Etc.

I wonder if it is possible to make infinitely many combinations of quarks/antiquarks and leptons/antileptons... Unmatter can combine with matter and/or antimatter and the result may be any of these three. Some unmatter could be in the strong force, hence part of hadrons.

8 Unmatter charge

The charge of unmatter may be positive as in the pentaquark theta-plus, 0 (as in positronium), or negative as in anti-ρ-meson (uˆd) (M. Jordan).

9 Containment

I think for the containment of antimatter and unmatter it would be possible to use electromagnetic fields (a container whose walls are electromagnetic fields). But its duration is unknown.

10 Further research

Let's start from neutrosophy [13], which is a generalization of dialectics, i. e. not only the opposites are combined but also the neutralities. Why? Because when an idea is launched, a category of people will accept it, others will reject it, and a third one will ignore it (don't care). But the dynamics between these three categories changes, so somebody accepting it might later reject or ignore it, or an ignorant will accept it or reject it, and so on. Similarly the dynamicity of <A>, <antiA>, <neutA>, where <neutA> means neither <A> nor <antiA>, but in between (neutral). Neutrosophy considers a kind not of di-alectics but tri-alectics (based on three components: <A>, <antiA>, <neutA>). Hence unmatter is a kind of neutrality (not referring to the charge) between matter and antimatter, i. e. neither one, nor the other.

Upon the model of unmatter we may look at ungravity, unforce, unenergy, etc.

Ungravity would be a mixture between gravity and antigravity (for example attracting and rejecting simultaneously or alternatively; or a magnet which changes the + and − poles frequently).

Unforce. We may consider positive force (in the direction we want), and negative force (repulsive, opposed to the previous). There could be a combination of both positive and negative forces in the same time, or alternating positive and negative, etc.

Unenergy would similarly be a combination between positive and negative energies (as the alternating current, a. c., which periodically reverses its direction in a circuit and whose frequency, f, is independent of the circuit's constants). Would it be possible to construct an alternating-energy generator?

To conclusion: According to the Universal Dialectic the unity is manifested in duality and the duality in unity. "Thus, Unmatter (unity) is experienced as duality (matter vs antimatter). Ungravity (unity) as duality (gravity vs antigravity). Unenergy (unity) as duality (positive energy vs negative energy) and thus also ... between duality of being (existence) vs nothingness (antiexistence) must be 'unexistence' (or pure unity)" (R. Davic).

References

1. Smarandache F. A new form of matter — unmatter, composed of particles and anti-particles. *Progr. in Phys.*, 2005, v. 1, 9–11.
2. Smarandache F. Matter, antimatter, and unmatter. *Infinite Energy*, v. 11, No. 62, 50–51, (July/August 2005).
3. Chubb S. Breaking through editorial. *Infinite Energy*, v. 11, No. 62, 6–7 (July/August 2005).
4. Alt C. et al., (NA49 Collaboration). *Phys. Rev. Lett.*, 2004, v. 92, 042003.
5. Carman D. S., Experimental evidence for the pentaquark. *Eur. Phys. A*, 2005, v. 24, 15–20.
6. Gray L., Hagerty P., Kalogeropoulos T. E. Evidence for the existence of a narrow p-barn bound state. *Phys. Rev. Lett.*, 1971, v. 26, 1491–1494.
7. Carrol A. S. et al. Observation of structure in $\bar{p}p$ and $\bar{p}d$ total cross sections below 1.1 GeV/s. *Phys. Rev. Lett.*, 1974, v.32, 247–250.
8. Kalogeropoulos T. E., Vayaki A., Grammatikakis G., Tsilimigras T., Simopoulou E. Observation of excessive and direct gamma production in $\bar{p}d$ annihilations at rest. *Phys. Rev. Lett.*, 1974, v.33, 1635–1637.
9. Chapiro I. S. *Physics-Uspekhi*, 1973, v.109, 431.
10. Bogdanova L. N., Dalkarov O. D., Chapiro I. S. Quasinuclear systems of nucleons and antinucleons. *Annals of Physics*, 1974, v.84, 261–284.
11. Chapiro I. S. New "nuclei" built on nucleons and anti-nucleons. *Nature (Russian)*, 1975, No. 12, 68–73.
12. Davic R., John K., Jordan M., Rabounski D., Borissova L., Levin B., Panchelyuga V., Shnoll S., Private communications with author, June-July, 2005.
13. Smarandache F. A unifying field in logics, neutrosophic logic / neutrosophy, neutrosophic set, neutrosophic probability. Amer. Research Press, 1998.

The Journal on Advanced Studies in Theoretical and Experimental Physics, including Related Themes from Mathematics

PROGRESS IN PHYSICS

A quarterly issue scientific journal, registered with the Library of Congress (DC).
This journal is peer reviewed and included in the abstracting and indexing coverage of:
Mathematical Reviews and MathSciNet (AMS, USA), DOAJ of Lund University (Sweden),
Referativnyi Zhurnal VINITI (Russia), etc.

ISSN: 1555-5534 (print)
ISSN: 1555-5615 (online)
4 issues per annum

Electronic version of this journal:
http://www.geocities.com/ptep_online

To order printed issues of this journal, contact the Editor in Chief.

ISSN: 1555-5534 (print)
ISSN: 1555-5615 (online)

Standard Address Number: 297-5092
Printed in the United States of America

OCTOBER 2005 **VOLUME 3**

CONTENTS

Information for Authors and Subscribers

Progress in Physics has been created for publications on advanced studies in theoretical and experimental physics, including related themes from mathematics. All submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

All submissions should be designed in LaTeX format using *Progress in Physics* template. This template can be downloaded from *Progress in Physics* home page http://www.geocities.com/ptep_online. Abstract and the necessary information about author(s) should be included into the papers. To submit a paper, mail the file(s) to Chief Editor.

All submitted papers should be as brief as possible. Beginning from 2006 we accept only short papers, no longer than 8 journal pages. Short articles are preferable.

All that has been accepted for the online issue of *Progress in Physics* is printed in the paper version of the journal. To order printed issues, contact Chief Editor.

This journal is non-commercial, academic edition. It is printed from private donations.

Observational Cosmology: From High Redshift Galaxies to the Blue Pacific

Halton Arp

Max-Planck-Institut für Astrophysik, Karl Schwarzschild-Str.1, Postfach 1317, D-85741 Garching, Germany

E-mail: arp@mpa-garching.mpg.de

1 Birth of galaxies

Observed: *Ejection of high redshift, low luminosity quasars from active galaxy nuclei.*

Shown by radio and X-ray pairs, alignments and luminous connecting filaments. Emergent velocities are much less than intrinsic redshift. Stripping of radio plasmas. Probabilities of accidental association negligible. See Arp, 2003 [4] for customarily supressed details.

Observed: *Evolution of quasars into normal companion galaxies.*

The large number of ejected objects enables a view of empirical evolution from high surface brightness quasars through compact galaxies. From gaseous plasmoids to formation of atoms and stars. From high redshift to low.

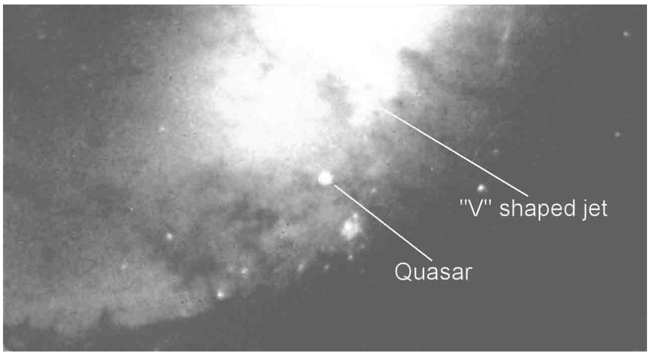


Fig. 1: Enhanced Hubble Space Telescope image showing ejection wake from the center of NGC 7319 (redshift $z = 0.022$) to within about 3.4 arcsec of the quasar (redshift $z = 2.11$)

Observed: *Younger objects have higher intrinsic redshifts.*

In groups, star forming galaxies have systematically higher redshifts, e. g. spiral galaxies. Even companions in evolved groups like our own Andromeda Group or the nearby M81 group still have small, residual redshift excesses relative to their parent.

Observed: *X-ray and radio emission generally indicate early evolutionary stages and intrinsic redshift.*

Plasmoids ejected from an active nucleus can fragment or ablate during passage through galactic and intergalactic medium which results in the forming of groups and clusters of proto galaxies. The most difficult result for astronomers to accept is galaxy clusters which have intrinsic redshifts. Yet the association of clusters with lower redshift parents is demonstrated in Arp and Russell, 2001 [1]. Individual cases of strong X-ray clusters are exemplified by elongations and connections as shown in the ejecting galaxy Arp 220, in Abell 3667 and from NGC 720 (again, summarized in Arp, 2003 [4]). Motion is confirmed by bow shocks and elongation is interpreted as ablation trails. In short — if a quasar evolves into a galaxy, a broken up quasar evolves into a group of galaxies.

2 Redshift is the key

Observed: *The whole quasar or galaxy is intrinsically redshifted.*

Objects with the same path length to the observer have much different redshifts and all parts of the object are shifted closely the same amount. Tired light is ruled out and also gravitational redshifting.

The fundamental assumption: *Are particle masses constant?*

The photon emitted in an orbital transition of an electron in an atom can only be redshifted if its mass is initially small. As time goes on the electron communicates with more and more matter within a sphere whose limit is expanding at velocity c. If the masses of electrons increase, emitted photons change from an initially high redshift to a lower redshift with time (see Narlikar and Arp, 1993 [6])

Predicted consequences: *Quasars are born with high redshift and evolve into galaxies of lower redshift.*

Near zero mass particles evolve from energy conditions in an active nucleus. (If particle masses have to be created sometime, it seems easier to grow things from a low mass state rather than producing them instantaneously in a finished state.)

DARK MATTER: *The establishment gets it right, sort of.*

In the Big Bang, gas blobs in the initial, hot universe have to condense into things we now see like quasars and galaxies. But we know hot gas blobs just go poof! Lots of dark matter (cold) had to be hypothesized to condense the gas cloud. They are still looking for it.

But low mass particles must slow their velocities in order to conserve momentum as their mass grows. Temperature is internal velocity. Thus the plasmoid cools and condenses its increasing mass into a compact quasar. So maybe we

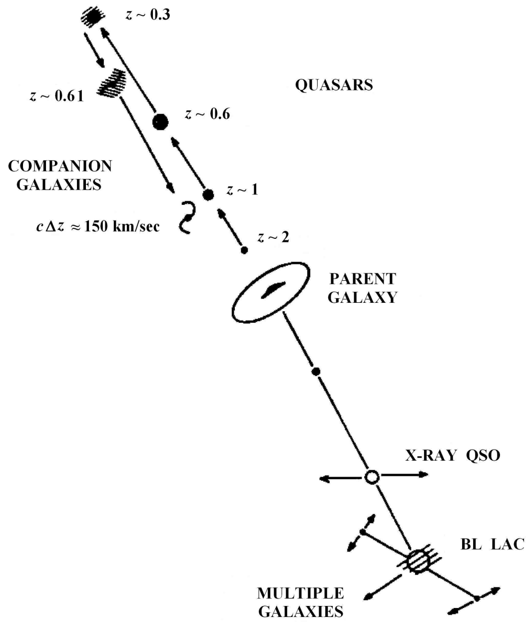


Fig. 2: Schematic representation of quasars and companion galaxies found associated with central galaxies from 1966 to present. The progression of characteristics is empirical but is also required by the variable mass theory of Narlikar and Arp, 1993 [6]

have been observing dark matter ever since the discovery of quasars! After all, what's in a name?

Observed: *Ambarzumian sees new galaxies.*

In the late 1950's when the prestigious Armenian astronomer, Viktor Ambarzumian was president of the International Astronomical Union he said that just looking at pictures convinced him that new galaxies were ejected out of old. Even now astronomers refuse to discuss it, saying that big galaxies cannot come out of other big galaxies. But we have just seen that the changing redshift is the key that unlocks the growth of new galaxies with time. They are small when they come from the small nucleus. Ambarzumian's superfluid just needed the nature of changing redshift. But Oort and conventional astronomers preferred to condense hot gas out of a hot expanding universe.

Observed: *The Hubble Relation.*

An article of faith in current cosmology is that the relation between faintness of galaxies and their redshift, the Hubble Relation, means that the more distant a galaxy is the faster it is receding from us. With our galaxy redshifts a function of age, however, the look back time to a distant galaxy shows it to us when it was younger and more intrinsically redshifted. No Doppler recession needed!

The latter non-expanding universe is even quantitative in that Narlikar's general solution of the General Relativistic equations ($m = t^2$) gives a Hubble constant directly in term of the age of our own galaxy. ($H_0 = 51$ km/sec×Mpc for age of our galaxy = 13 billion years). The Hubble constant observed from the most reliable Cepheid distances is $H_0 = 55$ (Arp, 2002 [3]). What are the chances of obtaining the correct Hubble constant from an incorrect theory with no adjustable parameters? If this is correct there is negligible room for expansion of the universe.

Observed: *The current Hubble constant is too large.*

A large amount of observing time on the Hubble Space Telescope was devoted to observing Cepheid variables whose distances divided into their redshifts gave a definitive value of $H_0 = 72$. That required the reintroduction of Einstein's cosmological constant to adjust to the observations. But $H_0 = 72$ was wrong because the higher redshift galaxies in the sample included younger (ScI) galaxies which had appreciable intrinsic redshifts.

Independent distances to these galaxies by means of rotational luminosity distances (Tully-Fisher distances) also showed this class of galaxies had intrinsic redshifts which gave too high a Hubble constant (Russell, 2002 [8]) In fact well known clusters of galaxies gives H_0's in the 90's (Russell, private communication) which clearly shows that neither do we have a correct distance scale or understanding of the nature of galaxy clusters.

DARK ENERGY: *Expansion now claimed to be acceleration.*

As distance measures were extended to greater distances by using Supernovae as standard candles it was found that the distant Supernovae were somewhat too faint. This led to a smaller H_0 and hence an acceleration compared to the supposed present day $H_0 = 72$. Of course the younger Supernovae could be intrinsically fainter and also we have seen the accepted present day H_0 is too large. Nevertheless astronomers have again added a huge amount of undetected substance to the universe to make it agree with properties of a disproved set of assumptions. This is called the accordance model but we could easily imagine another name for it.

3 Physics — local and universal

Instead of extrapolating our local phenomena out to the universe one might more profitably consider our local region as a part of the physics of the universe.

Note: *Flat space, no curves, no expansion.*

The general solution of energy/momentum conservation (relativistic field equations) which Narlikar made with $m = t^2$ gives a Euclidean, three dimensional, uncurved space. The usual assumption that particle masses are constant in time only projects our local, snapshot view onto the rest of the universe.

In any case it is not correct to solve the equations in a non-general case. In that case the usual procedure of assigning curvature and expansion properties to the mathematical term space (which has no physical attributes!) is only useful for excusing the violations with the observations caused by the inappropriate assumption of constant elementary masses.

Consequences: *Relativity theory can furnish no gravity.*

Space (nothing) can not be a "rubber sheet". Even if there could be a dimple — nothing would roll into it unless there was a previously existing pull of gravity. We need to find a plausible cause for gravity other than invisible bands pulling things together.

Required: *Very small wave/particles pushing against bodies.*

In 1747 the Genevoise philosopher-physicist George-Louis Le Sage postulated that pressure from the medium which filled space would push bodies together in accordance with the Newtonian Force $=1/r^2$ law. Well before the continuing fruitless effort to unify Relativistic gravity and quantum gravity, Le Sage had solved the problem by doing away with the need to warp space in order to account for gravity.

Advantages: *The Earth does not spiral into the Sun.*

Relativistic gravity is assigned an instantaneous component as well as a component that travels with the speed of light, c. If gravity were limited to c, the Earth would be rotating around the Sun where it was about 8 minutes ago. By calculating under the condition that no detectable reduction in the size of the Earth's orbit has been observed, Tom Van Flandern arrives at the minimum speed of gravity of $2\times10^{10}\,c$. We could call these extremely fast, extremely penetrating particles gravitons.

A null observation saves causality.

The above reasoning essentially means that gravity can act as fast as it pleases, but not instantaneously because that would violate causality. This is reassuring since causality seems to be an accepted property of our universe (except for some early forms of quantum theory).

Black holes into white holes.

In its usual perverse way all the talk has been about black holes and all the observations have been about white holes. Forget for a moment that from the observer's viewpoint it would take an infinity of time to form a black hole. The observations show abundant material being ejected from stars, nebulae, galaxies, quasars. What collimates these out of a region in which everything is supposed to fall into? (Even ephemeral photons of light.) After 30 years of saying nothing comes out of black holes, Stephen Hawking now approaches the observations saying maybe a little leaks out.

Question: *What happens when gravitons encounter a black hole?*

If the density inside the concentration of matter is very high the steady flux of gravitons absorbed will eventually heat the core and eventually this energy must escape. After all it is only a local concentration of matter against the continuous push of the whole of intergalactic space. Is it reasonable to say it will escape through the path of least resistance, for example through the flattened pole of a spinning sphere which is usual picture of the nucleus? Hence the directional nature of the observed plasmoid ejections.

4 Planets and people

In our own solar system we know the gas giant planets increase in size as we go in toward the Sun through Neptune, Uranus, Saturn and Jupiter. On the Earth's side of Jupiter, however, we find the asteroid belt. It does not take an advanced degree to come to the idea that the asteroids are the remains of a broken up planet. But how? Did something crash into it? What does it mean about our solar system?

Mars: *The Exploding Planet Hypothesis.*

We turn to a real expert on planets, Tom Van Flandern. For years he has argued in convincing detail that Mars, originally bigger than Earth, had exploded visibly scarring the surface of its moon, the object we now call Mars. One detail should be especially convincing, namely that the present Mars, unable to hold an atmosphere, had long been considered devoid of water, a completely arid desert. But recent up-close looks have revealed evidence for "water dumps", lots of water in the past which rapidly went away. Where else could this water have come from except the original, close-by Mars as it exploded?

For me the most convincing progression is the increasing masses of the planets from the edge of the planetary system toward Jupiter and then the decreasing masses from Jupiter through Mercury. Except for the present Mars! But that continuity would be preserved with an original Mars larger than Earth and its moon larger than the Earth's moon.

As for life on Mars, the Viking lander reported bacteria but the scientist said no. Then there was controversy about organic forms in meteorites from Mars. But the most straight forward statement that can be made is that features have now been observed that look "artificial" to some. Obviously no one is certain at this point but most scientists are trained to stop short of articulating the obvious.

Gravitons: *Are planets part of the universe?*

If a universal sea of very small, very high speed gravitons are responsible for gravity in galaxies and stars would not these same gravitons be passing through the solar system and the planets in it? What would be the effect if a small percentage were, over time, absorbed in the cores of planets?

Speculation: *What would we expect?*

Heating the core of a gas giant would cause the liquid/gaseous planet to expand in size. But if the core of a rocky planet would be too rigid to expand it would eventually explode. Was the asteroid planet the first to go? Then the original Mars? And next the Earth?

Geology: *Let's argue about the details.*

Originally it was thought the Earth was flat. Then spherical but with the continents anchored in rock. When Alfred Wegener noted that continents fitted together like jigsaw puzzle and therefore had been pulled apart, it was violently rejected because geologists said they were anchored in basaltic rock. Finally it was found that the Atlantic trench between the Americas and Africa/Europe was opening up at a rate of just about right for the Earth's estimated age (Kokus, 2002 [5]). So main stream geologists invented plate tectonics where the continents skated blythly around on top of this anchoring rock!

In 1958 the noted Geologist S. Warren Carey and in 1965 K. M. Creer (in the old, usefully scientific *Nature Magazine*) were among those who articulated the obvious, namely that the Earth is expanding. The controversy between plate tectonics and expanding Earth has been acrid ever since. (One recent conference proceedings by the latter adherents is "Why Expanding Earth?" (Scalera and Jacob, 2003 [7]).

Let's look around us.

The Earth is an obviously active place. volcanos, Earth quakes, island building. People seem to agree the Atlantic is widening and the continents separating. But the Pacific is violently contested with some satellite positioning claiming no expansion. I remember hearing S. Warren Carey painstakingly interpreting maps of the supposed subduction zone where the Pacific plate was supposed to be diving under the Andean land mass of Chile. He argued that there was no debris scraped off the supposedly diving Pacific Plate. But in any case, where was the energy coming from to drive a huge Pacific plate under the massive Andean plate?

My own suggestion about this is that the (plate) is stuck, not sliding under. Is it possible that the pressure from the Pacific Basin has been transmitted into the coastal ranges of the Americas where it is translated into mountain building? (Mountain building is a particularly contentious disagreement between static and expanding Earth proponents.)

It is an impressive, almost thought provoking sight, to see hot lava welling up from under the southwest edge of the Big Island of Hawaii forming new land mass in front of our eyes. All through the Pacific there are underground vents, volcanos, mountain and island building. Is it possible this upwelling of mass in the central regions of the Pacific is putting pressure on the edge? Does it represent the emergence of material comparable to that along the Mid Atlantic ridge on the other side of the globe?

The future: *Life as an escape from danger.*

The galaxy is an evolving, intermittently violent environment. The organic colonies that inhabit certain regions within it may or may not survive depending on how fast they recognize danger and how well they adapt, modify it or escape from it. Looking out over the beautiful blue Pacific one sees tropical paradises. On one mountain top, standing on barely cool lava, is the Earth's biggest telescope. Looking out in the universe for answers. Can humankind collectively understand these answers? Can they collectively ensure their continued appreciation of the beauty of existence.

References

1. Arp H. and Russell D. G. A possible relationship between quasars and clusters of galaxies. *Astrophysical Journal*, 2001, v. 549, 802–819
2. Arp H. *Pushing Gravity*, ed. by M. R. Edwards, 2002, 1.
3. Arp H. Arguments for a Hubble constant near H0 = 55. *Astrophysical Journal*, 2002, 571, 615–618.
4. Arp H. Catalog of discordant redshift associations. Apeiron, Montreal, 2003.
5. Kokus M. *Pushing Gravity*, ed. by M. R. Edwards, 2002, 285.
6. Narlikar J. and Arp H. Flat spacetime cosmology — a unified framework for extragalactic redshifts. *Astrophysical Journal*, 1993, 405, 51-56.
7. Scalera G. and Jacob K.-H. (editors). Why expanding Earth? Nazionale di Geofisica e Vulcanologoia, Technisch Univ. Berlin, publ. INGV Roma, Italy, 2003.
8. Russell D. G. Morphological type dependence in the Tully-Fisher relationship. *Astrophysical Journal*, 2004, v. 607, 241-246.
9. Van Flandern T. Dark matter, missing planets and new comets. 2nd ed., North Atlantic Books, Berkely, 1999.
10. Van Flandern T. *Pushing Gravity*, ed. by M. R. Edwards, 2002, 93.

On the General Solution to Einstein's Vacuum Field for the Point-Mass when $\lambda \neq 0$ and Its Consequences for Relativistic Cosmology

Stephen J. Crothers
Sydney, Australia
E-mail: thenarmis@yahoo.com

It is generally alleged that Einstein's theory leads to a finite but unbounded universe. This allegation stems from an incorrect analysis of the metric for the point-mass when $\lambda \neq 0$. The standard analysis has incorrectly assumed that the variable r denotes a radius in the gravitational field. Since r is in fact nothing more than a real-valued parameter for the actual radial quantities in the gravitational field, the standard interpretation is erroneous. Moreover, the true radial quantities lead inescapably to $\lambda = 0$ so that, cosmologically, Einstein's theory predicts an infinite, static, empty universe.

1 Introduction

It has been shown [1, 2, 3] that the variable r which appears in the metric for the gravitational field is neither a radius nor a coordinate in the gravitational field, and further [3], that it is merely a real-valued parameter in the pseudo-Euclidean spacetime (M_s, g_s) of Special Relativity, by which the Euclidean distance $D = |r - r_0| \in (M_s,\ g_s)$ is mapped into the non-Euclidean distance $R_p \in (M_g, g_g)$, where (M_g, g_g) denotes the pseudo-Riemannian spacetime of General Relativity. Owing to their invalid assumptions about the variable r, the relativists claim that $r = \sqrt{\frac{3}{\lambda}}$ defines a "horizon" for the universe (e.g. [4]), by which the universe is supposed to have a finite volume. Thus, they have claimed a finite but unbounded universe. This claim is demonstrably false.

The standard metric for the simple point-mass when $\lambda \neq 0$ is,

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{\lambda}{3}\, r^2\right) dt^2 - \left(1 - \frac{2m}{r} - \frac{\lambda}{3}\, r^2\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{1}$$

The relativists simply look at (1) and make the following assumptions.

(a) *The variable r is a radial coordinate in the gravitational field*;

(b) *r can go down to 0*;

(c) *A singularity in the gravitational field can occur only where the Riemann tensor scalar curvature invariant (or Kretschmann scalar) $f = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded*.

The standard analysis has never proved these assumptions, but nonetheless simply takes them as given. I have demonstrated elsewhere [3] that when $\lambda = 0$, these assumptions are false. I shall demonstrate herein that when $\lambda \neq 0$ these assumptions are still false, and further, that λ can only take the value of zero in Einstein's theory.

2 Definitions

As is well-known, the basic spacetime of the General Theory of Relativity is a metric space of the Riemannian geometry family, namely — the four-dimensional pseudo-Riemannian space with Minkowski signature. Such a space, like any Riemannian metric space, is strictly negative non-degenerate, i. e. the fundamental metric tensor $g_{\alpha\beta}$ of such a space has a determinant which is strictly negative: $g = \det \| g_{\alpha\beta} \| < 0$.

Space metrics obtained from Einstein's equations can be very different. This splits General Relativity's spaces into numerous families. The two main families are derived from the fact that the energy-momentum tensor of matter $T_{\alpha\beta}$, contained in the Einstein equations, can (1) be linearly proportional to the fundamental metric tensor $g_{\alpha\beta}$ or (2) have a more compound functional dependence. The first case is much more attractive to scientists, because in this case one can use $g_{\alpha\beta}$, taken with a constant numerical coefficient, instead of the usual $T_{\alpha\beta}$, in the Einstein equations. Spaces of the first family are known as *Einstein spaces*.

From the purely geometrical perspective, an Einstein space [5] is described by any metric obtained from

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = \kappa T_{\alpha\beta} - \lambda g_{\alpha\beta}\,,$$

where κ is a constant and $T_{\alpha\beta} \propto g_{\alpha\beta}$, and therefore includes all partially degenerate metrics. Accordingly, such spaces become non-Einstein only when the determinant g of the metric becomes

$$g = \det \| g_{\alpha\beta} \| = 0\,.$$

In terms of the required physical meaning of General Relativity I shall call a spacetime associated with a non-degenerate metric, an Einstein universe, and the associated metric an Einstein metric.

Cosmological models involving either $\lambda \neq 0$ or $\lambda = 0$, which do not result in a degenerate metric, I shall call relativistic cosmological models, which are necessarily Einstein universes, with associated Einstein metrics.

Thus, any "partially" degenerate metric where $g \neq 0$ is not an Einstein metric, and the associated space is not an Einstein universe. Any cosmological model resulting in a "partially" degenerate metric where $g \neq 0$ is neither a relativistic cosmological model nor an Einstein universe.

3 The general solution when $\lambda \neq 0$

The general solution for the simple point-mass [3] is,

$$ds^2 = \left(\frac{\sqrt{C_n} - \alpha}{\sqrt{C_n}}\right) dt^2 - \left(\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}\right) \frac{{C'_n}^2}{4C_n} dr^2 - C_n (d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

$$C_n(r) = \left[|r - r_0|^n + \alpha^n\right]^{\frac{2}{n}}, \; n \in \Re^+,$$

$$\alpha = 2m, \quad r_0 \in \Re,$$

where n and r_0 are arbitrary and r is a real-valued parameter in (M_s, g_s).

The most general static metric for the gravitational field [3] is,

$$ds^2 = A(D)dt^2 - B(D)dr^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{3}$$

$$D = |r - r_0|, \quad r_0 \in \Re,$$

where analytic $A, B, C > 0 \; \forall \; r \neq r_0$.

In relation to (3) I identify the coordinate radius D, the r-parameter, the radius of curvature R_c, and the proper radius (proper distance) R_p.

1. *The coordinate radius is* $D = |r - r_0|$.
2. *The r-parameter is the variable* r.
3. *The radius of curvature is* $R_c = \sqrt{C(D(r))}$.
4. *The proper radius is* $R_p = \int \sqrt{B(D(r))} dr$.

I remark that $R_p(D(r))$ gives the mapping of the Euclidean distance $D = |r - r_0| \in (M_s, g_s)$ into the non-Euclidean distance $R_p \in (M_g, g_g)$ [3]. Furthermore, the geometrical relations between the components of the metric tensor are inviolable and therefore hold for all metrics with the form of (3).

Thus, on the metric (2),

$$R_c = \sqrt{C_n(D(r))},$$

$$R_p = \int \sqrt{\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}} \, \frac{C'_n}{2\sqrt{C_n}} \, dr.$$

Transform (3) by setting,

$$r^* = \sqrt{C(D(r))}, \tag{4}$$

to carry (3) into,

$$ds^2 = A^*(r^*)dt^2 - B^*(r^*)dr^{*2} - r^{*2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5}$$

For $\lambda \neq 0$, one finds in the usual way that the solution to (5) is,

$$ds^2 = \left(1 - \frac{\alpha}{r^*} - \frac{\lambda}{3} r^{*2}\right) dt^2 - \left(1 - \frac{\alpha}{r^*} - \frac{\lambda}{3} r^{*2}\right)^{-1} dr^{*2} - r^{*2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{6}$$

$$\alpha = \text{const.}$$

Then by (4),

$$ds^2 = \left(1 - \frac{\alpha}{\sqrt{C}} - \frac{\lambda}{3} C\right) dt^2 - \left(1 - \frac{\alpha}{\sqrt{C}} - \frac{\lambda}{3} C\right)^{-1} \frac{C'^2}{4C} dr^2 - C\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{7}$$

$$C = C(D(r)), \quad D = D(r) = |r - r_0|, \quad r_0 \in \Re,$$

$$\alpha = \text{const},$$

where $r \in (M_s, g_s)$ is a real-valued parameter and also $r_0 \in (M_s, g_s)$ is an arbitrary constant which specifies the position of the point-mass in parameter space.

When $\alpha = 0$, (7) reduces to the empty de Sitter metric, which I write generally, in view of (7), as

$$ds^2 = \left(1 - \frac{\lambda}{3} F\right) dt^2 - \left(1 - \frac{\lambda}{3} F\right)^{-1} d\sqrt{F}^2 - F\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{8}$$

$$F = F(D(r)), \quad D = D(r) = |r - r_0|, \quad r_0 \in \Re.$$

If $F(D(r)) = r^2$, $r_0 = 0$, and $r \geqslant r_0$, then the usual form of (8) is obtained,

$$ds^2 = \left(1 - \frac{\lambda}{3} r^2\right) dt^2 - \left(1 - \frac{\lambda}{3} r^2\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{9}$$

The admissible forms for $C(D(r))$ and $F(D(r))$ must now be generally ascertained.

If $C' \equiv 0$, then $B(D(r)) = 0 \;\forall\; r$, in violation of (3). Therefore $C' \neq 0 \;\forall\; r \neq r_0$.

Now $C(D(r))$ must be such that when $r \to \pm\infty$, equation (7) must reduce to (8) asymptotically. So,

$$\text{as } r \to \pm\infty, \ \frac{C(D(r))}{F(D(r))} \to 1.$$

I have previously shown [3] that the condition for singularity on a metric describing the gravitational field of the point-mass is,

$$g_{00}(r_0) = 0\,. \tag{10}$$

Thus, by (7), it is required that,

$$1 - \frac{\alpha}{\sqrt{C(D(r_0))}} - \frac{\lambda}{3}C(D(r_0)) = 1 - \frac{\alpha}{\beta} - \frac{\lambda}{3}\beta^2 = 0\,, \tag{11}$$

having set $\sqrt{C(D(r_0))} = \beta$. Thus, β is a scalar invariant for (7) that must contain the independent factors contributing to the gravitational field, i.e. $\beta = \beta(\alpha, \lambda)$. Consequently it is required that when $\lambda = 0$, $\beta = \alpha = 2m$ to recover (2), when $\alpha = 0$, $\beta = \sqrt{\frac{3}{\lambda}}$ to recover (8), and when $\alpha = \lambda = 0$, and $\beta = 0$, $C(D(r)) = |r - r_0|^2$ to recover the flat spacetime of Special Relativity. Also, when $\alpha = 0$, $C(D(r))$ must reduce to $F(D(r))$. The value of $\beta = \beta(\lambda) = \sqrt{F(D(r_0))}$ in (8) is also obtained from,

$$g_{00}(r_0) = 0 = 1 - \frac{\lambda}{3}F(D(r_0)) = 1 - \frac{\lambda}{3}\beta^2\,.$$

Therefore,

$$\beta = \sqrt{\frac{3}{\lambda}}\,. \tag{12}$$

Thus, to render a solution to (7), $C(D(r))$ must at least satisfy the following conditions.

1. $C'(D(r)) \neq 0 \ \forall\ r \neq r_0$.
2. As $r \to \pm\infty$, $\frac{C(D(r))}{F(D(r))} \to 1$.
3. $C(D(r_0)) = \beta^2$, $\beta = \beta(\alpha, \lambda)$.
4. $\lambda = 0 \Rightarrow \beta = \alpha = 2m$ and $C = \left(|r - r_0|^n + \alpha^n\right)^{\frac{2}{n}}$.
5. $\alpha = 0 \Rightarrow \beta = \sqrt{\frac{3}{\lambda}}$ and $C(D(r)) = F(D(r))$.
6. $\alpha = \lambda = 0 \Rightarrow \beta = 0$ and $C(D(r)) = |r - r_0|^2$.

Both α and $\beta(\alpha, \lambda)$ must also be determined.

Since (11) is a cubic, it cannot be solved exactly for β. However, I note that the two positive roots of (11) are approximately α and $\sqrt{\frac{3}{\lambda}}$. Let $P(\beta) = 1 - \frac{\alpha}{\beta} - \frac{\lambda}{3}\beta^2$. Then according to Newton's method,

$$\beta_{m+1} = \beta_m - \frac{P(\beta_m)}{P'(\beta_m)} = \beta_m - \frac{\left(1 - \frac{\alpha}{\beta_m} - \frac{\lambda}{3}\beta_m^2\right)}{\left(\frac{\alpha}{\beta_m^2} - \frac{2\lambda}{3}\beta_m\right)}\,. \tag{13}$$

Taking $\beta_1 = \alpha$ into (13) gives,

$$\beta \approx \beta_2 = \frac{3\alpha - \lambda\alpha^3}{3 - 2\lambda\alpha^2}\,, \tag{14a}$$

and

$$\beta \approx \beta_3 = \frac{3\alpha - \lambda\alpha^3}{3 - 2\lambda\alpha^2} - \left[\frac{1 - \frac{\alpha\left(3 - 2\lambda\alpha^2\right)}{\left(3\alpha - \lambda\alpha^3\right)} - \frac{\lambda}{3}\left(\frac{3\alpha - \lambda\alpha^3}{3 - 2\lambda\alpha^2}\right)^2}{\alpha\left(\frac{3 - 2\lambda\alpha^2}{3\alpha - \lambda\alpha^3}\right)^2 - \frac{2\lambda}{3}\left(\frac{3\alpha - \lambda\alpha^3}{3 - 2\lambda\alpha^2}\right)}\right], \tag{14b}$$

etc., which satisfy the requirement that $\beta = \beta(\alpha, \lambda)$.

Taking $\beta_1 = \sqrt{\frac{3}{\lambda}}$ into (13) gives,

$$\beta \approx \beta_2 = \sqrt{\frac{3}{\lambda}} + \frac{\alpha}{\alpha\sqrt{\frac{\lambda}{3}} - 2}\,, \tag{15a}$$

and

$$\beta \approx \beta_3 = \sqrt{\frac{3}{\lambda}} + \frac{\alpha}{\alpha\sqrt{\frac{\lambda}{3}} - 2} - \left[\frac{1 - \frac{\alpha}{\left(\sqrt{\frac{3}{\lambda}} + \frac{\alpha}{\alpha\sqrt{\frac{\lambda}{3}} - 2}\right)} - \frac{\lambda}{3}\left(\sqrt{\frac{3}{\lambda}} + \frac{\alpha}{\alpha\sqrt{\frac{\lambda}{3}} - 2}\right)^2}{\frac{\alpha}{\left(\sqrt{\frac{3}{\lambda}} + \frac{\alpha}{\alpha\sqrt{\frac{\lambda}{3}} - 2}\right)^2} - \frac{2\lambda}{3}\left(\sqrt{\frac{3}{\lambda}} + \frac{\alpha}{\alpha\sqrt{\frac{\lambda}{3}} - 2}\right)}\right], \tag{15b}$$

etc., which satisfy the requirement that $\beta = \beta(\alpha, \lambda)$.

However, according to (14a) and (14b), when $\lambda = 0$, $\beta = \alpha = 2m$, and when $\alpha = 0$, $\beta \neq \sqrt{\frac{3}{\lambda}}$. According to (15a), (15b), when $\lambda = 0$, $\beta \neq \alpha = 2m$, and when $\alpha = 0$, $\beta = \sqrt{\frac{3}{\lambda}}$. The required form for β, and therefore the required form for $C(D(r))$, *cannot be constructed*, i.e. *it does not exist*. There is no way $C(D(r))$ can be constructed to satisfy all the required conditions to render an admissible solution to (7) in the form of (3). Therefore, the assumption that $\lambda \neq 0$ is incorrect, and so $\lambda = 0$. This can be confirmed in the following way.

The proper radius $R_p(r)$ of (8) is given by,

$$R_p(r) = \int \frac{d\sqrt{F}}{\sqrt{1 - \frac{\lambda}{3}F}} = \sqrt{\frac{3}{\lambda}}\arcsin\sqrt{\frac{\lambda}{3}F(r)} + K\,,$$

where K is a constant. Now, the following condition must be satisfied,

$$\text{as } r \to r_0^{\pm}, \ R_p \to 0^{+}\,,$$

and therefore,

$$R_p(r_0) = 0 = \sqrt{\frac{3}{\lambda}}\arcsin\sqrt{\frac{\lambda}{3}F(r_0)} + K\,,$$

and so,

$$R_p(r)=\sqrt{\frac{3}{\lambda}}\left[\arcsin\sqrt{\frac{\lambda}{3}F(r)}-\arcsin\sqrt{\frac{\lambda}{3}F(r_0)}\right]. \quad (16)$$

According to (8),

$$g_{00}(r_0)=0 \Rightarrow F(r_0)=\frac{3}{\lambda}.$$

But then, by (16),

$$\sqrt{\frac{\lambda}{3}F(r)}\equiv 1,$$

$$R_p(r)\equiv 0.$$

Indeed, by (16),

$$\sqrt{\frac{\lambda}{3}F(r_0)}\leqslant\sqrt{\frac{\lambda}{3}F(r)}\leqslant 1,$$

or

$$\sqrt{\frac{3}{\lambda}}\leqslant\sqrt{F(r)}\leqslant\sqrt{\frac{3}{\lambda}},$$

and so

$$F(r)\equiv\frac{3}{\lambda}, \quad (17)$$

and

$$R_p(r)\equiv 0. \quad (18)$$

Then $F'(D(r))\equiv 0$, and so there exists no function $F(r)$ which renders a solution to (8) in the form of (3) when $\lambda\neq 0$ and therefore there exists no function $C(D(r))$ which renders a solution to (7) in the form of (3) when $\lambda\neq 0$. Consequently, $\lambda=0$.

Owing to their erroneous assumptions about the r-parameter, the relativists have disregarded the requirement that $A, B, C>0$ in (3) must be met. If the required form (3) is relaxed, in which case the resulting metric is *non-Einstein*, and cannot therefore describe an Einstein universe, (8) can be written as,

$$ds^2=-\frac{3}{\lambda}\left(d\theta^2+\sin^2\theta d\varphi^2\right). \quad (8b)$$

This means that metric $(8)\equiv(8b)$ maps the whole of (M_s, g_s) into the *point* $R_p(D(r))\equiv 0$ of the de Sitter "space" (M_{ds}, g_{ds}).

Einstein, de Sitter, Eddington, Friedmann, and the modern relativists all, have incorrectly *assumed* that r is a radial coordinate in (8), and consequently think of the "space" associated with (8) as extended in the sense of having a volume greater than zero. This is incorrect.

The radius of curvature of the point $R_p(D(r))\equiv 0$ is,

$$R_c(D(r))\equiv\sqrt{\frac{3}{\lambda}}.$$

The "surface area" of the point is,

$$A=\frac{12\pi}{\lambda}.$$

De Sitter's empty spherical universe has zero volume. Indeed, by (8) and (8b),

$$V=\lim_{r\to\pm\infty}\frac{3}{\lambda}\int_{r_0}^{r}0\,dr\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\varphi=0,$$

consequently, de Sitter's empty spherical universe is indeed "empty"; and meaningless. It is *not* an Einstein universe.

On (8) and (8b) the ratio,

$$\frac{2\pi\sqrt{F(r)}}{R_p(r)}=\infty\ \forall\ r.$$

Therefore, the lone point which consitutes the empty de Sitter "universe" (M_{ds}, g_{ds}) is a quasiregular singularity and consequently cannot be extended.

It is the unproven and invalid assumptions about the variable r which have lead the relativists astray. They have carried this error through all their work and consequently have completely lost sight of legitimate scientific theory, producing all manner of nonsense along the way. Eddington [4], for instance, writes in relation to (1), $\gamma=1-\frac{2m}{r}-\frac{\alpha r^2}{3}$ for his equation (45.3), and said,

> *At a place where γ vanishes there is an impassable barrier, since any change dr corresponds to an infinite distance ids surveyed by measuring rods. The two positive roots of the cubic (45.3) are approximately*
>
> $$r=2m \quad and \quad r=\sqrt{\left(\tfrac{3}{\alpha}\right)}.$$
>
> *The first root would represent the boundary of the particle — if a genuine particle could exist — and give it the appearance of impenetrability. The second barrier is at a very great distance and may be described as the* horizon *of the world.*

Note that Eddington, despite these erroneous claims, did not admit the sacred black hole. His arguments however, clearly betray his assumption that r is a radius on (1). I also note that he has set the constant numerator of the middle term of his γ to $2m$, as is usual, however, like all the modern relativists, he did not indicate how this identity is to be achieved. This is just another assumption. As Abrams [6] has pointed out in regard to (1), one cannot appeal to far-field Keplerian orbits to fix the constant to $2m$ — but the issue is moot, since $\lambda=0$.

There is no black hole associated with (1). The Lake-Roeder black hole is inconsistent with Einstein's theory.

4 The homogeneous static models

It is routinely alleged by the relativists that the static homogeneous cosmological models are exhausted by the line-elements of Einstein's cylindrical model, de Sitter's spherical model, and that of Special Relativity. This is not correct, as I shall now demonstrate that the only homogeneous universe admitted by Einstein's theory is that of his Special Theory of Relativity, which is a static, infinite, pseudo-Euclidean, empty world.

The cosmological models of Einstein and de Sitter are composed of a single world line and a single point respectively, neither of which can be extended. Their line-elements therefore *cannot* describe any Einstein universe.

If the Universe is considered as a continuous distribution of matter of proper macroscopic density ρ_{00} and pressure P_0, the stress-energy tensor is,

$$T_1^1 = T_2^2 = T_3^3 = -P_0\,, \quad T_4^4 = \rho_{00}\,,$$

$$T_\nu^\mu = 0, \quad \mu \neq \nu\,.$$

Rewrite (5) by setting,

$$A^*(r^*) = e^\nu, \quad \nu = \nu(r^*)\,,$$

$$B^*(r^*) = e^\sigma, \quad \sigma = \sigma(r^*)\,. \tag{19}$$

Then (5) becomes,

$$ds^2 = e^\nu dt^2 - e^\sigma dr^{*2} - r^{*2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,. \tag{20}$$

It then follows in the usual way that,

$$8\pi P_0 = e^{-\sigma}\left(\frac{\bar{\nu}}{r^*} + \frac{1}{r^{*2}}\right) - \frac{1}{r^{*2}} + \lambda\,, \tag{21}$$

$$8\pi\rho_{00} = e^{-\sigma}\left(\frac{\bar{\sigma}}{r^*} - \frac{1}{r^{*2}}\right) + \frac{1}{r^{*2}} - \lambda\,, \tag{22}$$

$$\frac{dP_0}{dr^*} = -\frac{\rho_{00} + P_0}{2}\bar{\nu}\,, \tag{23}$$

where

$$\bar{\nu} = \frac{d\nu}{dr^*}, \quad \bar{\sigma} = \frac{d\sigma}{dr^*}\,.$$

Since P_0 is to be the same everywhere, (23) becomes,

$$\frac{\rho_{00} + P_0}{2}\bar{\nu} = 0\,.$$

Therefore, the following three possibilities arise,

1. $\frac{d\nu}{dr^*} = 0$;
2. $\rho_{00} + P_0 = 0$;
3. $\frac{d\nu}{dr^*} = 0$ and $\rho_{00} + P_0 = 0$.

The 1st possibility yields Einstein's so-called cylindrical model, the 2nd yields de Sitter's so-called spherical model, and the 3rd yields Special Relativity.

5 Einstein's cylindrical cosmological model

In this case, to reduce to Special Relativity,

$$\nu = \text{const} = 0.$$

Therefore, by (21),

$$8\pi P_0 = \frac{e^{-\sigma}}{r^{*2}} - \frac{1}{r^{*2}} + \lambda\,,$$

and by (19),

$$8\pi P_0 = \frac{1}{B^*(r^*)r^{*2}} - \frac{1}{r^{*2}} + \lambda\,,$$

and by (4),

$$8\pi P_0 = \frac{1}{BC} - \frac{1}{C} + \lambda\,,$$

so

$$\frac{1}{B} = 1 - \left(\lambda - 8\pi P_0\right)C\,,$$

$$C = C(D(r)), \quad D(r) = |r - r_0|, \quad B = B(D(r))\,,$$

$$r_0 \in \Re\,.$$

Consequently, Einstein's line-element can be written as,

$$\begin{aligned} ds^2 = dt^2 - \left[1 - \left(\lambda - 8\pi P_0\right)C\right]^{-1} d\sqrt{C}^{\,2} - \\ - C\left(d\theta^2 + \sin^2 d\varphi^2\right) = \\ = dt^2 - \left[1 - \left(\lambda - 8\pi P_0\right)C\right]^{-1}\frac{C'^2}{4C}dr^2 - \\ - C\left(d\theta^2 + \sin^2 d\varphi^2\right)\,, \end{aligned} \tag{24}$$

$$C = C(D(r)), \quad D(r) = |r - r_0|, \quad r_0 \in \Re\,,$$

where r_0 is arbitrary.

It is now required to determine the admissible form of $C(D(r))$.

Clearly, if $C' \equiv 0$, then $B = 0 \ \forall\ r$, in violation of (3). Therefore, $C' \neq 0 \ \forall\ r \neq r_0$.

When $P_0 = \lambda = 0$, (24) must reduce to Special Relativity, in which case,

$$P_0 = \lambda = 0 \Rightarrow C(D(r)) = |r - r_0|^2\,.$$

The metric (24) is singular when $g_{11}^{-1}(r_0) = 0$, i.e. when,

$$1 - \left(\lambda - 8\pi P_0\right)C(r_0) = 0\,,$$

$$\Rightarrow\ C(r_0) = \frac{1}{\lambda - 8\pi P_0}\,. \tag{25}$$

Therefore, for $C(D(r))$ to render an admissible solution to (24) in the form of (3), it must at least satisfy the following conditions:

1. $C' \neq 0 \ \forall \ r \neq r_0$;
2. $P_0 = \lambda = 0 \Rightarrow C(D(r)) = |r - r_0|^2$;
3. $C(r_0) = \frac{1}{\lambda - 8\pi P_0}$.

Now the proper radius on (24) is,

$$R_p(r) = \int \frac{d\sqrt{C}}{\sqrt{1 - \left(\lambda - 8\pi P_0\right) C}} = \\ = \frac{1}{\sqrt{\lambda - 8\pi P_0}} \arcsin \sqrt{(\lambda - 8\pi P_0) C(r)} + K ,$$

$$K = \text{const.} ,$$

which must satisfy the condition,

$$\text{as } r \to r_0^{\pm} , \ R_p \to 0^+ .$$

Therefore,

$$R_p(r_0) = 0 = \frac{1}{\sqrt{\lambda - 8\pi P_0}} \times \\ \times \arcsin \sqrt{(\lambda - 8\pi P_0) C(r_0)} + K ,$$

so

$$R_p(r) = \frac{1}{\sqrt{\lambda - 8\pi P_0}} \Big[\arcsin \sqrt{(\lambda - 8\pi P_0) C(r)} - \\ - \arcsin \sqrt{(\lambda - 8\pi P_0) C(r_0)} \ \Big] . \qquad (26)$$

Now if follows from (26) that,

$$\sqrt{(\lambda - 8\pi P_0) C(r_0)} \leqslant \sqrt{(\lambda - 8\pi P_0) C(r)} \leqslant 1 ,$$

so

$$C(r_0) \leqslant C(r) \leqslant \frac{1}{\left(\lambda - 8\pi P_0\right)} ,$$

and therefore by (25),

$$\frac{1}{\left(\lambda - 8\pi P_0\right)} \leqslant C(r) \leqslant \frac{1}{\left(\lambda - 8\pi P_0\right)} .$$

Thus,

$$C(r) \equiv \frac{1}{\left(\lambda - 8\pi P_0\right)} ,$$

and so $C'(r) \equiv 0 \Rightarrow B(r) \equiv 0$, in violation of (3). Therefore there exists no $C(D(r))$ to satisfy (24) in the form of (3) when $\lambda \neq 0$, $P_0 \neq 0$. Consequently, $\lambda = P_0 = 0$, and (24) reduces to,

$$ds^2 = dt^2 - \frac{C'^2}{4C} dr^2 - C \left(d\theta^2 + \sin^2 d\varphi^2 \right) . \qquad (27)$$

The form of $C(D(r))$ must still be determined.

Clearly, if $C' \equiv 0$, $B(D(r)) = 0 \ \forall \ r$, in violation of (3). Therefore, $C' \neq 0 \ \forall \ r \neq r_0$.

Since there is no matter present, it is required that,

$$C(r_0) = 0 \quad \text{and} \quad \frac{C(D(r))}{|r - r_0|^2} = 1 .$$

This requires trivially that,

$$C(D(r)) = |r - r_0|^2 .$$

Therefore (27) becomes,

$$ds^2 = dt^2 - \frac{(r - r_0)^2}{|r - r_0|^2} dr^2 - |r - r_0|^2 \left(d\theta^2 + \sin^2 d\varphi^2 \right) = \\ = dt^2 - dr^2 - |r - r_0|^2 \left(d\theta^2 + \sin^2 d\varphi^2 \right) ,$$

which is precisely the metric of Special Relativity, according to the natural reduction on (2).

If the required form (3) is relaxed, in which case the resulting metric is *not* an Einstein metric, Einstein's cylindrical line-element is,

$$ds^2 = dt^2 - \frac{1}{\left(\lambda - 8\pi P_0\right)} \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right) . \qquad (28)$$

This is a line-element which cannot describe an Einstein universe. The Einstein space described by (28) consists of only one "world line", through the point,

$$R_p(r) \equiv 0 .$$

The spatial extent of (28) is a single point. The radius of curvature of this point space is,

$$R_c(r) \equiv \frac{1}{\sqrt{\lambda - 8\pi P_0}} .$$

For all r, the ratio $\frac{2\pi R_c}{R_p}$ is,

$$\frac{\frac{2\pi}{\sqrt{\lambda - 8\pi P_0}}}{R_p(r)} = \infty .$$

Therefore $R_p(r) \equiv 0$ is a quasiregular singular point and consequently cannot be extended.

The "surface area" of this point space is,

$$A = \frac{4\pi}{\lambda - 8\pi P_0} .$$

The volume of the point space is,

$$V = \lim_{r \to \pm\infty} \frac{1}{\left(\lambda - 8\pi P_0\right)} \int_{r_0}^{r} 0 \ dr \int_0^{\pi} \sin\theta \ d\theta \int_0^{2\pi} d\varphi = 0 .$$

Equation (28) maps the whole of (M_s, g_s) into a quasiregular singular "world line".

Einstein's so-called "cylindrical universe" is meaningless. It does not contain a black hole.

6 De Sitter's spherical cosmological model

In this case,

$$\rho_{00} + P_0 = 0\,.$$

Adding (21) to (22) and setting to zero gives,

$$8\pi\left(\rho_{00} + P_0\right) = e^{-\sigma}\left(\frac{\bar{\sigma}}{r^*} + \frac{\bar{\nu}}{r^*}\right) = 0\,,$$

or

$$\bar{\nu} = -\bar{\sigma}\,.$$

Therefore,

$$\nu(r^*) = -\sigma(r^*) + \ln K_1\,, \tag{29}$$

$$K_1 = \text{const.}$$

Since ρ_{00} is required to be a constant independent of position, equation (22) can be immediately integrated to give,

$$e^{-\sigma} = 1 - \frac{\lambda + 8\pi\rho_{00}}{3}\,r^{*2} + \frac{K_2}{r^*}\,, \tag{30}$$

$$K_2 = \text{const.}$$

According to (30),

$$-\sigma = \ln\left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}\,r^{*2} + \frac{K_2}{r^*}\right),$$

and therefore, by (29),

$$\nu = \ln\left[\left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}\,r^{*2} + \frac{K_2}{r^*}\right)K_1\right].$$

Substituting into (20) gives,

$$ds^2 = \left[\left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}\,r^{*2} + \frac{K_2}{r^*}\right)K_1\right]dt^2 - \left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}\,r^{*2} + \frac{K_2}{r^*}\right)^{-1}dr^{*2} - r^{*2}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

which is, by (4),

$$ds^2 = \left[\left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}C + \frac{K_2}{\sqrt{C}}\right)K_1\right]dt^2 - \left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}C + \frac{K_2}{\sqrt{C}}\right)^{-1}\frac{C'^2}{4C}dr^2 - C\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{31}$$

Now, when $\lambda = \rho_{00} = 0$, equation (31) must reduce to the metric for Special Relativity. Therefore,

$$K_1 = 1, \quad K_2 = 0\,,$$

and so de Sitter's line-element is,

$$ds^2 = \left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}C\right)dt^2 - \left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}C\right)^{-1}\frac{C'^2}{4C}\,dr^2 - C\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{32}$$

$$C = C(D(r)), \quad D(r) = |r - r_0|, \quad r_0 \in \Re\,,$$

where r_0 is arbitrary.

It remains now to determine the admissible form of $C(D(r))$ to render a solution to equation (32) in the form of equation (3).

If $C' \equiv 0$, then $B(D(r)) = 0\ \forall\ r$, in violation of (3). Therefore $C' \neq 0\ \forall\ r \neq r_0$.

When $\lambda = \rho_{00} = 0$, (32) must reduce to that for Special Relativity. Therefore,

$$\lambda = \rho_{00} = 0 \Rightarrow C(D(r)) = |r - r_0|^2\,.$$

Metric (32) is singular when $g_{00}(r_0) = 0$, i.e. when

$$1 - \frac{\lambda + 8\pi\rho_{00}}{3}C(r_0) = 0\,,$$

$$\Rightarrow C(r_0) = \frac{3}{\lambda + 8\pi\rho_{00}}\,. \tag{33}$$

Therefore, to render a solution to (32) in the form of (3), $C(D(r))$ must at least satisfy the following conditions:

1. $C' \neq 0\ \forall\ r \neq r_0$;
2. $\lambda = \rho_{00} = 0 \Rightarrow C(D(r)) = |r - r_0|^2$;
3. $C(r_0) = \frac{3}{\lambda + 8\pi\rho_{00}}$.

The proper radius on (32) is,

$$R_p(r) = \int \frac{d\sqrt{C}}{\sqrt{1 - \left(\frac{\lambda + 8\pi\rho_{00}}{3}\right)C}} = \sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}}\arcsin\sqrt{\left(\frac{\lambda + 8\pi\rho_{00}}{3}\right)C(r)} + K\,, \tag{34}$$

$$K = \text{const}\,,$$

which must satisfy the condition,

$$\text{as } r \to r_0^{\pm},\ R_p(r) \to 0^+\,.$$

Therefore,

$$R_p(r_0) = 0 = \sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}}\arcsin\sqrt{\left(\frac{\lambda + 8\pi\rho_{00}}{3}\right)C(r_0)} + K\,,$$

so (34) becomes,

$$R_p(r) = \sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}\left[\arcsin\sqrt{\left(\frac{\lambda+8\pi\rho_{00}}{3}\right)C(r)} - \right.$$
$$\left. - \arcsin\sqrt{\left(\frac{\lambda+8\pi\rho_{00}}{3}\right)C(r_0)}\,\right]. \tag{35}$$

It then follows from (35) that,

$$\sqrt{\left(\frac{\lambda+8\pi\rho_{00}}{3}\right)C(r_0)} \leqslant \sqrt{\left(\frac{\lambda+8\pi\rho_{00}}{3}\right)C(r)} \leqslant 1\,,$$

or

$$C(r_0) \leqslant C(r) \leqslant \frac{3}{\lambda+8\pi\rho_{00}}\,.$$

Then, by (33),

$$\frac{3}{\lambda+8\pi\rho_{00}} \leqslant C(r) \leqslant \frac{3}{\lambda+8\pi\rho_{00}}\,.$$

Therefore, $C(r)$ is a constant function for all r,

$$C(r) \equiv \frac{3}{\lambda+8\pi\rho_{00}}\,, \tag{36}$$

and so,

$$C'(r) \equiv 0\,,$$

which implies that $B(D(r)) \equiv 0$, in violation of (3). Consequently, there exists no function $C(D(r))$ to render a solution to (32) in the form of (3). Therefore, $\lambda = \rho_{00} = 0$, and (32) reduces to the metric of Special Relativity in the same way as does (24).

If the required form (3) is relaxed, in which case the resulting metric is *not* an Einstein metric, de Sitter's line-element is,

$$ds^2 = -\frac{3}{\lambda+8\pi\rho_{00}}\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{37}$$

This line-element cannot describe an Einstein universe. The Einstein space described by (37) consists of only one point:

$$R_p(r) \equiv 0\,.$$

The radius of curvature of this point is,

$$R_c(r) \equiv \sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}\,,$$

and the "surface area" of the point is,

$$A = \frac{12\pi}{\lambda+8\pi\rho_{00}}\,.$$

The volume of de Sitter's "spherical universe" is,

$$V = \left(\frac{3}{\lambda+8\pi\rho_{00}}\right)\lim_{r\to\pm\infty}\int_{r_0}^{r} 0\,dr\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\varphi = 0\,.$$

For all values of r, the ratio,

$$\frac{2\pi\sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}}{R_p(r)} = \infty\,.$$

Therefore, $R_p(r) \equiv 0$ is a quasiregular singular point and consequently cannot be extended.

According to (32), metric (37) maps the whole of (M_s, g_s) into a quasiregular singular point.

Thus, de Sitter's spherical universe is meaningless. It does not contain a black hole.

When $\rho_{00} = 0$ and $\lambda \neq 0$, de Sitter's empty universe is obtained from (37). I have already dealt with this case in section 3.

7 The infinite static homogeneous universe of special relativity

In this case, by possibility 3 in section 4,

$$\bar{\nu} = \frac{d\nu}{dr^*} = 0, \quad \text{and} \quad \rho_{00} + P_0 = 0\,.$$

Therefore,

$$\nu = \text{const} = 0 \quad \text{by section 5}$$

and

$$\bar{\sigma} = -\bar{\nu} \quad \text{by section 6}\,.$$

Hence, also by section 6,

$$\sigma = -\nu = 0\,.$$

Therefore, (20) becomes,

$$ds^2 = dt^2 - dr^{*2} - r^{*2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

which becomes, by using (4),

$$ds^2 = dt^2 - \frac{C'^2}{4C}dr^2 - C\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$C = C(D(r)), \quad D(r) = |r - r_0|, \quad r_0 \in \Re\,,$$

which, by the analyses in sections 5 and 6, becomes,

$$ds^2 = dt^2 - \frac{\left(r-r_0\right)^2}{|r-r_0|^2}dr^2 - |r-r_0|^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{38}$$
$$r_0 \in \Re\,,$$

which is the flat, empty, and infinite spacetime of Special Relativity, obtained from (2) by natural reduction.

When $r_0 = 0$ and $r \geqslant r_0$, (38) reduces to the usual form used by the relativists,

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

The radius of curvature of (38) is,

$$D(r) = |r - r_0|.$$

The proper radius of (38) is,

$$R_p(r) = \int_0^{|r-r_0|} d|r - r_0| = \int_{r_0}^{r} \frac{(r - r_0)}{|r - r_0|} dr = |r - r_0| \equiv D.$$

The ratio,

$$\frac{2\pi D(r)}{R_p(r)} = \frac{2\pi |r - r_0|}{|r - r_0|} = 2\pi \;\forall\; r.$$

Thus, only (38) can represent a static homogeneous universe in Einstein's theory, contrary to the claims of the modern relativists. However, since (38) contains no matter it cannot model the universe other than locally.

8 Cosmological models of expansion

In view of the foregoing it is now evident that the models proposed by the relativists purporting an expanding universe are also untenable in the framework of Einstein's theory. The line-element obtained by the Abbé Lemaître and by Robertson, for instance, is inadmissible. Under the false assumption that r is a radius in de Sitter's spherical universe, they proposed the following transformation of coordinates on the metric (32) (with $\rho_{00} \neq 0$ in the misleading form given in formula 9),

$$\bar{r} = \frac{r}{\sqrt{1 - \frac{r^2}{W^2}}} e^{-\frac{t}{W}}, \quad \bar{t} = t + \frac{1}{2} W \ln\left(1 - \frac{r^2}{W^2}\right), \tag{39}$$

$$W^2 = \frac{\lambda + 8\pi\rho_{00}}{3},$$

to get

$$ds^2 = d\bar{t}^2 - e^{\frac{2\bar{t}}{W}}\left(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\varphi^2\right),$$

or, by dropping the bar and setting $k = \frac{1}{W}$,

$$ds^2 = dt^2 - e^{2kt}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right). \tag{40}$$

Now, as I have shown, (32) has no solution in $C(D(r))$ in the form (3), so transformations (39) and metric (40) are meaningless concoctions of mathematical symbols. Owing to their false assumptions about the parameter r, the relativists mistakenly think that $C(D(r)) \equiv r^2$ in (32). Furthermore, if the required form (3) is relaxed, thereby producing *non-Einstein metrics*, de Sitter's "spherical universe" is given by (37), and so, by (35), (36), and (40),

$$C(D(r)) = r^2 \equiv \frac{\lambda + 8\pi\rho_{00}}{3},$$

and the transformations (39) and metric (40) are again utter nonsense. The Lemaître-Robertson line-element is inevitably, unmitigated claptrap. This can be proved generally as follows.

The most general non-static line-element is

$$\begin{aligned} ds^2 = A(D,t)dt^2 - B(D,t)dD^2 - \\ - C(D,t)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \end{aligned} \tag{41}$$

$$D = |r - r_0|, \quad r_0 \in \Re$$

where analytic $A, B, C > 0 \;\forall\; r \neq r_0$ and $\forall\; t$.

Rewrite (41) by setting,

$$A(D,t) = e^{\nu}, \quad \nu = \nu(G(D),t),$$

$$B(D,t) = e^{\sigma}, \quad \sigma = \sigma(G(D),t),$$

$$C(D,t) = e^{\mu}G^2(D), \quad \mu = \mu(G(D),t),$$

to get

$$ds^2 = e^{\nu}dt^2 - e^{\sigma}dG^2 - e^{\mu}G^2(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{42}$$

Now set,

$$r^* = G(D(r)), \tag{43}$$

to get

$$ds^2 = e^{\nu}dt^2 - e^{\sigma}dr^{*2} - e^{\mu}r^{*2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{44}$$

$$\nu = \nu(r^*, t), \quad \sigma = \sigma(r^*, t), \quad \mu = \mu(r^*, t).$$

One then finds in the usual way that the solution to (44) is,

$$\begin{aligned} ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1 + \frac{k}{4}r^{*2}\right)^2} \times \\ \times \left[dr^{*2} + r^{*2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \end{aligned} \tag{45}$$

where k is a constant.

Then by (43) this becomes,

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1 + \frac{k}{4}G^2\right)^2}\left[dG^2 + G^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

or,

$$\begin{aligned} ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1 + \frac{k}{4}G^2\right)^2} \times \\ \times \left[G'^2 dr^2 + G^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \end{aligned} \tag{46}$$

$$G' = \frac{dG}{dr} \,,$$

$$G = G(D(r))\,, \quad D(r) = |r - r_0|\,, \quad r_0 \in \Re\,.$$

The admissible form of $G(D(r))$ must now be determined.

If $G' \equiv 0$, then $B(D,t) = 0 \;\forall\; r$ and $\forall\; t$, in violation of (41). Therefore $G' \neq 0 \;\forall\; r \neq r_0$.

Metric (46) is singular when,

$$1 + \frac{k}{4} G^2(r_0) = 0\,,$$

$$\Rightarrow \;\; G(r_0) = \frac{2}{\sqrt{-k}} \Rightarrow k < 0\,. \tag{47}$$

The proper radius on (46) is,

$$R_p(r,t) = e^{\frac{1}{2}g(t)} \int \frac{dG}{1 + \frac{k}{4}G^2} = $$

$$= e^{\frac{1}{2}g(t)} \left(\frac{2}{\sqrt{k}} \arctan \frac{\sqrt{k}}{2} G(r) + K \right),$$

$$K = \text{const}\,,$$

which must satisfy the condition,

$$\text{as } r \to r_0^{\pm},\; R_p \to 0^{+}\,.$$

Therefore,

$$R_p(r_0,t) = e^{\frac{1}{2}g(t)} \left(\frac{2}{\sqrt{k}} \arctan \frac{\sqrt{k}}{2} G(r_0) + K \right) = 0\,,$$

and so

$$R_p(r,t) = e^{\frac{1}{2}g(t)} \frac{2}{\sqrt{k}} \left[\arctan \frac{\sqrt{k}}{2} G(r) - \arctan \frac{\sqrt{k}}{2} G(r_0) \right]. \tag{48}$$

Then by (47),

$$R_p(r,t) = e^{\frac{1}{2}g(t)} \frac{2}{\sqrt{k}} \left[\arctan \frac{\sqrt{k}}{2} G(r) - \arctan \sqrt{-1} \right], \tag{49}$$

$$k < 0\,.$$

Therefore, there exists no function $G(D(r))$ rendering a solution to (46) in the required form of (41).

The relativists however, owing to their invalid assumptions about the parameter r, write equation (46) as,

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1 + \frac{k}{4} r^2\right)^2} \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right], \tag{50}$$

having assumed that $G(D(r)) \equiv r$, and erroneously take r as a radius on the metric (50), valid down to 0. Metric (50) is a meaningless concoction of mathematical symbols. Nevertheless, the relativists transform this meaningless expression with a meaningless change of "coordinates" to obtain the Robertson-Walker line-element, as follows.

Transform (46) by setting,

$$\bar{G}(\bar{r}) = \frac{G(r)}{1 + \frac{k}{4} G^2}\,.$$

This carries (46) into,

$$ds^2 = dt^2 - e^{g(t)} \left[\frac{d\bar{G}^2}{\left(1 - \kappa \bar{G}^2\right)} + \bar{G}^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right]. \tag{51}$$

This is easily seen to be the familiar Robertson-Walker line-element if, following the relativists, one incorrectly assumes $\bar{G} \equiv \bar{r}$, disregarding the fact that the admissible form of $\bar{G}$ *must be ascertained*. In any event (51) is meaningless, owing to the meaninglessness of (50), which I confirm as follows.

$\bar{G}' \equiv 0 \Rightarrow \bar{B} = 0 \;\;\forall\;\; \bar{r}$, in violation of (41). Therefore $\bar{G}' \neq 0 \;\forall\; \bar{r} \neq \bar{r}_0$.

Equation (51) is singular when,

$$1 - k\bar{G}^2(\bar{r}_0) = 0 \;\Rightarrow \bar{G}(\bar{r}_0) = \frac{1}{\sqrt{k}} \;\Rightarrow k > 0\,. \tag{52}$$

The proper radius on (51) is,

$$\bar{R}_p = e^{\frac{1}{2}g(t)} \int \frac{d\bar{G}}{\sqrt{1 - k\bar{G}^2}}$$

$$= e^{\frac{1}{2}g(t)} \left(\frac{1}{\sqrt{k}} \arcsin \sqrt{k}\bar{G}(\bar{r}) + K \right),$$

$$K = \text{const.}\,,$$

which must satisfy the condition,

$$\text{as } \;\bar{r} \to \bar{r}_0^{\pm},\; \bar{R}_p \to 0^{+}\,,$$

so

$$\bar{R}_p(\bar{r}_0,t) = 0 = e^{\frac{1}{2}g(t)} \left(\frac{1}{\sqrt{k}} \arcsin \sqrt{k}\bar{G}(\bar{r}_0) + K \right).$$

Therefore,

$$\bar{R}_p(\bar{r},t) = e^{\frac{1}{2}g(t)} \frac{1}{\sqrt{k}} \times \left[\arcsin \sqrt{k}\bar{G}(\bar{r}) - \arcsin \sqrt{k}\bar{G}(\bar{r}_0) \right]. \tag{53}$$

Then

$$\sqrt{k}\bar{G}(\bar{r}_0) \leqslant \sqrt{k}\bar{G}(\bar{r}) \leqslant 1\,,$$

or

$$\bar{G}(\bar{r}_0) \leqslant \bar{G}(\bar{r}) \leqslant \frac{1}{\sqrt{k}}.$$

Then by (52),

$$\frac{1}{\sqrt{k}} \leqslant \bar{G}(\bar{r}) \leqslant \frac{1}{\sqrt{k}},$$

so

$$\bar{G}(\bar{r}) \equiv \frac{1}{\sqrt{k}}.$$

Consequently, $\bar{G}'(\bar{r}) = 0 \ \forall \ \bar{r}$ and $\forall \ t$, in violation of (41). Therefore, there exists no function $\bar{G}(\bar{D}(\bar{r}))$ to render a solution to (51) in the required form of (41).

If the conditions on (41) are relaxed in the fashion of the relativists, non-Einstein metrics with expanding radii of curvature are obtained. Nonetheless the associated spaces have zero volume. Indeed, equation (40) becomes,

$$ds^2 = dt^2 - e^{2kt}\frac{(\lambda + 8\pi\rho_{00})}{3}\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \quad (54)$$

This is not an Einstein universe. The radius of curvature of (54) is,

$$R_c(r,t) = e^{kt}\sqrt{\frac{\lambda + 8\pi\rho_{00}}{3}},$$

which expands or contracts with the sign of the constant k. Even so, the proper radius of the "space" of (54) is,

$$R_p(r,t) = \lim_{r \to \pm\infty}\int_{r_0}^{r} 0 \, dr \equiv 0.$$

The volume of this point-space is,

$$V = \lim_{r \to \pm\infty} e^{2kt}\frac{(\lambda + 8\pi\rho_{00})}{3}\int_{r_0}^{r} 0 \, dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} \equiv 0.$$

Metric (54) consists of a single "world line" through the point $R_p(r,t) \equiv 0$. Furthermore, $R_p(r,t) \equiv 0$ is a quasi-regular singular point-space since the ratio,

$$\frac{2\pi e^{kt}\sqrt{\lambda + 8\pi\rho_{00}}}{\sqrt{3}R_p(r,t)} \equiv \infty.$$

Therefore, $R_p(r,t) \equiv 0$ cannot be extended.

Similarly, equation (51) becomes,

$$ds^2 = dt^2 - \frac{e^{g(t)}}{k}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (55)$$

which is not an Einstein metric. The radius of curvature of (55) is,

$$R_c(r,t) = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}},$$

which changes with time. The proper radius is,

$$R_p(r,t) = \lim_{r \to \pm\infty}\int_{r_0}^{r} 0 \, dr \equiv 0,$$

and the volume of the point-space is

$$V = \lim_{r \to \pm\infty}\frac{e^{g(t)}}{k}\int_{r_0}^{r} 0 \, dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} \equiv 0.$$

Metric (55) consists of a single "world line" through the point $R_p(r,t) \equiv 0$. Furthermore, $R_p(r,t) \equiv 0$ is a quasi-regular singular point-space since the ratio,

$$\frac{2\pi e^{\frac{1}{2}g(t)}}{\sqrt{k}R_p(r,t)} \equiv \infty.$$

Therefore, $R_p(r,t) \equiv 0$ cannot be extended.

It immediately follows that the Friedmann models are all invalid, because the so-called Friedmann equation, with its associated equation of continuity, $T^{\mu\nu}_{;\mu} = 0$, is based upon metric (51), which, as I have proven, has *no solution* in $\overline{G}(\overline{r})$ in the required form of (41). Furthermore, metric (55) cannot represent an Einstein universe and therefore has no cosmological meaning. Consequently, the Friedmann equation is also nothing more than a meaningless concoction of mathematical symbols, destitute of any physical significance whatsoever. Friedmann incorrectly assumed, just as the relativists have done all along, that the parameter r is a radius in the gravitational field. Owing to this erroneous assumption, his treatment of the metric for the gravitational field violates the inherent geometry of the metric and therefore violates the geometrical form of the pseudo-Riemannian spacetime manifold. The same can be said of Einstein himself, who did not understand the geometry of his own creation, and by making the same mistakes, failed to understand the implications of his theory.

Thus, the Friedmann models are all invalid, as is the Einstein-de Sitter model, and all other general relativistic cosmological models purporting an expansion of the universe. Furthermore, there is no general relativistic substantiation of the Big Bang hypothesis. Since the Big Bang hypothesis rests solely upon an invalid interpretation of General Relativity, it is abject nonsense. The standard interpretations of the Hubble-Humason relation and the cosmic microwave background are not consistent with Einstein's theory. Einstein's theory cannot form the basis of a cosmology.

9 Singular points in Einstein's universe

It has been pointed out before [7, 8, 3] that singular points in Einstein's universe are quasiregular. No curvature type singularities arise in Einstein's universe. The oddity of a point being associated with a non-zero radius of curvature is an inevitable consequence of Einstein's geometry. There is *nothing* more pointlike in Einstein's universe, and nothing more pointlike in the de Sitter point world or the Einstein cylindrical world line. A point as it is usually conceived of in Minkowski space *does not exist* in Einstein's universe. The modern relativists have not understood this inescapable fact.

Acknowledgements

I would like to extend my thanks to Dr. D. Rabounski and Dr. L. Borissova for their kind advice as to the clarification of my definitions and my terminology, manifest as section 2 herein.

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

References

1. Stavroulakis N. On a paper by J. Smoller and B. Temple. *Annales de la Fondation Louis de Broglie*, 2002, v. 27, 3 (see also in www.geocities.com/theometria/Stavroulakis-1.pdf).
2. Stavroulakis N. On the principles of general relativity and the $S\Theta(4)$-invariant metrics. *Proc. 3rd Panhellenic Congr. Geometry*, Athens, 1997, 169 (see also in www.geocities.com/theometria/Stavroulakis-2.pdf).
3. Crothers S. J. On the geometry of the general solution for the vacuum field of the point-mass. *Progress in Physics*, 2005, v. 2, 3–14.
4. Eddington A. S. The mathematical theory of relativity. Cambridge University Press, Cambridge, 2nd edition, 1960.
5. Petrov A. Z. Einstein spaces. Pergamon Press, London, 1969.
6. Abrams L. S. The total space-time of a point-mass when $\Lambda \neq 0$, and its consequences for the Lake-Roeder black hole. *Physica A*, v. 227, 1996, 131–140 (see also in arXiv: gr-qc/0102053).
7. Brillouin M. The singular points of Einstein's Universe. *Journ. Phys. Radium*, 1923, v. 23, 43 (see also in arXiv: physics/0002009).
8. Abrams L. S. Black holes: the legacy of Hilbert's error. *Can. J. Phys.*, 1989, v. 67, 919 (see also in arXiv: gr-qc/0102055).

The First Crisis in Cosmology Conference

Monção, Portugal, June 23–25 2005

Hilton Ratcliffe

Astronomical Society of Southern Africa

E-mail: ratcliff@iafrica.com

The author attended the first Crisis in Cosmology Conference of the recently associated Alternative Cosmology Group, and makes an informal report on the proceedings with some detail on selected presentations.

In May 2004, a group of about 30 concerned scientists published an open letter to the global scientific community in *New Scientist* in which they protested the stranglehold of Big Bang theory on cosmological research and funding. The letter was placed on the Internet* and rapidly attracted wide attention. It currently has about 300 signatories representing scientists and researchers of disparate backgrounds, and has led to a loose association now known as the Alternative Cosmology Group†. This writer was one of the early signatories to the letter, and holding the view that the Big Bang explanation of the Universe is scientifically untenable, patently illogical, and without any solid observational support whatsoever, became involved in the organisation of an international forum where we could share ideas and plan our way forward. That idea became a reality with the staging of the *First Crisis in Cosmology Conference* (CCC-1) in the lovely, medieval walled village of Monção, far northern Portugal, over 3 days in June of this year.

It was sponsored in part by the University of Minho in Braga, Portugal, and the Institute for Advanced Studies at Austin, Texas. Professor José Almeida of the Department of Physics at the University of Minho was instrumental in the organisation and ultimate success of an event that is now to be held annually. The conference was arranged in 3 sessions. On the first day, papers were presented on observations that challenge the present model, the second day dealt with conceptual difficulties in the standard model, and we concluded with alternative cosmological world-views. Since it is not practicable here to review all the papers presented (some 34 in total, plus 6 posters), I'll selectively confine my comments to those that interested me particularly. The American Institute of Physics will publish the proceedings of the conference in their entirety in due course for those interested in the detail.

First up was professional astronomer Dr. Riccardo Scarpa of the European Southern Observatory, Santiago, Chile. His job involves working with the magnificent Very Large Telescope array at Paranal, and I guess that makes him the envy of just about every astronomer with blood in his veins! His paper was on Modified Newtonian Dynamics (MOND), which I had eagerly anticipated and thoroughly appreciated. MOND is a very exciting development in observational astronomy used to make Dark Matter redundant in the explanation of cosmic gravitational effects like the anomalous rotational speeds of galaxies. Mordehai Milgrom of the Weizmann Institute in Israel first noticed that mass discrepancies in stellar systems are detected only when the internal acceleration of gravity falls below the well-established value $a_0 = 1.2\times10^{-8}$ cm$\times$s^{-2}. The standard Newtonian gravitational values fit perfectly above this threshold, and below a_0 MOND posits a breakdown of Newton's law. The dependence then becomes linear with an asymptotic value of acceleration $a = (\mathrm{a}_0\,\mathrm{g})^{1/2}$, where g is the Newtonian value. Scarpa has called this the *weak gravitational regime*, and he and colleagues Marconi and Gilmozzi have applied it extensively to globular clusters with 100% success. What impressed me most was that the clear empirical basis of MOND has been thoroughly tested, and is now in daily use by professional astronomers at what is arguably the most sophisticated and advanced optical-infrared observatory in the world. In practice, there is no need to invoke Dark Matter. Quote from Riccardo: "*Dark Matter is the craziest idea we've ever had in astronomy. It can appear when you need it, it can do what you like, be distributed in any way you like. It is the fairy tale of astronomy*".

Big Bang theory depends critically on three first principles: that the Universe is holistically and systematically expanding as per the Friedmann model; that General Relativity correctly describes gravitation; and that Milne's Cosmological Principle, which declares that the Universe at some arbitrary "large scale" is isotropic and homogeneous, is true. The falsification of any one of these principles would lead to the catastrophic failure of the theory. We saw at the conference that all three can be successfully challenged on the basis of empirical science. Retired electrical engineer Tom Andrews presented a novel approach to the validation (or rather, invalidation) of the expanding Universe model. It is well known that type 1A supernovae (SNe) show measurable anomalous dimming (with distance or remoteness in time) in a flat expanding Universe model. Andrews used

*http://www.cosmologystatement.org/

†http://www.cosmology.info/

observational data from two independent sets of measurements of brightest cluster galaxies (defined as the brightest galaxy in a cluster). It was expected, since the light from the SNe and the bright galaxies traverses the same space to get to us, that the latter should also be anomalously dimmed. They clearly are not. The orthodox explanation for SNe dimming — that it is the result of the progressive expansion of space — is thereby refuted. He puts a further nail in the coffin by citing Goldhaber's study of SNe light curves, which did not reveal the second predicted light-broadening effect due to time dilation. Says Andrews: "*The Hubble redshift of Fourier harmonic frequencies [for SNe] is shown to broaden the light curve at the observer by* $(1+z)$. *Since this broadening spreads the total luminosity over a longer time period, the apparent luminosity at the observer is decreased by the same factor. This accounts quantitatively for the dimming of SNe. On the other hand, no anomalous dimming occurs for galaxies since the luminosity remains constant over time periods much longer than the light travel time to the observer. This effect is consistent with the non-expanding Universe model. The expanding model is logically falsified*".

Professor Mike Disney of the School of Physics and Astronomy at Cardiff University calls a spade a spade. He has created an interesting benchmark for the evaluation of scientific models — he compares the number of free parameters in a theory with the number of independent measurements, and sets an arbitrary minimum of $+3$ for the excess of measurements over free parameters to indicate that the theory is empirically viable. He ran through the exercise for the Big Bang model, and arrived at a figure of -3 (17 free parameters against 14 measured). He therefore argued that the there is little statistical significance in the good fits claimed by Big Bang cosmologists since the surfeit of free parameters can easily mould new data to fit a desired conclusion. Quote: "*The study of some 60 cultures, going back 12,000 years, shows that, like it or not, we will always have a cosmology, and there have always been more free parameters than independent measurements. The best model is a compromise between parsimony (Occam's razor) and goodness-of-fit*".

Disney has a case there, and it is amply illustrated when it comes to Big Bang Nucleosynthesis (which depends initially on an arbitrarily set baryon/photon ratio), and the abundances of chemical elements. Dr. Tom van Flandern is another straight talking, no frills man of science. He opened his abstract with the words "*The Big Bang has never achieved a true prediction success where the theory was placed at risk of falsification before the results were known*". Ten years ago, Tom's web site listed the Top Ten Problems with the Big Bang, and today he has limited it to the Top Fifty. He pointed out the following contradictions in predicted light element abundances: observed deuterium abundances don't tie up with observed abundances of ^{4}He and ^{7}Li, and attempts to explain this inconsistency have failed. The ratio of deuterium to hydrogen near the centre of the Milky Way is 5 orders of magnitude higher than the Standard Model predicts, and measuring either for quasars produces deviation from predictions. Also problematic for BBN are barium and beryllium, produced assumedly as secondary products of supernovae by the process of spallation. However, observations of metal-poor stars show greater abundance of Be than possible by spallation. Van Flandern: "*It should be evident to objective minds that nothing about the Universe interpreted with the Big Bang theory is necessarily right, not even the most basic idea in it that the Universe is expanding*".

Problems in describing the geometry of the Universe were dealt with by several speakers, and we must here of course drill down a bit to where the notion came from (in the context of Big Bang theory). The theory originated in Father Georges Lemaître's extensions to Friedmann's solution of the Einstein General Relativity (GR) field equations, which showed that the Universe described in GR could not be static as Einstein believed. From this starting point emerged some irksome dilemmas regarding the fundamental nature of space and the distribution of matter within it. It was here more than anywhere that the rich diversity of opinion and approach within the Alternative Cosmology Group was demonstrated. Professor Yurij Baryshev of the Institute of Astronomy at St. Petersburg State University quietly presented his argument against the Cosmological Principle: large-scale structure is not possible in the Friedmann model, yet observation shows it for as far as we can see. I had recently read Yurij's book *The Discovery of Cosmic Fractals*, and knew that he had studied the geometric fractals of Yale's famous Professor Benoit Mandelbrot, which in turn led to his extrapolation of a fractal (inhomogeneous, anisotropic) non-expanding large-scale universe. Baryshev discussed gravitation from the standpoint that the physics of gravity should be the focus of cosmological research. General Relativity and the Feynman field are different at all scales, although to date, all relativistic tests cannot distinguish between them. He pointed out that if one reversed the flow and shrunk the radius, eventually the point would be reached where the energy density of the Universe would exceed the rest mass, and that is logically impossible. He left us with this gem: Feynman to his wife (upon returning from a conference) "*Remind me not to attend any more gravity conferences*!"

Conference co-ordinator Professor José Almeida presented a well-argued case for an interesting and unusual world-view: a hyperspherical Universe of 4-D Euclidean space (called 4-Dimensional Optics or 4DO) rather than the standard non-Euclidean Minkowski space. Dr. Franco Selleri of the Università di Bari in Italy provided an equally interesting alternative — the certainty that the Universe in which we live and breathe is a construction in simple 3-D Euclidean space precludes the possibility of the Big Bang model. He says: "*No structure in three dimensional space, born from an explosion that occurred 10 to 20 billion years ago, could* *resemble the Universe we observe*". The key to Selleri's theory is absolute simultaneity, obtained by using a term e_1 (the coefficient of x in the transformation of time) in the Lorentz transformations, so that $e_1 = 0$. Setting $e_1 = 0$ separates time and space, and a conception of reality is introduced in which no room is left for a fourth dimension. Both Big Bang and its progenitor General Relativity depend critically on 4-D Minkowski space, so the argument regressed even further to the viability of Relativity itself. And here is where the big guns come in!

World-renowned mathematical physicist Professor Huseyin Yilmaz, formerly of the Institute for Advanced Studies at Princeton University, and his hands-on experimentalist colleague Professor Carrol Alley of the University of Maryland, introduced us to the Yilmaz cosmology. Altogether 4 papers were presented at CCC-1 on various aspects of Yilmaz theory, and a fifth, by Dr. Hal Puthoff of the Institute for Advanced Studies at Austin, was brought to the conference but not presented. It is no longer controversial to suggest that GR has flaws, although I still feel awkward saying it out loud! Professor Yilmaz focussed on the fact that GR excludes gravitational stress-energy as a source of curvature. Consequently, stress-energy is merely a coordinate artefact in GR, whereas in the Yilmaz modification it is a true tensor. Hal Puthoff described the GR term to me as a "*pseudo-tensor, which can appear or disappear depending on how you treat mass*". The crucial implication of this, in the words of Professor Alley, is that since "*interactions are carried by the field stress energy, there are no interactive n-body solutions to the field equations of General Relativity*". In plain language, GR is a single-body description of gravity! The Yilmaz equations contain the correct terms, and they have been applied with success to various vexing problems, for example the precession of Mercury's perihelion, lunar laser ranging measurements, the flying of atomic clocks in aircraft, the relativistic behaviour of clocks in the GPS, and the predicted *Sagnac effect* in the one-way speed of light on a rotating table. Anecdote from Professor Alley: at a lecture by Einstein in the 1920's, Professor Sagnac was in the audience. He questioned Einstein on the *gedanken* experiment regarding contra-radiating light on a rotating plate. Einstein thought for a while and said, "That has got nothing to do with relativity". Sagnac loudly replied, "In that case, Dr. Einstein, relativity has got nothing to do with reality!"

The great observational "proof" of Big Bang theory is undoubtedly the grandly titled Cosmic Microwave Background Radiation, stumbled upon by radio engineers Penzias and Wilson in 1965, hijacked by Princeton cosmologist Jim Peebles, and demurely described by UC's COBE data analyser Dr. George Smoot as "*like looking at the fingerprint of God*". Well, it's come back to haunt them! I was delighted that despite some difficulties Glenn Starkman of Case Western Reserve University was able to get his paper presented at the conference as I had been keenly following his work on the Wilkinson Microwave Anisotropy Probe (WMAP) data. Dr. Starkman has discovered some unexpected (for Big Bangers) characteristics (he describes them as "bizarre") in the data that have serious consequences for the Standard Model. Far from having the smooth, Gaussian distribution predicted by Big Bang, the microwave picture has distinct anisotropies, and what's more says Starkman, they are clearly aligned with local astrophysical structures, particularly the ecliptic of the Solar System. Once the dipole harmonic is stripped to remove the effect of the motion of the Solar System, the other harmonics, quadrupole, octopole, and so on reveal a distinct alignment with local objects, and show also a preferred direction towards the Virgo supercluster. Conference chair, plasma physicist Eric Lerner concurred in his paper. He suggested that the microwave background is nothing more than a radio fog produced by plasma filaments, which has reached a natural isotropic thermal equilibrium of just under 3K. The radiation is simply starlight that has been absorbed and re-radiated, and echoes the anisotropies of the world around us. These findings correlate with the results of a number of other independent studies, including that of Larson and Wandelt at the University of Illinois, and also of former Cambridge *enfant terrible* and current Imperial College theoretical physics prodigy, Professor João Magueijo. Quote from Starkman: "*This suggests that the reported microwave background fluctuations on large angular scales are not in fact cosmic, with important consequences*". Phew!

The final day saw us discussing viable alternative cosmologies, and here one inevitably leans towards personal preferences. My own bias is unashamedly towards scientists who adopt the classical empirical method, and there is no better example of this than Swedish plasma physics pioneer and Nobel laureate Hannes Alfven. Consequently, I favoured the paper on Plasma Cosmology presented by Eric Lerner, and as a direct result of that inclination find it very difficult here to be brief! Lerner summarised the basic premises: most of the Universe is plasma, so the effect of electromagnetic force on a cosmic scale is at least comparable to gravitation. Plasma cosmology assumes no origin in time for the Universe, and can therefore accommodate the conservation of energy/matter. Since we see evidence of evolution all around us, we can assume evolution in the Universe, though not at the pace or on the scale of the Big Bang. Lastly, plasma cosmology tries to explain as much of the Universe as possible using known physics, and does not invoke assistance from supernatural elements. Plasmas are scale invariant, so we can safely infer large-scale plasma activity from what we see terrestrially. Gravity acts on filaments, which condense into "blobs" and disks form. As the body contracts, it gets rid of angular momentum which is conducted away by plasma. Lerner's colleague Anthony Peratt of Los Alamos Laboratory modelled plasma interaction on a computer and has arrived

The First Crisis in Cosmology Conference, Monção, Portugal, June 23–25 2005
Schedule of Presentations

Name	Location	Paper Title
Antonio Alfonso-Faus aalfonsofaus@yahoo.es	Madrid Polytech. Univ., Spain	Mass boom vs Big Bang
Carrol Alley coalley@physics.umd.edu	Univ. of Maryland, USA	Going “beyond Einstein” with Yilmaz theory
José Almeida bda@fisica.uminho.pt	Universidade do Minho	Geometric drive of Universal expansion
Thomas Andrews tba@xoba.com	USA	Falsification of the expanding Universe model
Yurij Baryshev yuba@astro.spbu.ru	St. Petersburg Univ., Russia	Conceptual problems of the standard cosmological model
Yurij Baryshev yuba@astro.spbu.ru	St. Petersburg Univ., Russia	Physics of gravitational interaction
Alain Blanchard alain.blanchard@ast.obs-mip.fr	Lab. d’Astrophys. Toulouse, France	The Big Bang picture: a wonderful success of modern science
M. de Campos campos@dfis.ufrr.br	Univ. Federal de Roraima, Brazil	The Dyer-Roeder relation
George Chapline chapline1@llnl.gov	Lawrence Livermore National Lab., USA	Tommy Gold revisited
Mike Disney mike.disney@astro.cf.ac.uk	Univ. of Cardiff, Great Britain	The insignificance of current cosmology
Anne M. Hofmeister and R. E. Criss hofmeist@wustl.edu	Washington Univ., USA	Implications of thermodynamics on cosmologic models
Michael Ibison ibison@earthtech.org	Inst. for Adv. Studies, Austin, USA	The Yilmaz cosmology
Michael Ibison ibison@earthtech.org	Inst. for Adv. Studies, Austin, USA	The steady-state cosmology
Michael Ivanov ivanovma@gw.bsuir.unibel.by	Belarus State Univ., Belarus	Low-energy quantum gravity
Moncy John moncyjohn@yahoo.co.uk	St. Thomas College, India	Decelerating past for the Universe?
Christian Joos and Josef Lutz jooss@ump.gwdg.de; josef.lutz@etit.tu-chemnitz.de	Univ. of Göttingen; Chemnitz Univ., Germany	Quantum redshift
Christian Joos and Josef Lutz jooss@ump.gwdg.de; josef.lutz@etit.tu-chemnitz.de	Univ. of Göttingen; Chemnitz Univ., Germany	Evolution of Universe in high-energy physics
S. P. Leaning		High redshift Supernovae data show no time dilation
Eric Lerner elerner@igc.org	Lawrenceville Plasma Physics, USA	Is the Universe expanding? Some tests of physical geometry

The First Crisis in Cosmology Conference, Monção, Portugal, June 23–25 2005
Schedule of Presentations (*continue*)

Eric Lerner elerner@igc.org	Lawrenceville Plasma Physics, USA	Overview of plasma cosmology
Sergey Levshakov lev@astro.ioffe.rssi.ru	Ioffe Phys. Tech. Inst., St. Petersburg, Russia	The cosmological variability of the fine-structure constant
Martin López-Corredoira martinlc@iac.es	Inst. de Astrofisica de Canarias, Spain	Research on non-cosmological redshifts
Oliver Manuel om@umr.edu	University of Missouri, USA	Isotopes tell Sun's origin and operation
Jaques Moret-Bailly Jacques.Moret-Bailly@u-bourgogne.fr	France	Parametric light-matter interactions
Frank Potter and Howard Preston drpotter@lycos.com	Univ. of California; Preston Research, USA	Large-scale gravitational quantisation states
Eugene Savov eugenesavov@mail.orbitel.bg	Bulgarian Acad. of Sciences	Unique firework Universe and 3-D spiral code
Riccardo Scarpa rscarpa@eso.org	European Southern Observatory, Chile	Modified Newtonian Dynamics: alternative to non-baryonic dark matter
Riccardo Scarpa, Gianni Marconi, and Roberto Gilmozzi rscarpa@eso.org; gmarconi@eso.org; rgilmozz@eso.org	European Southern Observatory, Chile	Using globular clusters to test gravity
Donald Scott dascott2@cox.net	USA	Real properties of magnetism and plasma
Franco Selleri Franco.Selleri@ba.infn.it	Università di Bari, Italy	Absolute simultaneity forbids Big Bang
Glenn Starkman starkman@balin.cwru.edu	Case Western Reserve Univ., USA	Is the low-lambda microwave background cosmic?
Glenn Starkman starkman@balin.cwru.edu	Case Western Reserve Univ., USA	Differentiating between modified gravity and dark energy
Tuomo Suntola tuomo.suntola@sci.fi	Finland	Spherically closed dynamic space
Francesco Sylos Labini	E. Fermi Centre, Italy	Non-linear structures in gravitation and cosmology
Y. P. Varshni ypvsj@uottawa.ca	Univ. of Ottawa, Canada	Common absorption lines in two quasars
Y. P. Varshni, J. Talbot and Z. Ma ypvsj@uottawa.ca	Univ. of Ottawa; Chin. Acad. of Sci. (China)	Peaks in emission lines in the spectra of quasars
Thomas van Flandern tomvf@metaresearch.org	Meta Research, USA	Top problems with Big Bang: the light elements
Mogens Wegener mwegener@aarhusmail.dk	University of Aarhus, Denmark	Kinematic cosmology
Huseyin Yilmaz	Princeton Univ., USA	Beyond Einstein

at a compelling simulation of the morphogenesis of galaxies. Since plasma cosmology has no time constraints, the development of large-scale structures — so problematic for Big Bang — is accommodated. Lerner admits that there's still a lot of work to be done, but with the prospect of more research funding coming our way, he foresees the tidying up of the theory into a workable cosmological model.

Dr. Alain Blanchard of the Laboratoire d'Astrophysique in Toulouse had come to CCC-1 explicitly to defend Big Bang, and he did so admirably. My fears that the inclusion of a single speaker against the motion might amount to mere tokenism were entirely unfounded. Despite the fact that many of us disagreed with much of what he said, he acquitted himself most competently and I would say ended up making a number of good friends at the conference. Two quotes from Dr. Blanchard: "*We are all scientists, and we all want to progress. Where we differ is in our own prejudice*." "*When you do an experiment, you can get a 'yes' or 'no' answer from your equipment. When you work with astrophysical data, you are dealing with an altogether more complex situation, infused with unknowns*."

No account of CCC-1 would be near complete without a summary of a paper that caught all of us by complete surprise. Professor Oliver Manuel is not an astronomer. Nor indeed is he a physicist. He is a nuclear chemist, chairman of the Department of Chemistry at the University of Missouri, and held in high enough esteem to be one of a handful of scientists entrusted with the job of analysing Moon rock brought back by the Apollo missions. His "telescope" is a mass spectrometer, and he uses it to identify and track isotopes in the terrestrial neighbourhood. His conclusions are astonishing, yet I can find no fault with his arguments. The hard facts that emerge from Professor Manuel's study indicate that the chemical composition of the Sun beneath the photosphere is predominantly iron! Manuel's thesis has passed peer review in several mainstream journals, including *Nature*, *Science*, and the *Journal of Nuclear Fusion*. He derives a completely revolutionary Solar Model, one which spells big trouble for BBN. Subsequent investigation has shown that it is likely to represent a major paradigm shift in solar physics, and has implications also for the field of nuclear chemistry. He makes the following claims:

1. The chemical composition of the Sun is predominantly iron.
2. The energy of the Sun is *not* derived from nuclear fusion, but rather from neutron repulsion.
3. The Sun has a solid, electrically conducting ferrite surface beneath the photosphere, and rotates uniformly at all latitudes.
4. The solar system originated from a supernova about 5 billion years ago, and the Sun formed from the neutron star that remained.

Manuel's study contains much more than the sample points mentioned above. Data freely available from NASA's SOHO and TRACE satellites graphically and unambiguously support Manuel's contentions (to the extent of images illustrating fixed surface formations revolving with a period of 27.3 days), and suggest that the standard Solar Model is grossly inaccurate. The implications, if Manuel's ideas are validated, are exciting indeed. His words: "*The question is, are neutron stars 'dead' nuclear matter, with tightly bound neutrons at minus 93 MeV relative to the free neutron, as widely believed? Or are neutron stars the greatest known source of nuclear energy, with neutrons at plus 10 to 22 MeV relative to free neutrons, as we conclude from the properties of the 2,850 known isotopes*?"

The conference concluded with a stirring concert by a 3-piece baroque chamber music ensemble, and it gave me cause to reflect that it appeared that only in our appreciation of music did we find undiluted harmony. That the Big Bang theory will pass into history as an artefact of man's obsession with dogma is a certainty; it will do so on its own merits, however, because it stands on feet of clay. For a viable replacement theory to emerge solely from the efforts of the Alternative Cosmology Group is unlikely unless the group can soon find cohesive direction, and put into practice the undertaking that we become completely interdisciplinary in our approach. Nonetheless, that there is a crisis in the world of science is now confirmed. Papers presented at the conference by some of the world's leading scientists showed beyond doubt that the weight of scientific evidence clearly indicates that the dominant theory on the origin and destiny of the Universe is deeply flawed. The implications of this damning consensus are serious indeed, and will in time fundamentally affect not only the direction of many scientific disciplines, but also threaten to change the very way that we do science.

The Michelson and Morley 1887 Experiment and the Discovery of Absolute Motion

Reginald T. Cahill

School of Chemistry, Physics and Earth Sciences, Flinders University, Adelaide 5001, Australia

E-mail: Reg.Cahill@flinders.edu.au

Physics textbooks assert that in the famous interferometer 1887 experiment to detect absolute motion Michelson and Morley saw no rotation-induced fringe shifts — the signature of absolute motion; it was a null experiment. However this is incorrect. Their published data revealed to them the expected fringe shifts, but that data gave a speed of some 8 km/s using a Newtonian theory for the calibration of the interferometer, and so was rejected by them solely because it was less than the 30 km/s orbital speed of the Earth. A 2002 post relativistic-effects analysis for the operation of the device however gives a different calibration leading to a speed > 300 km/s. So this experiment detected both absolute motion and the breakdown of Newtonian physics. So far another six experiments have confirmed this first detection of absolute motion in 1887.

1 Introduction

The first detection of absolute motion, that is motion relative to space itself, was actually by Michelson and Morley in 1887 [1]. However they totally bungled the reporting of their own data, an achievement that Michelson managed again and again throughout his life-long search for experimental evidence of absolute motion.

The Michelson interferometer was a brilliantly conceived instrument for the detection of absolute motion, but only in 2002 [2] was its principle of operation finally understood and used to analyse, for the first time ever, the data from the 1887 experiment, despite the enormous impact of that experiment on the foundations of physics, particularly as they were laid down by Einstein. So great was Einstein's influence that the 1887 data was never re-analysed post-1905 using a proper relativistic-effects based theory for the interferometer. For that reason modern-day vacuum Michelson interferometer experiments, as for example in [3], are badly conceived, and their null results continue to cause much confusion: only a Michelson interferometer in gas-mode can detect absolute motion, as we now see. So as better and better vacuum interferometers were developed over the last 70 years the rotation-induced fringe shift signature of absolute motion became smaller and smaller. But what went unnoticed until 2002 was that the gas in the interferometer was a key component of this instrument when used as an "absolute motion detector", and over time the experimental physicists were using instruments with less and less sensitivity; and in recent years they had finally perfected a totally dud instrument. Reports from such experiments claim that absolute motion is not observable, as Einstein had postulated, despite the fact that the apparatus is totally insensitive to absolute motion. It must be emphasised that absolute motion is not inconsistent with the various well-established relativistic effects; indeed the evidence is that absolute motion is the cause of these relativistic effects, a proposal that goes back to Lorentz in the 19th century. Then of course one must use a relativistic theory for the operation of the Michelson interferometer. What also follows from these experiments is that the Einstein-Minkowski spacetime ontology is invalidated, and in particular that Einstein's postulates regarding the invariant speed of light have always been in disagreement with experiment from the beginning. This does not imply that the use of a mathematical spacetime is not permitted; in quantum field theory the mathematical spacetime encodes absolute motion effects. An ongoing confusion in physics is that absolute motion is incompatible with Lorentz symmetry, when the evidence is that it is the cause of that dynamical symmetry.

2 Michelson interferometer

The Michelson interferometer compares the change in the difference between travel times, when the device is rotated, for two coherent beams of light that travel in orthogonal directions between mirrors; the changing time difference being indicated by the shift of the interference fringes during the rotation. This effect is caused by the absolute motion of the device through 3-space with speed v, and that the speed of light is relative to that 3-space, and not relative to the apparatus/observer. However to detect the speed of the apparatus through that 3-space gas must be present in the light paths for purely technical reasons. A theory is required to calibrate this device, and it turns out that the calibration of gas-mode Michelson interferometers was only worked out in 2002. The post relativistic-effects theory for this device is remarkably simple. The Fitzgerald-Lorentz contraction effect causes the arm AB parallel to the absolute velocity to be physically contracted to length

$$L_{||} = L\sqrt{1-\frac{v^2}{c^2}}. \qquad (1)$$

The time t_{AB} to travel AB is set by $Vt_{AB} = L_{||} + vt_{AB}$, while for BA by $Vt_{BA} = L_{||} - vt_{BA}$, where $V = c/n$ is the speed of light, with n the refractive index of the gas present (we ignore here the Fresnel drag effect for simplicity — an effect caused by the gas also being in absolute motion). For the total ABA travel time we then obtain

$$t_{ABA} = t_{AB} + t_{BA} = \frac{2LV}{V^2 - v^2}\sqrt{1-\frac{v^2}{c^2}}\,. \qquad (2)$$

For travel in the AC direction we have, from the Pythagoras theorem for the right-angled triangle in Fig. 1 that $(Vt_{AC})^2 = L^2 + (vt_{AC})^2$ and that $t_{CA} = t_{AC}$. Then for the total ACA travel time

$$t_{ACA} = t_{AC} + t_{CA} = \frac{2L}{\sqrt{V^2 - v^2}}\,. \qquad (3)$$

Then the difference in travel time is

$$\Delta t = \frac{(n^2-1)L}{c}\frac{v^2}{c^2} + \mathrm{O}\left(\frac{v^4}{c^4}\right). \qquad (4)$$

after expanding in powers of v/c (here the sign O means for "order"). This clearly shows that the interferometer can only operate as a detector of absolute motion when not in vacuum ($n = 1$), namely when the light passes through a gas, as in the early experiments (in transparent solids a more complex phenomenon occurs and rotation-induced fringe shifts from absolute motion do not occur). A more general analysis [2, 9, 10], including Fresnel drag, gives

$$\Delta t = k^2 \frac{Lv_P^2}{c^3}\cos\left[2(\theta - \psi)\right], \qquad (5)$$

where $k^2 \approx n(n^2-1)$, while neglect of the Fitzgerald-Lorentz contraction effect gives $k^2 \approx n^3 \approx 1$ for gases, which is essentially the Newtonian calibration that Michelson used. All the rotation-induced fringe shift data from the 1887 Michelson-Morley experiment, as tabulated in [1], is shown in Fig. 2. The existence of this data continues to be denied by the world of physics.

The interferometers are operated with the arms horizontal, as shown by Miller's interferometer in Fig. 3. Then in (5) θ is the azimuth of one arm (relative to the local meridian), while ψ is the azimuth of the absolute motion velocity projected onto the plane of the interferometer, with projected component v_P. Here the Fitzgerald-Lorentz contraction is a real dynamical effect of absolute motion, unlike the Einstein spacetime view that it is merely a spacetime perspective artefact, and whose magnitude depends on the choice of observer. The instrument is operated by rotating at a rate of one rotation over several minutes, and observing the shift in the fringe pattern through a telescope during the rotation.

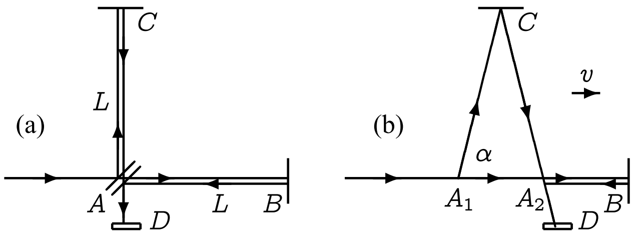


Fig. 1: Schematic diagrams of the Michelson Interferometer, with beamsplitter/mirror at A and mirrors at B and C on arms from A, with the arms of equal length L when at rest. D is the detector screen. In (a) the interferometer is at rest in space. In (b) the interferometer is moving with speed v relative to space in the direction indicated. Interference fringes are observed at D. If the interferometer is rotated in the plane through 90°, the roles of arms AC and AB are interchanged, and during the rotation shifts of the fringes are seen in the case of absolute motion, but only if the apparatus operates in a gas. By measuring fringe shifts the speed v may be determined.

Then fringe shifts from six (Michelson and Morley) or twenty (Miller) successive rotations are averaged, and the average sidereal time noted, giving in the case of Michelson and Morley the data in Fig. 2, or the Miller data like that in Fig. 4. The form in (5) is then fitted to such data, by varying the parameters v_P and ψ. However Michelson and Morley implicitly assumed the Newtonian value $k = 1$, while Miller used an indirect method to estimate the value of k, as he understood that the Newtonian theory was invalid, but had no other theory for the interferometer. Of course the Einstein postulates have that absolute motion has no meaning, and so effectively demands that $k = 0$. Using $k = 1$ gives only a nominal value for v_P, being some 8 km/s for the Michelson and Morley experiment, and some 10 km/s from Miller; the difference arising from the different latitude of Cleveland and Mt. Wilson. The relativistic theory for the calibration of gas-mode interferometers was first used in 2002 [2].

3 Michelson-Morley data

Fig.2 shows all the Michelson and Morley air-mode interferometer fringe shift data, based upon a total of only 36 rotations in July 1887, revealing the nominal speed of some 8 km/s when analysed using the prevailing but incorrect Newtonian theory which has $k = 1$ in (5); and this value was known to Michelson and Morley. Including the Fitzgerald-Lorentz dynamical contraction effect as well as the effect of the gas present as in (5) we find that $n_{air} = 1.00029$ gives $k^2 = 0.00058$ for air, which explains why the observed fringe shifts were so small. We then obtain the speeds shown in Fig. 2. In some cases the data does not have the expected form in (5); because the device was being operated at almost the limit of sensitivity. The remaining fits give a speed in excess of 300 km/s. The often-repeated statement that Michelson and Morley did not see any rotation-induced fringe shifts

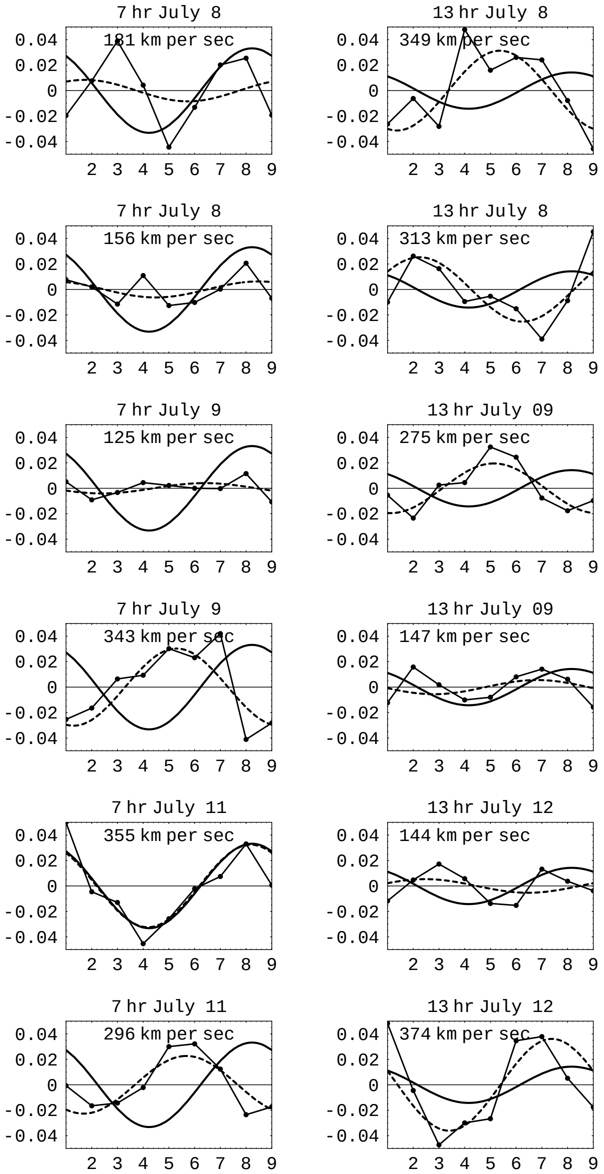


Fig. 2: Shows all the Michelson-Morley 1887 data after removal of the temperature induced linear fringe drifts. The data for each 360° full turn (the average of 6 individual turns) is divided into the 1st and 2nd 180° parts and plotted one above the other. The dotted curve shows a best fit to the data using (5), while the full curves show the expected forms using the Miller direction for **v** and the location and times of the Michelson-Morley observations in Cleveland, Ohio in July, 1887. While the amplitudes are in agreement in general with the Miller based predictions, the phase varies somewhat. Miller also saw a similar effect. This may be related to the Hick's effect [4] when, necessarily, the mirrors are not orthogonal, or may correspond to a genuine fluctuation in the direction of **v** associated with wave effects. We see that this data corresponds to a speed in excess of 300 km/s, and not the 8 km/s reported in [1], which was based on using Newtonian physics to calibrate the interferometer.

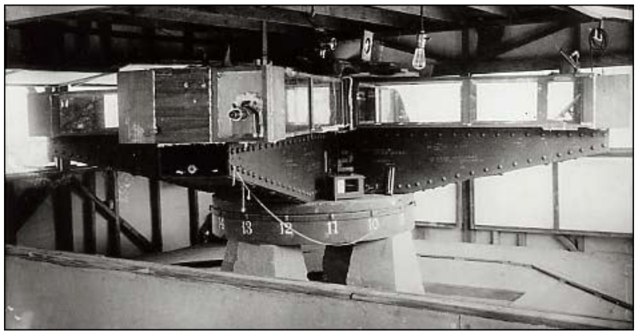


Fig. 3: Miller's interferometer with an effective arm length of $L = 32$ m achieved by multiple reflections. Used by Miller on Mt.Wilson to perform the 1925–1926 observations of absolute motion. The steel arms weighed 1200 kilograms and floated in a tank of 275 kilograms of Mercury. From Case Western Reserve University Archives.

is completely wrong; all physicists should read their paper [1] for a re-education, and indeed their paper has a table of the observed fringe shifts. To get the Michelson-Morley Newtonian based value of some 8 km/s we must multiply the above speeds by $k = \sqrt{0.00058} = 0.0241$. They rejected their own data on the sole but spurious ground that the value of 8 km/s was smaller than the speed of the Earth about the Sun of 30 km/s. What their result really showed was that (i) absolute motion had been detected because fringe shifts of the correct form, as in (5), had been detected, and (ii) that the theory giving $k^2 = 1$ was wrong, that Newtonian physics had failed. Michelson and Morley in 1887 should have announced that the speed of light did depend of the direction of travel, that the speed was relative to an actual physical 3-space. However contrary to their own data they concluded that absolute motion had not been detected. This bungle has had enormous implications for fundamental theories of space and time over the last 100 years, and the resulting confusion is only now being finally corrected.

4 Miller interferometer

It was Miller [4] who saw the flaw in the 1887 paper and realised that the theory for the Michelson interferometer must be wrong. To avoid using that theory Miller introduced the scaling factor k, even though he had no theory for its value. He then used the effect of the changing vector addition of the Earth's orbital velocity and the absolute galactic velocity of the solar system to determine the numerical value of k, because the orbital motion modulated the data, as shown in Fig. 5. By making some 12,000 rotations of the interferometer at Mt. Wilson in 1925/26 Miller determined the first estimate for k and for the absolute linear velocity of the solar system. Fig. 4 shows typical data from averaging the fringe shifts from 20 rotations of the Miller interferometer, performed over a short period of time, and clearly shows the expected

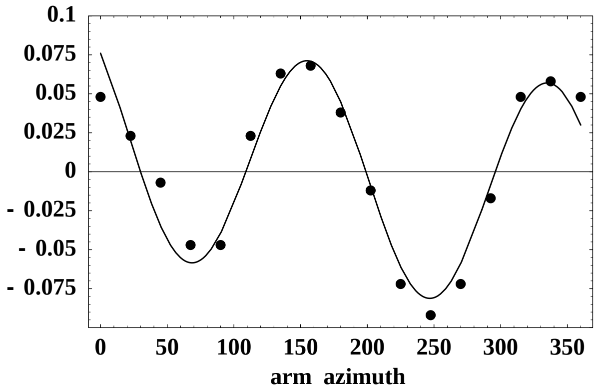


Fig. 4: Typical Miller rotation-induced fringe shifts from average of 20 rotations, measured every 22.5°, in fractions of a wavelength $\Delta\lambda/\lambda$, vs azimuth θ (deg), measured clockwise from North, from Cleveland Sept. 29, 1929 16:24 UT; 11:29 average sidereal time. This shows the quality of the fringe data that Miller obtained, and is considerably better than the comparable data by Michelson and Morley in Fig. 2. The curve is the best fit using the form in (5) but including a Hick's [4] $\cos(\theta-\beta)$ component that is required when the mirrors are not orthogonal, and gives $\psi = 158°$, or 22° measured from South, and a projected speed of $v_P = 351$ km/s. This value for v is different from that in Fig. 2 because of the difference in latitude of Cleveland and Mt. Wilson. This process was repeated some 12,000 times over days and months throughout 1925/1926 giving, in part, the data in Fig. 5.

form in (5) (only a linear drift caused by temperature effects on the arm lengths has been removed — an effect also removed by Michelson and Morley and also by Miller). In Fig. 4 the fringe shifts during rotation are given as fractions of a wavelength, $\Delta\lambda/\lambda = \Delta t/T$, where Δt is given by (5) and T is the period of the light. Such rotation-induced fringe shifts clearly show that the speed of light is different in different directions. The claim that Michelson interferometers, operating in gas-mode, do not produce fringe shifts under rotation is clearly incorrect. But it is that claim that lead to the continuing belief, within physics, that absolute motion had never been detected, and that the speed of light is invariant. The value of ψ from such rotations together lead to plots like those in Fig. 5, which show ψ from the 1925/1926 Miller [4] interferometer data for four different months of the year, from which the RA $= 5.2$ hr is readily apparent. While the orbital motion of the Earth about the Sun slightly affects the RA in each month, and Miller used this effect do determine the value of k, the new theory of gravity required a reanalysis of the data [9, 11], revealing that the solar system has a large observed galactic velocity of some 420±30 km/s in the direction (RA $= 5.2$ hr, Dec $=-67$ deg). This is different from the speed of 369 km/s in the direction (RA $= 11.20$ hr, Dec $=-7.22$ deg) extracted from the Cosmic Microwave Background (CMB) anisotropy, and which describes a motion relative to the distant universe, but not relative to the local 3-space. The Miller velocity is explained

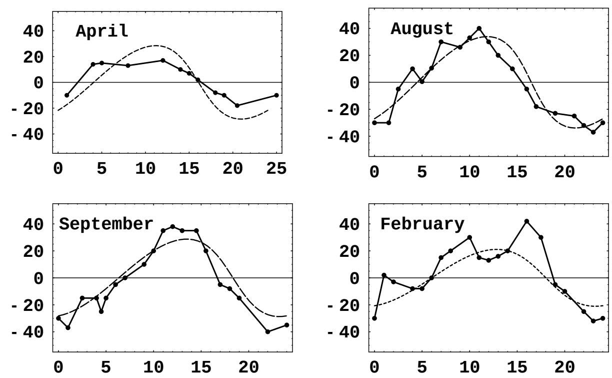


Fig. 5: Miller azimuths ψ, measured from south and plotted against sidereal time in hrs, showing both data and best fit of theory giving $v = 433$ km/s in the direction ($\alpha = 5.2^{\text{hr}}, \delta = -67°$), using $n = 1.000226$ appropriate for the altitude of Mt. Wilson. The variation form month to month arises from the orbital motion of the Earth about the Sun: in different months the vector sum of the galactic velocity of the solar system with the orbital velocity and sun in-flow velocity is different. As shown in Fig. 6 DeWitte using a completely different experiment detected the same direction and speed.

by galactic gravitational in-flows*.

Two old interferometer experiments, by Illingworth [5] and Joos [6], used helium, enabling the refractive index effect to be recently confirmed, because for helium, with $n = 1.000036$, we find that $k^2 = 0.00007$. Until the refractive index effect was taken into account the data from the helium-mode experiments appeared to be inconsistent with the data from the air-mode experiments; now they are seen to be consistent. Ironically helium was introduced in place of air to reduce any possible unwanted effects of a gas, but we now understand the essential role of the gas. The data from an interferometer experiment by Jaseja *et al* [7], using two orthogonal masers with a He-Ne gas mixture, also indicates that they detected absolute motion, but were not aware of that as they used the incorrect Newtonian theory and so considered the fringe shifts to be too small to be real, reminiscent of the same mistake by Michelson and Morley. The Michelson interferometer is a 2nd order device, as the effect of absolute motion is proportional to $(v/c)^2$, as in (5).

5 1st order experiments

However much more sensitive 1st order experiments are also possible. Ideally they simply measure the change in the one-way EM travel-time as the direction of propagation is changed. Fig. 6 shows the North-South orientated coaxial cable Radio Frequency (RF) travel time variations measured by DeWitte in Brussels in 1991 [9, 10, 11], which gives the same RA of absolute motion as found by Miller. That ex-

*See online papers http://www.mountainman.com.au/process_physics/ http://www.scieng.flinders.edu.au/cpes/people/cahill_r/processphysics.html

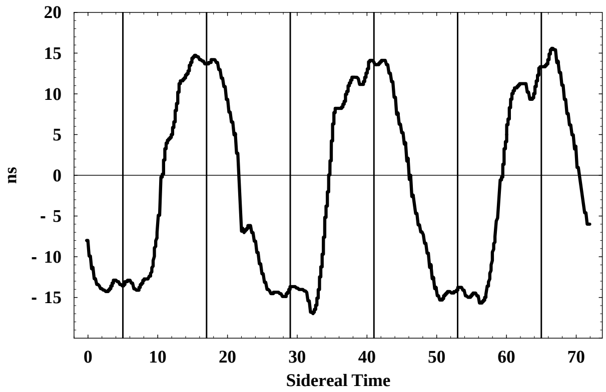


Fig. 6: Variations in twice the one-way travel time, in ns, for an RF signal to travel 1.5 km through a coaxial cable between Rue du Marais and Rue de la Paille, Brussels. An offset has been used such that the average is zero. The cable has a North-South orientation, and the data is the difference of the travel times for NS and SN propagation. The sidereal time for maximum effect of $\sim$5 hr and $\sim$17 hr (indicated by vertical lines) agrees with the direction found by Miller. Plot shows data over 3 sidereal days and is plotted against sidereal time. DeWitte recorded such data from 178 days, and confirmed that the effect tracked sidereal time, and not solar time. Miller also confirmed this sidereal time tracking. The fluctuations are evidence of turbulence in the flow.

periment showed that RF waves travel at speeds determined by the orientation of the cable relative to the Miller direction. That these very different experiments show the same speed and RA of absolute motion is one of the most startling discoveries of the twentieth century. Torr and Kolen [8] using an East-West orientated nitrogen gas-filled coaxial cable also detected absolute motion. It should be noted that analogous optical fibre experiments give null results for the same reason, apparently, that transparent solids in a Michelson interferometer also give null results, and so behave differently to coaxial cables.

Modern resonant-cavity interferometer experiments, for which the analysis leading to (5) is applicable, use vacuum with $n = 1$, and then $k = 0$, predicting no rotation-induced fringe shifts. In analysing the data from these experiments the consequent null effect is misinterpreted, as in [3], to imply the absence of absolute motion. But it is absolute motion which causes the dynamical effects of length contractions, time dilations and other relativistic effects, in accord with Lorentzian interpretation of relativistic effects. The detection of absolute motion is not incompatible with Lorentz symmetry; the contrary belief was postulated by Einstein, and has persisted for over 100 years, since 1905. So far the experimental evidence is that absolute motion and Lorentz symmetry are real and valid phenomena; absolute motion is motion presumably relative to some substructure to space, whereas Lorentz symmetry parameterises dynamical effects caused by the motion of systems through that substructure. There are novel wave phenomena that could also be studied; see footnote on page 28. In order to check Lorentz symmetry we can use vacuum-mode resonant-cavity interferometers, but using gas within the resonant-cavities would enable these devices to detect absolute motion with great precision.

6 Conclusions

So absolute motion was first detected in 1887, and again in at least another six experiments over the last 100 years. Had Michelson and Morley been as astute as their younger colleague Miller, and had been more careful in reporting their *non-null* data, the history of physics over the last 100 years would have totally different, and the spacetime ontology would never have been introduced. That ontology was only mandated by the mistaken belief that absolute motion had not been detected. By the time Miller had sorted out that bungle, the world of physics had adopted the spacetime ontology as a model of reality because that model appeared to be confirmed by many relativistic phenomena, mainly from particle physics, although these phenomena could equally well have been understood using the Lorentzian interpretation which involved no spacetime. We should now understand that in quantum field theory a mathematical spacetime encodes absolute motion effects upon the elementary particle systems, but that there exists a physically observable foliation of that spacetime into a geometrical model of time and a separate geometrical model of 3-space.

References

1. Michelson A. A. and Morley A. A. *Philos. Mag.*, S. 5, 1887, v. 24, No. 151, 449–463.
2. Cahill R. T. and Kitto K. Michelson-Morley experiments revisited and the cosmic background radiation preferred frame. *Apeiron*, 2003, v. 10, No. 2, 104–117.
3. Müller H. *et al.* Modern Michelson-Morley experiment using cryogenic optical resonators. *Phys. Rev. Lett.*, 2003, v. 91(2), 020401-1.
4. Miller D. C. *Rev. Mod. Phys.*, 1933, v. 5, 203–242.
5. Illingworth K. K. *Phys. Rev.*, 1927, v. 3, 692–696.
6. Joos G. *Annalen der Physik*, 1930, Bd. 7, 385.
7. Jaseja T. S. *et al. Phys. Rev. A*, 1964, v. 133, 1221.
8. Torr D. G. and Kolen P. *Precision Measurements and Fundamental Constants*, ed. by Taylor B. N. and Phillips W. D. Nat. Bur. Stand. (U.S.), Spec. Pub., 1984, v. 617, 675.
9. Cahill R.T. *Relativity, Gravitation, Cosmology*, Nova Science Pub., NY, 2004, 168–226.
10. Cahill R. T. Absolute motion and gravitational effects. *Apeiron*, 2004, v. 11, No. 1, 53–111.
11. Cahill R. T. Process Physics: from information theory to quantum space and matter. Nova Science Pub., NY, 2005.

Novel Gravity Probe B Frame-Dragging Effect

Reginald T. Cahill

School of Chemistry, Physics and Earth Sciences, Flinders University, Adelaide 5001, Australia

E-mail: Reg.Cahill@flinders.edu.au

The Gravity Probe B (GP-B) satellite experiment will measure the precession of on-board gyroscopes to extraordinary accuracy. Such precessions are predicted by General Relativity (GR), and one component of this precession is the "frame-dragging" or Lense-Thirring effect, which is caused by the rotation of the Earth. A new theory of gravity, which passes the same extant tests of GR, predicts, however, a second and much larger "frame-dragging" precession. The magnitude and signature of this larger effect is given for comparison with the GP-B data.

1 Introduction

The Gravity Probe B (GP-B) satellite experiment was launched in April 2004. It has the capacity to measure the precession of four on-board gyroscopes to unprecedented accuracy [1, 2, 3, 4]. Such a precession is predicted by the Einstein theory of gravity, General Relativity (GR), with two components (i) a geodetic precession, and (ii) a "frame-dragging" precession known as the Lense-Thirring effect. The latter is particularly interesting effect induced by the rotation of the Earth, and described in GR in terms of a "gravitomagnetic" field. According to GR this smaller effect will give a precession of 0.042 arcsec per year for the GP-B gyroscopes. However a recently developed theory gives a different account of gravity. While agreeing with GR for all the standard tests of GR this theory gives a dynamical account of the so-called "dark matter" effect in spiral galaxies. It also successfully predicts the masses of the black holes found in the globular clusters M15 and G1. Here we show that GR and the new theory make very different predictions for the "frame-dragging" effect, and so the GP-B experiment will be able to decisively test both theories. While predicting the same earth-rotation induced precession, the new theory has an additional much larger "frame-dragging" effect caused by the observed translational motion of the Earth. As well the new theory explains the "frame-dragging" effect in terms of vorticity in a "substratum flow". Herein the magnitude and signature of this new component of the gyroscope precession is predicted for comparison with data from GP-B when it becomes available.

2 Theories of gravity

The Newtonian "inverse square law" for gravity,

$$F = \frac{Gm_1m_2}{r^2}, \tag{1}$$

was based on Kepler's laws for the motion of the planets. Newton formulated gravity in terms of the gravitational acceleration vector field $\mathbf{g}(\mathbf{r},t)$, and in differential form

$$\nabla\cdot\mathbf{g} = -4\pi G\rho\,, \tag{2}$$

where $\rho(\mathbf{r},t)$ is the matter density. However there is an alternative formulation [5] in terms of a vector "flow" field $\mathbf{v}(\mathbf{r},t)$ determined by

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{v}) + \nabla\cdot\big[(\mathbf{v}\cdot\nabla)\,\mathbf{v}\big] = -4\pi G\rho\,, \tag{3}$$

with $\mathbf{g}$ now given by the Euler "fluid" acceleration

$$\mathbf{g} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = \frac{d\mathbf{v}}{dt}\,. \tag{4}$$

Trivially this $\mathbf{g}$ also satisfies (2). External to a spherical mass M of radius R a velocity field solution of (2) is

$$\mathbf{v}(\mathbf{r}) = -\sqrt{\frac{2GM}{r}}\,\hat{\mathbf{r}}\,, \qquad r > R\,, \tag{5}$$

which gives from (4) the usual inverse square law $\mathbf{g}$ field

$$\mathbf{g}(\mathbf{r}) = -\frac{GM}{r^2}\,\hat{\mathbf{r}}\,, \qquad r > R\,. \tag{6}$$

However the flow equation (2) is not uniquely determined by Kepler's laws because

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{v}) + \nabla\cdot\big[(\mathbf{v}\cdot\nabla)\mathbf{v}\big] + C(\mathbf{v}) = -4\pi G\rho\,, \tag{7}$$

where

$$C(\mathbf{v}) = \frac{\alpha}{8}\big[(\mathrm{tr}D)^2 - \mathrm{tr}\,(D^2)\big]\,, \tag{8}$$

and

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{9}$$

also has the same external solution (5), because $C(\mathbf{v}) = 0$ for the flow in (5). So the presence of the $C(\mathbf{v})$ would not have manifested in the special case of planets in orbit about the massive central sun. Here α is a dimensionless constant — a new gravitational constant, in addition to usual the Newtonian gravitational constant G. However inside a spherical mass we find [5] that $C(\mathbf{v}) \neq 0$, and using the Greenland borehole g anomaly data [4] we find that $\alpha^{-1} = 139 \pm 5$, which gives the fine structure constant $\alpha = e^2\hbar/c \approx 1/137$ to within experimental error. From (4) we can write

$$\nabla \cdot \mathbf{g} = -4\pi G\rho - 4\pi G\rho_{DM}\,, \tag{10}$$

where

$$\rho_{DM}(\mathbf{r}) = \frac{\alpha}{32\pi G}\left[(\mathrm{tr}D)^2 - \mathrm{tr}(D^2)\right], \tag{11}$$

which introduces an effective "matter density" representing the flow dynamics associated with the $C(\mathbf{v})$ term. In [5] this dynamical effect is shown to be the "dark matter" effect. The interpretation of the vector flow field $\mathbf{v}$ is that it is a manifestation, at the classical level, of a quantum substratum to space; the flow is a rearrangement of that substratum, and not a flow *through* space. However (7) needs to be further generalised [5] to include vorticity, and also the effect of the motion of matter through this substratum via

$$\mathbf{v}_R\{\mathbf{r}_0(t), t\} = \mathbf{v}_0(t) - \mathbf{v}\{\mathbf{r}_0(t), t\}\,, \tag{12}$$

where $\mathbf{v}_0(t)$ is the velocity of an object, at $\mathbf{r}_0(t)$, relative to the same frame of reference that defines the flow field; then $\mathbf{v}_R$ is the velocity of that matter relative to the substratum. The flow equation (7) is then generalised to, with $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ the Euler fluid or total derivative,

$$\begin{aligned} &\frac{dD_{ij}}{dt} + \frac{\delta_{ij}}{3}\mathrm{tr}(D^2) + \frac{\mathrm{tr}D}{2}\left(D_{ij} - \frac{\delta_{ij}}{3}\mathrm{tr}D\right) + \\ &+ \frac{\delta_{ij}}{3}\frac{\alpha}{8}\left[(\mathrm{tr}D)^2 - \mathrm{tr}(D^2)\right] + (\Omega D - D\Omega)_{ij} = \\ &= -4\pi G\rho\left(\frac{\delta_{ij}}{3} + \frac{v_R^i v_R^j}{2c^2} + \ldots\right), \quad i,j = 1,2,3, \end{aligned} \tag{13}$$

$$\nabla \times (\nabla \times \mathbf{v}) = \frac{8\pi G\rho}{c^2}\mathbf{v}_R\,, \tag{14}$$

$$\begin{aligned} \Omega_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = \\ &= -\frac{1}{2}\epsilon_{ijk}\,\omega_k = -\frac{1}{2}\epsilon_{ijk}(\nabla \times \mathbf{v})_k\,, \end{aligned} \tag{15}$$

and the vorticity vector field is $\vec{\omega} = \nabla \times \mathbf{v}$. For zero vorticity and $v_R \ll c$ (13) reduces to (7). We obtain from (14) the Biot-Savart form for the vorticity

$$\vec{\omega}(\mathbf{r}, t) = \frac{2G}{c^2}\int d^3r'\,\frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|^3}\mathbf{v}_R(\mathbf{r}', t) \times (\mathbf{r} - \mathbf{r}')\,. \tag{16}$$

The path $\mathbf{r}_0(t)$ of an object through this flow is obtained by extremising the relativistic proper time

$$\tau[\mathbf{r}_0] = \int dt\left(1 - \frac{\mathbf{v}_R^2}{c^2}\right)^{1/2} \tag{17}$$

giving, as a generalisation of (4), the acceleration

$$\begin{aligned} \frac{d\mathbf{v}_0}{dt} &= \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] + (\nabla \times \mathbf{v}) \times \mathbf{v}_R - \\ &- \frac{\mathbf{v}_R}{1 - \frac{\mathbf{v}_R^2}{c^2}}\frac{1}{2}\frac{d}{dt}\left(\frac{\mathbf{v}_R^2}{c^2}\right). \end{aligned} \tag{18}$$

Formulating gravity in terms of a flow is probably unfamiliar, but General Relativity (GR) permits an analogous result for metrics of the Panlevé-Gullstrand class [7],

$$d\tau^2 = g_{\mu\nu}\,dx^\mu dx^\nu = dt^2 - \frac{1}{c^2}\left[d\mathbf{r} - \mathbf{v}(\mathbf{r}, t)dt\right]^2. \tag{19}$$

The external-Schwarzschild metric belongs to this class [8], and when expressed in the form of (19) the $\mathbf{v}$ field is identical to (5). Substituting (19) into the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \frac{8\pi G}{c^2}T_{\mu\nu}\,, \tag{20}$$

gives

$$\begin{aligned} G_{00} &= \sum_{i,j=1,2,3} v_i\,\mathcal{G}_{ij}\,v_j - c^2\sum_{j=1,2,3}\mathcal{G}_{0j}\,v_j - \\ &\quad - c^2\sum_{i=1,2,3} v_i\,\mathcal{G}_{i0} + c^2\mathcal{G}_{00}\,, \\ G_{i0} &= -\sum_{j=1,2,3}\mathcal{G}_{ij}\,v_j + c^2\mathcal{G}_{i0}\,, \\ G_{ij} &= \mathcal{G}_{ij}\,, \qquad\qquad i,j = 1,2,3, \end{aligned} \tag{21}$$

where the $\mathcal{G}_{\mu\nu}$ are given by

$$\begin{aligned} \mathcal{G}_{00} &= \frac{1}{2}\left[(\mathrm{tr}D)^2 - \mathrm{tr}(D^2)\right], \\ \mathcal{G}_{i0} &= \mathcal{G}_{0i} = -\frac{1}{2}\left[\nabla \times (\nabla \times \mathbf{v})\right]_i\,, \\ \mathcal{G}_{ij} &= \frac{d}{dt}\left(D_{ij} - \delta_{ij}\,\mathrm{tr}D\right) + \left(D_{ij} - \frac{1}{2}\delta_{ij}\,\mathrm{tr}D\right)\mathrm{tr}D - \\ &\quad - \frac{1}{2}\delta_{ij}\,\mathrm{tr}(D^2) + (\Omega D - D\Omega)_{ij}\,, \quad i,j = 1,2,3 \end{aligned} \tag{22}$$

and so GR also uses the Euler "fluid" derivative, and we obtain a set of equations analogous but not identical to (13)–(14). In vacuum, with $T_{\mu\nu} = 0$, we find that (22) demands that

$$\left[(\mathrm{tr}D)^2 - \mathrm{tr}(D^2)\right] = 0\,. \tag{23}$$

This simply corresponds to the fact that GR does not permit the "dark matter" dynamical effect, namely that $\rho_{DM} = 0$, according to (10). This happens because GR was forced to agree with Newtonian gravity, in the appropriate limits, and that theory also has no such effect. The predictions from (13)–(14) and from (22) for the Gravity Probe B experiment are different, and provide an opportunity to test both gravity theories.

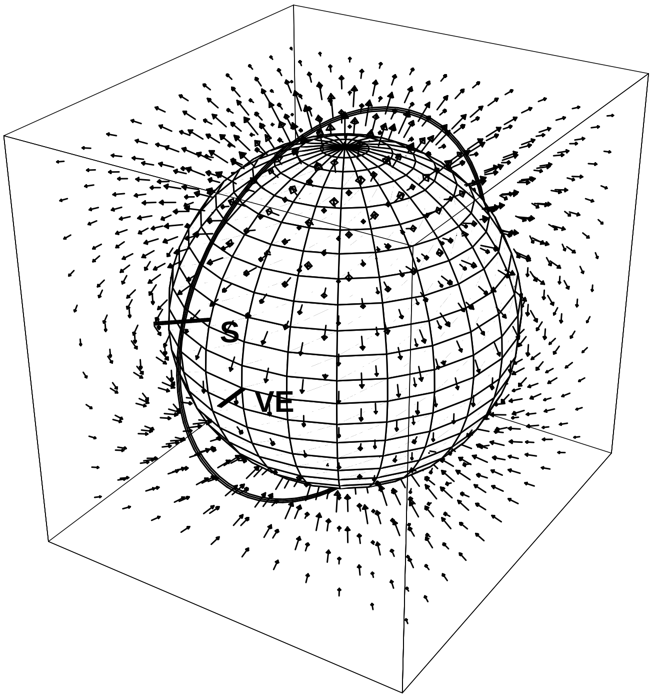


Fig. 1: Shows the Earth (N is up) and vorticity vector field component $\vec{\omega}$ induced by the rotation of the Earth, as in (24). The polar orbit of the GP-B satellite is shown, $\mathbf{S}$ is the gyroscope starting spin orientation, directed towards the guide star IM Pegasi, RA $= 22^{\mathrm{h}}53'2.26''$, Dec $= 16°50'28.2''$, and $\mathbf{VE}$ is the vernal equinox.

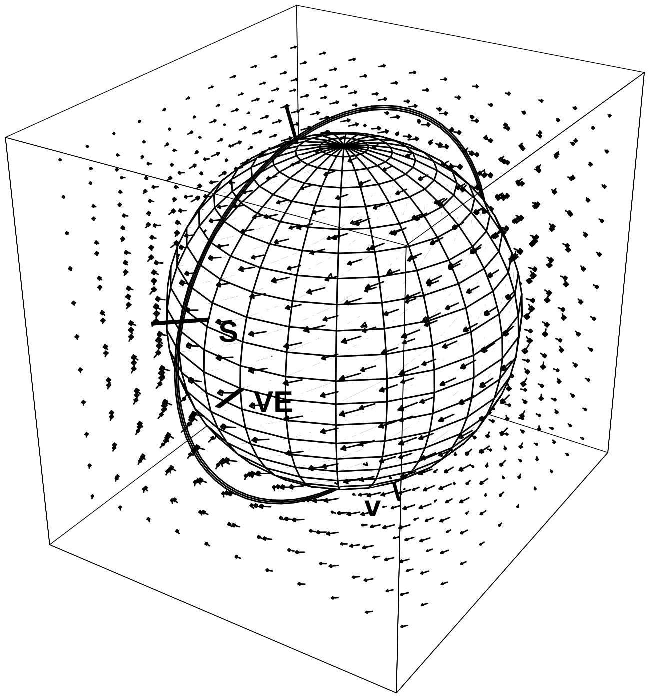


Fig. 2: Shows the Earth (N is up) and the much larger vorticity vector field component $\vec{\omega}$ induced by the translation of the Earth, as in (27). The polar orbit of the GP-B satellite is shown, and $\mathbf{S}$ is the gyroscope starting spin orientation, directed towards the guide star IM Pegasi, RA $= 22^{\mathrm{h}}53'2.26''$, Dec $= 16°50'28.2''$, $\mathbf{VE}$ is the vernal equinox, and $\mathbf{V}$ is the direction RA $= 5.2^{\mathrm{h}}$, Dec $= -67°$ of the translational velocity $\mathbf{v}_c$.

3 "Frame-dragging" as a vorticity effect

Here we consider one difference between the two theories, namely that associated with the vorticity part of (18), leading to the "frame-dragging" or Lense-Thirring effect. In GR the vorticity field is known as the "gravitomagnetic" field $\mathbf{B} = -c\,\vec{\omega}$. In both GR and the new theory the vorticity is given by (16) but with a key difference: in GR $\mathbf{v}_R$ is *only* the rotational velocity of the matter in the Earth, whereas in (13)–(14) $\mathbf{v}_R$ is the vector sum of the rotational velocity and the translational velocity of the Earth through the substratum. At least seven experiments have detected this translational velocity; some were gas-mode Michelson interferometers and others coaxial cable experiments [8, 9, 10], and the translational velocity is now known to be approximately 430 km/s in the direction RA $= 5.2^{\mathrm{h}}$, Dec $= -67°$. This direction has been known since the Miller [11] gas-mode interferometer experiment, but the RA was more recently confirmed by the 1991 DeWitte coaxial cable experiment performed in the Brussels laboratories of Belgacom [9]. This flow is related to galactic gravity flow effects [8, 9, 10], and so is different to that of the velocity of the Earth with respect to the Cosmic Microwave Background (CMB), which is 369 km/s in the direction RA $= 11.20^{\mathrm{h}}$, Dec $= -7.22°$.

First consider the common but much smaller rotation induced "frame-dragging" or vorticity effect. Then $\mathbf{v}_R(\mathbf{r}) = \mathbf{w} \times \mathbf{r}$ in (16), where $\mathbf{w}$ is the angular velocity of the Earth, giving

$$\vec{\omega}(\mathbf{r}) = 4\,\frac{G}{c^2}\,\frac{3(\mathbf{r}\cdot\mathbf{L})\mathbf{r} - r^2\mathbf{L}}{2\,r^5}\,, \qquad (24)$$

where $\mathbf{L}$ is the angular momentum of the Earth, and $\mathbf{r}$ is the distance from the centre. This component of the vorticity field is shown in Fig. 1. Vorticity may be detected by observing the precession of the GP-B gyroscopes. The vorticity term in (18) leads to a torque on the angular momentum $\mathbf{S}$ of the gyroscope,

$$\vec{\tau} = \int d^3r\,\rho(\mathbf{r})\,\mathbf{r} \times \left[\vec{\omega}(\mathbf{r}) \times \mathbf{v}_R(\mathbf{r})\right], \qquad (25)$$

where ρ is its density, and where $\mathbf{v}_R$ is used here to describe the rotation of the gyroscope. Then $d\mathbf{S} = \vec{\tau}dt$ is the change in $\mathbf{S}$ over the time interval dt. In the above case $\mathbf{v}_R(\mathbf{r}) = \mathbf{s} \times \mathbf{r}$, where $\mathbf{s}$ is the angular velocity of the gyroscope. This gives

$$\vec{\tau} = \frac{1}{2}\,\vec{\omega} \times \mathbf{S} \qquad (26)$$

and so $\vec{\omega}/2$ is the instantaneous angular velocity of precession of the gyroscope. This corresponds to the well known fluid

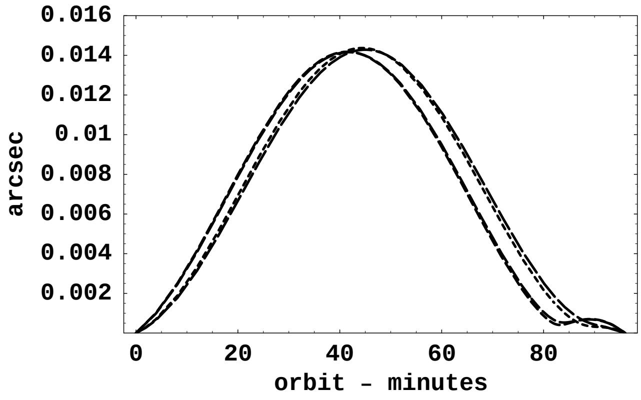


Fig. 3: Predicted variation of the precession angle $\Delta\Theta = |\Delta\mathbf{S}(t)|/|\mathbf{S}(0)|$, in arcsec, over one 97 minute GP-B orbit, from the vorticity induced by the translation of the Earth, as given by (28). The orbit time begins at location **S**. Predictions are for the months of April, August, September and February, labeled by increasing dash length. The "glitches" near 80 minutes are caused by the angle effects in (28). These changes arise from the effects of the changing orbital velocity of the Earth about the Sun. The GP-B expected angle measurement accuracy is 0.0005 arcsec. Novel gravitational waves will affect these plots.

result that the vorticity vector is twice the angular velocity vector. For GP-B the direction of **S** has been chosen so that this precession is cumulative and, on averaging over an orbit, corresponds to some $7.7{\times}10^{-6}$ arcsec per orbit, or 0.042 arcsec per year. GP-B has been superbly engineered so that measurements to a precision of 0.0005 arcsec are possible.

However for the unique translation-induced precession if we use $v_R \approx v_C = 430$ km/s in the direction RA $= 5.2^{\text{hr}}$, Dec $= -67^\circ$, namely ignoring the effects of the orbital motion of the Earth, the observed flow past the Earth towards the Sun, and the flow into the Earth, and effects of the gravitational waves, then (16) gives

$$\vec{\omega}(\mathbf{r}) = \frac{2GM}{c^2}\,\frac{\mathbf{v}_C \times \mathbf{r}}{r^3}\,. \qquad (27)$$

This much larger component of the vorticity field is shown in Fig. 2. The maximum magnitude of the speed of this precession component is $\omega/2 = g v_C/c^2 = 8{\times}10^{-6}$ arcsec/s, where here g is the gravitational acceleration at the altitude of the satellite. This precession has a different signature: it is not cumulative, and is detectable by its variation over each single orbit, as its orbital average is zero, to first approximation. Fig. 3 shows $\Delta\Theta = |\Delta\mathbf{S}(t)|/|\mathbf{S}(0)|$ over one orbit, where, as in general,

$$\Delta\mathbf{S}(t) = \left[\int_0^t dt'\,\frac{1}{2}\,\vec{\omega}\big(\mathbf{r}(t')\big)\right] \times \mathbf{S}(t') \approx \approx \left[\int_0^t dt'\,\frac{1}{2}\,\vec{\omega}\big(\mathbf{r}(t')\big)\right] \times \mathbf{S}(0)\,. \qquad (28)$$

Here $\Delta\mathbf{S}(t)$ is the integrated change in spin, and where the approximation arises because the change in $\mathbf{S}(t')$ on the RHS of (28) is negligible. The plot in Fig. 3 shows this effect to be some $30\times$ larger than the expected GP-B errors, and so easily detectable, if it exists as predicted herein. This precession is about the instantaneous direction of the vorticity $\vec{\omega}\big(\mathbf{r}(t)\big)$ at the location of the satellite, and so is neither in the plane, as for the geodetic precession, nor perpendicular to the plane of the orbit, as for the earth-rotation induced vorticity effect.

Because the yearly orbital rotation of the Earth about the Sun slightly effects $\mathbf{v}_C$ [9] predictions for four months throughout the year are shown in Fig. 3. Such yearly effects were first seen in the Miller [11] experiment.

References

1. Schiff L. I. *Phys. Rev. Lett.*, 1960, v. 4, 215.
2. Van Patten R. A. and Everitt C. W. F. *Phys. Rev. Lett.*, 1976, v. 36, 629.
3. Everitt C. W. F. et al. *Near Zero: Festschrift for William M. Fairbank*, ed. by Everitt C. W. F., Freeman, S. Francisco, 1986.
4. Turneaure J. P., Everitt C. W. F., Parkinson B. W. et al. The Gravity Probe B relativity gyroscope experiment. *Proc. of the Fourth Marcell Grossmann Meeting in General Relativity*, ed. by R. Ruffini, Elsevier, Amsterdam, 1986.
5. Cahill R. T. "Dark matter" as a quantum foam in-flow effect. *Trends In Dark Matter Research*, ed. by Blain J. Val, Nova Science Pub, NY, 2005; Cahill R. T. Gravitation, the "dark matter" effect and the fine structure constant. *Apeiron*, 2005, v. 12, No. 2, 144–177.
6. Ander M. E. et al. *Phys. Rev. Lett.*, 1989, v. 62, 985.
7. Panlevé P. *Com. Rend. Acad. Sci.*, 1921, v. 173, 677; Gullstrand A. *Ark. Mat. Astron. Fys.*, 1922, v. 16, 1.
8. Cahill R. T. Quantum foam, gravity and gravitational waves. *Relativity, Gravitation, Cosmology*, ed. by Dvoeglazov V. V. and Espinoza Garrido A. A. Nova Science Pub., NY, 2004, 168–226.
9. Cahill R. T. Absolute motion and gravitational effects. *Apeiron*, 2004, v. 11, No. 1, 53–111.
10. Cahill R. T. Process Physics: from information theory to quantum space and matter. Nova Science Pub., NY, 2005.
11. Miller D. C. *Rev. Mod. Phys.*, 1993, v. 5, 203–242.

Relations Between Physical Constants

Roberto Oros di Bartini*

This article discusses the main analytic relationship between physical constants, and applications thereof to cosmology. The mathematical bases herein are group theoretical methods and topological methods. From this it is argued that the Universe was born from an Inversion Explosion of the primordial particle (pre-particle) whose outer radius was that of the classical electron, and inner radius was that of the gravitational radius of the electron. All the mass was concentrated in the space between the radii, and was inverted outside the particle through the pre-particle's surface (the inversion classical radius). This inversion process continues today, determining evolutionary changes in the fundamental physical constants.

As is well known, group theoretical methods, and also topological methods, can be effectively employed in order to interpret physical problems. We know of studies setting up the discrete interior of space-time, and also relationships between atomic quantities and cosmological quantities.

However, no analytic relationship between fundamental physical quantities has been found. They are determined only by experimental means, because there is no theory that could give a theoretical determination of them.

Roberto di Bartini, 1920's
(*in Italian Air Force uniform*)

In this brief article we give the results of our own study, which, employing group theoretical methods and topological methods, gives an analytic relationship between physical constants.

Let us consider a predicative unbounded and hence unique specimen A. Establishing an identity between this specimen A and itself

$$A \equiv A\,, \qquad A\frac{1}{A} = 1\,,$$

is the mapping which transfers images of A in accordance with the pre-image of A.

The specimen A, by definition, can be associated only with itself. For this reason it's inner mapping can, according to Stoilow's theorem, be represented as the superposition of a topological mapping and subsequently by an analytic mapping.

The population of images of A is a point-containing system, whose elements are equivalent points; an n-dimensional affine spread, containing $(n+1)$-elements of the system, transforms into itself in linear manner

$$x'_i = \sum_{k=1}^{n+1} a_{ik} x_k\,.$$

With all a_{ik} real numbers, the unitary transformation

$$\sum_k a^*_{ik} a_{lk} = \sum_k a^*_{ki} a_{kl}\,, \qquad i,k = 1,2,3\ldots,n+1\,,$$

is orthogonal, because $\det a_{ik} = \pm 1$. Hence, this transformation is rotational or, in other words, an inversion twist.

A projective space, containing a population of all images of the object A, can be metrizable. The metric spread R^n (coinciding completely with the projective spread) is closed, according to Hamel's theorem.

A coincidence group of points, drawing elements of the set of images of the object A, is a finite symmetric system, which can be considered as a topological spread mapped into the spherical space R^n. The surface of an $(n+1)$-dimensional sphere, being equivalent to the volume of an n-dimensional torus, is completely and everywhere densely filled by the n-dimensional excellent, closed and finite point-containing system of images of the object A.

The dimension of the spread R^n, which consists only of the set of elements of the system, can be any integer n inside the interval $(1-N)$ to $(N-1)$ where N is the number of entities in the ensemble.

We are going to consider sequences of stochastic transitions between different dimension spreads as stochastic vector

*Brief contents of this paper was presented by Prof. Bruno Pontecorvo to the Proceedings of the Academy of Sciences of the USSR (*Doklady Acad. Sci. USSR*), where it was published in 1965 [19]. Roberto di Bartini (1897–1974), the author, was an Italian mathematician and aircraft engineer who, from 1923, worked in the USSR where he headed an aircraft project bureau. Because di Bartini attached great importance to this article, he signed it with his full name, including his titular prefix and baronial name Oros — from Orosti, the patrimony near Fiume (now Rijeka, located in Croatian territory near the border), although he regularly signed papers as Roberto Bartini. The limited space in the Proceedings did not permit publication of the whole article. For this reason Pontecorvo acquainted di Bartini with Prof. Kyril Stanyukovich, who published this article in his bulletin, in Russian. Pontecorvo and Stanyukovich regarded di Bartini's paper highly. Decades later Stanyukovich suggested that it would be a good idea to publish di Bartini's article in English, because of the great importance of his idea of applying topological methods to cosmology and the results he obtained. (Translated by D. Rabounski and S. J. Crothers.) — Editor's remark.

quantities, i. e. as fields. Then, given a distribution function for frequencies of the stochastic transitions dependent on n, we can find the most probable number of the dimension of the ensemble in the following way.

Let the differential function of distribution of frequencies ν in the spectra of the transitions be given by

$$\varphi(\nu) = \nu^n \exp[-\pi\nu^2]\,.$$

If $n \gg 1$, the mathematical expectation for the frequency of a transition from a state n is equal to

$$m(\nu) = \frac{\displaystyle\int_0^\infty \nu^n \exp[-\pi\nu^2]\,d\nu}{\displaystyle 2\int_0^\infty \exp[-\pi\nu^2]\,d\nu} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{\frac{n+1}{2}}}\,.$$

The statistical weight of the time duration for a given state is a quantity inversely proportional to the probability of this state to be changed. For this reason the most probable dimension of the ensemble is that number n under which the function $m(\nu)$ has its minimum.

The inverse function of $m(\nu)$, is

$$\Phi_n = \frac{1}{m(\nu)} = S_{(n+1)} = {}_{\rm T}V_n\,,$$

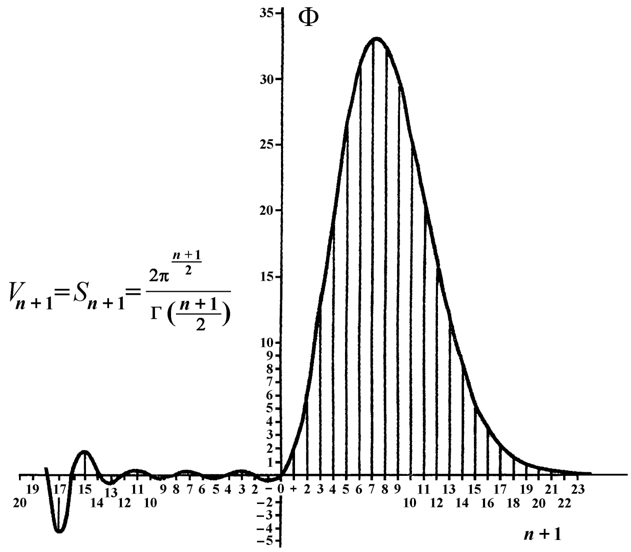
where the function Φ_n is isomorphic to the function of the surface's value $S_{(n+1)}$ of a unit radius hypersphere located in an $(n+1)$-dimensional space (this value is equal to the volume of an n-dimensional hypertorus). This isomorphism is adequate for the ergodic concept, according to which the spatial and time spreads are equivalent aspects of a manifold. So, this isomorphism shows that realization of the object A as a configuration (a form of its real existence) proceeds from the objective probability of the existence of this form.

The positive branch of the function Φ_n is unimodal; for negative values of $(n+1)$ this function becomes sign-alternating (see the figure).

The formation takes its maximum length when $n = \pm 6$, hence the most probable and most unprobable extremal distributions of primary images of the object A are presented in the 6-dimensional closed configuration: the existence of the total specimen A we are considering is 6-dimensional.

Closure of this configuration is expressed by the finitude of the volume of the states, and also the symmetry of distribution inside the volume.

Any even-dimensional space can be considered as the product of two odd-dimensional spreads, which, having the same odd-dimension and the opposite directions, are embedded within each other. Any spherical formation of n dimensions is directed in spaces of $(n+1)$ and higher dimensions. Any odd-dimensional projective space, if immersed in its own dimensions, becomes directed, while any even-dimensional projective space is one-sided. Thus the form of the real existence of the object A we are considering is a $(3+3)$-dimensional complex formation, which is the product of the 3-dimensional spatial-like and 3-dimensional time-like spreads (each of them has its own direction in the $(3+3)$-dimensional complex formation).



One of the main concepts in dimension theory and combinatorial topology is nerve. Using this term, we come to the statement that any compact metric space of n dimensions can be mapped homeomorphicly into a subset located in a Euclidean space of $(2n+1)$ dimensions. And conversely, any compact metric space of $(2n+1)$ dimensions can be mapped homeomorphicly way into a subset of n dimensions. There is a unique correspondence between the mapping $7 \to 3$ and the mapping $3 \to 7$, which consists of the geometrical realization of the abstract complex A.

The geometry of the aforementioned manifolds is determined by their own metrics, which, being set up inside them, determines the quadratic interval

$$\Delta s^2 = \Phi_n^2 \sum_{ik}^{n} g_{ik}\,\Delta x^i \Delta x^k, \qquad i,k = 1,2,\ldots,n\,,$$

which depends not only on the function g_{ik} of coordinates i and k, but also on the function of the number of independent parameters Φ_n.

The total length of a manifold is finite and constant, hence the sum of the lengths of all formations, realized in the manifold, is a quantity invariant with respect to orthogonal transformations. Invariance of the total length of the formation is expressed by the quadratic form

$$N_i r_i^2 = N_k r_k^2\,,$$

where N is the number of entities, r is the radial equivalent of the formation. From here we see, the ratio of the radii is

$$\frac{R\rho}{r^2} = 1\,,$$

where R is the largest radius; ρ is the smallest radius, realised in the area of the transformation; r is the radius of spherical inversion of the formation (this is the calibre of the area). The transformation areas are included in each other, the inversion twist inside them is cascaded

$$\sqrt{\frac{Rr}{2\pi}} = R_e\,, \qquad \sqrt{R\rho} = r\,, \qquad \sqrt{\frac{r\rho}{2\pi}} = \rho_e\,.$$

Negative-dimensional configurations are inversion images, corresponding to anti-states of the system. They have mirror symmetry if $n = l(2m-1)$ and direct symmetry if $n = 2(2m)$, where $m = 1, 2, 3$. Odd-dimensional configurations have no anti-states. The volume of the anti-states is

$$V_{(-n)} = 4\,\frac{-1}{V_n}\,.$$

Equations of physics take a simple form if we use the LT kinematic system of units, whose units are two aspects l and t of the radius through which areas of the space R^n undergo inversion: l is the element of the spatial-like spread of the subspace L, and t is the element of time-like spread of the subspace T. Introducing homogeneous coordinates permits reduction of projective geometry theorems to algebraic equivalents, and geometrical relations to kinematic relations.

The kinematic equivalent of the formation corresponds the following model.

An elementary $(3+3)$-dimensional image of the object A can be considered as a wave or a rotating oscillator, which, in turn, becomes the sink and source, produced by the singularity of the transformation. There in the oscillator polarization of the background components occurs — the transformation $L \to T$ or $T \to L$, depending on the direction of the oscillator, which makes branching L and T spreads. The transmutation $L \leftrightarrow T$ corresponds the shift of the field vector at $\pi/2$ in its parallel transfer along closed arcs of radii R and r in the affine coherence space R^n.

The effective abundance of the pole is

$$e = \frac{1}{2}\,\frac{1}{4\pi}\int_s E ds\,.$$

A charge is an elementary oscillator, making a field around itself and inside itself. There in the field a vector's length depends only on the distance r_i or $1/r_i$ from the centre of the peculiarity. The inner field is the inversion map of the outer field; the mutual correspondence between the outer spatial-like and the inner time-like spreads leads to torsion of the field.

The product of the space of the spherical surface and the strength in the surface is independent of r_i; this value depends only on properties of the charge q

$$4\pi q = S\dot{V} = 4\pi r^2 \frac{d^2 l}{dt^2}\,.$$

Because the charge manifests in the spread R^n only as the strength of its field, and both parts of the equations are equivalent, we can use the right side of the equation instead of the left one.

The field vector takes its ultimate value

$$c = \frac{l}{t} = \sqrt{\frac{S\dot{V}}{4\pi r_i}} = 1$$

in the surface of the inversion sphere with the radius r. The ultimate value of the field strength lt^{-2} takes a place in the same surface; $\nu = t^{-1}$ is the fundamental frequency of the oscillator. The effective (half) product of the sphere surface space and the oscillation acceleration equals the value of the pulsating charge, hence

$$4\pi q = \frac{1}{2}\,4\pi\nu r_i^2 \frac{l}{t} = 2\pi r_i c^2.$$

In LT kinematic system of units the dimension of a charge (both gravitational and electric) is

$$\dim m = \dim e = L^3 T^{-2}.$$

In the kinematic system LT, exponents in structural formulae of dimensions of all physical quantities, including electromagnetic quantities, are integers.

Denoting the fundamental ratio l/t as C, in the kinematic system LT we obtain the generalized structural formula for physical quantities

$$D^{\Sigma n} = c^{\gamma} T^{n-\gamma},$$

where $D^{\Sigma n}$ is the dimensional volume of a given physical quantity, Σn is the sum of exponents in the formula of dimensions (see above), T is the radical of dimensions, n and γ are integers.

Thus we calculate dimensions of physical quantities in the kinematic LT system of units (see Table 1).

Physical constants are expressed by some relations in the geometry of the ensemble, reduced to kinematic structures. The kinematic structures are aspects of the probability and configuration realization of the abstract complex A. The most stable form of a kinematic state corresponds to the most probable form of the stochastic existence of the formation.

The value of any physical constant can be obtained in the following way.

The maximum value of the probability of the state we are considering is the same as the volume of a 6-dimensional torus,

$$V_6 = \frac{16\pi^3}{15}\, r^3 = 33.0733588\, r^6.$$

The extreme numerical values — the maximum of the positive branch and the minimum of the negative branches of the function Φ_n are collected in Table 2.

Table 1

Parameter	Σn	Quantity $D^{\Sigma n}$, taken under γ equal to:							
		5	4	3	2	1	0	−1	−2
		C^5T^{n-5}	C^4T^{n-4}	C^3T^{n-3}	C^2T^{n-2}	C^1T^{n-1}	C^0T^{n-0}	$C^{-1}T^{n+1}$	$C^{-2}T^{n+2}$
Surface power				L^3T^{-5}					
Pressure					L^2T^{-4}				
Current density	−2					L^1T^{-3}			
Mass density, angular acceleration							L^0T^{-2}		
Volume charge density								$L^{-1}T^{-1}$	
Electromagnetic field strength					L^2T^{-3}				
Magnetic displacement, acceleration	−1					L^1T^{-2}			
Frequency							L^0T^{-1}		
Power		L^5T^{-5}							
Force			L^4T^{-4}						
Current, loss mass				L^3T^{-3}					
Potential difference	0				L^2T^{-2}				
Velocity						L^1T^{-1}			
Dimensionless constants							L^0T^0		
Conductivity								$L^{-1}T^1$	
Magnetic permittivity									$L^{-2}T^2$
Force momentum, energy		L^5T^{-4}							
Motion quantity, impulse			L^4T^{-3}						
Mass, quantity of magnetism or electricity	+1			L^3T^{-2}					
Two-dimensional abundance					L^2T^{-1}				
Length, capacity, self-induction						L^1T^0			
Period, duration							L^0T^1		
Angular momentum, action		L^5T^{-3}							
Magnetic momentum			L^4T^{-2}						
Loss volume	+2			L^3T^{-1}					
Surface					L^2T^0				
						L^1T^1			
							L^0T^2		
Moment of inertia		L^5T^{-2}							
			L^4T^{-1}						
Volume of space	+3			L^3T^0					
Volume of time							L^0T^3		

Table 2

$n+1$	$+7.256946404$	-4.99128410
S_{n+1}	$+33.161194485$	-0.1209542108

The ratio between the ultimate values of the function S_{n+1} is

$$\bar{E} = \frac{\left|+S_{(n+1)_{max}}\right|}{\left|-S_{(n+1)_{min}}\right|} = 274.163208\, r^{12}.$$

On the other hand, a finite length of a spherical layer of R^n, homogeneously and everywhere densely filled by doublets of the elementary formations A, is equivalent to a vortical torus, concentric with the spherical layer. The mirror image of the layer is another concentric homogeneous double layer, which, in turn, is equivalent to a vortical torus coaxial with the first one. Such formations were studied by Lewis and Larmore for the $(3+1)$-dimensional case.

Conditions of stationary vortical motion are realized if

$$V \times \operatorname{rot} V = \operatorname{grad} \varphi\,, \qquad 2v ds = d\Gamma\,,$$

where φ is the potential of the circulation, Γ is the main kinematic invariant of the field. A vortical motion is stable only if the current lines coincide with the trajectory of the vortex core. For a $(3+1)$-dimensional vortical torus we have

$$V_x = \frac{\Gamma}{2\pi D}\left[\ln\frac{4D}{r} - \frac{1}{4}\right],$$

where r is the radius of the circulation, D is the torus diameter.

The velocity at the centre of the formation is

$$V_{\odot} = \frac{u\pi D}{2r}\,.$$

The condition $V_x = V_{\odot}$, in the case we are considering, is true if $n = 7$

$$\ln\frac{4D}{r} = (2\pi + 0.25014803)\,\frac{2n+1}{2n} = \\ = 2\pi + 0.25014803 + \frac{n}{2n+1} = 7\,,$$

$$\frac{D}{r} = \bar{E} = \frac{1}{4}\,e^{7} = 274.15836\,.$$

In the field of a vortical torus, with Bohr radius of the charge, $r = 0.999\,9028$, the quantity π takes the numerical value $\pi^* = 0.999\,9514\,\pi$. So $E = \frac{1}{4}e^{6.9996968} = 274.074996$. In the LT kinematic system of units, and introducing the relation $B = V_6 E/\pi = 2885.3453$, we express values of all constants by prime relations between E and B

$$K = \delta\, \widetilde{E}^{\alpha}\, \widetilde{B}^{\beta},$$

where δ is equal to a quantized turn, α and β are integers.

Table 3 gives numerical values of physical constants, obtained analytically and experimentally. The appendix gives experimental determinations in units of the CGS system (cm, gramme, sec), because they are conventional quantities, not physical constants.

The fact that the theoretically and experimentally obtained values of physical constants coincide permits us to suppose that all metric properties of the considered total and unique specimen A can be identified as properties of our observed World, so the World is identical to the unique "particle" A. In another paper it will be shown that a $(3+3)$-dimensional structure of space-time can be proven in an experimental way, and also that this 6-dimensional model is free of logical difficulties derived from the $(3+1)$-dimensional concept of the space-time background*.

In the system of units we are using here the gravitational constant is

$$\kappa = \frac{1}{4\pi}\left[\frac{l^0}{t^0}\right].$$

If we convert its dimensions back to the CGS system, so that $G = \left[\frac{l^3}{mt^2}\right]$, appropriate numerical values of the physical quantities will be determined in another form (Column 5 in Table 3). Reduced physical quantities are given in Column 8. Column 9 gives evolutionary changes of the physical quantities with time according to the theory, developed by Stanyukovich [17]†.

The gravitational "constant", according to his theory, increases proportionally to the space radius (and also the world-time) and the number of elementary entities, according to Dirac [18], increases proportional to the square of the space radius (and the square of world-time as well). Therefore we obtain $N = T_m^2 \simeq B^{24}$, hence $B \approx T_m^{\frac{1}{12}}$.

Because $T_m = t_0\omega_0 \simeq 10^{40}$, where $t_0 \simeq 10^{17}$ sec is the space age of our Universe and $\omega_0 = \frac{c}{\rho} = 10^{23}\ \text{sec}^{-1}$ is the frequency of elementary interactions, we obtain $B \simeq 10^{\frac{10}{3}} = = 10^{\frac{1}{3}} \times 1000$.

In this case we obtain $m \sim e^2 \sim \hbar \sim T_m^{-2} \sim B^{-24}$, which is in good agreement with the evolution concept developed by Stanyukovich.

Appendix

Here is a determination of the quantity 1 cm in the CGS system of units. The analytic value of Rydberg constant is

*Roberto di Bartini died before he prepared the second paper. He died sitting at his desk, looking at papers with drawings of vortical tori and draft formulae. According to Professor Stanyukovich, Bartini was not in the habit of keeping many drafts, so unfortunately, we do not know anything about the experimental statement that he planned to provide as the proof to his concept of the $(3+3)$-dimensional space-time background. — D. R.

†Stanyukovich's theory is given in Part II of his book [17]. Here T_{0m} is the world-time moment when a particle (electron, nucleon, etc.) was born, T_m is the world-time moment when we observe the particle. — D. R.

Table 3

Parameter	Notation	Structural formula	$K=\delta E^{\alpha}B^{\beta}$	Analytically obtained numerical values: LT-system of units		Analytically obtained numerical values: CGS-system	Observed numerical values in CGS-system	Structural formula in CGS	Dependence on time
Sommerfeld constant	$1/\alpha$	$1/2E$	$2^{-1}\pi^0E^0B^0$	1.370375×10^{2}	l^0t^0	1.370375×10^{2}	$1.370374\times10^{2}\,\mathrm{cm^0gm^0sec^0}$	$\frac{1}{2}E$	const
Gravitational constant	κ	$1/4\pi F^*$	$2^{-2}\pi^{-1}E^0B^0$	7.986889×10^{-2}	l^0t^0	6.670024×10^{-8}	$6.670\times10^{-8}\,\mathrm{cm^3gm^{-1}sec^{-1}}$	κ	$\kappa\frac{T_m}{T_{0m}}$
Fundamental velocity	c	l/t	$2^0\pi^0E^0B^0$	1.000000×10^{0}	l^1t^{-1}	2.997930×10^{10}	$2.997930\times10^{10}\,\mathrm{cm^1gm^0sec^{-1}}$	C	const
Mass basic ratio	n/m	$2B/\pi$	$2^1\pi^{-1}E^0B^1$	1.836867×10^{3}	l^0t^0	1.836867×10^{3}	$1.83630\times10^{3}\,\mathrm{cm^0gm^0sec^0}$	$\frac{n}{m}$	$\frac{n}{m}\left(\frac{T_m}{T_{0m}}\right)^{\frac{1}{12}}$
Charge basic ratio	e/m	B^6	$2^0\pi^0E^0B^6$	5.770146×10^{20}	l^0t^0	5.273048×10^{17}	$5.273058\times10^{17}\,\mathrm{cm^{\frac{2}{3}}gm^{-2}sec^{\frac{1}{2}}}$	$\frac{e}{\sqrt{\kappa m}}$	$\frac{e}{\sqrt{\kappa m}}\left(\frac{T_m}{T_{0m}}\right)^{\frac{1}{2}}$
Gravitational radius of electron	ρ	$r/2\pi B^{12}$	$2^{-1}\pi^{-1}E^0B^{-12}$	4.7802045×10^{-43}	l^1t^0	1.346990×10^{-55}	$1.348\times10^{-55}\,\mathrm{cm^1gm^0sec^0}$	S	const
Electric radius of electron	ρ_{e}	$r/2\pi B^{6}$	$2^{-1}\pi^{-1}E^0B^{-6}$	2.753248×10^{-21}	l^1t^0	7.772329×10^{-35}	—	S_{e}	$S_{\mathrm{e}}\left(\frac{T_{0m}}{T_m}\right)^{\frac{1}{2}}$
Classical radius of inversion	r	$\sqrt{R\rho}$	$2^0\pi^0E^0B^0$	1.000000×10^{0}	l^1t^0	2.817850×10^{-13}	$2.817850\times10^{-13}\,\mathrm{cm^1gm^0sec^0}$	r	const
Space radius	R	$2\pi B^{12}r$	$2^1\pi^1E^0B^{12}$	2.091961×10^{42}	l^1t^0	5.894831×10^{29}	$10^{29}>10^{28}\,\mathrm{cm^1gm^0sec^0}$	R	$R\frac{T_m}{T_{0m}}$
Electron mass	m	$2\pi\rho c^2$	$2^0\pi^0E^0B^{-12}$	3.003491×10^{-42}	l^3t^{-2}	9.108300×10^{-28}	$9.1083\times10^{-28}\,\mathrm{cm^0gm^1sec^0}$	κm	$\kappa m\frac{T_{0m}}{T_m}$
Nucleon mass	n	$2rc^2/\pi B^{11}$	$2^1\pi^{-1}E^0B^{-11}$	5.517016×10^{-39}	l^3t^{-2}	1.673074×10^{-24}	$1.67239\times10^{-24}\,\mathrm{cm^0gm^1sec^0}$	κn	$\kappa n\left(\frac{T_{0m}}{T_m}\right)^{\frac{11}{12}}$
Electron charge	e	$2\pi\rho_{\mathrm{e}}c^2$	$2^0\pi^0E^0B^{-6}$	1.733058×10^{-21}	l^3t^{-2}	4.802850×10^{-10}	$4.80286\times10^{-10}\,\mathrm{cm^{\frac{3}{2}}gm^{\frac{1}{2}}sec^{-1}}$	$\sqrt{\kappa e}$	$\sqrt{\kappa e}\left(\frac{T_{0m}}{T_m}\right)^{\frac{1}{2}}$
Space mass	M	$2\pi Rc^2$	$2^2\pi^2E^0B^{12}$	1.314417×10^{43}	l^3t^{-2}	3.986064×10^{57}	$10^{57}>10^{56}\,\mathrm{cm^0gm^1sec^0}$	κM	$\kappa M\frac{T_{0m}}{T_m}$
Space period	T	$2\pi B^{12}t$	$2^1\pi^1E^0B^{12}$	2.091961×10^{42}	l^0t^1	1.966300×10^{19}	$10^{19}>10^{17}\,\mathrm{cm^0gm^0sec^1}$	T	$T\frac{T_{0m}}{T_m}$
Space density	γ_{k}	$M/2\pi^2R^3$	$2^{-2}\pi^{-3}E^0B^{-24}$	7.273495×10^{-86}	l^0t^{-2}	9.858261×10^{-34}	$\sim10^{-31}\,\mathrm{cm^{-3}gm^1sec^0}$	$\kappa\gamma_{\mathrm{k}}$	$\kappa\gamma_{\mathrm{k}}\left(\frac{T_{0m}}{T_m}\right)^{2}$
Space action	H	$Mc2\pi R$	$2^4\pi^4E^0B^{24}$	1.727694×10^{86}	l^5t^{-3}	4.426057×10^{98}	— $\mathrm{cm^2gm^1sec^{-1}}$	H	const
Number of actual entities	N	R/ρ	$2^2\pi^2E^0B^{24}$	4.376299×10^{84}	l^0t^0	4.376299×10^{84}	$>10^{82}\,\mathrm{cm^0gm^0sec^0}$	N	$N\frac{T_m^2}{T_{0m}^2}$
Number of primary interactions	A	NT	$2^3\pi^3E^0B^{36}$	9.155046×10^{126}	l^0t^0	9.155046×10^{126}	— $\mathrm{cm^0gm^0sec^0}$	NT	$NM\left(\frac{T_m}{T_{0m}}\right)^{3}$
Planck constant	$\hbar$	$mc\pi Er$	$2^0\pi^1E^1B^{-12}$	2.586100×10^{-39}	l^5t^{-3}	6.625152×10^{-27}	$6.62517\times10^{-27}\,\mathrm{cm^2gm^1sec^{-1}}$	$\kappa\hbar$	$\kappa\hbar\frac{T_{0m}}{T_m}$
Bohr magneton	μ_{b}	$Er^2c^2/4B^6$	$2^{-2}\pi^0E^1B^{-6}$	1.187469×10^{-19}	l^4t^{-2}	9.273128×10^{-21}	$9.2734\times10^{-21}\,\mathrm{cm^{\frac{5}{2}}gm^{\frac{1}{2}}sec^{-1}}$	$\sqrt{\kappa\mu}$	$\sqrt{\kappa\mu}\left(\frac{T_{0m}}{T_m}\right)^{\frac{1}{2}}$
Compton frequency	ν_{c}	$c/2\pi Er$	$2^{-1}\pi^{-1}E^{-1}B^0$	5.806987×10^{-4}	l^0t^{-1}	6.178094×10^{19}	$6.1781\times10^{19}\,\mathrm{cm^0gm^0sec^{-1}}$	$\sqrt{c}$	const

$^*F=E/(E-1)=1.003662$

$[R_\infty] = (1/4\pi E^3)l^{-1} = 3.0922328{\times}10^{-8} l^{-1}$, the experimentally obtained value of the constant is $(R_\infty) = 109737.311 \pm \pm 0.012 \text{cm}^{-1}$. Hence 1 cm is determined in the CGS system as $(R_\infty)/[R_\infty] = 3.5488041{\times}10^{12} l$.

Here is a determination of the quantity 1 sec in the CGS system of units. The analytic value of the fundamental velocity is $[c] = l/t = 1$, the experimentally obtained value of the velocity of light in vacuum is $(c) = 2.997930 \pm \pm 0.0000080{\times}10^{-10} \text{cm}{\times}\text{sec}^{-1}$. Hence 1 sec is determined in the CGS system as $(c)/l\,[c] = 1.0639066{\times}10^{23} t$.

Here is a determination of the quantity 1 gramme in the CGS system of units. The analytic value of the ratio e/mc is $[e/mc] = \widetilde{B}^6 = 5.7701460{\times}10^{20} l^{-1} t$. This quantity, measured in experiments, is $(e/mc) = 1.758897 \pm 0.000032{\times}10^7$ $(\text{cm}{\times}\text{gm}^{-1})^{\frac{1}{2}}$. Hence 1 gramme is determined in the CGS system as $\dfrac{(e/mc)^2}{l[e/mc]^2} = 3.297532510{\times}10^{-15} l^3 t^{-2}$, so CGS' one gramme is 1 gm (CGS) $= 8.351217{\times}10^{-7} \text{cm}^3 \text{sec}^{-2}$ (CS).

References

1. Pauli W. Relativitätstheorie. *Encyclopäedie der mathematischen Wissenschaften*, Band V, Heft IV, Art. 19, 1921 (Pauli W. Theory of Relativity. Pergamon Press, 1958).
2. Eddington A. S. The mathematical theory of relativity, Cambridge University Press, Cambridge, 2nd edition, 1960.
3. Hurewicz W. and Wallman H. Dimension theory. Foreign Literature, Moscow, 1948.
4. Zeivert H. and Threphall W. Topology. GONTI, Moscow, 1938.
5. Chzgen Schen-Schen (Chern S. S.) Complex manifolds. Foreign Literature, Moscow, 1961
6. Pontriagine L. Foundations of combinatory topology. OGIZ, Moscow, 1947.
7. Busemann G. and Kelley P. Projective geometry. Foreign Literature, Moscow, 1957.
8. Mors M. Topological methods in the theory of functions. Foreign Literature, Moscow, 1951.
9. Hilbert D. und Cohn-Vossen S. Anschauliche Geometrie. Springer Verlag, Berlin, 1932 (Hilbert D. and Kon-Fossen S. Obvious geometry. GTTI, Moscow, 1951).
10. Vigner E. The theory of groups. Foreign Literature, Moscow, 1961.
11. Lamb G. Hydrodynamics. GTTI, Moscow, 1947.
12. Madelunge E. The mathematical apparatus in physics. PhysMathGiz, Moscow, 1960.
13. Bartlett M. Introduction into probability processes theory. Foreign Literature, Moscow, 1958.
14. McVittie G. The General Theory of Relativity and cosmology. Foreign Literature, Moscow, 1961.
15. Wheeler D. Gravitation, neutrino, and the Universe. Foreign Literature, Moscow, 1962.
16. Dicke R. *Review of Modern Physics*, 1957, v. 29, No. 3.
17. Stanyukovich K. P. Gravitational field and elementary particles. Nauka, Moscow, 1965.
18. Dirac P. A. M. *Nature*, 1957, v. 139, 323; *Proc. Roy. Soc. A*, 1938, v. 6, 199.
19. Oros di Bartini R. Some relations between physical constants. *Doklady Acad. Nauk USSR*, 1965, v. 163, No. 4, 861–864.

Introducing Distance and Measurement in General Relativity: Changes for the Standard Tests and the Cosmological Large-Scale

Stephen J. Crothers

Sydney, Australia

E-mail: thenarmis@yahoo.com

Relativistic motion in the gravitational field of a massive body is governed by the external metric of a spherically symmetric extended object. Consequently, any solution for the point-mass is inadequate for the treatment of such motions since it pertains to a fictitious object. I therefore develop herein the physics of the standard tests of General Relativity by means of the generalised solution for the field external to a sphere of incompressible homogeneous fluid.

1 Introduction

The orthodox treatment of physics in the vicinity of a massive body is based upon the Hilbert [1] solution for the point-mass, a solution which is neither correct nor due to Schwarzschild [2], as the latter is almost universally claimed.

In previous papers [3, 4] I derived the correct general solution for the point-mass and the point-charge in all their standard configurations, and demonstrated that the Hilbert solution is invalid. The general solution for the point-mass is however, inadequate for any real physical situation since the material point (and also the material point-charge) is a fictitious object, and so quite meaningless. Therefore, I avail myself of the general solution for the external field of a sphere of incompressible homogeneous fluid, obtained in a particular case by K. Schwarzschild [5] and generalised by myself [6] to,

$$ds^2 = \left[\frac{(\sqrt{C_n}-\alpha)}{\sqrt{C_n}}\right] dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n}-\alpha)}\right]\frac{{C'_n}^2}{4C_n}dr^2 - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{1}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \epsilon^n\right)^{\frac{2}{n}},$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3\left|\chi_a - \chi_0\right|\,,$$

$$R_{c_a} = \sqrt{\frac{3}{\kappa\rho_0}}\sin\left|\chi_a - \chi_0\right|\,,$$

$$\epsilon = \sqrt{\frac{3}{\kappa\rho_0}}\left\{\frac{3}{2}\sin^3\left|\chi_a - \chi_0\right| - \frac{9}{4}\cos\left|\chi_a - \chi_0\right|\left[\left|\chi_a - \chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a - \chi_0\right|\right]\right\}^{\frac{1}{3}},$$

$$r_0 \in \Re\,,\ r \in \Re\,,\ n \in \Re^+\,,\ \chi_0 \in \Re\,,\ \chi_a \in \Re\,,$$

$$\arccos\frac{1}{3} < |\chi_a - \chi_0| < \frac{\pi}{2}\,,$$

$$|r_a - r_0| \leqslant |r - r_0| < \infty\,,$$

where ρ_0 is the constant density of the fluid, k^2 is Gauss' gravitational constant, the sign a denotes values at the surface of the sphere, $|\chi - \chi_0|$ parameterizes the radius of curvature of the interior of the sphere centred arbitrarily at χ_0, $|r - r_0|$ is the coordinate radius in the spacetime manifold of Special Relativity which is a parameter space for the gravitational field external to the sphere centred arbitrarily at r_0.

To eliminate the infinite number of coordinate systems admitted by (1), I rewrite the said metric in terms of the only measurable distance in the gravitational field, i .e. the circumference G of a great circle, thus

$$ds^2 = \left(1 - \frac{2\pi\alpha}{G}\right)dt^2 - \left(1 - \frac{2\pi\alpha}{G}\right)^{-1}\frac{dG^2}{4\pi^2} - \frac{G^2}{4\pi^2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,, \tag{2}$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0|\,,$$

$$2\pi\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0| \leqslant G < \infty\,,$$

$$\arccos\frac{1}{3} < |\chi_a - \chi_0| < \frac{\pi}{2}\,.$$

2 Distance and time

According to (1), if t is constant, a three-dimensional manifold results, having the line-element,

$$ds^2 = \left[\frac{\sqrt{C_n}}{(\sqrt{C_n}-\alpha)}\right]\frac{{C'_n}^2}{4C_n}dr^2 + C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{3}$$

If $\alpha = 0$, (1) reduces to the line-element of flat spacetime,

$$ds^2 = dt^2 - dr^2 - |r - r_0|^2 (d\theta^2 + \sin^2\theta d\varphi^2)\,, \qquad (4)$$
$$0 \leqslant |r - r_0| < \infty\,,$$

since then $r_a \equiv r_0$.

The introduction of matter makes $r_a \neq r_0$, owing to the extended nature of a real body, and introduces distortions from the Euclidean in time and distance. The value of α is effectively a measure of this distortion and therefore fixes the spacetime.

When $\alpha = 0$, the distance $D = |r - r_0|$ is the radius of a sphere centred at r_0. If $r_0 = 0$ and $r \geqslant 0$, then $D \equiv r$ and is then both a radius and a coordinate, as is clear from (4).

If r is constant in (3), then $C_n(r) = R_c^2$ is constant, and so (3) becomes,

$$ds^2 = R_c^2 (d\theta^2 + \sin^2\theta d\varphi^2)\,, \qquad (5)$$

which describes a sphere of constant radius R_c embedded in Euclidean space. The infinitesimal tangential distances on (5) are simply,

$$ds = R_c \sqrt{d\theta^2 + \sin^2\theta d\varphi^2}\,.$$

When θ and φ are constant, (3) yields the proper radius,

$$R_p = \int \sqrt{\frac{\sqrt{C_n(r)}}{\sqrt{C_n(r)} - \alpha}\, \frac{C_n'(r)}{2\sqrt{C_n(r)}}}\, dr = \int \sqrt{\frac{\sqrt{C_n(r)}}{\sqrt{C_n(r)} - \alpha}}\, d\sqrt{C_n(r)}\,, \qquad (6)$$

from which it clearly follows that the parameter r does not measure radial distances in the gravitational field.

Integrating (6) gives,

$$R_p(r) = \sqrt{\sqrt{C_n}(\sqrt{C_n} - \alpha)} + \alpha \ln \left| \sqrt{\sqrt{C_n}} + \sqrt{\sqrt{C_n} - \alpha} \right| + K\,,$$
$$K = \text{const}\,,$$

which must satisfy the condition,

$$r \to r_a^{\pm} \Rightarrow R_p \to R_{p_a}^{+}\,,$$

where r_a is the parameter value at the surface of the body and R_{p_a} the indeterminate proper radius of the sphere from outside the sphere. Therefore,

$$R_p(r) = R_{p_a} + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} - \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a} - \alpha\right)} + \alpha \ln \left| \frac{\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}}{\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}} \right|, \qquad (7)$$

which, by the use of (1) and (2), becomes

$$R_p(r) = R_{p_a} + \sqrt{\frac{G}{2\pi}\left(\frac{G}{2\pi} - \alpha\right)} - \sqrt{\sqrt{\frac{3}{\kappa\rho_0}} \sin|\chi_a - \chi_0| \left(\sqrt{\frac{3}{\kappa\rho_0}} \sin|\chi_a - \chi_0| - \alpha\right)} + \alpha \ln \left| \frac{\sqrt{\frac{G}{2\pi}} + \sqrt{\frac{G}{2\pi} - \alpha}}{\sqrt{\sqrt{\frac{3}{\kappa\rho_0}} \sin|\chi_a - \chi_0|} + \sqrt{\sqrt{\frac{3}{\kappa\rho_0}} \sin|\chi_a - \chi_0| - \alpha}} \right|, \qquad (8)$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}} \sin^3|\chi_a - \chi_0|\,.$$

According to (1), the proper time is related to the coordinate time by,

$$d\tau = \sqrt{g_{00}}\, dt = \sqrt{1 - \frac{\alpha}{\sqrt{C_n(r)}}}\, dt\,. \qquad (9)$$

When $\alpha = 0$, $d\tau = dt$ so that proper time and coordinate time are one and the same in flat spacetime. With the introduction of matter, proper time and coordinate time are no longer the same. It is evident from (9) that both τ and t are finite and non-zero, since according to (1),

$$\frac{1}{9} < 1 - \frac{\alpha}{\sqrt{C_n(r_a)}} \leqslant 1 - \frac{\alpha}{\sqrt{C_n(r)}}\,,$$

i.e.

$$\frac{1}{9} < \cos^2|\chi_a - \chi_0| \leqslant 1 - \frac{\alpha}{\sqrt{C_n(r)}}\,,$$

or

$$\frac{1}{3}\, dt \leqslant d\tau \leqslant dt\,,$$

since In the far field, according to (9),

$$\sqrt{C_n(r)} \to \infty \Rightarrow d\tau \to dt\,,$$

recovering flat spacetime asymptotically.

Therefore, if a body falls from rest from a point distant from the gravitating mass, it will reach the surface of the mass in a finite coordinate time and a finite proper time. According to an external observer, time does not stop at the surface of the body, where $dt = 3d\tau$, contrary to the orthodox analysis based upon the fictitious point-mass.

3 Radar sounding

Consider an observer in the field of a massive body. Let the observer have coordinates, $(r_1, \theta_0, \varphi_0)$. Let the coordinates of a small body located between the observer and the massive body along a radial line be $(r_2, \theta_0, \varphi_0)$. Let the observer emit a radar pulse towards the small body. Then by (1),

$$\left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right)dt^2=\left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right)^{-1}\frac{C_n'^2(r)}{4C_n(r)}\,dr^2=$$
$$=\left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right)^{-1}d\sqrt{C_n(r)}^{\,2},$$

so

$$\frac{d\sqrt{C_n(r)}}{dt}=\pm\left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right),$$

or

$$\frac{dr}{dt}=\pm\frac{2\sqrt{C_n(r)}}{C_n'(r)}\left(1-\frac{\alpha}{\sqrt{C_n(r)}}\right).$$

The coordinate time for the pulse to travel to the small body and return to the observer is,

$$\Delta t=-\int_{\sqrt{C_n(r_1)}}^{\sqrt{C_n(r_2)}}\frac{d\sqrt{C_n}}{1-\frac{\alpha}{\sqrt{C_n}}}+\int_{\sqrt{C_n(r_2)}}^{\sqrt{C_n(r_1)}}\frac{d\sqrt{C_n}}{1-\frac{\alpha}{\sqrt{C_n}}}=$$
$$=2\int_{\sqrt{C_n(r_2)}}^{\sqrt{C_n(r_1)}}\frac{d\sqrt{C_n}}{1-\frac{\alpha}{\sqrt{C_n}}}.$$

The proper time lapse is, according to the observer, by formula (1),

$$\Delta\tau=\sqrt{1-\frac{\alpha}{\sqrt{C_n}}}\,dt=2\sqrt{1-\frac{\alpha}{\sqrt{C_n}}}\int_{\sqrt{C_n(r_2)}}^{\sqrt{C_n(r_1)}}\frac{d\sqrt{C_n}}{1-\frac{\alpha}{\sqrt{C_n}}}=$$
$$=2\sqrt{1-\frac{\alpha}{\sqrt{C_n}}}\left(\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}+\alpha\ln\left|\frac{\sqrt{C_n(r_1)}-\alpha}{\sqrt{C_n(r_2)}-\alpha}\right|\right).$$

The proper distance between the observer and the small body is,

$$R_p=\int_{\sqrt{C_n(r_2)}}^{\sqrt{C_n(r_1)}}\sqrt{\frac{\sqrt{C_n}}{\sqrt{C_n}-\alpha}}\,d\sqrt{C_n}$$
$$=\sqrt{\sqrt{C_n(r_1)}\left(\sqrt{C_n(r_1)}-\alpha\right)}-$$
$$-\sqrt{\sqrt{C_n(r_2)}\left(\sqrt{C_n(r_2)}-\alpha\right)}+$$
$$+\alpha\ln\left|\frac{\sqrt{\sqrt{C_n(r_1)}}+\sqrt{\sqrt{C_n(r_1)}-\alpha}}{\sqrt{\sqrt{C_n(r_2)}}+\sqrt{\sqrt{C_n(r_2)}-\alpha}}\right|.$$

Then according to classical theory, the round trip time is

$$\Delta\bar{\tau}=2R_p\,,$$

so $\Delta\tau\neq\Delta\bar{\tau}$.

If $\frac{\alpha}{\sqrt{C_n(r)}}$ is small for

$$\sqrt{C_n(r_2)}<\sqrt{C_n(r)}<\sqrt{C_n(r_1)}\,,$$

then

$$\Delta\tau\approx 2\Bigg[\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}-$$
$$-\frac{\alpha\left(\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}\right)}{2\sqrt{C_n(r_1)}}+\alpha\ln\sqrt{\frac{\sqrt{C_n(r_1)}}{\sqrt{C_n(r_2)}}}\Bigg],$$

$$\Delta\bar{\tau}\approx 2\left[\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}+\frac{\alpha}{2}\ln\sqrt{\frac{\sqrt{C_n(r_1)}}{\sqrt{C_n(r_2)}}}\right].$$

Therefore,

$$\Delta\tau-\Delta\bar{\tau}\approx\alpha\Bigg[\ln\sqrt{\frac{\sqrt{C_n(r_1)}}{\sqrt{C_n(r_2)}}}-$$
$$-\frac{\left(\sqrt{C_n(r_1)}-\sqrt{C_n(r_2)}\right)}{\sqrt{C_n(r_1)}}\Bigg]=$$
$$=\alpha\left(\ln\sqrt{\frac{G_1}{G_2}}-\frac{G_1-G_2}{G_1}\right)=\tag{10}$$
$$=\sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a-\chi_0|\left(\ln\sqrt{\frac{G_1}{G_2}}-\frac{G_1-G_2}{G_1}\right),$$

$$G=G(r)=2\pi\sqrt{C_n(D(r))}\,.$$

Equation (10) gives the time delay for a radar signal in the gravitational field.

4 Spectral shift

Let an emitter of light have coordinates $(t_E, r_E, \theta_E, \varphi_E)$. Let a receiver have coordinates $(t_R, r_R, \theta_R, \varphi_R)$. Let u be an affine parameter along a null geodesic with the values u_E and u_R at emitter and receiver respectively. Then,

$$\left(1-\frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{du}\right)^2=\left(1-\frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{du}\right)^2+$$
$$+C_n\left(\frac{d\theta}{du}\right)^2+C_n\sin^2\theta\left(\frac{d\varphi}{du}\right)^2,$$

so

$$\frac{dt}{du}=\left[\left(1-\frac{\alpha}{\sqrt{C_n}}\right)^{-1}\bar{g}_{ij}\frac{dx^i}{du}\frac{dx^j}{du}\right]^{\frac{1}{2}},$$

where $\bar{g}_{ij} = -g_{ij}$. Then,

$$t_R - t_E = \int_{u_E}^{u_R} \left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1} \bar{g}_{ij} \frac{dx^i}{du} \frac{dx^j}{du} \right]^{\frac{1}{2}} du\,,$$

and so, for spatially fixed emitter and receiver,

$$t_R^{(1)} - t_E^{(1)} = t_R^{(2)} - t_E^{(2)}\,,$$

and therefore,

$$\Delta t_R = t_R^{(2)} - t_R^{(1)} = t_E^{(2)} - t_E^{(1)} = \Delta t_E\,. \tag{11}$$

Now by (1), the proper time is,

$$\Delta\tau_E = \sqrt{1 - \frac{\alpha}{\sqrt{C_n(r_E)}}}\,\Delta t_E\,,$$

and

$$\Delta\tau_R = \sqrt{1 - \frac{\alpha}{\sqrt{C_n(r_R)}}}\,\Delta t_R\,.$$

Then by (11),

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left[\frac{1 - \frac{\alpha}{\sqrt{C_n(r_R)}}}{1 - \frac{\alpha}{\sqrt{C_n(r_E)}}} \right]^{\frac{1}{2}}. \tag{12}$$

If z regular pulses of light are emitted, the emitted and received frequencies are,

$$\nu_E = \frac{z}{\Delta\tau_E}\,, \quad \nu_R = \frac{z}{\Delta\tau_R}\,,$$

so by (12),

$$\frac{\Delta\nu_R}{\Delta\nu_E} = \left[\frac{1 - \frac{\alpha}{\sqrt{C_n(r_E)}}}{1 - \frac{\alpha}{\sqrt{C_n(r_R)}}} \right]^{\frac{1}{2}} \approx$$

$$\approx 1 + \frac{\alpha}{2}\left(\frac{1}{\sqrt{C_n(r_R)}} - \frac{1}{\sqrt{C_n(r_E)}} \right),$$

whence,

$$\frac{\Delta\nu}{\nu_E} = \frac{\nu_R - \nu_E}{\nu_E} \approx \frac{\alpha}{2}\left(\frac{1}{\sqrt{C_n(r_R)}} - \frac{1}{\sqrt{C_n(r_E)}} \right) =$$

$$= \pi\alpha\left(\frac{1}{G_R} - \frac{1}{G_E}\right) =$$

$$= \pi\sqrt{\frac{3}{\kappa\rho_0}}\,\sin^3\left|\chi_a - \chi_0\right|\left(\frac{1}{G_R} - \frac{1}{G_E}\right).$$

5 Advance of the perihelia

Consider the Lagrangian,

$$\begin{aligned} L = {} & \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{d\tau}\right)^2 \right] - \\ & - \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 \right] - \\ & - \frac{1}{2}\left[C_n\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 \right) \right], \end{aligned} \tag{13}$$

where τ is the proper time. Restricting motion, without loss of generality, to the equatorial plane, $\theta = \frac{\pi}{2}$, the Euler-Lagrange equations for (13) are,

$$\begin{aligned} & \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1} \frac{d^2\sqrt{C_n}}{d\tau^2} + \frac{\alpha}{2C_n}\left(\frac{dt}{d\tau}\right)^2 - \\ & - \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-2} \frac{\alpha}{2C_n}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - \sqrt{C_n}\left(\frac{d\varphi}{d\tau}\right)^2 = 0\,, \end{aligned} \tag{14}$$

$$\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\frac{dt}{d\tau} = \text{const} = K\,, \tag{15}$$

$$C_n \frac{d\varphi}{d\tau} = \text{const} = h\,, \tag{16}$$

and $ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$ becomes,

$$\begin{aligned} & \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{d\tau}\right)^2 - \\ & - \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - C_n\left(\frac{d\varphi}{d\tau}\right)^2 = 1\,. \end{aligned} \tag{17}$$

Rearrange (17) for,

$$\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\frac{\dot{t}^2}{\dot{\varphi}^2} - \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{d\sqrt{C_n}}{d\varphi}\right)^2 - C_n = \frac{1}{\dot{\varphi}^2}\,. \tag{18}$$

Substituting (15) and (16) into (18) gives,

$$\left(\frac{d\sqrt{C_n}}{d\varphi}\right)^2 + C_n\left(1 + \frac{C_n}{h^2}\right)\left(1 - \frac{\alpha}{\sqrt{C_n}}\right) - \frac{K^2}{h^2}C_n^2 = 0\,.$$

Setting $u = \frac{1}{\sqrt{C_n}}$ reduces (18) to,

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = E + \frac{\alpha}{h^2}u + \alpha u^3\,, \tag{19}$$

where $E = \frac{(K^2-1)}{h^2}$. The term αu^3 represents the general-relativistic perturbation of the Newtonian orbit.

Aphelion and perihelion occur when $\frac{du}{d\varphi} = 0$, so by (19),

$$\alpha u^3 - u^2 + \frac{\alpha}{h^2}u + E = 0\,, \tag{20}$$

Let $u = u_1$ at aphelion and $u = u_2$ at perihelion, so $u_1 \leqslant u \leqslant u_2$. One then finds in the usual way that the angle $\Delta\varphi$ between aphelion and subsequent perihelion is,

$$\Delta\varphi = \left[1 + \frac{3\alpha}{4}\left(u_1 + u_2\right)\right]\pi\,.$$

Therefore, the angular advance ψ between successive perihelia is,

$$\psi = \frac{3\alpha\pi}{2}\left(u_1 + u_2\right) = \frac{3\alpha\pi}{2}\left(\frac{1}{\sqrt{C_n(r_1)}} + \frac{1}{\sqrt{C_n(r_2)}}\right) = 3\alpha\pi^2\left(\frac{1}{G_1} + \frac{1}{G_2}\right), \tag{21}$$

where G_1 and G_2 are the measurable circumferences of great circles at aphelion and at perihelion. Thus, to correctly determine the value of ψ, the values of the said circumferences must be ascertained by direct measurement. Only the circumferences are measurable in the gravitational field. The radii of curvature and the proper radii must be calculated from the circumference values.

If the field is weak, as in the case of the Sun, one may take $G \approx 2\pi r$, for r as an approximately "measurable" distance from the gravitating sphere to a spacetime event. In such a situation equation (21) becomes,

$$\psi \approx \frac{3\alpha\pi}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right). \tag{22}$$

In the case of the Sun, $\alpha \approx 3000$ m, and for the planet Mercury, the usual value of $\psi \approx 43$ arcseconds per century is obtained from (22). I emphasize however, that this value is a Euclidean approximation for a weak field. In a strong field equation (22) is entirely inappropriate and equation (21) must be used. Unfortunately, this means that accurate solutions cannot be obtained since there is no obvious way of obtaining the required circumferences in practise. This aspect of Einstein's theory seriously limits its utility. Since the relativists have not detected this limitation the issue has not previously arisen in general.

6 Deflection of light

In the case of a photon, equation (17) becomes,

$$\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - C_n\left(\frac{d\varphi}{d\tau}\right)^2 = 1\,,$$

which leads to,

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = F + \alpha u^3\,. \tag{23}$$

Let the radius of curvature of a great circle at closest approach be $\sqrt{C_n(r_c)}$. Now when there is no mass present, (23) becomes

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = F\,,$$

and has solution,

$$u = u_c \sin\varphi \Rightarrow \sqrt{C_n(r_c)} = \sqrt{C_n(r)}\sin\varphi\,,$$

and

$$u_c^2 = \frac{1}{\sqrt{C_n(r_c)}} = F\,.$$

If

$$\sqrt{C_n(r)} \gg \alpha, \quad \sqrt{C_n(r)} > \sqrt{C_n(r_a)}\,,$$

and $u = u_c > u_a$ at closest approach, then

$$\frac{du}{d\varphi} = 0 \quad \text{at} \quad u = u_c\,,$$

so $F = u_c^2\left(1 - u_c\alpha\right)$, and (23) becomes,

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = u_c^2\left(1 - u_c\alpha\right) + \alpha u^3\,. \tag{24}$$

Equation (24) must have a solution close to flat spacetime, so let

$$u = u_c \sin\varphi + \alpha w(\varphi)\,.$$

Putting this into (24) and working to first order in α, gives

$$2\left(\frac{dw}{d\varphi}\right)\cos\varphi + 2w\sin\varphi = u_c^2\left(\sin^3\varphi - 1\right),$$

or

$$\frac{d}{d\varphi}\left(w\sec\varphi\right) = \frac{1}{2}u_c^2\left(\sec\varphi\tan\varphi - \sin\varphi - \sec^2\varphi\right),$$

and so,

$$w = \frac{1}{2}u_c^2\left(1 + \cos^2\varphi - \sin\varphi\right) + A\cos\varphi\,,$$

where A is an integration constant. If the photon originates at infinity in the direction $\varphi = 0$, then $w(0) = 0$, so $A = -u_c^2$, and

$$u = u_c\left(1 - \frac{1}{2}\alpha u_c\right)\sin\varphi + \frac{1}{2}\alpha u_c^2\left(1 - \cos\varphi\right)^2, \tag{25}$$

to first order in α. Putting $u = 0$ and $\varphi = \pi + \Delta\varphi$ into (25), then to first order in $\Delta\varphi$,

$$0 = -u_c\Delta\varphi + 2\alpha u_c^2\,,$$

so the angle of deflection is,

$$\Delta\varphi = 2\alpha u_c = \frac{2\alpha}{\sqrt{C_n(r_c)}} = \frac{2\alpha}{\left(\left|r_c - r_0\right|^n + \epsilon^n\right)^{\frac{1}{n}}} = \frac{4\pi\alpha}{G_c}\,,$$

$$G_c \geqslant G_a .$$

At a grazing trajectory to the surface of the body,

$$G_c = G_a = 2\pi\sqrt{C_n(r_a)} ,$$

$$\sqrt{C_n(r_a)} = \sqrt{\frac{3}{\kappa\rho_0}} \sin\left|\chi_a - \chi_0\right| ,$$

so then

$$\Delta\varphi = \frac{2\sqrt{\frac{3}{\kappa\rho_0}} \sin^3\left|\chi_a - \chi_0\right|}{\sqrt{\frac{3}{\kappa\rho_0}} \sin\left|\chi_a - \chi_0\right|} = 2\sin^2\left|\chi_a - \chi_0\right| . \quad (26)$$

For the Sun [5],

$$\sin\left|\chi_a - \chi_0\right| \approx \frac{1}{500} ,$$

so the deflection of light grazing the limb of the Sun is,

$$\Delta\varphi \approx \frac{2}{500^2} \approx 1.65'' .$$

Equation (26) is an interesting and quite surprising result, for $\sin\left|\chi_a - \chi_0\right|$ gives the ratio of the "naturally measured" fall velocity of a free test particle falling from rest at infinity down to the surface of the spherical body, to the speed of light in vacuo. Thus,

> *the deflection of light grazing the limb of a spherical gravitating body is twice the square of the ratio of the fall velocity of a free test particle falling from rest at infinity down to the surface, to the speed of light in vacuo, i.e.,*

$$\Delta\varphi = 2\sin^2\left|\chi_a - \chi_0\right| = 2\left(\frac{v_a}{c}\right)^2 = \frac{4GM_g}{c^2 R_{c_a}} ,$$

where R_{c_a} is the radius of curvature of the body, M_g the active mass, and G is the gravitational constant. The quantity v_a is the escape velocity,

$$v_a = \sqrt{\frac{2GM_g}{R_{c_a}}} .$$

7 Practical constraints and general comment

Owing to their invalid assumptions about the r-parameter [7], the relativists have not recognised the practical limitations associated with the application of General Relativity. It is now clear that the fundamental element of distance in the gravitational field is the circumference of a great circle, centred at the heart of an extended spherical body and passing through a spacetime event external thereto. Heretofore the orthodox theorists have incorrectly taken the r-parameter, not just as a radius in the gravitational field, but also as a *measurable* radius in the field. This is not correct. The only measurable distance in the gravitational field is the aforesaid circumference of a great circle, from which the radius of curvature $\sqrt{C_n(r)}$ and the proper radius $R_p(r)$ must be calculated, thus,

$$\sqrt{C_n(r)} = \frac{G}{2\pi} ,$$

$$R_p(r) = \int \sqrt{-g_{11}}\, dr .$$

Only in the weak field, where the spacetime curvature is very small, can $\sqrt{C_n(r)}$ be taken approximately as the Euclidean value r, thereby making $R_p(r) \equiv \sqrt{C_n(r)} \equiv r$, as in flat spacetime. In a strong field this *cannot* be done. Consequently, the problem arises as to how to accurately measure the required great circumference? The correct determination, for example, of the circumferences of great circles at aphelion and perihelion seem to be beyond practical determination. Any method adopted for determining the required circumference must be completely independent of any Euclidean quantity since, other than the great circumference itself, only non-Euclidean distances are valid in the gravitational field, being determined by it. Therefore, anything short of physically measuring the great circumference will fail. Consequently, General Relativity, whether right or wrong as theories go, suffers from a serious practical limitation.

The value of the r-parameter is coordinate dependent and is rightly determined from the coordinate independent value of the circumference of the great circle associated with a spacetime event. One cannot obtain a circumference for the great circle of a given spacetime event, and hence the related radius of curvature and associated proper radius, from the specification of a coordinate radius, because the latter is not unique, being conditioned by arbitrary constants. The coordinate radius is therefore superfluous. It is for this reason that I completely eliminated the coordinate radius from the metric for the gravitational field, to describe the metric in terms of the only quantity that is measurable in the gravitational field — the great circumference (see also [6]). The presence of the r-parameter has proved misleading to the relativists. Stavroulakis [8, 9, 10] has also completely eliminated the r-parameter from the equations, but does not make use of the great circumference. His approach is formally correct, but rather less illuminating, because his resulting line element is in terms of the a quantity which is not measurable in the gravitational field. One cannot obtain an explicit expression for the great circumference in terms of the proper radius.

As to the cosmological large-scale, I have proved elsewhere [11] that General Relativity adds nothing to Special Relativity. Einstein's field equations do not admit of solutions when the cosmological constant is not zero, and they do not admit of the expanding universe solutions alleged by the relativists. The lambda "solutions" and the expanding universe "solutions" are the result of such a muddleheadedness that it is difficult to apprehend the kind of thoughtlessness that gave them birth. Since Special Relativity describes an empty world (no gravity) it cannot form a basis for any cosmology. This theoretical result is all the more interesting owing to its agreement with observation. Arp [12], for instance, has adduced considerable observational data which is consistent on the large-scale with a flat, infinite, non-expanding Universe in Heraclitian flux. Bearing in mind that both Special Relativity and General Relativity *cannot* yield a spacetime on the cosmological "large-scale", there is currently no theoretical replacement for Newton's cosmology, which accords with deep-space observations for a flat space, infinite in time and in extent. The all pervasive rolê given heretofore by the relativists to General Relativity, can be justified no longer. General Relativity is a theory of only *local* phenonomea, as is Special Relativity.

Another serious shortcoming of General Relativity is its current inability to deal with the gravitational interaction of two comparable masses. It is not even known if Einstein's theory admits of configurations involving two or more masses [13]. This shortcoming seems rather self evident, but apprently not so for the relativists, who routinely talk of black hole binary systems and colliding black holes (e.g. [14]), aside of the fact that no theory predicts the existence of black holes to begin with, but to the contrary, precludes them.

Acknowledgement

I am indebted to Dr. Dmitri Rabounski for drawing my attention to a serious error in a preliminary draft of this paper, and for additional helpful suggestions.

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

References

1. Hilbert D. *Nachr. Ges. Wiss. Gottingen, Math. Phys. Kl.*, 1917, 53 (arXiv: physics/0310104, www.geocities.com/theometria/hilbert.pdf).
2. Schwarzschild K. On the gravitational field of a mass point according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 189 (arXiv: physics/9905030, www.geocities.com/theometria/schwarzschild.pdf).
3. Crothers S. J. On the general solution to Einstein's vacuum field and it implications for relativistic degeneracy. *Progress in Physics*, 2005, v. 1, 68–73.
4. Crothers S. J. On the ramifications of the Schwarzschild spacetime metric. *Progress in Physics*, 2005, v. 1, 74–80.
5. Schwarzschild K. On the gravitational field of a sphere of incompressible fluid according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 424 (arXiv: physics/9912033, www.geocities.com/theometria/Schwarzschild2.pdf).
6. Crothers S. J. On the vacuum field of a sphere of incompressible fluid. *Progress in Physics*, 2005, v. 2, 43–47.
7. Crothers S. J. On the geometry of the general solution for the vacuum field of the point-mass. *Progress in Physics*, 2005, v. 2, 3–14.
8. Stavroulakis N. A statical smooth extension of Schwarzschild's metric. *Lettere al Nuovo Cimento*, 1974, v. 11, 8 (www.geocities.com/theometria/Stavroulakis-3.pdf).
9. Stavroulakis N. On the Principles of General Relativity and the $S\Theta(4)$-invariant metrics. *Proc. 3rd Panhellenic Congr. Geometry*, Athens, 1997, 169 (www.geocities.com/theometria/Stavroulakis-2.pdf).
10. Stavroulakis N. On a paper by J. Smoller and B. Temple. *Annales de la Fondation Louis de Broglie*, 2002, v. 27, 3 (www.geocities.com/theometria/Stavroulakis-1.pdf).
11. Crothers S. J. On the general solution to Einstein's vacuum field for the point-mass when $\lambda \neq 0$ and its implications for relativistic cosmology, *Progress in Physics*, 2005, v. 3, 7–18.
12. Arp, H. Observational cosmology: from high redshift galaxies to the blue pacific, *Progress in Physics*, 2005, v. 3, 3–6.
13. McVittie G. C. Laplace's alleged "black hole". *The Observatory*, 1978, v. 98, 272 (www.geocities.com/theometria/McVittie.pdf).
14. Misner C. W., Thorne K. S., Wheeler J. A. Gravitation. W. H. Freeman and Company, New York, 1973.

A Re-Examination of Maxwell's Electromagnetic Equations

Jeremy Dunning-Davies

Department of Physics, University of Hull, Hull, England

E-mail: J.Dunning-Davies@hull.ac.uk

It is pointed out that the usual derivation of the well-known Maxwell electromagnetic equations holds only for a medium at rest. A way in which the equations may be modified for the case when the mean flow of the medium is steady and uniform is proposed. The implication of this for the problem of the origin of planetary magnetic fields is discussed.

1 Introduction

Maxwell's electromagnetic equations are surely among the best known and most widely used sets of equations in physics. However, possibly because of this and since they have been used so successfully in so many areas for so many years, they are, to some extent, taken for granted and used with little or no critical examination of their range of validity. This is particularly true of the two equations

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

and

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\,.$$

Both these equations are used widely but, although the point is made quite clearly in most elementary, as well as more advanced, textbooks, it is often forgotten that these equations apply *only* when the medium involved is assumed to be at rest. This assumption is actually crucial in the derivation of these equations since it is because of it that it is allowable to take the operator d/dt inside the integral sign as a partial derivative and so finally derive each of the above equations. This leaves open the question of what happens if the medium is not at rest?

As is well known, for a non-conducting medium at rest, Maxwell's electromagnetic equations, when no charge is present, reduce to

$$\nabla \cdot \mathbf{E} = 0\,, \quad \nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}\,,$$

$$\nabla \cdot \mathbf{H} = 0\,, \quad \nabla \times \mathbf{H} = -\frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t}\,,$$

where $\mathbf{D} = \varepsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}$ and μ, ε are assumed constant in time.

The first two equations are easily seen to lead to

$$\nabla^2 \mathbf{E} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}\,,$$

and the latter two to

$$\nabla^2 \mathbf{H} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}\,.$$

Therefore, in this special case, *provided* the medium is at rest, both $\mathbf{E}$ and $\mathbf{H}$ satisfy the well-known wave equation. However, it has been shown [1] that, if the mean flow is steady and uniform, and, therefore, both homentropic and irrotational, the system of equations governing small-amplitude homentropic irrotational wave motion in such a flow reduces to the equation

$$\nabla^2 \varphi = \frac{1}{c^2}\frac{D^2 \varphi}{Dt^2}\,.$$

which is sometimes referred to as the convected, or progressive, wave equation. The question which remains is, for the case of a medium not at rest, should Maxwell's electromagnetic equations be modified so as to reduce to this progressive wave equation in the case of a non-conducting medium with no charge present?

2 Generalisation of Maxwell's equations

In the derivation of

$$\nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}$$

it proves necessary to consider the integral

$$-\frac{\mu}{c}\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S}$$

and interchange the derivative and the integral. This operation may be carried out only for a medium at rest. However, if the medium is moving, then the surface $\mathbf{S}$ in the integral will be moving also, and the mere change of $\mathbf{S}$ in the field $\mathbf{B}$ will cause changes in the flux. Hence, following Abraham and Becker [2], a new kind of differentiation with respect to time is defined by the symbol $\dot{\mathbf{B}}$ as follows:

$$\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S} = \int \dot{\mathbf{B}}\cdot d\mathbf{S}\,. \qquad \text{(a)}$$

Here, $\dot{\mathbf{B}}$ is a vector, the flux of which across the moving surface equals the rate of increase with time of the flux of $\mathbf{B}$ across the same surface. In order to find $\dot{\mathbf{B}}$, the exact details of the motion of the surface concerned must be known. Suppose this motion described by a vector $\mathbf{u}$, which is assumed given for each element $d\mathbf{S}$ of the surface and is the velocity of the element.

Let S_1 be the position of the surface S at time $(t - dt)$ and S_2 the position at some later time t. S_2 may be obtained from S_1 by giving each element of S_1 a displacement $\mathbf{u}dt$. The surfaces S_1 and S_2, together with the strip produced during the motion, bound a volume $dt \int \mathbf{u} \cdot d\mathbf{S}$.

The rate of change with time of the flux of $\mathbf{B}$ across S may be found from the difference between the flux across S_2 at time t and that across S_1 at time $(t - dt)$; that is

$$\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = \frac{\int \mathbf{B}_t \cdot d\mathbf{S}_2 - \int \mathbf{B}_{t-dt} \cdot d\mathbf{S}_1}{dt} ,$$

where the subscript indicates the time at which the flux is measured.

The divergence theorem may be applied at time t to the volume bounded by S_1, S_2 and the strip connecting them. Here the required normal to S_2 will be the outward pointing normal and that to S_1 the inward pointing normal. Also, a surface element of the side face will be given by $d\mathbf{s} \times \mathbf{u}dt$. Then, the divergence theorem gives

$$\int_{S_2} \mathbf{B}_t \cdot d\mathbf{S}_2 + dt \oint \mathbf{B} \cdot d\mathbf{s} \times \mathbf{u} - \int_{S_1} \mathbf{B}_t \cdot d\mathbf{S}_1 = dt \int (\nabla \cdot \mathbf{B})\, \mathbf{u} \cdot d\mathbf{S}.$$

Also

$$\int \mathbf{B}_{t-dt} \cdot d\mathbf{S}_1 = \int \mathbf{B}_t \cdot d\mathbf{S}_1 - \int \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S}_1 dt .$$

Hence,

$$\int \mathbf{B}_t \cdot d\mathbf{S}_2 - \int \mathbf{B}_{t-dt} \cdot d\mathbf{S}_1 = dt \Big\{ \int \dot{\mathbf{B}} \cdot d\mathbf{S}_1 + \int (\nabla \cdot \mathbf{B}) \mathbf{u} \cdot d\mathbf{S}_1 - \oint \mathbf{B} \cdot d\mathbf{s} \times \mathbf{u} \Big\} .$$

Using Stokes' theorem, the final term on the right-hand side of this equation may be written

$$\oint \mathbf{B} \cdot d\mathbf{s} \times \mathbf{u} = \oint \mathbf{u} \times \mathbf{B} \cdot d\mathbf{s} = \int \Big\{ \nabla \times (\mathbf{u} \times \mathbf{B}) \Big\} \cdot d\mathbf{S} ,$$

and so finally

$$\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = \int \left\{ \frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} (\nabla \cdot \mathbf{B}) - \nabla \times (\mathbf{u} \times \mathbf{B}) \right\} \cdot d\mathbf{S} .$$

Therefore, the $\dot{\mathbf{B}}$, introduced in (a) above, is given by

$$\dot{\mathbf{B}} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} (\nabla \cdot \mathbf{B}) - \nabla \times (\mathbf{u} \times \mathbf{B})$$

or, noting that

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{u} (\nabla \cdot \mathbf{B}) - \mathbf{B} (\nabla \cdot \mathbf{u}) + (\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B} ,$$

$$\dot{\mathbf{B}} = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} + \mathbf{B} (\nabla \cdot \mathbf{u}) - (\mathbf{B} \cdot \nabla) \mathbf{u} .$$

However, if the mean flow is steady and uniform and, therefore, both homentropic and irrotational, the fluid velocity, $\mathbf{u}$, will be constant and this latter equation will reduce to

$$\dot{\mathbf{B}} = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} = \frac{D\mathbf{B}}{Dt} ,$$

that is, for such flow, $\dot{\mathbf{B}}$ becomes the well-known Euler derivative. It might be noted, though, that, for more general flows, the expression for $\dot{\mathbf{B}}$ is somewhat more complicated.

It follows that, if the mean flow is steady and uniform, the Maxwell equation, mentioned above, becomes

$$\nabla \times \mathbf{E} = -\frac{\mu}{c} \frac{D\mathbf{H}}{Dt} = -\frac{\mu}{c} \left[\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{H} \right] .$$

Also, in this particular case, the remaining three Maxwell equations will be

$$\nabla \cdot \mathbf{E} = 0 , \quad \nabla \cdot \mathbf{H} = 0 ,$$

$$\nabla \times \mathbf{H} = \frac{\varepsilon}{c} \frac{D\mathbf{E}}{Dt} = \frac{\varepsilon}{c} \left[\frac{\partial \mathbf{E}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{E} \right] ,$$

with this form for the final equation following in a manner similar to that adopted above when noting that, for a steady, uniform mean flow, $\partial/\partial t$ is replaced by D/Dt in the equation for $\nabla \times \mathbf{E}$.

These four modified Maxwell equations lead to both $\mathbf{E}$ and $\mathbf{H}$ satisfying the above mentioned progressive wave equation, as they surely must.

3 The origin of planetary magnetic fields

It is conceivable that use of these modified Maxwell electromagnetic equations could provide new insight into the problem of the origin of planetary magnetic fields. This is a problem which has existed, without a really satisfactory explanation, for many years. It would seem reasonable to expect all such fields to arise from the same physical mechanism, although the minute detail might vary from case to case. The mechanism generally favoured as providing the best explanation for the origin of these fields was the dynamo mechanism, although the main reason for its adoption was the failure of the alternatives to provide a consistent explanation. However, Cowling [3] showed that there is a limit to the degree of symmetry encountered in a steady dynamo mechanism; this result, based on the traditional electromagnetic equations of Maxwell, shows that the steady maintenance of a poloidal field is simply not possible — the result is in reality an anti-dynamo theorem which raises difficulties in understanding the observed symmetry of the dipole field.

Following Alfvén [4], it might be noted that, in a stationary state, there is no electromagnetic field along a neutral line because that would imply a non-vanishing $\nabla\times\mathbf{E}$, and so a time varying $\mathbf{B}$. The induced electric field $\mathbf{v}\times\mathbf{B}$ vanishes on the neutral line since $\mathbf{B}$ does. Thus, there can be no electromotive force along the neutral line, and therefore the current density in the stationary state vanishes, the conductivity being infinite. On the other hand, $\nabla\times\mathbf{B}$ does not vanish on the neutral line. By Maxwell's usual equations, the non-vanishing $\nabla\times\mathbf{B}$ and the vanishing current density are in contradiction and so the existence of a rotationally symmetric steady-state dynamo is disproved. However, this conclusion may not be drawn if the modified Maxwell equations, alluded to earlier, are used, since, even in the steady state where the partial derivatives with respect to time will all be zero, the equation for $\nabla\times\mathbf{B}$ will reduce to

$$\nabla\times\mathbf{B} = \frac{1}{\mu}\left[\mathbf{j} + \varepsilon\frac{\partial\mathbf{E}}{\partial t} + \varepsilon\mathbf{v}\cdot\nabla\mathbf{E}\right] \to \frac{\varepsilon}{\mu}\mathbf{v}\cdot\nabla\mathbf{E}$$

and there is no reason why this extra term on the right-hand side should be identically equal to zero. Also, the non-vanishing of $\nabla\times\mathbf{E}$ will not imply a time varying $\mathbf{B}$ since, once again, there is an extra term $-\mathbf{v}\cdot\nabla\mathbf{B}$ remaining to equate with the $\nabla\times\mathbf{E}$. It follows that an electromagnetic field may exist along the neutral line under these circumstances. Hence, no contradiction occurs; instead, a consistent system of differential equations remains to be solved.

References

1. Thornhill C. K. *Proc. Roy. Soc. Lond.*, 1993, v. 442, 495.
2. Abraham M. , Becker, R. The Classical Theory of Electricity and Magnetism. Blackie and Son Ltd. , London, 1932.
3. Cowling T. G. *Monthly Notices of the Royal Astronomical Society*, 1934, v. 94, 39.
4. Alfvén H., Fälthammar C. G. Cosmical Electrodynamics. Oxford at the Clarendon Press, 1963.

Black Holes in Elliptical and Spiral Galaxies and in Globular Clusters

Reginald T. Cahill

School of Chemistry, Physics and Earth Sciences, Flinders University, Adelaide 5001, Australia

E-mail: Reg.Cahill@flinders.edu.au

Supermassive black holes have been discovered at the centers of galaxies, and also in globular clusters. The data shows correlations between the black hole mass and the elliptical galaxy mass or globular cluster mass. It is shown that this correlation is accurately predicted by a theory of gravity which includes the new dynamics of self-interacting space. In spiral galaxies this dynamics is shown to explain the so-called "dark matter" rotation-curve anomaly, and also explains the Earth based bore-hole g anomaly data. Together these effects imply that the strength of the self-interaction dynamics is determined by the fine structure constant. This has major implications for fundamental physics and cosmology.

4 Introduction

Our understanding of gravity is based on Newton's modelling of Kepler's phenomenological laws for the motion of the planets within the solar system. In this model Newton took the gravitational acceleration field to be the fundamental dynamical degree of freedom, and which is determined by the matter distribution; essentially via the "universal inverse square law". However the observed linear correlation between masses of black holes with the masses of the "host" elliptical galaxies or globular clusters suggests that either the formation of these systems involves common evolutionary dynamical processes or that perhaps some new aspect to gravity is being revealed. Here it is shown that if rather than an acceleration field a velocity field is assumed to be fundamental to gravity, then we immediately find that these black hole effects arise as a space self-interaction dynamical effect, and that the observed correlation is simply that $M_{BH}/M = \alpha/2$ for spherical systems, where α is the fine structure constant ($\alpha = e^2/\hbar c = 1/137.036$), as shown in Fig. 1. This dynamics also manifests within the Earth, as revealed by the bore hole g anomaly data, as in Fig. 2. It also offers an explanation of the "dark matter" rotation-velocity effect, as illustrated in Fig. 3. This common explanation for a range of seemingly unrelated effects has deep implications for fundamental physics and cosmology.

5 Modelling gravity

Let us phenomenologically investigate the consequences of using a velocity field $\mathbf{v}(\mathbf{r}, t)$ to be the fundamental dynamical degree of freedom to model gravity. The gravitational acceleration field is then defined by the Euler form

$$\mathbf{g}(\mathbf{r},t) \equiv \lim_{\Delta t\to 0}\frac{\mathbf{v}(\mathbf{r}+\mathbf{v}(\mathbf{r},t)\Delta t, t+\Delta t)-\mathbf{v}(\mathbf{r},t)}{\Delta t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v}. \quad (1)$$

This form is mandated by Galilean covariance under change of observer. A minimalist non-relativistic modelling of the dynamics for this velocity field gives a direct account of the various phenomena noted above; basically the Newtonian formulation of gravity missed a key dynamical effect that did not manifest within the solar system.

In terms of the velocity field Newtonian gravity dynamics involves using $\nabla.$ to construct a rank-0 tensor that can be related to the matter density ρ. The coefficient turns out to be the Newtonian gravitational constant G.

$$\nabla.\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v}\right) = -4\pi G\rho\,. \quad (2)$$

This is clearly equivalent to the differential form of Newtonian gravity, $\nabla.\mathbf{g} = -4\pi G\rho$. Outside of a spherical mass M (2) has solution*

$$\mathbf{v}(\mathbf{r}) = -\sqrt{\frac{2GM}{r}}\,\hat{\mathbf{r}}\,, \quad (3)$$

for which (1) gives the usual inverse square law

$$\mathbf{g}(\mathbf{r}) = -\frac{GM}{r^2}\,\hat{\mathbf{r}}\,. \quad (4)$$

The simplest non-Newtonian dynamics involves the two rank-0 tensors constructed at 2nd order from $\partial v_i/\partial x_j$

$$\nabla.\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v}\right) + \frac{\alpha}{8}(\mathrm{tr}D)^2 + \frac{\beta}{8}\,\mathrm{tr}(D^2) = -4\pi G\rho, \quad (5)$$

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad (6)$$

and involves two arbitrary dimensionless constants. The velocity in (3) is also a solution to (5) if $\beta = -\alpha$, and we then define

$$C(\mathbf{v},t) = \frac{\alpha}{8}\left((\mathrm{tr}D)^2 - \mathrm{tr}(D^2)\right). \quad (7)$$

*We assume $\nabla\times\mathbf{v} = \mathbf{0}$, then $(\mathbf{v}.\nabla)\,\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v}^2)$.

Hence the modelling of gravity by (5) and (1) now involves two gravitational constants G and α, with α being the strength of the self-interaction dynamics, but which was not apparent in the solar system dynamics. We now show that all the various phenomena discussed herein imply that α is the fine structure constant $\approx 1/137$ up to experimental errors [1]. Hence non-relativistic gravity is a more complex phenomenon than currently understood. The new key feature is that (5) has a one-parameter μ class of vacuum ($\rho = 0$) "black hole" solutions in which the velocity field self-consistently maintains the singular form

$$\mathbf{v}(\mathbf{r}) = -\mu r^{-\alpha/4}\,\hat{\mathbf{r}}\,. \tag{8}$$

This class of solutions will be seen to account for the "black holes" observed in galaxies and globular cluster. As well this velocity field, from (1), gives rise to a non-"inverse square law" acceleration

$$\mathbf{g}(\mathbf{r}) = -\frac{\alpha\mu}{4}\,r^{-(1+\alpha/4)}\,\hat{\mathbf{r}}\,. \tag{9}$$

This turns out to be the cause of the so-called "dark-matter" effect observed in spiral galaxies. For this reason we define

$$\rho_{DM}(\mathbf{r}) = \frac{\alpha}{32\pi G}\Big((\mathrm{tr}D)^2 - \mathrm{tr}(D^2)\Big), \tag{10}$$

so that (5) and (1) can be written as

$$\nabla.\mathbf{g} = -4\pi G\rho - 4\pi G\rho_{DM}\,, \tag{11}$$

which shows that we can think of the new self-interaction dynamics as generating an effective "dark matter" density.

6 Spherical systems

It is sufficient here to consider time-independent and spherically symmetric solutions of (5) for which v is radial. Then we have the integro-differential form for (5)

$$v^2(r) = 2G\int d^3s\,\frac{\rho(s) + \rho_{DM}\big(v(s)\big)}{|\,\mathbf{r}-\mathbf{s}\,|}, \tag{12}$$

$$\rho_{DM}\big(v(r)\big) = \frac{\alpha}{8\pi G}\left(\frac{v^2}{2r^2} + \frac{vv'}{r}\right). \tag{13}$$

as $\nabla^2\frac{1}{|\mathbf{r}-\mathbf{s}|} = -4\pi\delta^4(\mathbf{r}-\mathbf{s})$. This then gives

$$\begin{aligned} v^2(r) = \frac{8\pi G}{r}\int_0^r s^2 ds\Big[\rho(s) + \rho_{DM}\big(v(s)\big)\Big] + \\ + 8\pi G\int_r^\infty s ds\Big[\rho(s) + \rho_{DM}\big(v(s)\big)\Big] \end{aligned} \tag{14}$$

on doing the angle integrations. We can also write (5) as a non-linear differential equation

$$2\,\frac{vv'}{r} + (v')^2 + vv'' = -4\pi G\rho(r) - 4\pi G\rho_{DM}\big(v(r)\big)\,. \tag{15}$$

7 Minimal black hole systems

There are two classes of solutions when matter is present. The simplest is when the black hole forms as a consequence of the velocity field generated by the matter, this generates what can be termed an induced minimal black hole. This is in the main applicable to systems such as planets, stars, globular clusters and elliptical galaxies. The second class of solutions correspond to non-minimal black hole systems; these arise when the matter congregates around a pre-existing "vacuum" black hole. The minimal black holes are simpler to deal with, particularly when the matter system is spherically symmetric. In this case the non-Newtonian gravitational effects are confined to within the system. A simple way to arrive at this property is to solve (14) perturbatively. When the matter density is confined to a sphere of radius R we find on iterating (14) that the "dark matter" density is confined to that sphere, and that consequently $g(r)$ has an inverse square law behaviour outside of the sphere. Iterating (14) once we find inside radius R that

$$\rho_{DM}(r) = \frac{\alpha}{2r^2}\int_r^R s\rho(s)ds + \mathrm{O}(\alpha^2). \tag{16}$$

and that the total "dark matter"

$$\begin{aligned} M_{DM} \equiv 4\pi\int_0^R r^2 dr\rho_{DM}(r) = \\ = \frac{4\pi\alpha}{2}\int_0^R r^2 dr\rho(r) + \mathrm{O}(\alpha^2) = \frac{\alpha}{2}\,M + \mathrm{O}(\alpha^2)\,, \end{aligned} \tag{17}$$

where M is the total amount of (actual) matter. Hence, to $\mathrm{O}(\alpha)$, $M_{DM}/M = \alpha/2$ independently of the matter density profile. This turns out to be a very useful property as knowledge of the density profile is then not required in order to analyse observational data. Fig. 1 shows the value of M_{BH}/M for, in particular, globular clusters $M15$ and $G1$ and highly spherical "elliptical" galaxies M32, M87 and NGC 4374, showing that this ratio lies close to the "$\alpha/2$-line", where α is the fine structure constant $\approx 1/137$. However for the spiral galaxies their M_{DM}/M values do not cluster close to the $\alpha/2$-line. Hence it is suggested that these spherical systems manifest the minimal black hole dynamics outlined above. However this dynamics is universal, so that any spherical system must induce such a minimal black hole mode, but for which outside of such a system only the Newtonian inverse square law would be apparent. So this mode must also apply to the Earth, which is certainly a surprising prediction. However just such an effect has manifested in measurements of g in mine shafts and bore holes since the 1980's. It will now be shown that data from these geophysical measurements give us a very accurate determination of the value of α in (5).

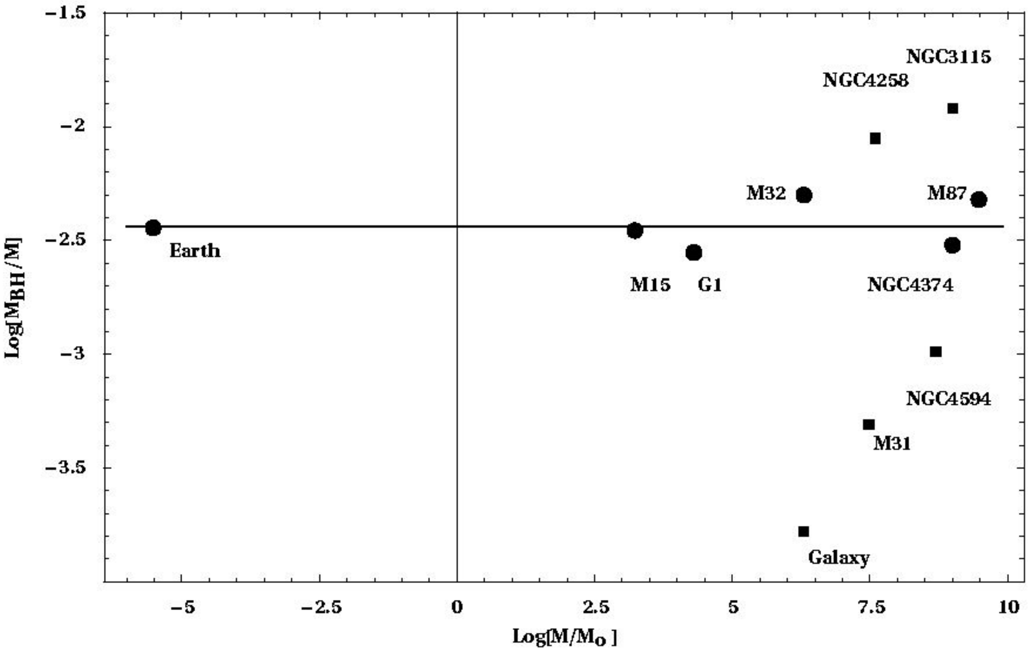


Fig. 1: The data shows $\mathrm{Log}_{10}[M_{BH}/M]$ for the "black hole" or "dark matter" masses M_{BH} for a variety of spherical matter systems with masses M, shown by solid circles, plotted against $\mathrm{Log}_{10}[M/M_0]$, where M_0 is the solar mass, showing agreement with the "$\alpha/2$-line" ($\mathrm{Log}_{10}[\alpha/2] = -2.44$) predicted by (17), and ranging over 15 orders of magnitude. The "black hole" effect is the same phenomenon as the "dark matter" effect. The data ranges from the Earth, as observed by the bore hole g anomaly, to globular cluster M15 [5, 6] and G1 [7], and then to spherical "elliptical" galaxies M32 (E2), NGC 4374 (E1) and M87 (E0). Best fit to the data from these star systems gives $\alpha = 1/134$, while for the Earth data in Fig. 2 $\alpha = 1/139$. A best fit to all the spherical systems in the plot gives $\alpha = 1/136$. In these systems the "dark matter" or "black hole" spatial self-interaction effect is induced by the matter. For the spiral galaxies, shown by the filled boxes, where here M is the bulge mass, the black hole masses do not correlate with the "$\alpha/2$-line". This is because these systems form by matter in-falling to a primordial black hole, and so these systems are more contingent. For spiral galaxies this dynamical effect manifests most clearly via the non-Keplerian rotation-velocity curve, which decrease asymptotically very slowly, as shown in Fig. 3, as determined by the small value of $\alpha \approx 1/137$. The galaxy data is from Table 1 of [8, updated].

8 Bore hole g anomaly

To understand this bore hole anomaly we need to compute the expression for $g(r)$ just beneath and just above the surface of the Earth. To lowest order in α the "dark-matter" density in (16) is substituted into (14) finally gives via (1) the acceleration

$$g(r) = \begin{cases} \dfrac{(1+\frac{\alpha}{2})GM}{r^2}, \quad r > R, \\ \dfrac{4\pi G}{r^2}\displaystyle\int_0^r s^2 ds\, \rho(s) + \\ + \dfrac{2\pi\alpha G}{r^2}\displaystyle\int_0^r \left(\int_s^R s' ds' \rho(s')\right) ds, \\ r < R. \end{cases} \tag{18}$$

This gives Newton's "inverse square law" for $r > R$, but in which we see that the effective Newtonian gravitational constant is $G_N = (1+\frac{\alpha}{2})G$, which is different to the fundamental gravitational constant G in (2). This caused by the additional "dark matter mass" in (17). Inside the Earth we see that (18) gives a $g(r)$ different from Newtonian gravity. This has actually been observed in mine/borehole measurements of $g(r)$ [2, 3, 4], though of course there had been no explanation for the effect, and indeed the reality of the effect was eventually doubted. The effect is that g decreases more slowly with depth than predicted by Newtonian gravity. Here the corresponding Newtonian form for $g(r)$ is

$$g(r)_{Newton} = \begin{cases} \dfrac{G_N M}{r^2}, \quad r > R, \\ \dfrac{4\pi G_N}{r^2}\displaystyle\int_0^r s^2 ds\, \rho(s), \quad r < R, \end{cases} \tag{19}$$

with $G_N = (1+\frac{\alpha}{2})G$. The gravity residual is defined as the difference between the Newtonian $g(r)$ and the measured $g(r)$, which we here identify with the $g(r)$ from (18),

$$\Delta g(r) \equiv g(r)_{Newton} - g(r)_{observed}. \tag{20}$$

Then $\Delta g(r)$ is found to be, to 1st order in $R - r$, i. e.

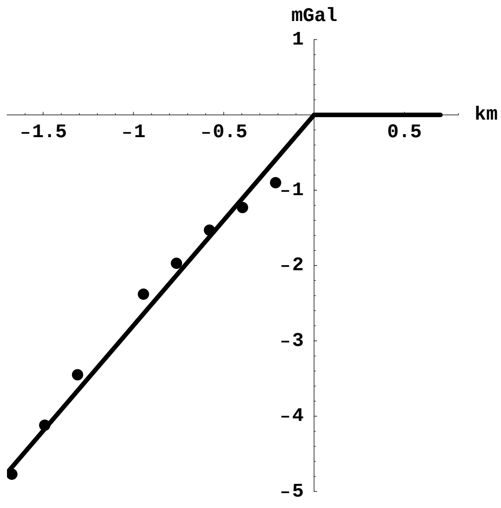


Fig. 2: The data shows the gravity residuals for the Greenland Ice Cap [4] measurements of the $g(r)$ profile, defined as $\Delta g(r) = g_{Newton} - g_{observed}$, and measured in mGal ($1\,\text{mGal} = 10^{-3}$ cm/sec^2), plotted against depth in km. Using (21) we obtain $\alpha^{-1} = 139 \pm 5$ from fitting the slope of the data, as shown.

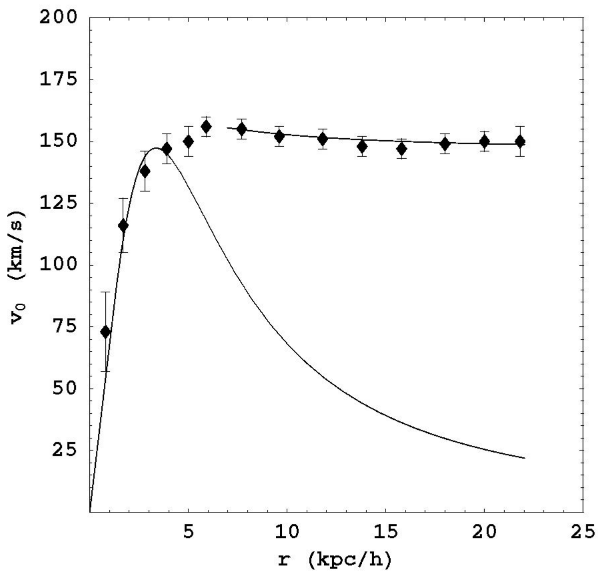


Fig. 3: Data shows the non-Keplerian rotation-speed curve $v_\circ$ for the spiral galaxy NGC 3198 in km/s plotted against radius in kpc/h. Lower curve is the rotation curve from the Newtonian theory for an exponential disk, which decreases asymptotically like $1/\sqrt{r}$. The upper curve shows the asymptotic form from (24), with the decrease determined by the small value of α. This asymptotic form is caused by the primordial black holes at the centres of spiral galaxies, and which play a critical role in their formation. The spiral structure is caused by the rapid in-fall towards these primordial black holes.

near the surface,

$$\Delta g(r) = \begin{cases} 0, & r > R\,, \\ -2\pi\alpha G_N \rho(R)(R-r)\,, & r < R\,, \end{cases} \tag{21}$$

which is the form actually observed [4], as shown in Fig. 2.

Gravity residuals from a bore hole into the Greenland Ice Cap were determined down to a depth of 1.5km. The ice had a measured density of $\rho = 930$ kg/m^3, and from (21), using $G_N = 6.6742 \times 10^{-11}$ m^3 s^{-2} kg^{-1}, we obtain from a linear fit to the slope of the data points in Fig. 2 that $\alpha^{-1} = 139 \pm 5$, which equals the value of the fine structure constant $\alpha^{-1} = 137.036$ to within the errors, and for this reason we identify the constant α in (5) as being the fine structure constant. Then we arrive at the conclusion that there is indeed "black hole" or "dark matter" dynamics within the Earth, and that from (17) we have again for the Earth that $M_{BH}/M = \alpha/2$, as is also shown in Fig. 1.

This "minimal black hole" effect must also occur within stars, although that could only be confirmed by indirect observations. This effect results in $g(r)$ becoming large at the center, unlike Newtonian gravity, which would affect nuclear reaction rates. This effect may already have manifested in the solar neutrino count problem [9, 10]. To study this will require including the new gravity dynamics into solar models.

9 Spiral galaxies

We now consider the situation in which matter in-falls around an existing primordial black hole. Immediately we see some of the consequences of this time evolution: (i) because the acceleration field falls off much slower than the Newtonian inverse square law, as in (9), this in-fall would happen very rapidly, and (ii) the resultant in-flow would result in the matter rotating much more rapidly than would be predicted by Newtonian gravity, (iii) so forming a quasar which, after the in-fall of some of the matter into the black hole has ceased, would (iv) result in a spiral galaxy exhibiting non-Keplerian rotation of stars and gas clouds, *viz* the so-called "dark matter" effect. The study of this time evolution will be far from simple. Here we simply illustrate the effectiveness of the new theory of gravity in explaining this "dark matter" or non-Keplerian rotation-velocity effect.

We can determine the star orbital speeds for highly non-spherical galaxies in the asymptotic region by solving (15), for asymptotically where $\rho \approx 0$ the velocity field will be approximately spherically symmetric and radial; nearer in we would match such a solution to numerically determined solutions of (5). Then (15) has an exact non-perturbative two-parameter (K and R_S) analytic solution,

$$v(r) = K \left(\frac{1}{r} + \frac{1}{R_S} \left(\frac{R_S}{r} \right)^{\frac{\alpha}{2}} \right)^{1/2}; \tag{22}$$

this velocity field then gives using (1) the non-Newtonian asymptotic acceleration

$$g(r) = \frac{K^2}{2}\left(\frac{1}{r^2} + \frac{\alpha}{2rR_S}\left(\frac{R_S}{r}\right)^{\frac{\alpha}{2}}\right), \tag{23}$$

applicable to the outer regions of spiral galaxies.

We then compute circular orbital speeds using $v_\circ(r) = \sqrt{rg(r)}$ giving the predicted "universal rotation-speed curve"

$$v_\circ(r) = \frac{K}{2}\left(\frac{1}{r} + \frac{\alpha}{2R_S}\left(\frac{R_S}{r}\right)^{\frac{\alpha}{2}}\right)^{1/2}. \tag{24}$$

Because of the α dependent part this rotation-speed curve falls off extremely slowly with r, as is indeed observed for spiral galaxies. This is illustrated in Fig. 3 for the spiral galaxy NGC 3198.

10 Interpretation and discussion

Section 2 outlines a model of space developed in [1, 11] in which space has a "substratum" structure which is in differential motion. This means that the substratum in one region may have movement relative to another region. The substratum is not embedded in a deeper space; the substratum itself defines space, and requiring that, at some level of description, it may be approximately described by a "classical" 3-vector velocity field $\mathbf{v}(\mathbf{r}, t)$. Then the dynamics of space involves specifying dynamical equations for this vector field. Here the coordinates $\mathbf{r}$ is not space itself, but a means of labelling points in space. Of course in dealing with this dynamics we are required to define $\mathbf{v}(\mathbf{r}, t)$ relative to some set of observers, and then the dynamical equations must be such that the vector field transforms covariantly with respect to changes of observers. As noted here Newtonian gravity itself may be written in terms of a vector field, as well as in terms of the usual acceleration field $\mathbf{g}(\mathbf{r}, t)$. General Relativity also has a special class of metric known as the Panlevé-Gullstrand metrics in which the metrics are specified by a velocity field. Most significantly the major tests of General Relativity involved the Schwarzschild metric, and this metric belongs to the Panlevé-Gullstrand class. So in both cases these putatively successful models of gravity involved, in fact, velocity fields, and so the spacetime metric description was not essential. As well there are in total some seven experiments that have detected this velocity field [12], so that it is more than a choice of dynamical degree of freedom: indeed it is more fundamental in the sense that from it the acceleration field or metric may be mathematically constructed.

Hence the evidence, both experimental and theoretical, is that space should be described by a velocity field. This implies that space is a complex dynamical system which is best thought of as some kind of "flow system". However the implicit question posed in this paper is that, given the physical existence of such a velocity field, are the Newtonian and/or General Relativity formalisms the appropriate descriptions of the velocity field dynamics? The experimental evidence herein implies that a different dynamics is required to be developed, because when we generalise the velocity field modelling to include a spatial self-interaction dynamics, the experimental evidence is that the strength of this dynamics is determined by the fine structure constant, α. This is an extraordinary outcome, implying that gravity is determined by two fundamental constants, G and α. As α clearly is not in Newtonian gravity nor in General Relativity the various observational and experimental data herein is telling us that neither of these theories of gravity is complete. The modelling discussed here is non-relativistic, and essentially means that Newtonian gravity was incomplete from the very beginning. This happened because the self-interaction dynamics did not manifest in the solar system planetary orbit motions, and so neither Kepler nor later Newton were aware of the intrinsic complexity of the phenomenon of gravity. General Relativity was of course constructed to agree with Newtonian gravity in the non-relativistic limit, and so missed out on this key non-relativistic self-interaction effect.

Given both the experimental detection of the velocity field, including in particular the recent discovery [11] of an in-flow velocity component towards the Sun in the 1925/26 Miller interferometer data, and in agreement with the speed value from (3) for the Sun, together with the data from various observations herein, all showing the presence of the α dependent effect, we should also discuss the physical interpretaion of the vacuum "black hole" solutions. These are different in character from the so-called "black holes" of General Relativity: we use the same name only because these new "black holes" have an event horizon, but otherwise they are completely different. In particular the mathematical existence of such vacuum "black holes" in General Relativity is doubtful. In the new theory of gravity these black holes are exact mathematical solutions of the velocity equations and correspond to self-sustaining in-flow singularities, that is, where the in-flow speed becomes very large within the classical description. This singularity would then require a quantum description to resolve and explain what actually happens there. The in-flow does not involve any conserved measure, and there is no notion of this in-flow connecting to wormholes etc. The in-flow is merely a self-destruction of space, and in [11] it is suggested that space is in essence an "information" system, in which case the destruction process is easier to comprehend. As for the in-flow into the Earth, which is completely analogous to the observed in-flow towards the Sun, the in-flow singularities or "black holes" are located at the centre of the Earth, but it is unclear whether there is one such singularity or multiple singularities. The experimental existence of the Earth-centred in-flow singularity is indirect, as it is inferred solely by the anomalous variation of g with depth, and that this variation is determined by the value of α. In the case of the globular clusters and elliptical galaxies, the in-flow singularities are observed by means of the large accelerations of stars located near the centres of such systems and so are more apparent, and as shown here in all case the effective mass of the in-flow singularity is $\alpha/2$ times the total mass of these systems. It is important to note here that even if we disregard the theoretical velocity field theory, we would still be left with the now well established $\alpha/2$ observational effect. But then this velocity field theory gives a simple explanation for this data, although that in itself does not exclude other theories offering a different explanation. It is hard to imagine however how either Newtonian gravity or General Relativity could offer such a simple explanation, seeing that neither involves α, and involve only G. As well we see that the new theory of gravity offers a very effective explanation for the rotation characteristics of spiral galaxies; the effect here being that the vacuum black hole(s) at the centres of such galaxies do not generate an acceleration field that falls off with distance according the inverse square law, but rather according to (23). Remarkably this is what the spiral galaxy data shows. This means that the so-called "dark matter" effect is not about a new and undetected form of matter. So the success of the new velocity field dynamics is that one theory explains a whole range of phenomena: this is the hallmark of any theory, namely economy of explanation.

11 Conclusion

The observational and experimental data confirm that the massive black holes in globular clusters and galaxies are necessary phenomena within a theory for gravity which uses a velocity field as the fundamental degree of freedom. This involves two constants G and α and the data reveals that α is the fine structure constant. This suggests that the spatial self-interaction dynamics, which is missing in the Newtonian theory of gravity, may be a manifestation at the classical level of the quantum behaviour of space. It also emerges that the "black hole" effect and the "dark matter" effect are one phenomenon, namely the non-Newtonian acceleration caused by singular solutions. This effect must manifest in planets and stars, and the bore hole g anomaly confirms that for planets. For stars it follows that the structure codes should be modified to include the new spatial self-interaction dynamics, and to determine the effect upon neutrino count rates. The data shows that spherical systems with masses varying over 15 orders of magnitude exhibit the α-dependent dynamical effect. The non-Newtonian gravitational acceleration of primordial black holes will cause rapid formation of quasars and stars, explaining why recent observations have revealed that these formed very early in the history of the universe. In this way the new theory of gravity makes the big bang theory compatible with these recent observations. These developments clearly have major implications for cosmology and fundamental physics. The various experiments that detected the velocity field are discussed in [11, 12].

This research is supported by an Australian Research Council Discovery Grant.

References

1. Cahill R. T. *Trends in Dark Matter Research*, ed. by Blain J. Val, Nova Science Pub, NY, 2005; *Apeiron*, 2005, v. 12, No. 2, 155–177.
2. Stacey F. D. et al. *Phys. Rev. D*, 1981, v. 23, 2683.
3. Holding S. C., Stacey F. D. & Tuck G. J. *Phys. Rev. D*, 1986, v. 33, 3487.
4. Ander M. E. et al. *Phys. Rev. Lett.*, 1989, v. 62, 985.
5. Gerssen J., van der Marel R. P., Gebhardt K., Guhathakurta P., Peterson R. & Pryor C. *Astrophys. J.*, 2002, v. 124, 3270; Addendum 2003, v. 125, 376.
6. Murphy B. W., Cohn H. N., Lugger P. N., & Dull J. D. *Bull. of American Astron. Society*, 1994, v. 26, No. 4, 1487.
7. Gebhardt K., Rich R. M., & Ho L. C. *Astrophys. J.*, 2002, v. L41, 578.
8. Kormendy J. & Richstone D. *Astronomy and Astrophysics*, 1995, v. 33, 581.
9. Davies R. *Phys. Rev. Lett.*, 1964, v. 12, 300.
10. Bahcall J. *Phys. Rev. Lett.*, 1964, v. 12, 303.
11. Cahill R. T., Process Physics: From Information Theory to Quantum Space and Matter. Nova Science Pub., NY, 2005.
12. Cahill R. T. *Progress in Physics*, 2005, v. 3, 25–29.

Is the Biggest Paradigm Shift in the History of Science at Hand?

Eit Gaastra
Groningen, The Netherlands
E-mail: eitgaastra@freeler.nl

According to a growing number of scientists cosmology is at the end of an era. This era started 100 years ago with the publication of Albert Einstein's special theory of relativity and came to its height in the 1920s when the theory of relativity was used to develop the big bang model. However, at this moment there is a crisis within cosmology. More and more scientists openly doubt the big bang. There are alternatives for the theory of relativity as well as for the big bang model, but so far most scientists are scared to pass over Einstein.

1 Introduction

The big bang model rests on three pillars [1]. This trinity is the cosmology of the twentieth century.

The first pillar is the Theory of General Relativity. In 1905 Einstein came with his Theory of Special Relativity which describes the behaviour of light and in 1916 he published a theory about gravity, the Theory of General Relativity. In publications in 1922 and 1924 the Russian mathematician Alexander Friedmann used the formulae of the General Theory of Relativity to prove that the universe was dynamic: either it expanded or it shrunk. In 1927 it was the Belgian priest and astronomer-cosmologist Georges Lemaître, using the cosmological equations of Friedmann, who suggested for the first time that the universe once could have sprung from a point of very high-density, the *primaeval atom*. Another link in the realization of the big bang model was the Dutch astronomer-cosmologist Willem de Sitter, who suggested in 1917, together with Einstein, the de-Sitter-universe, which was based on the formulae of the General Theory of Relativity. The de-Sitter-universe has no mass, but has the feature that mass particles that form in it will accelerate away from each other.

The second pillar on which the big bang model rests is the stretching of light in an expanding universe. In the 1920s Edwin Hubble discovered that certain dots in the night sky are not stars but galaxies instead. From 1924 on he measured the distances of the galaxies and in 1929 he announced that the wavelength of light of galaxies is shifted towards a longer wavelength. The further away the galaxy the more "stretched" the light. At the time this stretching of light was explained with the big bang model of Lemaître. The universe could have sprung from a point of very high-density mass and ever since the universe would expand as a balloon. Because of the expansion of the universe space in the universe would stretch and in that case light would stretch along with space. The stretching of light of faraway galaxies is still explained this way, although a lot of astronomers customarily to refer to this stretching as if it is caused by the recessional velocity of galaxies in the big bang universe.

The third pillar was discovered in 1965. In 1948 a group of cosmologists calculated that in the case of a big bang certain radiation still had to be left over from a period shortly after the big bang. In 1965 such radiation was measured. This radiation (of 3 Kelvin) is now known as the cosmic background radiation and since 1965 it is seen as the big proof of the big bang model.

2 Alternatives for the theory of relativity

Einstein unfolded his special theory of relativity in an article in 1905, in which he states that the velocity of light is always constant relative to an observer. But the apparent constancy of the velocity of light can be explained differently.

Gravitons or other not yet detected particles may act as the medium that is needed by light to propagate itself. This is somewhat comparable to air molecules that are needed as a medium by sound to propagate itself. A theory that calls a medium into existence to explain the propagation of light is called an aether theory. Aether theories created a furore in the nineteenth century, but fell into oblivion after 1905, because of the rise of the theory of relativity. However, the last decennium the aether concept is making a come back and is getting more and more advocates, among whom is the Italian professor of physics Selleri [2]. (Also more advocates because despite the announcements by Michelson and Morley about the "null result", their famous interferometer 1887 experiment actually may have detected both absolute motion and the breakdown of Newtonian physics [3].)

Albert Einstein's theory of General Relativity of 1916 describes the movement of light and matter with the curvature of space-time more accurately than Isaac Newton's universal law of gravitation from the seventeenth century. There are alternatives, both for the Theory of General Relativity and Newtonian gravity. The physics professors Assis [4] and

Ghosh [5] look at inertia and gravity as forces that are caused by all the matter in the universe. This is called the extended Mach principle, after Ernst Mach who suggested in the nineteenth century that the inertia of any body is caused by its interaction with the rest of the universe.

There is also the so-called pushing gravity concept, a gravity model with gravitons going in and out of matter and by doing so pushing objects towards each other (on a macro-scale, for instance a teacup that falls to the ground or stars that are pushed towards each other; on a subatomic level things are different). Pushing gravity too is an alternative for both the Theory of General Relativity and Newtonian gravity. The pushing gravity concept was first suggested by Nicolas Fatio de Duillier in the seventeenth century [6].

An aether theory, the extended Mach principle as well as pushing gravity, takes the line that smaller particles (like gravitons) that we cannot yet detect do exist. The three theories can stand alone, but can be combined as well. The pushing gravity concept for instance, can be used as an explanation for the extended Mach principle.

In a bizarre way individual photons and individual atoms seem to interfere with themselves in the famous two-slit experiment in Quantum Mechanics. An aether theory can explain the baffling interference in a very simple way [7, 8]. That is why, with an aether theory, Quantum Mechanics may also be unsettled. Next to that the intriguing black holes, sprung from the mathematics of the theory of relativity, may vanish by embracing the pushing gravity concept. (Besides, black holes may not be predicted by General Relativity [9, 10].)

3 Alternatives for the big bang

Fritz Zwicky suggested in 1929 that photons may lose energy while travelling through space, but so far his idea has always been overshadowed by the big bang explanation with stretching space. Zwicky's explanation is known as the tired light concept and it is used by alternative thinking scientists as part of a model that looks at the universe as infinite in time and space. In a tired light theory photons lose energy by interaction with gravitons or other small particles. The tired light model can be combined with an aether theory, the extended Mach principle and pushing gravity.

Next to alternatives for the theory of relativity and the stretching of light, scientists have found alternatives for the third pillar of current conventional cosmology, the cosmic background radiation discovered in 1965. That a cosmic background radiation can originate as a result of the equilibrium temperature of the universe was already suggested by many scientists in the half century preceding 1948, the year in which cosmologists predicted the cosmic background radiation of the big bang universe [11]. In a space and time infinite universe many old cooled down remnants (amongst which are dust and asteroids) of planets and stars may exist between the stars, between galaxies and between clusters of galaxies. Such remnants will eventually reach the very cold temperature (3 Kelvin) of the universe and send out radiation that corresponds with that temperature. Other examples of alternatives that can explain an equilibrium temperature are direct energy exchange between photons or indirect energy exchange between photons via gravitons or other small particles. A growing number of scientists looks at the cosmic background radiation as a result of the equilibrium temperature of a universe infinite in space and time.

In the sixteenth century Thomass Digges was the first scientist to advance a universe filled with an infinite number of stars. In the last decennium more and more scientists have taken the line of an infinite universe filled with an infinite number of galaxies. (Also because, despite all beliefs to the contrary, General Relativity may not predict an expanding universe; the Friedmann models and the Einstein-de Sitter model may be invalid [12].)

4 Clusters of galaxies at large distances?

If there was no big bang, and if we live in an infinite universe, then distances of faraway galaxies are much larger than presently thought. A few years back big bang cosmologists concluded that the big bang ought to have taken place 13.7 billion years ago. Therefore within the big bang model objects are always less than 13.7 years old. Big bang astronomers observe certain galaxies with enormous shifts of the wavelength of light and therefore think these objects sent out their light very long ago, for instance 13 billion years. With the tired light model in an infinite universe objects with such large shifts of the wavelength of light will be at distances of more than 70 billion light-years. The galaxies, which big bang astronomers now think they observe at these large distances, may therefore be clusters of galaxies in reality.

In the 1920s Edwin Hubble inaugurated a new era by finding that certain dots in the night sky are not stars, but galaxies instead. Only then did scientists realize that certain objects are at much larger distances than accepted at the time. Within the years to come new telescopes will deliver sharper images of faraway objects which are now addressed as galaxies. The big bang model already has difficulty explaining galaxies in the very early universe, because in the big bang formed, loose matter, needs time to aggregate into stars and galaxies. If it turns out that not only galaxies but also big clusters of galaxies exist in the very early universe the big bang model will probably go down. In that case there will be a lot of change within cosmology, and also the theory of relativity will then be highly questioned. With the festivities of 100 years of relativity we may have come close to the end of a scientific era.

5 Knowledge and power

If the big bang model goes down then of course the first question is: What will replace it? If the here named alternatives break through then also another question rises: Why did the alternatives need so much time to break through?

A good theory needing a lot of time to break through has happened before. In the third century BC the Greek philosopher and scientist Aristarchus published a book in which he proposed that the Earth rotates daily and revolves annually about the Sun. Eighteen hundred years later Copernicus was aware of the proposition by Aristarchus. Aristarchus and Copernicus were the heroes of the Copernican Revolution that followed after the publication of Copernicus' book *Revolutions of the Celestial Spheres* in 1543 [1]. The power of the Sun-centred model was its simplicity compared to the epicycles of the Earth-centred model.

It took a long time, after the publication of Copernicus' greatest work, before the Earth-centred model was left *en masse* for the Sun-centred model. One of the reasons for this was that, for a long time, the Earth-centred model described the movement of planets more accurately than the Sun-centred model of Copernicus. Formulae of wrong models stay dominant when alternatives are not sufficiently developed. The gravity formulae of the theory of relativity and the law of universal gravitation by Newton don't explain how gravity works, but they can be used to calculate with. The pushing gravity model explains, in a very simple way, how gravity works, but when it comes to formulae the concept is, as was the model of Copernicus four centuries ago, still in its infancy. The same applies for aether theories, the extended Mach principle, the tired light model and the equilibrium temperature of the universe as an explanation for the cosmic background radiation. The power of the aforementioned alternatives is that they form, in a very simple way, a coherent whole within an infinite universe model.

Another reason for the late definitive capitulation of the Sun-centred model was that the new model endangered the position of authority held by the Catholic Church. Four centuries ago scientific knowledge was dictated by the Catholic Church. Those who wanted to make a career as a scientist, or just wanted to stay alive as a human, were forced to canonize the Earth-centred model.

Right now established science institutes dictate knowledge when it comes to the fields of physics, cosmology and astronomy. Physics professors Assis (Brazil) and Ghosh (India) independently developed the same alternative for the theory of relativity. Both have published their work, but within the established science institutes they don't find an audience. Professor of physics, the late Paul Marmet (Canada), attached questions to the fundamental laws of nature (like the theory of relativity) and had to leave the science institute where he did his research. Right now students learn to canonize the big bang and the theory of relativity. At this moment career-fear is the big obstacle when it comes to progress in physics, cosmology and astronomy.

6 Are time and space properties of our reason?

Isaac Newton (1642–1726) thought that there was something like "absolute space" and "absolute time" and two centuries later Albert Einstein (1879–1955) melted these two together in the "space-time" concept. Newton and Einstein argued that space and time do exist physically, and ever since conventional scientists think that way too. However, it has been argued for centuries by scientists and philosophers (often scientists and philosophers at the same time) that space and time are not physically existing entities. Examples of such alternative thinkers are the Frenchman Rene Descartes (1596–1650), the Dutchman Christiaan Huygens (1629–1695), the German Gottfried Leibniz (1646–1716), the Irishman George Berkeley (1685–1753), the East-Prussian Immanuel Kant (1724–1804) and the already mentioned Austrian, Ernst Mach (1838–1916).

Our current natural sciences have their origin in Newton's laws and formulae. Many physicists, cosmologists and astronomers dismiss philosophy because they think it is misty. They feel safe with the basics and mathematics of the current conventional standard theories. Still, though mathematics is needed to do good predictions, sooner or later the whole bastion falls apart if mathematics is based upon wrong principles. Thinking about basic principles needs philosophy. Centuries ago it was the generalists, with philosophy and all the natural sciences in their package, who advocated that space and time were properties of our reason in the first place and not properties of the world. The theory of relativity has time as the fourth dimension. If time does not exist then the theory of relativity can be dismissed, and also the string theory, which has run wild with the mathematics of the theory of relativity and works with eleven dimensions.

Processes in an atomic clock slow down when the clock moves fast, and often this is seen as evidence for the existence of time. But in the case of an aether, processes in fast moving atomic clocks slow down because more aether slows down the processes in the clock. Our brains use time to compare the movement of mass with the movement of other mass. For instance the rotation of our Earth (24 hours or one day) and the orbit of our Earth around the Sun (365 days or one year). That is all; it does not mean that time really exists. If time does not exist physically then the whole scientific bastion as we have known it since Newton and, especially, as we have known it the last 100 years, falls apart.

7 Revolution by computer?

One can draw a parallel between what is happening now and what happened four centuries ago. Before Copernicus entered the scene, the Catholic Church had passed on more or less definitely settled knowledge for more than thousand years. However, where knowledge did not change much with respect to its contents, a strong development took place with respect to the passing on and propagation of the knowledge. In the early Middle Ages convents arose, in the twelfth century came the cathedral-schools and around 1200 the first universities were founded. In the course of centuries these universities gained an ever more independent position with respect to the church, which finally made the church lose its position of authority with respect to science.

Next to that in the late Middle Ages the church lost its monopoly with respect to knowledge, faster, because of the invention of the art of printing. From that moment on more people could master knowledge themselves and could have their own thoughts about it and propagate those thoughts by printing and distributing their own books.

The third development, at the end of the Middle Ages, that would help the Copernican Revolution, was the invention of the telescope, which brought new possibilities for astronomy.

A few decennia ago the computer was developed. It brought the internet, which split itself from science and obtained its own independent position. The internet brings knowledge to a lot of people all over the world. Now people can publish their ideas with respect to physics, cosmology and astronomy, independently of the universities and established periodicals. The universities lose more and more their monopoly as guardians of science, and the same goes for the periodicals that serve as their extension piece. Before the internet alternative thinking scientists were unknown isolated islands who could not publish their ideas and did not know of each other's existence. Now there are web pages which form a vibrating net of interacting alternative models, a net that grows every day. Next to that it is thanks to the computer that very strong telescopes have been put into use these last decennia, and that ever stronger and better telescopes are on their way. Perhaps the science historians of the future will conclude that it was the computer that brought the Second Copernican Revolution.

8 Conclusions

Established conventional physicists and cosmologists behave as the church at the time of Galileo. Not by threatening with the death penalty, but simply by sniffing at alternative ideas. This will change as soon as the concerning noses smell funding money instead of career-fear. In our current society money and careers are the central issues where it comes to our necessities of life. Like four centuries ago the worries about the necessities of life are the driving forces behind the impasse. Still, just as at the time of Copernicus and Galileo: under the surface of the current standard theories the revolution may be going on at full speed. In June 2005 dissidents argued at the first ever crisis in cosmology conference in Monção, Portugal [13] that the big bang theory fails to explain certain observations. The biggest revolution in the history of science may be at hand.

References

1. Harrison E. R. Cosmology: the science of the universe. Cambridge University Press, Cambridge, 2000.
2. Selleri F. Lezioni di relativita' da Einstein all' etere di Lorentz. Progedit, Bari, 2003.
3. Cahill R. T. The Michelson and Morley 1887 experiment and the discovery of absolute motion. *Progress in Physics*, 2005, v. 3, 25–29.
4. Assis A. K. T. Relational Mechanics. Apeiron, Montreal, 1999.
5. Ghosh A. Origin of Inertia. Apeiron, Montreal, 2000.
6. Van Lunteren F. *Pushing Gravity*, ed. by M. R. Edwards, 2002, 41.
7. Edwards M. R. *Pushing Gravity*, ed. by M. R. Edwards, 2002, 137.
8. Buonomano V. *Pushing Gravity*, ed. by M. R. Edwards, 2002, 303.
9. Crothers S. J. On the general solution to Einstein's vacuum field and its implications for relativistic degeneracy. *Progress in Physics*, 2005, v. 1, 68–73.
10. Crothers S. J. On the ramifications of the Schwarzschild space-time metric. *Progress in Physics*, 2005, v. 1, 74–80.
11. Assis A. K. T. and Neves M. C. D. History of the 2.7 K temperature prior to Penzias and Wilson. *Apeiron*, 1995, v. 2, 79–84.
12. Crothers S. J. On the general solution of Einstein's vacuum field for the point-mass when $\lambda \neq 0$ and its implications for relativistic cosmology. *Progress in Physics*, 2005, v. 3, 7–18.
13. Ratcliffe H. The first crisis in cosmology conference. *Progress in Physics*, 2005, v. 3, 19–24.

Sources of Stellar Energy and the Theory of the Internal Constitution of Stars

Nikolai Kozyrev*

This is a presentation of research into the inductive solution to the problem on the internal constitution of stars. The solution is given in terms of the analytic study of regularities in observational astrophysics. Conditions under which matter exists in stars are not the subject of a priori suppositions, they are the objects of research.

In the first part of this research we consider two main correlations derived from observations: "mass-luminosity" and "period — average density of Cepheids". Results we have obtained from the analysis of the correlations are different to the standard theoretical reasoning about the internal constitution of stars. The main results are: (1) in any stars, including even super-giants, the radiant pressure plays no essential part — it is negligible in comparison to the gaseous pressure; (2) inner regions of stars are filled mainly by hydrogen (the average molecular weight is close to $^1/_2$); (3) absorption of light is derived from Thomson dispersion in free electrons; (4) stars have an internal constitution close to polytropic structures of the class $^3/_2$.

The results obtained, taken altogether, permit calculation of the physical conditions in the internal constitution of stars, proceeding from their observational characteristics L, M, and R. For instance, the temperature obtained for the centre of the Sun is about 6 million degrees. This is not enough for nuclear reactions.

In the second part, the Russell-Hertzsprung diagram, transformed according to physical conditions inside stars shows: the energy output inside stars is a simple function of the physical conditions. Instead of the transection line given by the heat output surface and the heat radiation surface, stars fill an area in the plane of density and temperature. The surfaces coincide, being proof of the fact that there is only one condition — the radiation condition. Hence stars generate their energy not in any reactions. Stars are machines, directly generating radiations. The observed diagram of the heat radiation, the relation "mass-luminosity-radius", cannot be explained by standard physical laws. Stars exist in just those conditions where classical laws are broken, and a special mechanism for the generation of energy becomes possible. Those conditions are determined by the main direction on the diagram and the main point located in the direction. Physical coordinates of the main point have been found using observational data. The constants (physical coordinates) should be included in the theory of the internal constitution of stars which pretend to adequately account for observational data. There in detail manifests the inconsistency of the explanations of stellar energy as given by nuclear reactions, and also calculations as to the percentage of hydrogen and helium in stars.

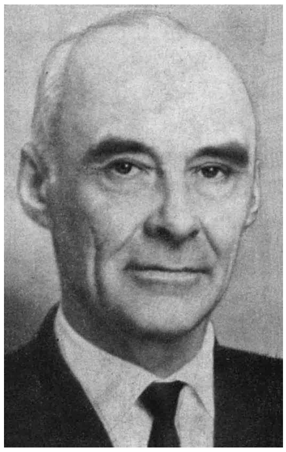
Also considered are peculiarities of some sequences in the Russell-Hertzsprung diagram, which are interesting from the theoretical viewpoint.

*Editor's remark: This is the doctoral thesis of Nikolai Aleksandrovich Kozyrev (1908–1983), the famous astronomer and experimental physicist — one of the founders of astrophysics in the 1930's, the discoverer of lunar volcanism (1958), and the atmosphere of Mercury (1963) (see the article *Kozyrev* in the *Encyclopaedia Britannica*). Besides his studies in astronomy, Kozyrev contributed many original experimental and theoretical works in physics, where he introduced the "causal or asymmetrical mechanics" which takes the physical properties of time into account. See his articles reporting on his many years of experimental research into the physical properties of time, *Time in Science and Philosophy* (Prague, 1971) and *On the Evolution of Double Stars*, *Comptes rendus* (Bruxelles, 1967). Throughout his scientific career Kozyrev worked at the Pulkovo Astronomical Observatory near St. Petersburg (except for the years 1946–1957 when he worked at the Crimean branch of the Observatory). In 1936 he was imprisoned for 10 years without judicial interdiction, by the communist regime in the USSR. Set free in 1946, he completed the draft of this doctoral thesis and published it in Russian in the local bulletin of the Crimean branch of the Observatory (*Proc. Crimean Astron. Obs.*, 1948, v. 2, and 1951, v. 6). Throughout the subsequent years he continued to expand upon his thesis. Although this research was started in the 1940's, it remains relevant today, because the basis here is observational data on stars of regular classes. This data has not changed substantially during the intervening decades. (Translated from the final Russian text by D. Rabounski and S. J. Crothers.)

The author dedicates this paper to the blessed memory of Prof. Aristarch A. Belopolski

Contents

Introduction



Prof. Nikolai Kozyrev, 1970's

Energy, radiated by the Sun and stars into space, is maintained by special sources which should keep stars radiating light during at least a few billion years. The energy sources should be dependent upon the physical conditions of matter inside stars. It follows from this fact that stars are stable space bodies. During the last decade, nuclear physics discovered thermonuclear reactions that could be the energy source satisfying the above requirements. The reactions between protons and numerous light nuclei, which result in transformations of hydrogen into helium, can be initiated under temperatures close to the possible temperature of the inner regions of stars — about 20 million degrees. Comparing different thermonuclear reactions, Bethe concluded that the energy of the Sun and other stars of the main sequence is generated in cyclic reactions where the main part is played by nitrogen and carbon nuclei, which capture protons and then produce helium nuclei [1]. This theory, developed by Bethe and widely regarded in recent years, has had no direct astrophysical verification until now. Stars produce various amounts of energy, e. g. stars of the giants sequence have temperatures much lower than that which is necessary for thermonuclear reactions, and the presence of bulk convection in upper shells of stars, supernova explosions, peculiar ultra-violet spectra lead to the conclusion that energy is generated even in the upper shells of stars and, sometimes, it is explosive. It is quite natural to inquire as to a general reason for all the phenomena. Therefore we should be more accurate in our attempts to apply the nuclear reaction theory to stars. It is possible to say (without exaggeration), that during the last century, beginning with Helmholtz's contraction hypothesis, every substantial discovery in physics led to new attempts to explain stellar energy. Moreover, after every attempt it was claimed that this problem was finally solved, despite the fact that there was no verification in astrophysical data. It is probable that there is an energy generation mechanism of a particular kind, unknown in an Earthly laboratory. At the same time, this circumstance cannot be related to a hypothesis that some exclusive conditions occur inside stars. Conditions inside many stars (e. g. the infrared satellite of ε Aurigae) are close to those that can be realized in the laboratory. The reason that such an energy generation mechanism remained elusive in experiments is due to peculiarities in the experiment statement and, possibly, in the necessity for large-scale considerations in the experiment. Considering physical theories, it is possible that their inconsistency in the stellar energy problem arises for the reason that the main principles of interaction between matter and radiant energy need to be developed further.

Much of the phenomena and empirical correlations discovered by observational astrophysics are linked to the problem of the origin of stellar energy, hence the observational data have no satisfying theoretical interpretation. First, it is related to behaviour of a star as a whole, i. e. to problems associated with the theory of the internal constitution of stars. Today's theories of the internal constitution of stars are built upon a priori assumptions about the behaviour of matter and energy in stars. One tests the truth or falsity of the theories by comparing the results of the theoretical analysis to observational data. This is one way to build various models of stars, which is very popular nowadays. But such an approach cannot be very productive, because the laws of Nature are sometimes so unexpected that many such trials, in order to guess them, cannot establish the correct solution. Because empirical correlations, characterizing a star as a whole, are surely obtained from observations, we have therein a possibility of changing the whole statement of the problem, formulating it in another way — considering the world of stars as a giant laboratory, where matter and radiant energy can be in enormously different scales of states, and proceeding from our analysis of observed empirical correlations obtained in the stellar laboratory, having made no arbitrary assumptions, we can find conditions governing the behaviour of matter and energy in stars as some unknown terms in the correlations, formulated as mathematical equations. Such a problem can seems hopelessly intractable, owing to so many unknown terms. Naturally, we do not know: (1) the phase state of matter — Boltzmann gas, Fermi gas, or something else; (2) the manner of energy transfer — radiation or convection — possible under some mechanism of energy generation; (3) the rôle of the radiant pressure inside stars, and other factors linked to the radiant pressure, namely — (4) the value of the absorption coefficient; (5) chemical composition of stars, i. e. the average numerical value of the molecular weight inside stars, and finally, (6) the mechanism generating stellar energy. To our good fortune is the fact that the main correlation of observational astrophysics, that between mass and luminosity of stars, although giving no answer as to the origin of stellar energy, gives data about the other unknowns. Therefore, employing the relation "period — average density of Cepheids", we make more precise our conclusions about the internal constitution of stars. As a result there is a possibility, even without knowledge of the origin of stellar energy, to calculate the physical conditions inside stars by proceeding from their observable characteristics: luminosity L, mass M, and radius R. On this basis we can interpret another correlation of observational astrophysics, the Russell-Hertzsprung diagram — the correlation between temperature and luminosity of stars, which depends almost exclusively on the last unknown (the mechanism generating stellar energy). The formulae obtained are completely unexpected from the viewpoint of theoretical physics. At the same time they are so typical that we have in them a possibility of studying the physical process which generates stellar energy.

This gives us an inductive method for determining a solution to the problem of the origin of stellar energy. Following this method we use some standard physical laws in subsequent steps of this research, laws which may be violated by phenomenology . However this circumstance cannot invalidate this purely astrophysical method. It only leads to the successive approximations so characteristic of the phenomenological method. Consequently, the results we have obtained in Part I can be considered as the first order of approximation.

The problem of the internal constitution of stars has been very much complicated by many previous theoretical studies. Therefore, it is necessary to consider this problem from the outset with the utmost clarity. Observations show that a star, in its regular duration, is in a balanced or quasi-balanced state. Hence matter inside stars should satisfy conditions of mechanical equilibrium and heat equilibrium. From this we obtain two main equations, by which we give a mathematical formulation of our problem. Considering the simplest case, we neglect the rotation of a star and suppose it spherically symmetric.

PART I

Chapter 1

Deducing the Main Equations of Equilibrium in Stars

1.1 Equation of mechanical equilibrium

Let us denote by P the total pressure, i. e. the sum of the gaseous pressure p and the radiant energy pressure B, taken at a distance r from the centre of a star. The mechanical equilibrium condition requires that the change of P in a unit of distance along the star's radius must be kept in equilibrium by the weight of a unit of the gas volume

$$\frac{dP}{dr} = -g\rho\,, \tag{1.1}$$

where ρ is the gas density, g is the gravity force acceleration. If φ is the gravitational potential

$$g = -\operatorname{grad}\varphi\,, \tag{1.2}$$

and the potential satisfies Poisson equation

$$\nabla^2\varphi = -4\pi G\rho\,,$$

where $G = 6.67{\times}10^{-8}$ is the gravitational constant. For spherical symmetry,

$$\nabla^2\varphi = \operatorname{div}\operatorname{grad}\varphi = \frac{1}{r^2}\frac{dr^2\operatorname{grad}\varphi}{dr}\,. \tag{1.4}$$

Comparing the equalities, we obtain the equation of mechanical equilibrium for a star

$$\frac{1}{\rho r^2}\frac{d}{dr}\left[\frac{r^2 dP}{\rho dr}\right] = -4\pi G\,, \qquad (1.5)$$

where

$$P = p + B\,. \qquad (1.6)$$

Radiations are almost isotropic inside stars. For this reason B equals one third of the radiant energy density. As we show in the next paragraph, we can put the radiant energy density, determined by the Stephan-Boltzmann law, in a precise form. Therefore,

$$B = \frac{1}{3}\,\alpha T^4, \qquad (1.7)$$

where $\alpha = 7.59{\times}10^{-15}$ is Stephan's constant, T is the absolute temperature. The pressure P depends, generally speaking, upon the matter density and the temperature. This correlation is given by the matter phase state. If the gas is ideal, it is

$$p = nkT = \frac{\Re T}{\mu}\,\rho\,. \qquad (1.8)$$

Here n is the number of particles in a unit volume of the gas, $k{=}1.372{\times}10^{-16}$ is Boltzmann's constant, $\Re{=}8.313{\times}10^{7}$ is Clapeyron's constant, μ is the average molecular weight.

For example, in a regular Fermi gas the pressure depends only on the density

$$p = K\rho^{5/3}, \quad K = \mu_e^{5/3} K_{\text{H}}\,, \quad K_{\text{H}} = 9.89{\times}10^{12}, \quad (1.9)$$

where μ_e is the number of the molecular weight units for each free electron.

We see that the pressure distribution inside a star can be obtained from (1.5) only if we know the temperature distribution. The latter is determined by the heat equilibrium condition.

1.2 Equation of heat equilibrium

Let us denote by ε the quantity of energy produced per second by a unit mass of stellar matter. The quantity ε is dependent upon the physical conditions of the matter in a star, so ε is a function of the radius r of a star. To study ε is the main task of this research. The heat equilibrium condition (known also as the energy balance condition) can be written as follows

$$\operatorname{div} F = \varepsilon\,\rho\,, \qquad (1.10)$$

where F is the total flow of energy, being the sum of the radiant energy flow F_R, the energy flow F_c dragged by convection currents, and the heat conductivity flow F_T

$$F = F_R + F_c + F_T\,. \qquad (1.11)$$

First we determine F_R. Radiations, being transferred through a layer of thickness ds, change their intensity I through the layer of thickness ds, according to Kirchhoff's law

$$\frac{dI}{ds} = -\kappa\rho\big(I - E\big)\,, \qquad (1.12)$$

where κ is the absorption coefficient per unit mass, E is the radiant productivity of an absolute black body (calculated per unit of solid angle ω). In polar coordinates this equation is

$$\cos\theta\,\frac{\partial I}{\partial r} - \frac{\sin\theta}{r}\frac{\partial I}{\partial\theta} = -\kappa\rho\big(I - E\big)\,, \qquad (1.12a)$$

where θ is the angle between the direction of the normal to the layer (the direction along the radius r) and the radiation direction (the direction of the intensity I). The flow F_R and the radiant pressure B are connected to the radiation intensity by the relations

$$F_R = \int I\cos\theta d\omega\,, \qquad Bc = \int I\cos^2\theta\, d\omega\,, \qquad (1.13)$$

where c is the velocity of light, while the integration is taken over all solid angles. We denote

$$\int I d\omega = J\,. \qquad (1.14)$$

Multiplying (1.12a) by $\cos\theta$ and taking the integral over all solid angles $d\omega$, we have

$$c\,\frac{dB}{dr} - \frac{1}{r}\,(J - 3Bc) = -\kappa\rho F_R\,.$$

In order to obtain F_R we next apply Eddington's approximation

$$3Bc = J = 4\pi E\,, \qquad (1.15)$$

thereby taking F_R to within high order terms. Then

$$F_R = -\frac{c}{\kappa\rho}\frac{dB}{dr}\,. \qquad (1.16)$$

Let us consider the convective energy flow F_c. Everyday we see huge convection currents in the surface of the Sun (it is possible this convection is forced by sudden production of energy). To make the convective energy flow F_c substantial, convection currents of matter should be rapid and cause transfer of energy over long distances in a star. Such conditions can be in regions of unstable convection of matter, where free convection can be initiated. Schwarzschild's pioneering research [2], and subsequent works by other astrophysicists (Unsöld, Cowling, Bierman and others) showed that although a star is in the state of stable mechanical and heat equilibrium as a whole, free convection can start in regions where (1) stellar energy sources rapidly increase their power, or (2) the ionization energy is of the same order as the heat energy of the gas.

We assume convection currents flowing along the radius of star. We denote by Q the total energy per unit of convection current mass. Hence, Q is the sum of the inner energy of the gas, the heat function, the potential and kinetic energies. We regularly assume that a convection current retains its own energy along its path, i. e. it changes adiabatically, and dissipation of its energy occurs only when the current stops. Then the energy flow transferred by the convection, according to Schmidt [3], is

$$F_c = -A\rho \frac{dQ}{dr}\,, \qquad A = \bar{v}\bar{\lambda}\,. \tag{1.17}$$

The quantity A is the convection coefficient, $\bar{\lambda}$ is the average length travelled by the convection current, $\bar{v}$ is the average velocity of the current. If the radiant pressure is negligible in comparison to the gaseous pressure, in an ideal gas (according to the 1st law of thermodynamics) we have

$$\frac{dQ}{dr} = c_v \frac{dT}{dr} + p\,\frac{d\,\frac{1}{\rho}}{dr}\,, \tag{1.18}$$

or, in another form,

$$\frac{dQ}{dr} = c_p \frac{dT}{dr} - \frac{1}{\rho}\frac{dp}{dr}\,, \tag{1.18}$$

where c_v is the heat capacity of the gas under constant volume, $c_p$is the heat capacity under constant pressure

$$c_p = c_v + \frac{\Re}{\mu}\,.$$

Denoting

$$\frac{c_p}{c_v} = \Gamma\,,$$

we have

$$c_p = \frac{\Gamma}{\Gamma - 1}\frac{\Re}{\mu}\,. \tag{1.20}$$

After an obvious transformation we arrive at the formulae

$$\frac{dQ}{dr} = -\frac{1}{\rho}\frac{dp}{dr}\,u\,, \qquad u = 1 - \frac{\Gamma}{4(\Gamma-1)}\frac{pdB}{Bdp}\,, \tag{1.21}$$

(for a monatomic gas $\Gamma = 5/3$).

The heat conductivity flow has a formula analogous to (1.17). Because particles move in any direction in a gas, in the formula for A we have one third of the average velocity of particles instead of $\bar{v}$. In this case dQ/dr is equal to only the first term of equation (1.18), and so dQ/dr has the same-order numerical value that it has in the energy convective flow F_c. Therefore, taking A from F_c (1.17) into account, we see that F_c is much more that F_T. In only very rare exceptions, like a degenerate gas, can the heat conductivity flow F_T be essential for energy transfer.

Using formulae (1.10), (1.16), (1.17), (1.21), we obtain the heat equilibrium equation

$$\frac{1}{\rho r^2}\frac{1}{dr}\left[\frac{r^2 db}{\kappa\rho dr}\right] - \frac{1}{c\rho r^2}\frac{1}{dr}\left[r^2 Au\frac{dp}{dr}\right] = -\frac{\varepsilon}{c}\,. \tag{1.22}$$

We finally note that, because ε is tiny value in comparison to the radiation per mass unit, even tiny changes in the state of matter should break the equalities. Therefore even for large regions in stars the heat equilibrium condition (1.10) can be locally broken. The same can be said about the equation for the convective energy flow, because huge convections in stars can be statistically interpreted in only large surfaces like that of a whole star. Therefore the equations we have obtained can be supposed as the average along the whole radius of a star, and taken over a long time. Then the equations are true.

The aforementioned limitations do not matter in our analysis because we are interested in understanding the behaviour of a star as a whole.

1.3 The main system of the equations. Transformation of the variables

In order to focus our attention on the main task of this research, we begin by considering the equations obtained for equilibrium in the simplest case: (1) in the mechanical equilibrium equation we assume the radiant pressure B negligible in comparison to the gaseous pressure p, while (2) in the heat equilibrium equation we assume the convection term negligible. Then we obtain the main system of the equations in the form

$$\left.\begin{aligned} \frac{1}{\rho r^2}\frac{d}{dr}\left[\frac{r^2 dp}{\rho dr}\right] &= -4\pi G\,, \\ \frac{1}{\rho r^2}\frac{d}{dr}\left[\frac{r^2 dB}{\kappa\rho dr}\right] &= -\frac{\varepsilon}{c}\,. \end{aligned}\right\} \tag{I}$$

The radiant pressure depends only on the gas temperature T, according to formula (1.7). The absorption coefficient κ (taken per unit mass) depends p and B. This correlation is unknown. Also unknown is the energy ε produced by a unit mass of gas. Let us suppose the functions known. Then in order to solve the system we need to have the state equation of matter, connecting ρ, p, and B. In this case only two functions remain unknown: for instance p and B, whose dependence on the radius r is fully determined by equations (I). These functions should satisfy the following boundary conditions. In the surface of a star the total energy flow is $F_0 = F_{R_0}$ ($F_c = F_T = 0$). According formula (1.13),

$$F_{R_0} = \frac{1}{2}\,J_0 = \frac{3}{2}\,cB_0\,,$$

so, taking formula (1.16) into account, we obtain the condition in the surface of a star

$$\text{under } p = 0 \text{ we have } B = -\frac{2}{3}\frac{dB}{\kappa\rho dr}\,, \tag{1.23}$$

From equations (I) we see that the finite solution condition under $r = 0$ is the same as

$$\text{under } r = 0 \text{ we have } \frac{dp}{dr} = 0\,, \quad \frac{dB}{dr} = 0\,. \tag{1.24}$$

The boundary conditions are absolutely necessary, they are true at the centre of any real star. The theory of the inner constitution of stars by Milne [4], built on solutions which do not satisfy these boundary conditions, does not mean that the boundary conditions are absolutely violated by the theory. In layers located far from the centre the boundary solutions can be realized, if derivatives of physical characteristics of matter are not continuous functions of the radius, but have breaks. Hence, Milne's theory permits a break a priori in the state equation of matter, so the theory permits stellar matter to exist in at least two different states. Following this hypothetical approach as to the properties of stellar matter, we can deduce conclusions about high temerpatures and pressures in stars. Avoiding the view that "peculiar" conditions exist in stars, we obtain a natural way of starting our research into the problem by considering the phase state equations of matter.

Hence we carry out very important transformations of the variables in the system (I). Instead of r and other variables we introduce dimensionless quantities bearing the same physical conditions. We denote by index c the values of the functions in the centre of a star ($r = 0$). Instead of r we introduce a dimensionless quantity x according to the formula

$$x = ar\,, \qquad a = \rho_c\sqrt{\frac{4\pi G}{p_c}}\,, \tag{1.25}$$

and we introduce functions

$$\rho_1 = \frac{\rho}{\rho_c}\,, \quad p_1 = \frac{p}{p_c}\,, \quad B_1 = \frac{B}{B_c}\,, \quad \dots \tag{1.26}$$

Then, as it is easy to check, the system (I) transforms to the form

$$\begin{aligned} &\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dp_1}{\rho_1 dx}\right] = -1\,, \\ &\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dB_1}{\kappa_1 \rho_1 dx}\right] = -\lambda\varepsilon_1\,, \end{aligned} \tag{Ia}$$

where

$$\lambda = \frac{\varepsilon_c \kappa_c}{4\pi G c \gamma_c}\,, \qquad \gamma_c = \frac{B_c}{p_c}\,. \tag{1.27}$$

Numerical values of all functions in the system (Ia) are between 0 and 1. Then the conditions at in the centre of a star ($x = 0$) take the form

$$p_1 = 1\,, \quad \frac{dp_1}{dx} = 0\,, \quad B_1 = 1\,, \quad \frac{dB_1}{dx} = 0\,. \tag{1.28}$$

In the surface of a star ($x = x_0$), instead of (1.23), we can use the simple conditions

$$B_1 = 0\,, \qquad p_1 = 0\,. \tag{1.29}$$

Here we can write the main system of the equations in terms of the new variables (Ia), taking convection into account. Because of (1.22), we obtain

$$\begin{aligned} &\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dp_1}{\rho_1 dx}\right] = -1\,, \\ &\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dB_1}{\kappa_1 \rho_1 dx}\right] - \frac{\kappa_c \rho_c}{c\gamma_c}\frac{1}{\rho_1 x^2}\left[x^2 A u \frac{dp_1}{dx}\right] = -\lambda\varepsilon_1\,. \end{aligned} \tag{II}$$

For an ideal gas, equation (1.21) leads to a very simple formula for u

$$u = 1 - \frac{\Gamma}{4(\Gamma - 1)}\frac{p_1\, dB_1}{B_1\, dp_1}\,. \tag{1.30}$$

Owing to (1.5) and (1.6) it follows at last that the main system of the equations, taking the radiant pressure into account in the absence of convection, takes the form

$$\begin{aligned} &\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 d(p_1 + \gamma_c B_1)}{\rho_1 dx}\right] = -1\,, \\ &\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dB_1}{\kappa_1 \rho_1 dx}\right] = -\lambda\varepsilon_1\,. \end{aligned} \tag{III}$$

Chapter 2

Analysis of the Main Equations and the Relation "Mass-Luminosity"

2.1 Observed characteristics of stars

Astronomical observations give the following quantities characterizing star: radius R, mass M, and luminosity L (the total energy radiated by a star per second). We are going to consider correlations between the quantities and parameters of the main system of the star equilibrium equations. As a result, the main system of the equations considered under any phase state of stellar matter includes only two parameters characterizing matter and radiation inside a star: B_c and p_c.

Because of formula (1.25), we obtain

$$R = \frac{1}{\rho_c}\sqrt{\frac{p_c}{4\pi G}}\, x_0\,, \tag{2.1}$$

where x_0 is the value of x at the surface of a star, where $p_1 = B_1 = 0$. With this formula, and introducing a state equation of matter, we can easily obtain the correlation $R = f(B_c, p_c)$. It should be noted that in the general case the value of x_0 in formula (2.1) is dependent on B_c and p_c. At the same time, because the equation system consists of functions variable between 0 and 1, the value of x_0 should be of the same order (i. e. close to 1). Therefore the first multiplier in (2.1) plays the main rôle.

Because of

$$M = 4\pi \int_0^R \rho r^2 dr \,,$$

we have

$$M = \frac{p_c^{3/2}}{G^{3/2}\sqrt{4\pi}\,\rho_c^2}\, M_{x_0} \,, \tag{2.2}$$

where

$$M_{x_0} = \int_0^{x_0} \rho_1 x^2 dx \,.$$

At last, the total luminosity of star is

$$L = 4\pi \int_0^R \varepsilon \rho r^2 dr \,,$$

and we obtain

$$\frac{L}{M} = \varepsilon_c \frac{L_{x_0}}{M_{x_0}} \,, \qquad L_{x_0} = \int_0^{x_0} \varepsilon_c \rho_1 x^2 dx \,. \tag{2.3}$$

Values of the quantities M_{x_0} and L_{x_0} should change a little under changes of p_c and B_c, remaining close to 1. If x_0, M_{x_0}, and L_{x_0} are the same for numerous stars, such stars are homological, so the stars actually have the *same structure*.

As it is easy to see, the average density $\bar{\rho}$ of star is connected to ρ_c by the formula

$$\bar{\rho} = \rho_c \frac{3M_{x_0}}{x_0^3} \,. \tag{2.4}$$

We find a formula for the total potential energy Ω of star thus

$$\Omega = -G \int_0^R \frac{M_r}{r}\, dM_r \,.$$

Multiplying the term under the integral by R, and dividing by M^2, we obtain

$$\Omega = -\frac{GM^2}{R}\, \Omega_{x_0} \tag{2.5}$$

and also

$$\Omega_{x_0} = \frac{x_0}{M_{x_0}^2} \int_0^{x_0} x \rho_1 M_x dx \,.$$

Under low radiant pressure, taking the equation of mechanical equilibrium into account, the system (I) gives

$$\int_0^{x_0} x \rho_1 M_x dx = -\int_0^{x_0} x^3 dp_1 = 3\int_0^{x_0} x^2 p_1 dx \,, \tag{2.5a}$$

from which we obtain

$$\Omega_{x_0} = \frac{3x_0 \displaystyle\int_0^{x_0} p_1 x^2 dx}{\left[\displaystyle\int_0^{x_0} \rho_1 x^2 dx\right]^2} \,. \tag{2.6}$$

Because all the functions included in the main system of equations can be expressed through B_1 and p_1, we can find the functions from the system of the differential equations with respect to two parameters B_c and p_c. Boundary conditions (1.28) are enough to find the solutions at the centre of a star. Hence, boundary conditions at the surface of a star (1.29) are true under only some relations between B_c and p_c. Therefore all quantities characterizing a star are functions of only one of two parameters, for instance B_c: $R = f_1(B_c)$, $M = f_2(B_c)$, $L = f_3(B_c)$. This circumstance, with the same chemical composition of stars, gives the relations: (1) "mass-luminosity" $L = \varphi_1(M)$ and (2) the Russell-Hertzsprung diagram $L = \varphi_2(R)$.

From the above we see that the equilibrium of stars has this necessary consequence: correlations between M, L, and R. Thus the correlations discovered by observational astrophysics can be predicted by the theory of the inner constitution of stars.

2.2 Stars of polytropic structure

Solutions to the main system of the equations give functions $p_1(x)$ and $B_1(x)$. Hence, solving the system we can as well obtain $B_1(p_1)$. If we set up a phase state, we can as well obtain the function $p_1(\rho_1)$.

Let us assume $p_1(\rho_1)$ as $p_1(\rho_1^{\Gamma})$, where Γ is a constant. Such a structure for a star is known as *polytropic*. Having stars of polytropic structure, we can easily find all the functions of x. Therefore, in order to obtain a representation of the solutions in the first instance, we are going to consider stars of polytropic structure. Emden's pioneering research on the internal constitution of stars was done in this way.

The aforementioned polytropic correlation can be used instead of the heat equilibrium equation, so only the first equation remains in the system. We introduce a new variable T_1 which, in an ideal gas, equals the reduced temperature

$$\frac{p_1}{\rho_1} = \rho_1^{\Gamma-1} = T_1 \,, \tag{2.7}$$

or, in another form,

$$\rho_1 = T_1^n \,, \qquad n = \frac{1}{\Gamma - 1} \,, \qquad p_1 = T_1^{n+1} \,, \tag{2.7a}$$

so that we obtain

$$dp_1 = (n+1)\, T_1^n dT \,.$$

Substituting the formulae into the first equation of the main system (I), we obtain

$$E\left[T_1'\right] = \frac{1}{x_1^2} \frac{1}{dx_1} \left[x_1^2 \frac{dT_1}{dx_1}\right] = -T_1^n \,, \tag{2.8}$$

where a new variable x_1 is introduced instead of x

$$x = \sqrt{n+1}\, x_1 \,. \tag{2.9}$$

Emden's equation (2.8) can be integrated very easily if $n=0$ or $n=1$. Naturally, under $n=0$ (a star of constant density) we obtain

$$p_1 = T_1 = 1 - \frac{x_1^2}{6}\,, \qquad (2.10)$$

so the remaining characteristics can be calculated just as easily. Under $n=1$ the substitution $n = T_1\, x_1$ reduces the differential equation (2.8) to the simple form $n'' = -n$. Hence, under $n=1$, we have

$$T_1 = \frac{\sin x_1}{x_1}\,, \qquad p_1 = \frac{\sin^2 x_1}{x_1^2}\,. \qquad (2.11)$$

With other polytropic indices n, we obtain solutions which are in series. All odd derivatives of the operator E should become zero under $x_1 = 0$. For even derivatives, we have

$$E_0^{(2i)}\left[T_1'\right] = \frac{2i+3}{2i+1}\, T_1^{(2i+2)}(0)\,. \qquad (2.12)$$

Now, differentiating equation (2.8), we obtain derivatives in different orders of the function T_1 under $x_1 = 0$, so we obtain the coefficients of the series expansion. As a result we obtain the series

$$\begin{aligned} T_1 &= 1 - \frac{x_1^2}{3!} + \frac{n}{5!}\,x_1^4 - \frac{n\,(8n-5)}{3{\times}7!}\,x_1^6 + \\ &+ \frac{n\,(122n^2 - 183n + 70)}{9{\times}9!}\,x_1^8 + \ldots \end{aligned} \qquad (2.13)$$

Using (2.13), we move far away from the special point $x_1 = 0$. Subsequent solutions can be obtained by numerical integration. As a result we construct a table containing characteristics of stellar structures under different n (see Table 1).

The case of $n = 3/2$ corresponds to an adiabatic change of the state of monatomic ideal gas ($\Gamma = 5/3$) and also a regular Fermi gas (1.9). If $n = 3$, we get a relativistic Fermi gas or an ideal gas under $B_1 = p_1$ (the latter is known as Eddington's solution).

In polytropic structures we can calculate exact values of Ω_{x_0}. Naturally, the integral of the numerator of (2.6) can be transformed to

$$\int_0^{x_0} p_1 x^2 dx = \int_0^{x_0} T_1\, dM_x = -\int_0^{x_0} M_x \frac{dT_1}{dx}\, dx\,.$$

Emden's equation leads to

$$M_x = -(n+1)\, x^2 \frac{dT_1}{dx}\,, \qquad (2.14)$$

so we obtain

$$\begin{aligned} \int_0^{x_0} p_1 x^2 dx &= \frac{1}{n+1}\int_0^{x_0} \frac{M_x^2}{x^2}\, dx = \\ &= -\frac{M_{x_0}^2}{x_0\,(n+1)} + \frac{2}{n+1}\int_0^{M_{x_0}} \frac{M_x}{x}\, dM_x\,. \end{aligned}$$

Table 1

n	x_0	M_{x_0}	$\frac{x_0^2}{3M_{x_0}}$	Ω_{x_0}
0	2.45	4.90	1.0	3/5
1	4.52	9.04	3.4	3/4
3/2	5.81	11.1	5.9	6/7
2	7.65	12.7	11.4	1
2.5	10.2	14.4	24.1	6/5
3	13.8	16.1	54.4	3/2
3.25	17.0	17.5	88.2	12/7

Formula (2.5a) leads to another relation between the integrals. As a result we obtain

$$\left[1 - \frac{6}{n+1}\right]\int_0^{x_0} p_1 x^2 dx = -\frac{M_{x_0}^2}{x_0\,(n+1)}\,,$$

and, substituting this into (2.6), we obtain Ritter's formula

$$\Omega_{x_0} = \frac{3}{5-n}\,. \qquad (2.15)$$

This formula, in addition to other conclusions, leads to the fact that a star can have a finite radius only if $n < 5$.

2.3 Solution to the simplest system of the equations

To begin, we consider the system (Ia), which is true in the absence of convection and if the radiant pressure is low. The absorption coefficient κ, the quantity of produced energy ε, and the phase state equation of matter, can be represented as products of different power functions p, B, ρ. Then the functions $\kappa_1 = \kappa/\kappa_c$, $\varepsilon_1 = \varepsilon/\varepsilon_c$, and the phase state equation, are dependent only on p_1, B_1, ρ_1; they have no parameters p_c, B_c, ρ_c. In this case the coefficient λ remains the sole parameter of the system. In this simplest case we study the system (Ia) under further limitations: we assume an ideal gas and κ independent of physical conditions. Thus, we have the correlations

$$\kappa = \text{const:}\;\; \kappa_1 = 1,\;\; p_1 = B_1^{1/4}\rho_1\,,\;\; \varepsilon_1 = f\,(p_1, B_1)\,, \qquad (2.16)$$

$$\begin{aligned} \frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dp_1}{\rho_1 dx}\right] &= -1\,, \\ \frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dB_1}{\rho_1 dx}\right] &= -\lambda\varepsilon_1\,, \end{aligned} \qquad (2.17)$$

where

$$\lambda = \frac{\varepsilon_c \kappa_c}{4\pi G c\,\gamma_c} \qquad \gamma_c = \frac{B_c}{p_c}\,. \qquad (2.18)$$

Taking integrals on the both parts of (2.17), we obtain

$$\frac{x^2}{\rho_1}\frac{dB_1}{dx} = -\lambda L_x\,, \qquad \frac{x}{\rho_1}\frac{dp_1}{dx} = -M_x\,, \qquad (2.19)$$

where we have introduced the notation

$$L_x = \int_0^x \varepsilon_1 \rho_1 x^2 dx\,, \qquad M_x = \int_0^x \rho_1 x^2 dx\,. \tag{2.20}$$

Integrating (2.19) using boundary conditions, we obtain

$$\lambda = \frac{l}{\displaystyle\int_0^{x_0} L_x \frac{\rho_1}{x^2}\,dx}\,, \qquad l = \int_0^{x_0} M_x \frac{\rho_1}{x^2}\,dx\,,$$

hence

$$\lambda = \frac{\displaystyle\int_0^{x_0} M_x \frac{\rho_1}{x^2}\,dx}{\displaystyle\int_0^{x_0} L_x \frac{\rho_1}{x^2}\,dx}\,. \tag{2.21}$$

From formulae (2.21) and (2.20) we conclude that the more concentrated are the sources of stellar energy, the greater is λ. If the source's productivity ε increases towards the centre of a star, $\lambda > 1$. If $\varepsilon = \text{const}$ along the radius, $\varepsilon_1 = 1$ and hence $\lambda = 1$. If stellar energy is generated mostly in the surface layers of a star, $\lambda < 1$. Equations (2.19) lead to

$$\frac{dB_1}{dp_1} = \frac{\lambda L_x}{M_x}\,. \tag{2.22}$$

Because of the boundary conditions $p_1 = 0$, $B_1 = 0$ and $p_1 = 1$, $B_1 = 1$, the derivative dB_1/dp_1 always takes the average value 1. Owing to

$$\left(\frac{dB_1}{dp_1}\right)_{x=0} = \lambda\,, \qquad \left(\frac{dB_1}{dp_1}\right)_{x=x_0} = \frac{\lambda L_{x_0}}{M_{x_0}}\,,$$

we come to the following conclusions: if energy sources are located at the centre of a star, $\lambda L_{x_0}/M_{x_0} < 1$; if energy sources are located on the surface, $\lambda L_{x_0}/M_{x_0} > 1$. If energy sources are homogeneously distributed inside a star, $\lambda L_{x_0}/M_{x_0} = 1$ and $B_1 = p_1$, so we have polytropic class 3, considered in the previous paragraph. This particular solution is the basis of Eddington's theory of the internal constitution of stars. If $n > 3$, $(dB_1/dp_1)_{x_0} \to \infty$ so we have $L_{x_0} \to \infty$. Therefore we conclude that polytropic classes $n > 3$ characterize stars where energy sources concentrate near the surface. Polytropic classes $n < 3$ correspond to stars where energy sources concentrate at the centre. Therefore the data of Table 1 characterize the most probable structures of stars. It should be noted that if $n < 3$, formulae (2.7) and (2.7a) lead to $(dB_1/dp_1)_{x_0} = 0$, and hence $L_{x_0} = 0$. So polytropic structures of stars where energy sources concentrate at the centre can exist only if there is an energy drainage in the upper layer of a star.

Differentiating formula (2.22) step-by-step and using the system (2.17) gives derivatives of $B_1(p_1)$ under $p_1 = 1$ and, hence, expansion of $B_1(p_1)$ into a Taylor series. The first terms of the expansion take the form

$$B_1 = 1 + \lambda(p_1 - 1) + \frac{3}{10}\lambda\left[\frac{\partial\varepsilon_1}{\partial p_1} + \lambda\frac{\partial\varepsilon_1}{\partial B_1}\right]_1 (p_1 - 1)^2 + \ldots$$

The surface condition $B_1 = 0$, being applied to this formula under $p_1 = 0$, gives an equation determining λ. This method gives a numerical value of λ which can be refined by numerical integration of the system (2.17). This integration can be done step-by-step.

The centre of a star, i.e. the point where $x = 0$, is the singular point of the differential equations (2.17). We can move far away from the singular point using series and then (as soon as their convergence becomes poor) we apply numerical integration. We re-write the system (2.7) as follows

$$\begin{aligned} E\left[\frac{B_1^{1/4}}{p_1}\frac{dp_1}{dx}\right] &= -p_1 B_1^{-1/4}\,, \\ E\left[\frac{B_1^{1/4}}{p_1}\frac{dB_1}{dx}\right] &= -\lambda\varepsilon_1 p_1 B_1^{-1/4}\,. \end{aligned} \tag{2.23}$$

Formula (2.12) gives

$$E_0^{(2i)}[u] = \frac{2i+3}{2i+1}[u]_0^{(2i+1)}\,. \tag{2.24}$$

Then, differentiating formula (2.23) step-by-step using (2.24), we obtain different order derivatives of the functions $p_1(x)$ and $B_1(x)$ under $x = 0$ that yields the possibility of expanding the functions into Laurent series. Here are the first few terms of the expansions

$$\begin{aligned} p_1 &= 1 - \frac{1}{3}\frac{x^2}{2!} + \frac{2}{15}[4-\lambda]\frac{x^4}{4!} - \ldots \\ B_1 &= 1 - \frac{\lambda}{3}\frac{x^2}{2!} + \\ &+ \frac{2\lambda}{15}\left[(4-\lambda) + \frac{3}{2}\left(\frac{\partial\varepsilon_1}{\partial p_1} + \lambda\frac{\partial\varepsilon_1}{\partial B_1}\right)_0\right]\frac{x^4}{4!} - \ldots \end{aligned} \tag{2.25}$$

In order to carry out numerical integration we use formulae which can be easily obtained from the system (2.23), namely

$$\begin{aligned} p_1'' &= -p_1^2 B_1^{-1/2} + p_1'\left[\frac{p_1'}{p_1} - \frac{B_1'}{4B_1} - \frac{2}{x}\right], \\ B_1'' &= -\lambda\varepsilon_1 p_1^2 B_1^{-1/2} + B_1'\left[\frac{p_1'}{p_1} - \frac{B_1'}{4B_1} - \frac{2}{x}\right]. \end{aligned} \tag{2.23a}$$

In this system, we introduce the reduced temperature T_1 instead of B_1, and a new variable $u_1 = p_1^{1/4}$ instead of p_1

$$\begin{aligned} u_1'' &= -\frac{u_1^5}{4T_1^2} + u_1'\left[\left(\frac{u_1'}{u_1} - \frac{T_1'}{T_1}\right) - \frac{2}{x}\right], \\ T_1'' &= -\frac{\lambda\varepsilon_1 u_1^8}{4T_1^5} + T_1'\left[4\left(\frac{u_1'}{u_1} - \frac{T_1'}{T_1}\right) - \frac{2}{x}\right]. \end{aligned} \tag{2.23b}$$

This substitution gives a great advantage, because of small slow changes of the functions T_1 and u_1.

A numerical solution can be obtained close to the surface layer, but not in the surface itself, because the equations (2.23) can be integrated in the upper layers without problems. Naturally, assuming $M_x = M_{x_0} = \text{const}$ and $L_x = L_{x_0} = \text{const}$ in formula (2.19), we obtain

$$\frac{dp_1}{\rho_1} = -\frac{M_{x_0}}{x^2}\,dx\,, \qquad \frac{dB_1}{\rho_1} = -\frac{\lambda L_{x_0}}{x^2}\,dx\,, \qquad B_1 = \frac{\lambda L_{x_0}}{M_{x_0}}\,p_1\,. \tag{2.26}$$

The ideal gas equation and the last relation of (2.26) permit us to write down

$$\frac{dp_1}{\rho_1} = B_1^{1/4}\frac{dp_1}{p_1} = B_1^{-3/4}dB_1\,.$$

Integrating the first equation of (2.26), we obtain

$$4T_1 = M_{x_0}\frac{x_0 - x}{x_0 x}\,, \tag{2.27}$$

which gives a linear law for the temperature increase within the uppermost layers of a star.

To obtain λ by step-by-step integration, we need to have a criterion by which the resulting value is true. It is easy to see from (2.26) that such a criterion can be a constant value for the quotient B_1/p_1 starting from x located far away from the centre of a star. Solutions are dependent on changes of λ, therefore an exact numerical value of this parameter should be found. Performing the numerical integration, values of the functions near the surface of a star are not well determined. Therefore, in order to calculate L_{x_0} and M_{x_0} in would be better to use their integral formulae (2.20). If energy sources increase their productivity towards the centre of a star, we obtain an exact value for L_{x_0} even in a very rough solution for the system. The calculation of x_0 is not as good, but it can be obtained for fixed M_{x_0} and x far away from the centre through formula (2.27)

$$x_0 = \frac{x}{1 - \frac{4T_1}{M_{x_0}}\,x}\,. \tag{2.27a}$$

Using the above method, exact solutions to the system are obtained. Table 2 contains the characteristics of the solutions in comparison to the characteristics of Eddington's model*.

The last column contains a characteristic that is very important for the "mass-luminosity" relation (as we will see later).

Let us determine what changes are expected in the characteristics of the internal constitution of stars if the absorption coefficient κ is variable. If κ is dependent on the physical conditions, equation (2.22) takes the form

$$\frac{dB_1}{dp_1} = \frac{\kappa_1 \lambda L_x}{M_x}\,. \tag{2.22a}$$

*In his model $\varepsilon_1 = 1$, so the energy sources productivity is $\varepsilon = \text{const}$ along the radius (see the first row in the table). — Editor's remark.

Table 2

ε_1	λ	x_0	M_{x_0}	L_{x_0}	$\frac{\lambda L_{x_0}}{M_{x_0}^3}$
1	1	13.8	16.1	16.1	$3.8{\times}10^{-3}$
B_1	1.76	10	12.4	2.01	$1.8{\times}10^{-3}$
$B_1 p_1$	2.32	9	11.5	1.57	$2.2{\times}10^{-3}$

The variability of κ can be determined by a function of the general form

$$\kappa_1 = \frac{p_1^{\alpha}}{B_1^{\beta}}\,.$$

At first we consider the simplest case where energy sources are homogeneously distributed inside a star. In this case $\varepsilon_1 = 1$, $L_x = M_x$, and equation (2.22a) can be integrated

$$B_1^{1+\beta} = \lambda\,\frac{1+\beta}{1+\alpha}\,p_1^{1+\alpha}\,.$$

Proceeding from the conditions at the centre of any star ($B_1 = p_1 = 1$), we obtain

$$\lambda = \frac{1+\alpha}{1+\beta}\,, \qquad B_1 = p_1^{\lambda}\,.$$

Hence the star has polytropic structure of class

$$n = \frac{4}{\lambda} - 1\,.$$

Looking from the physical viewpoint, the most probable effects are: decrease in the absorption coefficient of a star with depth, and also $\alpha \geqslant -1$. Because

$$\kappa_1 = p_1^{\frac{\alpha-\beta}{1+\beta}} = B_1^{\frac{\alpha-\beta}{1+\alpha}}\,,$$

κ_1 decreases with increase of p_1 and B_1 only if $\alpha < \beta$. Then it is evident that $\lambda < 1$ and $n > 3$. Hence, variability of κ results in an increase of polytropic class. According to the theory of photoelectric absorption,

$$\kappa_1 = \frac{\rho_1}{T_1^{3.5}}\,.$$

In this case $\alpha = 1$, $\beta = 1.125$ and hence $n = 3.25$. Table 1 gives respective numerical values of the characteristics x_0 and M_{x_0}. Other calculated characteristics are $\lambda = 0.94$ and $\lambda L_{x_0}/M_{x_0}^3 = \lambda/M_{x_0}^2 = 3.06{\times}10^{-3}$. All the numerical values are close to those calculated in Table 2. This is expected, if variability of κ leads to the same order effect for the other classes of energy source distribution inside stars.

Looking at Table 1 and Table 2 we see that the characteristics $x_0 \simeq M_{x_0} \simeq 10$ and $\lambda L_{x_0}/M_{x_0}^3 \simeq 2{\times}10^{-3}$ have tiny changes under different suppositions about the internal constitution of stars (the internal distribution of energy sources)*. There are three main cases: (1) sources of stellar energy, homogeneously distributed inside a star, (2) energy sources are so strongly concentrated at the centre of star that their productivity is proportional to the 8th order of the temperature, (3) polytropic structures where energy sources are concentrated at the surface — there is a drainage in the surface layer of a star. It should be noted that we did not consider other possible cases of distributed energy sources in a star, such as production of energy in only an "energetically active" layer at a middle distance from the centre. In such distributed energy sources, as it is easy to see from the second equation of the main system, there should be an isothermal core inside a star, and such a star is close to polytropic structures higher than class 3. In this case, instead of the former ε_1, we can build $\varepsilon/\varepsilon_{\max}=\varepsilon_1$, $0\leqslant\varepsilon_1\leqslant 1$, which will be subsumed into λ. However in such a case ε_1, and hence all characteristics obtained as solutions to the system, is dependent on p_c and B_c, and the possibility to solve the system everywhere inside a star sets up as well correlations between the parameters. At last we reach the very natural conclusion that energy is generated inside a star only under specific relations between B and p in that quantity which is required by the compatibility of the equilibrium equations. In order to continue this research and draw conclusions, it is very important to note the fact that the characteristic $\lambda L_{x_0}/M^3_{x_0}$ is actually the same for any stellar structure (see the last column in Table 2). This characteristic remains almost constant under even exotic distributions of energy sources in stars (exotic sources of stellar energy), because of a parallel increase/decrease of its numerator and denominator. Following a line of successive approximations, we have a right to accept the tables as the first order approximation which can be compared to observational data. All the above conclusions show that it is not necessary to solve the main system of the equilibrium equations (2.17) for more detailed cases of the aforementioned structures of stars. Therefore we did not prove the uniqueness of the parameter λ.

2.4 Physical conditions at the centre of stars

The average density of the Sun is $\bar{\rho}_\odot=1.411$. Using this numerical value in (2.4), we obtain a formula determining the central density of stars

$$\rho_c = 0.470\,\frac{x_0^3}{M_{x_0}}\,\frac{\frac{M}{M_\odot}}{\left(\frac{R}{R_\odot}\right)^3}\,. \tag{2.28}$$

Taking this into account, formula (2.1) permits calculation of the gaseous pressure at the centre of a star

$$p_c = \frac{G}{4\pi}\left(\frac{M_\odot}{R^2_\odot}\right)^2 \frac{x_0^4}{M^2_{x_0}}\,\frac{\left(\frac{M}{M_\odot}\right)^2}{\left(\frac{R}{R_\odot}\right)^4}\,. \tag{2.29}$$

Because $M_\odot=1.985\times10^{33}$ and $R_\odot=6.95\times10^{10}$, we obtain

$$p_c = 8.9\times10^{14}\,\frac{x_0^4}{M^2_{x_0}}\,\frac{\left(\frac{M}{M_\odot}\right)^2}{\left(\frac{R}{R_\odot}\right)^4}\,. \tag{2.30}$$

Thus the pressure at the centre of the Sun should be about 10^{16} dynes/cm^2 (ten billion atmospheres). It should be noted, as we see from the deductive method, the formulae for ρ_c and p_c are applicable to any phase state of matter.

Let us assume stars consisting of an ideal gas. Then taking the ratio of (2.30) to (2.28) and using the ideal gas equation (1.8), we obtain the temperature at the centre of a star

$$T_c = 2.29\times10^7\mu\,\frac{x_0}{M_{x_0}}\,\frac{\frac{M}{M_\odot}}{\frac{R}{R_\odot}}\,. \tag{2.31}$$

Hence, the temperature at the centre of the Sun should be about 10 million degrees. As another example, consider the infrared satellite of ε Aurigae. For this star we have $M=24.6M_\odot$, $\log\left(L/L_\odot\right)=4.46$, $R=2{,}140R_\odot$ [5]. Calculating the central density and temperature by formulae (2.30) and (2.31), we obtain $T_c\simeq2\times10^5$ and $p_c\simeq2\times10^5$: thus the temperature is about two hundred thousand degrees and the pressure about one atmosphere. Because the star is finely located in the "mass-luminosity" diagram (Fig. 1) and the Russell-Hertzsprung diagram, we have reason to conclude: *the star has the internal constitution regular for all stars*. This conclusion can be the leading arrow pointing to the supposition that heat energy is generated in stars under physical conditions close to those which can be produced in an Earthly laboratory.

Let us prove that only inside white dwarfs (the stars of the very small radii — about one hundredth of $R_\odot$), the degenerate Fermi gas equation (1.9) can be valid. Naturally, if gas at the centre of a star satisfies the Fermi equation, we obtain $p_c=1\times10^{13}\rho_c^{5/3}\mu_e^{-5/3}$. Formulae (2.28) and (2.30) show that this condition is true only if

$$\frac{R}{R_\odot} = 3.16\times10^{-3}\,\frac{x_0 M^{1/3}_{x_0}}{\left(\frac{M}{M_\odot}\right)^{1/3}}\,\mu_e^{-5/3}\,. \tag{2.32}$$

This formula remains true independently of the state of matter in other parts of the star. The last circumstance can affect only the numerical value of the factor $x_0M^{1/3}_{x_0}$. At the same time, table 1 shows that we can assume the numerical value approximately equal to 10.† Formula (2.32) shows that

*It should be noted that the tables characterize the structure of stars only if the radiant pressure is low. In the opposite case all the characteristics x_0, M_{x_0}, and others are dependent on γ_c.

†For stars of absolutely different structure, including such boundary instances as the naturally impossible case of equally dense stars, and the cases where energy sources are located at the surface. — Editor's remark.

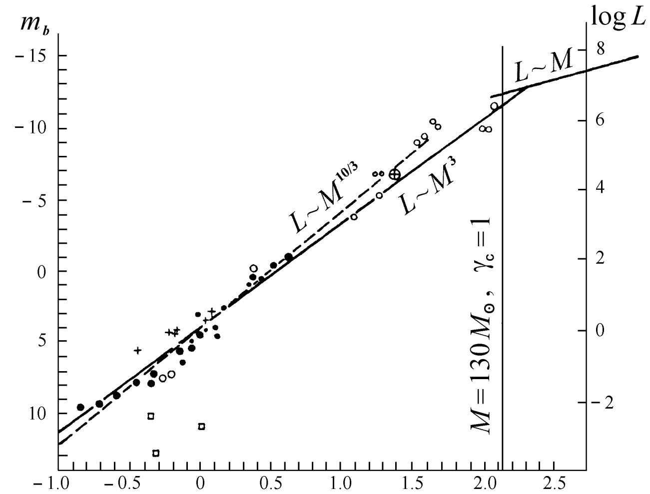


Fig. 1: The "mass-luminosity" relation. Here points are visual binaries, circles are spectral-binaries and eclipse variable stars, crosses are stars in Giades, squares are white dwarfs, the crossed circle is the satellite of ε Aurigae.

for regular degeneration of gas, stars (under $M = M_\odot$) should have approximately the same radius $R \simeq 2{\times}10^9$, i. e. about 20,000 km ($R = 0.03 R_\odot$). Such dimensions are attributed to white dwarfs. For example, the satellite of Sirius has $M = 0.94 M_\odot$ and $R = 0.035 R_\odot$ [6]. If the density is more than the above mentioned (if the radius is less than $R = 0.03 R_\odot$) and the mass of the star increases, formula (2.32) shows that regular degeneration can become relativistic degeneration

$$p = K\rho^{4/3}, \qquad K = K_{\rm H}\,\mu_e^{-4/3}, \qquad K_{\rm H} = 1.23{\times}10^{15}.$$

We apply these formulae to the centre of a star, and take equations (2.28) and (2.30) into account. As a result we see that the radius drops out of the formulae, so relativistic degeneration can be realized in a star solely in terms of the mass

$$\frac{M}{M_\odot} = 0.356 M_{x_0}\,, \qquad (\mu_e = 1). \tag{2.32a}$$

Because of Table 1, we see: $n = 0$ only if $M_{x_0} = 16.1$. Hence, the lower boundary of the mass of a non-degenerated gaseous star is $5.7 M_\odot$. In order to study degenerated gaseous stars in detail, we should use the phase state equation that includes the regular state, the boundary state between the regular and degenerated states, and the degenerated state. Applying formulae (2.28) and (2.30) to the above ratio, we obtain a correlation between the radius and the mass of a star, which is unbounded for small radii. It should be noted that introduction of a mass-radius correlation is the essence of Chandrasekhar's theory of white dwarfs [7]. On the other hand, having observable sizes of white dwarfs, equation (2.32) taken under $x_0 M_{x_0}^{1/3} = 10$ gives the same numerical values for radii as Chandrasekhar's table (his well-known relation between the radius and mass of star). The exact numerical value of the ultimate mass calculated by him coincides with our $5.7 M_\odot$. In Chandrasekhar's formula, as well as in our formula (2.32), radius is correlated opposite to mass. Today we surely know masses and radii of only three white dwarfs: the white dwarfs do not confirm the opposite correlation mass-radius. So, save for the radius of Sirius' satellite coinciding with our formula (2.32), we have no direct astrophysical confirmation about degeneration of gas inside white dwarfs.

Considering stars built on an ideal gas, we deduce a formula determining the mass of a star dependent on internal physical conditions. We can use formulae (2.30) and (2.31) or formula (2.2) directly. Applying the Boyle-Mariotte equation (1.8) to formula (2.2), and taking the Stephan-Boltzmann law (1.7) into account, we obtain

$$M = C\,\frac{\gamma_c^{1/2}}{\mu^2}\,M_{x_0}\,, \qquad C = \frac{\Re^2}{G^{3/2}\sqrt{\frac{4}{3}\pi\alpha}} = 2.251{\times}10^{33}. \tag{2.33}$$

Introducing the mass of the Sun $M_\odot = 1.985{\times}10^{33}$ into the equation, we obtain

$$M = 1.134 M_\odot\,\frac{\gamma_c^{1/2}}{\mu^2}\,M_{x_0}\,. \tag{2.34}$$

As we will see below, the "mass-luminosity" correlation shows γ_c is close to 1 for blue super-giants. Hence formula (2.34) gives the observed numerical values for masses of stars. The fact that we obtain true orders for numerical values of the masses of stars, proceeding only from numerical values of the fundamental constants G, $\Re$, α, is excellent confirmation of the theory.

2.5 The "mass-luminosity" relation

In deducing the "mass-luminosity" correlation, we assume: (1) the radiant pressure is negligible in comparison to the gaseous pressure everywhere inside a star; (2) stars consist of an ideal gas; (3) ε and κ can be approximated by functions like $p^\alpha B^\beta$. Then the main system of the equilibrium equations takes the form

$$\begin{aligned} \frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dp_1}{\rho_1 dx}\right] &= -1\,,\\ \frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dB_1}{\kappa_1\rho_1 dx}\right] &= -\lambda\varepsilon_1\,, \end{aligned} \tag{2.35}$$

where

$$\lambda = \frac{\varepsilon_c \kappa_c}{4\pi G c \gamma_c}\,, \qquad \gamma_c = \frac{B_c}{p_c}\,.$$

Solving the system, as we know, is possible under a numerical value of λ close to 1. Hence a star can be in equilibrium only if the energy generated inside it is determined by the formula

$$\varepsilon_c = \frac{\lambda 4\pi G c}{\kappa_c}\,\gamma_c\,. \tag{2.36}$$

If a star produces another quantity of energy, it will contract or expand until its new shape results in production of energy exactly by formula (2.36). Because γ_c determines the mass of a star (2.34) and ε_c determines the luminosity of a star, the "mass-luminosity" correlation should be contained in formula (2.36). In other words, the "mass-luminosity" correlation is the condition of equilibrium of stars.

Because of (2.3),

$$\varepsilon_c = \frac{L}{M}\frac{M_{x_0}}{L_{x_0}}\,.$$

Substituting this equation into (2.36), we obtain

$$L = \frac{4\pi G c}{\kappa_c}\frac{\lambda L_{x_0}}{M_{x_0}}\,M\gamma_c\,.$$

The quantity γ_c can be removed with the mass of a star by (2.33)

$$L = \frac{4\pi G^4 4\pi\alpha}{3\kappa_c \Re^4}\,\mu^4\left(\frac{\lambda L_{x_0}}{M^3_{x_0}}\right)M^3. \tag{2.37}$$

The luminosity of the Sun is $L_\odot = 3.78\times10^{33}$. Proceeding from formula (2.37), we obtain

$$\frac{L}{L_\odot} = 1.04\times10^4\,\frac{\mu^4}{\kappa_c}\left(\frac{\lambda L_{x_0}}{M^3_{x_0}}\right)\left(\frac{M}{M_\odot}\right)^3. \tag{2.38}$$

The formula (2.38) gives a very simple correlation: the luminosity of a star is proportional to the third order of its mass. In deducing this formula, we accepted that ε is determined by a function $\varepsilon \sim p^\alpha B^\alpha$, so ε_1 depends on p_1 and B_1. It is evident that rejection of this assumption cannot substantially change the obtained correlation (2.38). Naturally, under arbitrary ε, the quantity ε_1 depends on p_c and B_c. Thus the multiplier $\lambda L_{x_0}/M^3_{x_0}$ in formula (2.38) will have different numerical values for different stellar structures. At the same time Table 2 shows that this multiplier is approximately the same for absolutely different structures, including boundary structures which are exotic. Therefore the "mass-luminosity" correlation gives no information about sources of stellar energy — the correlation is imperceptible to their properties. However, other assumptions are very important. As we see from the deductive path to formula (2.33), the correlation between mass and luminosity can be deduced only if the pressure depends on temperature, so our formula (2.38) can be obtained only if the gas is ideal. It is also important to make the absorption coefficient κ constant for all stars. The rôle of the radiant pressure will be considered in the next paragraph.

And so forth we are going to compare formula (2.38) to observational data. Fig. 1 shows masses and luminosities of stars, according to today's data. The diagram has been built on masses of stars taken from Kuiper's data base [8], and the monograph by Russell and Moore [9]. We excluded Trumpler stars [10] from the Kuiper data, because their masses were measured uncertainly. Naturally, Trumpler calculated masses of such stars, located in stellar clusters, with the supposition that the K-term (the term for radiant velocities with respect to the whole cluster) is fully explained by Einstein's red shift. For this reason the calculated masses of Trumpler stars can be much more than their real masses. Instead of Trumpler stars, in order to fill the spaces of extremely bulky stars in the diagram, we used extremely bulky eclipse variable stars (VV Cephei, V 381 Scorpii) and data for Plasckett's spectral-variable star BD $+6^\circ$ 1309.

As we see in Fig. 1, our obtained correlation $L \sim M^3$ is in good accord with the observational data in all spectra of observed masses (having a small deviation inside $1.5m$). The dashed line $L \sim M^{10/3}$ is only a little different from our line. Parenago [11], Kuiper [8], Russell [9], and others accept this $L \sim M^{10/3}$ line as the best representation of observational data. Some researchers found the exponent of mass more than our's. For instance, Braize [12] obtained $L \sim M^{3.58}$. Even if such maximal deviation from our exponential index 3 is real, the theoretical result is excellent for most stars. The coefficient of proportionality in our formula (2.38) is very susceptible to μ. For this reason, coincidence of our theoretical correlation and observational data is evidence that the chemical composition of stars is the same on the average. The same should be said about the absorption coefficient κ: because physical conditions inside stars can be very different even under the same luminosity (for example, red giants and blue stars located in the main direction), it is an unavoidable conclusion that the absorption coefficient of stellar matter is independent of pressure and temperature. The conclusions justify our assumption in §1.3, when we solved the main system of equilibrium equations.

The fact that white dwarfs lie off the main sequence can be considered as a confirmation of degenerate gas inside them. Because a large increase of the absorption coefficient in white dwarfs in comparison to regular stars is not very plausible, another explanation can be given only if the structural multiplier $\lambda L_{x_0}/M_{x_0}$ in white dwarfs is $\sim$100 times more than in other stars. The location of white dwarfs in the Russell-Hertzsprung diagram can give a key to this problem.

At last, proceeding from observational data, we calculate the coefficient μ^4/κ_c in our theoretical formula (2.38). The line $L \sim M^3$, which is the best representation of observational data, lies a little above the point where the Sun is located. For this reason, under $M = M_\odot$, we should have $L = 1.8\,L_\odot$ in our formula (2.38). According to table 2, we assume $\lambda L_{x_0}/M^3_{x_0} = 2\times10^{-3}$. Then we obtain

$$\frac{\mu^4}{\kappa_c} = 0.08\,. \tag{2.39}$$

2.6 The radiant pressure inside stars

In the above we neglected the radiant pressure in comparison to the gaseous one in the equation of mechanical equilibrium of a star. Now we consider the main system of the equation (III), which takes the radiant pressure into account. If the absorption coefficient κ is constant ($\kappa_1 = 1$), this system takes the form

$$\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dp_1}{\rho_1 dx}\right] = -\left(1-\lambda\gamma_c\varepsilon_1\right), \qquad \frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dB_1}{\rho_1 dx}\right] = -\lambda\varepsilon_1\,. \tag{2.40}$$

After calculations analogous to those carried out in deducing formula (2.21), we obtain

$$\lambda\left(1+\gamma_c\right) = \frac{\displaystyle\int_0^{x_0} M_x \frac{\rho_1}{x^2}\,dx}{\displaystyle\int_0^{x_0} L_x \frac{\rho_1}{x^2}\,dx}\,. \tag{2.41}$$

The ratio of integrals in this formula depends on the distribution of energy sources inside a star, i. e. on the structure of a star. This ratio maintains a numerical value close to 1 under any conditions. Thus $\lambda\left(1+\gamma_c\right) \sim 1$. If energy sources are distributed homogeneously throughout the volume of a star, we have $\varepsilon_1 = 1$, $L_x = M_x$ and hence the exact equality $\lambda\left(1+\gamma_c\right) = 1$. If energy sources are concentrated at the centre of a star, $\lambda\left(1+\gamma_c\right) > 1$. In this case, if the radiant pressure takes high values ($\gamma_c > 1$), the internal constitution of a star becomes very interesting, because in this case $\lambda\gamma_c > 1$ and the right side term in the first equation of (2.40) is positive at the centre of a star, our formula (2.41) leads to $p_1'' > 0$, and hence at the centre of such a star the gaseous pressure and the density have a minimum, while their maximum is located at a distance from the centre*.

From this we conclude that extremely bulky stars having high γ_c can be in equilibrium only if $\lambda\left(1+\gamma_c\right) \sim 1$, or, in other words, if the next condition is true

$$\varepsilon_c \sim \frac{4\pi G c}{\kappa}\,. \tag{2.42}$$

Thus, starting from an extremely bulky stellar mass wherein $\gamma_c > 1$, the quantity of energy generated by a unit of the mass should be constant for all such extremely bulky stars. The luminosity of such stars, following formulae (2.3) and (2.2), should be directly proportional to their mass: $L \sim M$. This correlation is given by the straight line drawn in the upper right corner of Fig. 1. Original data due to Eddington [13] and others showed an inclination of the "mass-luminosity" line to this direction in the region of bulky stars (the upper right corner of the diagram). But further more exact data, as it was especially shown by Russell [9] and Baize [12], do not show the inclination for even extremely bulky stars (see our Fig. 2). Therefore we can conclude that there are no internal structures of stars for $\gamma_c > 1$; the ultimate case of possible masses of stars is the case where $\gamma_c = 1$. Having no suppositions about the origin of energy sources in stars†, it is very difficult to give an explanation of this fact proceeding from only the equilibrium of stars. The very exotic internal constitution of stars under $\gamma_c > 1$ suggests that if such stars really exist in nature, they are very rare exceptions.

In order to ascertain what influence γ_c has on the structure of a star, we consider the simplest (abstract) case where energy sources are distributed homogeneously throughout a star ($\varepsilon_1 = 1$). In this case, as we know,

$$\lambda = \frac{1}{1+\gamma_c}\,, \tag{2.43}$$

and the system (2.40) takes the form

$$\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dp_1}{\rho_1 dx}\right] = -\frac{1}{1+\gamma_c}\,, \qquad \frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dB_1}{\rho_1 dx}\right] = -\frac{1}{1+\gamma_c}\,. \tag{2.44}$$

Introducing a new variable $x_{\gamma_c=0}$ instead of x

$$x = \sqrt{1+\gamma_c}\, x_{\gamma_c=0}\,, \tag{2.45}$$

we obtain the main system in the same form as that in the absence of the radiant pressure. So, in this case the main characteristics of the internal constitution of star are

$$x_0 = x_{0(\gamma_c=0)}\left(1+\gamma_c\right)^{1/2}, \quad M_{x_0} = M_{x_0(\gamma_c=0)}\left(1+\gamma_c\right)^{3/2}, \quad L_{x_0} = L_{x_0(\gamma_c=0)}\left(1+\gamma_c\right)^{3/2}, \qquad \lambda = \frac{\lambda_{\gamma_c=0}}{1+\gamma_c}\,. \tag{2.46}$$

Characteristics indexed by $\gamma_c = 0$ are attributed to the structures of stars where $\gamma_c \ll 1$; their numerical values can be taken from our Table 2. Because Table 2 shows very small changes in M_{x_0} for very different structures of stars, formulae (2.46) should as well give an approximate picture for other structures of stars. Under high γ_c, the mass of a star (2.34) becomes

$$M \simeq 1.134\, M_\odot \frac{\gamma_c^{1/2}}{\mu^2}\left(1+\gamma_c\right)^{3/2} M_{x_0(\gamma_c=0)}\,. \tag{2.47}$$

*This amazing conclusion about the internal constitution of a star is true under only high values of the radiant pressure. In regular stars the radiant pressure is so low that we neglect it in comparison to the gaseous pressure (see previous paragraphs). — Editor's remark.

†That is the corner-stone of Kozyrev's research. — Editor's remark.

Astronomical observations show that maximum masses of stars reach $\sim 120 M_\odot$ — see Fig. 1, showing an inclination of the "mass-luminosity" correlation near $\log(M/M_\odot) = 2$. Assuming this mass in (2.47), and assuming $\gamma_c = 1$ and $M_{x_0} = 10$ for it, we obtain the average molecular weight $\mu = 0.51$.

Then in such stars, by formula (2.39), we obtain $\kappa = 0.8$. On the other hand, because the "mass-luminosity" correlation has a tendency to the line $L \sim M$ for extremely bulky masses (see Fig. 1), we obtain the ultimate value $\bar{\varepsilon} = 5\times 10^4$. For homogeneously distributed energy sources, formula (2.42) leads to $\kappa = 0.5$. If they are concentrated at the centre, $\varepsilon_c > \bar{\varepsilon} = \varepsilon_c (L_{x_0}/M_{x_0})$. Even in this case formula (2.42) leads to $\varepsilon_c > \bar{\varepsilon}$. There is some compensation, so the calculated numerical value of the absorption coefficient κ is true. An exact formula for $\bar{\varepsilon}$ can be easily obtained as

$$\frac{L}{M} = \bar{\varepsilon} = \frac{4\pi G c}{\kappa} \frac{L_{x_0}}{M_{x_0}} \frac{\int_0^{x_0} M_x \frac{\rho_1}{x^2}\, dx}{\int_0^{x_0} L_x \frac{\rho_1}{x^2}\, dx} \frac{\gamma_c}{1+\gamma_c} \,. \tag{2.48}$$

So, having considered the "mass-luminosity" correlation, we draw the following important conclusions:

1. All stars (except possibly for white dwarfs) are built on an ideal gas;
2. In their inner regions, where stellar energy is generated, all stars have the same chemical composition, $\mu = \text{const} = 1/2$, so they are built on a mix of protons and electrons without substantial percentage of other nuclei;
3. The absorption coefficient per unit of mass κ is independent of the physical conditions inside stars, it is a little less than 1.

Thomson dispersion of light in free electrons has the same properties. Naturally, the Thomson dispersion coefficient per electron is

$$\sigma_0 = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2} \right)^2 = 6.66\times 10^{-25}, \tag{2.49}$$

where e and m_e are the charge and the mass of the electron. In the mix of protons and electrons we obtain

$$\kappa_{\rm T} = \frac{\sigma_0}{m_{\rm H}} = \frac{6.66\times 10^{-25}}{1.66\times 10^{-24}} = 0.40\,. \tag{2.50}$$

The fact that our calculated approximate value of κ is close to $\kappa_{\rm T} = 0.40$ shows that the interaction between light and matter inside stars is determined mainly by the Thomson process — acceleration of free electrons by the electric field of light waves.

Because μ stays in the "mass-luminosity" correlation (2.38) in fourth degree, the obtained theoretical value of μ is quite exact with respect to the real one. If $\kappa = \kappa_{\rm T}$, as a result of (2.39) we have $\mu = 0.43$. Because μ cannot be less than $1/2$, the obtained ultimate value of $\kappa = 0.8$ is twice $\kappa_{\rm T} = 0.40$. This fact can be explained by the circumstance that, in this case of extremely bulky masses, the structural coefficient in formula (2.38) should be twice as small. It is evident that we can accept $\mu = 1/2$ to within 0.05. If all heavy nuclei are ionized, their average molecular weight is 2. If we assume the average molecular weight in a star to be 0.55 instead of $1/2$, the percentage of ionized atoms of hydrogen $\chi_{\rm H}$ becomes

$$2\chi_{\rm H} + \frac{1}{2}(1-\chi_{\rm H}) = \frac{1}{0.55}\,, \qquad \chi_{\rm H} \simeq 90\%\,.$$

Thus the maximum admissible composition of heavy nuclei inside stars, permitted by the "mass-luminosity" correlation, is only a few percent. Under $\mu = 1/2$ the mass of a star, where $\gamma_c = 1$, is obtained as $130 M_\odot$. This value is indicated by the vertical line in Fig. 1.

At last we calculate the radiant pressure at the centre of the Sun. Formula (2.34) leads to $\gamma_{c\odot} \simeq 10^{-3}$. In this case the radiant pressure term in the equation of mechanical equilibrium can be neglected.

2.7 Comparing the obtained results to results obtained by other researchers

To deduce the "mass-luminosity" correlation by the explanation according to the regular theory of the internal constitution of stars, becomes very complicated because the theoreticians take a priori the absorption coefficient as dependent on the physical conditions. They supposed the absorption of light inside stars due to free-connected transitions of electrons (absorption outside spectral series) or transitions of electrons from one hyperbolic orbit to another in the field of positive charged nuclei. The theory of such absorption was first developed by Kramers, and subsequently by Gaunt, and especially, by Chandrasekhar [14]. According to Chandrasekhar, the absorption coefficient depends on physical conditions as

$$\kappa_{\rm Ch} = 3.9\times 10^{25} \frac{\rho}{T^{3.5}} \left(1 - \chi_{\rm H}^2\right), \tag{2.51}$$

where $\chi_{\rm H}^2$ is the percentage of hydrogen, the numerical factor is obtained for Russell's composition of elements. In order to clarify the possible rôle of such absorption in the "mass-luminosity" correlation, we assume (for simplicity)

$$\kappa_{\rm Ch} = \frac{\kappa_0}{\gamma}\,. \tag{2.52}$$

In this case, having small γ_c, formulae (2.38) and (2.33) show $L \sim M^5$. This exponent is large, so we cannot neglect γ_c in comparison to 1. If γ_c is large, the formulae show $L \sim M^{3/2}$. Thus, in order to coordinate theory and observations, we are forced to consider "middle" numerical values of

γ_c and reject the linear correlation between $\log L$ and $\log M$. Formulae (2.47) and (2.48) show

$$\begin{aligned} M^2 &\sim \frac{\gamma_c\left(1+\gamma_c\right)^2}{\mu^4}\,, \qquad M^2 \sim \frac{1-\beta}{\mu^4\beta^4}\,, \\ L &\sim M\,\frac{\gamma_c^2}{1+\gamma_c}\,, \qquad L \sim M^{3/2}\left(1-\beta\right)^{3/2}\mu\,. \end{aligned} \tag{2.53}$$

Here are formulae where γ_c has been replaced with the constant β, one regularly uses in the theory of the internal constitution of stars

$$\beta = \frac{p_c}{p_0} = \frac{1}{1+\gamma_c}\,. \tag{2.54}$$

Thus the "mass-luminosity" correlation, described by the two formulae (2.53), becomes very complicated. The formulae are in approximate agreement with Eddington's formulae [15] and others. The exact formula for (2.51) introduces the central temperature T_c into them. Under large γ_c, as we see from formulae (2.46), the formula for T_c (2.31) includes the multiplier β

$$T_c = 2.29{\times}10^7\,\mu\,\beta\left(\frac{x_0}{M_{x_0}}\right)_{\gamma_c=0}\frac{\frac{M}{M_\odot}}{\frac{R}{R_\odot}}\,. \tag{2.55}$$

Then, through T_c, the radius and the reduced temperature of a star can be introduced into the "mass-luminosity" correlation. This is the way to obtain the well-known Eddington temperature correction.

In order to coordinate the considered case of "middle" γ_c, we should accept $\gamma_c = 1$ starting from masses $M \simeq 10 M_\odot$. So, for the Sun we obtain $\gamma_{c\odot} = 0.08$. As we see from formula (2.47), it is possible if $\mu \simeq 2$. Then formula (2.39), using the numerical value $\lambda L_{x_0}/M_{x_0}^3 = 3.8{\times}10^{-3}$ given by Eddington's model, gives $\kappa_{c\odot} = 170$ and $\kappa_0 = 14$. The theoretical value of κ_0 can be obtained by comparing (2.52) and (2.51); it is

$$\kappa_0 = \frac{\alpha\mu}{3\Re\sqrt{T_{c\odot}}}\,3.9{\times}10^{25}\left(1-\chi_{\text{H}}^2\right). \tag{2.56}$$

According to (2.55) we obtain $T_{c\odot} = 4{\times}10^7$. Then, by (2.56), we have $\kappa_0 = 0.4$. So, according to Eddington's model, the theoretically obtained value of the absorption coefficient $\kappa_0 = 14$ is 30 times less than the $\kappa_0 = 0.4$ required, consistent with the observational data*. This divergence is the well-known "difficulty" associated with Eddington's theory, already noted by Eddington himself. According to Strömgren [16], this difficulty can be removed if we accept the hypothesis that stars change their chemical composition with luminosity. Supposing the maximum hydrogen content, μ can vary within the boundaries $1/2 \leqslant \mu \leqslant 2$. Then, as we see from (2.56), the theoretical value of κ_0 decreases slightly. On the other hand, the previous paragraph showed that the value of κ_0, obtained from observations, decreases much more. As a result, the theoretical and observational values of κ can be matched (which is in accordance with Strömgren's conclusion). All theoretical studies by Strömgren's followers, who argued for evolutionary changes of relative amounts of hydrogen in stars, were born from the above hypothesis. The hypothesis became very popular, because it provided an explanation of stellar energy by means of thermonuclear reactions, as suggested by Bethe.

It is evident that the above theories are very strained. On the other hand, the simplicity of our theory and the general way it was obtained are evidence of its truth. It should be noted that our two main conclusions

$$(1)\ \ \mu = 1/2\,,\ \ \chi_{\text{H}}^2 = 1\,; \qquad (2)\ \ \kappa = \kappa_{\text{T}}\,,$$

obtained independently of each other, are physically connected. Naturally, if $\chi_{\text{H}}^2 = 1$, Chandrasekhar's formula (2.51) becomes inapplicable. Kramers absorption (free-connected transitions) becomes a few orders less; it scarcely reaches the Thomson process. At the same time, our main result is that $\gamma_c < 1$ for all stars, and this led to all the results of our theory. Therefore this result is so important that we mean to verify it by other astrophysical data. We will do it in the next chapter, analysing the correlation "period — average density of Cepheids". In addition, according to our theory, the central regions of stars, where stellar energy is generated, consist almost entirely of hydrogen. This conclusion, despite its seemingly paradoxical nature, must be considered as an empirically established fact. We will see further that study of the problem of the origin of stellar energy will reconcile this result with spectroscopic data about the presence of heavy elements in the surface layers of stars.

2.8 The rôle of convection inside stars

In §1.3 we gave the equations of equilibrium of stars (II), which take convective transfer of energy into account. Assuming the convection coefficient $A = \text{const}$, the second equation of the system (the heat equilibrium equation) can be written as

$$\frac{1}{\rho_1 x^2}\frac{d}{dx}\left[\frac{x^2 dB_1}{\rho_1 dx}\right] - \frac{\kappa_c\rho_c A}{c\gamma_c}\frac{1}{\rho_1 x^2}\left[x^2 u\frac{dp_1}{dx}\right] = -\lambda\varepsilon_1\,, \tag{2.57}$$

where

$$u = 1 - \frac{\Gamma}{4(\Gamma-1)}\frac{p_1}{B_1}\frac{dB_1}{dp_1}\,. \tag{2.58}$$

The convection term in (2.57) plays a substantial rôle only if

$$\frac{\kappa_c\rho_c A}{c\gamma_c} > 1\,, \qquad A > \frac{c\gamma_c}{\kappa_c\rho_c}\,. \tag{2.59}$$

*As it was shown in the previous paragraph, Kozyrev's theory gives $\kappa_0 = 0.5$–0.8 for stars having different internal constitutions, which corresponds well to observations. — Editor's remark.

Table 3

κ	x_1	M_{x_1}	λL_{x_0}	x_0	M_{x_0}	$\dfrac{x_0^3}{3M_{x_0}}$	$\dfrac{\lambda L_{x_0}}{M_{x_0}^3}$
const	2.4913	3.570	3.018	8.9	11.46	20.5	1.97×10^{-3}
$\kappa_{\rm Ch}$	1.88	1.25	1.25	11.2	12.4	37.0	0.65×10^{-3}

Hence the convection coefficient for the Sun should satisfy $A_\odot > 5\times10^7$. In super-giants, convection would be substantial only under $A > 10^{16}$. The convection coefficient A, as we see from formula (1.17), equals the product of the convective current velocity $\bar{v}$ and the average length $\bar{\lambda}$ travelled by the current. Thus convection can influence energy transfer inside super-giants if convection currents are about the size of the star (which seems improbable). At the same time, if a convection instability occurs in a star, the average length of travel of the current becomes the size of the whole convection zone. Then the coefficient A increases so much that it can reach values satisfying (2.59). If A is much more than the right side of (2.59), taking into account the fact that all terms of the equilibrium equation (2.57) are about 1, the term in square brackets is close to 0. Then, if A is large,

$$u = 0\,, \quad \text{hence} \quad B_1 = p_1^{\frac{4(\Gamma-1)}{\Gamma}}\,, \tag{2.60}$$

which is the equation of adiabatic changes of state. For a monatomic gas, $\Gamma = 5/3$ ($n = 3/2$) and hence

$$B_1 = p_1^{8/5}\,. \tag{2.61}$$

Because, according our conclusions, stars are built up almost entirely of hydrogen, Γ can be different from $5/3$ in only the upper layers of stars, which is insufficient in our consideration of a star as a whole. Therefore zones of free convection can appear because of an exotic distribution of energy sources.

Free convection can also start in another case, as soon as the temperature gradient of radiant equilibrium exceeds the temperature gradient of convective equilibrium. This is Schwarzschild's condition, and it can be written as

$$\left(\frac{d\log B_1}{d\log p_1}\right)_{\rm Rad} > \left(\frac{d\log B_1}{d\log p_1}\right)_{\rm Con},$$

which, taking (2.22) and (2.61) into account, leads to

$$\frac{\lambda L_x}{M_x} > 1.6\,\frac{B_1}{p_1}\,. \tag{2.62}$$

From this formula we see that free convection is impossible in the surface layers of stars. In central regions we obtain the next condition for free convection

$$\lambda > 1.6.$$

Table 2 shows that even when $\varepsilon_1 = B_1$, any star should contain a convective core. If ε_1 depends only on temperature and can be approximated by function $\varepsilon_1 = T^m$, the calculations show that λ reaches its critical value of 1.6 when $m = 3.5$. Thus a star has a convective core if $m > 3.5$. The radius x_1 of the convective core is determined by equality between the temperature gradients (see above). Writing (2.62) as an equality, we obtain

$$\lambda L_{x_1} = 1.6\,M_{x_1}\left(\frac{B_1}{p_1}\right)_{x_1} = 1.6 M_{x_1}\rho_{x_1}\,. \tag{2.63}$$

It is evident that the size of the convective core increases if the energy sources become more concentrated at the centre of a star. In the case of a strong concentration, all energy sources become concentrated inside the convective core. Then inside the region of radiant equilibrium we have $\lambda L_x = \lambda L_{x_1} = \text{const}$. Because the border of the convective core is determined by equality of the physical characteristics' gradients in regions of radiant equilibrium and convective equilibrium, not only are p_1 and T_1 continuous inside such stars but so are their derivatives. Therefore such a structure for a star can be finely calculated by solving the main system of the equilibrium equations under $\varepsilon_1 = 0$ and boundary conditions: (1) under some values of $x = x_1$, quantities p_1, B_1 and their derivatives should have numerical values satisfying the solution to Emden's equation under $n = 3/2$; (2) under some value $x = x_0$ we should have $p_1 = B_1 = 0$. The four boundary conditions fully determine the solution. We can find x_1 by step-by-step calculations as done in §2.3 for λ.

The formulated problem, known as the problem of the internal constitution of a star having a point-source of energy and low radiant pressure, was first set up by Cowling [17]. In his calculations the absorption coefficient was taken as variable according to Chandrasekhar's formula (2.51): $\kappa = \kappa_{\rm Ch}$. However in §2.6 we showed that $\kappa = \kappa_{\rm T}$ inside all stars. Only in the surface layers of a star should κ increase to $\kappa_{\rm T}$. But, because of very slow changes of physical conditions along the radius of a star, κ remains $\kappa_{\rm T}$ in the greater part of the volume of a star. Therefore it is very interesting to calculate the internal structure of a star under given values of $\kappa = \text{const}$. We did this, differing thereby from Cowling's model, so that there are two alternatives: our model ($\kappa = \text{const}$) and Cowling's model ($\kappa_{\rm T}$). All the calculations were carried out by numerical integration of (2.23b) assuming there that $\varepsilon_1 = 0$.

Table 4

x	T_1	p_1	ρ_1
0.00	1.000	1.000	1.000
0.50	0.983	0.958	0.975
1.00	0.935	0.845	0.904
1.50	0.856	0.677	0.791
2.00	0.762	0.507	0.665
2.50	0.652	0.346	0.530
3.00	0.544	0.211	0.388
3.50	0.451	0.117	0.259
4.00	0.370	0.598×10^{-1}	0.161
4.50	0.328	0.284×10^{-1}	0.936×10^{-1}
5.00	0.245	0.125×10^{-1}	0.510×10^{-1}
5.50	0.195	0.52×10^{-2}	0.266×10^{-1}
6.00	0.154	0.20×10^{-2}	0.129×10^{-1}
6.50	0.118	0.67×10^{-3}	0.57×10^{-2}
7.00	0.087	0.20×10^{-3}	0.23×10^{-2}
7.50	0.060	0.49×10^{-4}	0.82×10^{-3}
8.00	0.036	0.64×10^{-5}	0.18×10^{-3}
8.50	0.015	0.19×10^{-6}	0.79×10^{-4}
8.90	0.000	0.000	0.000

Table 3 gives the main characteristics of the "convective" model of a star under $\kappa = \text{const}$ and $\kappa = \kappa_{\text{Ch}}$. The κ_{Ch} are taken from Cowling's calculations. Values of λL_{x_0} were found by formula (2.62). In this model, distribution of energy sources inside the convective core does not matter. For this reason, the quantities λ and L_{x_0} are inseparable. If we would like to calculate them separately, we should set up the distribution function for them inside the convective core.

We see that the main characteristics of the structure of a star, the quantities x_0, M_{x_0}, and λL_{x_0}, are only a little different from those calculated in Table 2. The main difference between structures of stars under the two values $\kappa = \text{const}$ and $\kappa = \kappa_{\text{Ch}}$ is that under our $\kappa = \text{const}$ the convective core is larger, so such stars are close to polytropic structures of class 3/2, and there we obtain a lower concentration of matter at the centre: $\rho_c = 20.5\,\bar{\rho}$. Table 4 gives the full list of calculations for our convective model ($\kappa = \text{const}$).

Chapter 3

The Internal Constitution of Stars, Obtained from the Analysis of the Relation "Period — Average Density of Cepheids" and Other Observational Data

In the previous chapter we deduced numerous theoretical correlations, which give a possibility of calculating the physical characteristics of matter inside stars if their structural characteristics are known. In order to be sure of the calculations, besides our general theoretical considerations, it would be very important to obtain the structural characteristics proceeding from observational data, related at least to some classes of stars.

Properties of the internal structure of a star should manifest in its dynamical properties. Therefore we expect that the observed properties of variable stars would permit us to learn of their structures. For instance, the pulsation period of Cepheids should be dependent on both their physical characteristics and the distribution of the characteristics inside the stars. Theoretical deduction of this correlation can be done very strictly. Therefore we have a basis for this deduction in all its details.

Radiation of energy by an oscillating star must result in a dispersion of mechanical energy of its oscillations. It is most probable that the oscillation energy of variable stars is generated and supported by energy sources connected to the oscillation and radiation processes. In other words, such stars are self-inducing oscillating systems. Observable arcs of the oscillating luminosity and speed reveal a nonlinear nature for the oscillations, which is specific to self-inducing oscillating systems. The key point of a self-inducing oscillating system is a harmonic frequency equal to the natural frequency of the whole oscillating system. Therefore, making no attempt to understand the nature of the oscillations, we can deduce the oscillation period as the natural period of weak linear oscillations.

3.1 The main equation of pulsation

Typical Cepheids have masses less than 10 solar masses. For instance, δ Cephei has $M \simeq 9\,M_\odot$. In this case equation (2.34) leads to $\gamma_c < 0.1$, so Cepheids should satisfy $L \sim M^3$, i. e. the "mass-luminosity" relation. Therefore the radiant pressure plays no rôle in such stars, so considering their internal constitutions we should take into account only the gaseous pressure. In solving this problem we will consider linear oscillations, neglecting higher order terms. This problem becomes much simpler because temperature changes in such a star satisfy adiabatic oscillations in almost its whole volume, except only for the surface layer. Naturally, in order to obtain the ratio between observed temperature variations and adiabatic temperature variations close to 1, the average change of energy inside 1 gram in one second should be about $\bar{\varepsilon}$, i. e. $\sim 10^2$. This is 10^8 per half period. On the other hand, the heat energy of a unit of mass should be about Ω/M (according to the virial theorem), that is $\sim 10^{15}$ ergs by formula (2.5). Thus during the pulsation the relative change of the energy is only 10^7, so pulsations of stars are adiabatic, with high precision. We assume that the pulsation of a star can be determined by a simple standing wave with a frequency $n/2\pi$

$$V(r,t)=V(r)\sin nt, \quad a=\frac{\partial^2 V}{\partial t^2}=-n^2 V(r)\sin nt, \tag{3.1}$$

where $V(r)$ is the relative amplitude of the pulsation

$$V(r)=\frac{\delta r}{r}\,.$$

By making the above assumptions, Eddington had solved the problem of pulsation of a star.

Linking the coordinate r to the same particle inside a star, we have the continuity equation as follows

$$M_r=\text{const}\,, \qquad r^2\rho\, dr=\text{const}\,. \tag{3.2}$$

Using the condition of adiabatic changes $\frac{\delta p}{p}=\Gamma\frac{\delta\rho}{\rho}$ and taking variation from the second equality, we obtain

$$\frac{\delta p}{p}=-\Gamma\left[3V+r\,\frac{dV}{dr}\right]. \tag{3.3}$$

It is evident that the equations of motion

$$\frac{dp}{\rho dr}=-(g+a)\,, \qquad g=\frac{GM_r}{r^2}$$

give, neglecting higher order terms,

$$\frac{d\delta p}{dr}=-a\rho+4V\,\frac{dp}{dr}\,.$$

Substituting formula (3.3) into this equation, we obtain Eddington's equation of pulsation

$$\begin{gathered}\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}\left[4+\frac{r}{p}\frac{dp}{dr}\right]+\\+\frac{V}{r\Gamma}\frac{1}{p}\frac{dp}{dr}\left[(3\Gamma-4)-\frac{n^2 r}{g}\right]=0\,.\end{gathered} \tag{3.4}$$

We introduce a dimensionless variable x instead of r (we used this variable in our studies of the internal constitution of stars). As it is easy to see

$$\frac{g}{r}=4\pi G\,\frac{\bar\rho_r}{3}=4\pi G\,\rho_c\,\frac{M_x}{x^3}\,. \tag{3.5}$$

Substituting (3.5) into formula (3.4), we transform the pulsation equation to the form

$$\begin{gathered}\frac{d^2V}{dr^2}+\frac{1}{x}\frac{dV}{dr}\left[4+\frac{x}{p_1}\frac{dp_1}{dr}\right]-\\-\frac{V}{x\Gamma}\frac{1}{p_1}\frac{dp_1}{dr}\left[(4-3\Gamma)+\frac{n^2}{4\pi G\rho_c}\frac{x^3}{3M_x}\right]=0\,.\end{gathered} \tag{3.6}$$

We transform this equation to self-conjugated form. Multiplying it by $x^4 p_1$, we obtain

$$\frac{d}{dx}\left[x^4 p_1\,\frac{dV}{dx}\right]-Vx^3\,\frac{dp_1}{dx}\,\frac{(4-3\Gamma)}{\Gamma}\left[1-\lambda\,\frac{x^3}{3M_x}\right]=0, \tag{3.7}$$

where

$$\lambda=\frac{n^2}{4\pi G\rho_c\left(\Gamma-\frac{4}{3}\right)}\,. \tag{3.8}$$

So the problem of finding the pulsation period has been reduced to a search for those numerical values of λ by which the differential equation (3.7) has a solution satisfying the "natural" boundary conditions

$$x^4 p_1\,\frac{dV}{dx}\bigg|_0^{x_0}=0. \tag{3.9}$$

Formula (3.8) gives the correlation "period — average density of Cepheids" and, hence, the general correlation "period — average density of a star". It is evident that λ depends on the internal structure of a star. Its expected numerical value should be about 1. For a homogeneously dense star, $x^3/(3M_x)=1$ everywhere inside it. In this case the differential equation (3.7) has the solution: $V=\text{const}$, $\lambda=1$. This solution determines the main oscillation of such a star. In order to find the main oscillations of differently structured stars, we proceed from the solution by applying the method of perturbations.

3.2 Calculation of the mean values in the pulsation equation by the perturbation method

We write the pulsation equation in general form

$$\big(py'\big)'+qy\,\big(1-\lambda\rho\big)=0\,. \tag{3.10}$$

If we know a solution to this equation under another function $\rho=\rho_0$

$$\big(py_0'\big)'+qy_0\,\big(1-\lambda_0\rho_0\big)=0\,, \tag{3.11}$$

hence we know the function y_0 and the parameter λ_0. After multiplying (3.10) by y_0 and (3.11) by y, we subtract one from the other. Then we integrate the result, taking the limits 0 and x_0. So, we obtain

$$\int_0^{x_0} qyy_0\left[\lambda_0\rho_0-\lambda\rho\right]dx=0\,,$$

hence

$$\lambda=\lambda_0\,\frac{\displaystyle\int_0^{x_0} qyy_0\rho_0\,dx}{\displaystyle\int_0^{x_0} qyy_0\rho\,dx}\,. \tag{3.12}$$

If the oscillations are small, equation (3.10) is the same as (3.11) with only an infinitely small correction

$$\rho=\rho_0+\delta\rho\,, \quad y=y_0+\delta y\,, \quad \lambda=\lambda_0+\delta\lambda\,.$$

Then the exact formula for $\delta\lambda$

$$\delta\lambda = -\lambda_0 \frac{\int_0^{x_0} q y y_0\, \delta\rho\, dx}{\int_0^{x_0} q y y_0\, \rho dx}$$

can be replaced by

$$\delta\lambda = -\lambda_0 \frac{\int_0^{x_0} q y_0^2\, \delta\rho\, dx}{\int_0^{x_0} q y_0^2\, \rho dx},$$

and thus we have

$$\lambda = \lambda_0 \frac{\int_0^{x_0} q y_0^2 \rho_0\, dx}{\int_0^{x_0} q y_0^2\, \rho dx}. \tag{3.13}$$

In our case $y_0 = 1$ and $\lambda_0 = 1$. Comparing formulae (3.10) and (3.7), using (3.13), we obtain

$$\lambda = \lambda_0 \frac{3\int_0^{x_0} x \rho_1 M_x\, dx}{\int_0^{x_0} x^4 \rho_1\, dx}. \tag{3.14}$$

We re-write this equation, according to (2.5a), as follows

$$\lambda = \lambda_0 \frac{9\int_0^{x_0} p_1 x^2\, dx}{\int_0^{x_0} \rho_1 x^4\, dx}. \tag{3.15}$$

If we introduce the average density $\bar{\rho}$ into formula (3.8) instead of the central one ρ_c, then according to (2.4),

$$\bar{\lambda} = \frac{n^2}{4\pi G\bar{\rho}\left(\Gamma - \frac{4}{3}\right)}, \tag{3.16}$$

$$\bar{\lambda} = \lambda\, \frac{\rho_c}{\bar{\rho}} = \frac{x_0^3}{3M_{x_0}}\, \lambda\,. \tag{3.17}$$

Using formulae (2.6) and (3.17) we re-write (3.15) as

$$\bar{\lambda} = \frac{x_0^2\, \Omega_{x_0} M_{x_0}}{I_{x_0}}, \tag{3.18}$$

where I_{x_0} is the dimensionless moment of inertia

$$I_{x_0} = \int_0^{x_0} \rho_1 x^4\, dx\,. \tag{3.19}$$

Formulae (3.16) and (3.18) determine the oscillation period of a star, $P = 2\pi/n$, independently of its average density $\bar{\rho}$. This result was obtained by Ledoux [18] by a completely different method. It is interesting that our equations (3.16) and (3.18) coincide with Ledoux's formulae.

We next calculate λ for stars of polytropic structures. In such cases I_{x_0} is

$$I_{x_0} = x_0^2 M_{x_0} - 6\,(n+1)\int_0^{x_0} T_1\, x^2\, dx\,, \tag{3.20}$$

where n is the polytropic exponent. Thus

$$\frac{1}{\bar{\lambda}} = \frac{5-n}{3}\left[1 - 6\,(n+1)\,\frac{\int_0^{x_0} T_1\, x^2\, dx}{M_{x_0} x_0^2}\right]. \tag{3.21}$$

Calculations of the numerical values of $\bar{\lambda}$ for cases of different polytropic exponents are given in Table 5.

Table 5

n	$\bar{\lambda}$
0	1.00
1	1.91
3/2	2.52
2	3.85
2.5	7.00
3	13.1

Under large $\bar{\lambda}$, much different from 1, the calculations for Table 5 are less precise. Therefore, in order to check the calculated results, it is interesting to compare the results for $n = 3$ to those obtained by Eddington via his exact solution of his adiabatic oscillation equation for his stellar model. For the stars we consider, he obtained, $\frac{n^2}{\pi G\rho_c\Gamma} = \frac{3}{10}(3 - 4/\Gamma)$. Hence, comparing his result to our formula (3.8), we obtain $\lambda = 9/40$ and $\bar{\lambda} = \frac{9}{40}\frac{\rho_c}{\bar{\rho}} = 12.23$. This is in good agreement with our result $\bar{\lambda} = 13.1$ given in Table 5.

3.3 Comparing the theoretical results to observational data

We represent the "period — average density" correlation in the next form

$$P\sqrt{\bar{\rho}_0} = c_1\,, \tag{3.22}$$

where P is the period (days), $\bar{\rho}_0$ is the average density expressed in the multiples of the average density of the Sun

$$n = \frac{2\pi}{86{,}400\, P}\,, \qquad \bar{\rho} = 1.411\, \bar{\rho}_0\,.$$

Employing formula (3.16), it is easy to obtain a correlation between the coefficients $\bar{\lambda}$ and c_1

$$\bar{\lambda}\left(\Gamma - \frac{4}{3}\right) = 0.447\big(10c_1\big)^{-2}. \tag{3.23}$$

By analysis of the "mass-luminosity" relation we have previously shown that the radiant pressure is much less than the gaseous pressure in a star. Therefore, because the inner regions of a star are primarily composed of hydrogen, the heat energy there is much more than the energy of ionization. So we have all grounds to assume $\Gamma = 5/3$, the ratio of the heat capacities for a monatomic gas. Hence

$$\bar{\lambda} = 1.34\big(10c_1\big)^{-2}. \qquad (3.24)$$

In order to express c_1 in terms of the observed characteristics of a star, we replace $\bar{\rho}_0$ in formula (3.22) with the reduced temperature and the luminosity, via the "mass-luminosity" formula. The "mass-luminosity" correlation has a general form $L \sim M^{\alpha}$ for any star. We denote by $\bar{T}$ the reduced temperature of a star (with respect to the temperature of the Sun), and by M_b its reduced stellar magnitude. Then, by formula (3.22), we obtain

$$\left(0.30-\frac{1}{5\alpha}\right)\big(M_b-M_{\odot}\big)+\log P+3\log\bar{T} = \log c_1 \,. \qquad (3.25)$$

From this formula we see that, in order to find c_1, it is unnecessary to know the exact value of α (if, of course, α has a large numerical value). Eddington's formula for the "mass-luminosity" relation, taken for huge masses, gives $\alpha \sim 2$ (compare with 2.53). Therefore, Eddington's value of $c_1 = 0.100$ is overstated. Applying another correlation, $L \sim M^{10/3}$, Parenago [19] obtained $c_1 = 0.071$. Becker [20] carried out a precise analysis of observational data using Kuiper's empirical "mass-luminosity" arc. He obtained the average value of $c_1 = 0.076$ for Cepheids. Formula (2.4) gives $\bar{\lambda} = 2.7$ or $\bar{\lambda} = 2.3$, so that Table 5 leads us to conclude that Cepheids have structures close to the polytropic class 3/2, like all other stars. Hence Cepheids have a low concentration of matter at the centre: $\rho_c = 6\bar{\rho}$.

This result is in qualitative agreement with the "natural viewpoint" that sources of stellar energy increase their productivity towards the centre of a star. However (as we saw in §2.8) the model for a point-source of energy and for a constant absorption coefficient, giving stars of minimal average densities, leads to a strong concentration at the centre, $\rho_c/\bar{\rho} = 20.5$. Thus $\bar{\lambda}$ for such a model should be more than an observable one. Really, having $\int_0^{x_0} p_1 x^2 dx = 6.06$ and $I_{x_0} = 140.0$ calculated by Table 4, formulae (3.15) and (3.17) give $\bar{\lambda} = 8.0$ for models with the ultimate concentration of energy sources. So, such stars are of the polytropic class $n = 2.5$. If the absorption coefficient is variable (Cowling's model), calculations give even more: $\bar{\lambda} = 8.4$.

Eddington and others, in their theoretical studies of the pulsation period within the framework of Eddington's model, explain the deviation between the theoretical and observed values of $\bar{\lambda}$ by an effect of the radiant pressure. Studies of pulsations under γ_c close to 1 show that the obtained formula for the period under low γ_c is true even if Γ is the reduced ratio of the heat capacities (which is, depending on the rôle of the radiant pressure, $4/3 \leqslant \Gamma \leqslant 5/3$).

Equation (3.23) shows that when $\bar{\lambda} = 12.23$ and the observable $c_1 = 0.075$ we have $\Gamma_{\rm eff} = 1.40$. At the same time $\Gamma_{\rm eff}$ should undergo changes independently of γ_c, i. e. depending upon the rôle of the radiant pressure. For a monatomic gas, Eddington [21] and others obtained this correlation as

$$\Gamma_{\rm eff} - \frac{4}{3} = \frac{1}{3}\,\frac{1+4\gamma_c}{(1+\gamma_c)(1+8\gamma_c)}\,. \qquad (3.26)$$

Under $\Gamma_{\rm eff} = 1.40$ we obtain $\gamma_c = 1.5$. We accept this numerical value in accordance with the average period of Cepheids, $P = 10^{\rm d}$. Then, by the "mass-luminosity" relation, $M = 12\,M_{\odot}$. It is possible to think that this result is in good agreement with the conventional viewpoint on the rôle of the radiant pressure inside stars (see §2.7). However, because λ_c depends on the mass of a star, other periods give different $\Gamma_{\rm eff}$ (by formula 3.26) and hence other numerical values of c_1. Using formulae (3.26) and (3,23), we can calculate c_1 for variable stars having longer pulsations, with periods $20^{\rm d} < P < 30^{\rm d}$. Instead of the average value $\log c_1 = -1.12$ found by Becker for the stars, there should be $\log c_1 = -1.00$. Despite the small change, observations show no such increase of c_1 [20]. Therefore, our conclusion about the negligible rôle of the radiant pressure in stars, even inside super-giants, finds a new verification. This result verifies as well our results $\mu = 1/2$ and $\kappa = \kappa_{\rm T}$, obtained in chapter 3.

3.4 Additional data about the internal constitution of stars

Some indications of the internal structure of stars can be obtained from analysis of the elliptic effect in the luminosity arcs of eclipse variable stars. Observations of such binaries gives the ratio of diameters at the equator of a star, which becomes elliptic because of the flow-deforming effect in such binary systems. For synchronous rotations of the whole system and each star in it, the compressed polar diameter of each star should be different (in the first order approximation) from the average equatorial one with a multiplier dependent on their masses. Thus, proceeding from the observed compression we can calculate the meridian compression ϵ. According to Clairaut's theory ϵ is proportional to φ, the ratio of the centrifugal force at the equator to the force of gravity

$$\epsilon = \alpha\varphi\,, \qquad \varphi = \frac{\omega^2}{3\pi G\bar{\rho}}\,,$$

where α is a constant dependent on the structure of the star. This constant was calculated for stars of polytropic structures by numerous researchers: Russell, Chandrasekhar and others. If $n = 0$ (homogeneous star), $\alpha = 1.25$. If $n = 1$, we have $\alpha = 15/(2\pi^2) = 0.755$. If $n = 5$ (the ultimate concentration, Roche's model), $\alpha = 0.50$. We see that the constant α is sensitive to changes in the structure of a star. Therefore determination of the numerical values of n in this way requires extremely precise observations. The values of n so obtained are very uncertain, despite the simplicity of the theory. Shapley first concluded that stars are almost homogeneous. This was verified by Luiten [22] who found the average value $\alpha = 0.57$ for a large number of stars like β Lyrae, and $\alpha = 0.71$ for stars like Algol. His results correspond to the polytropic structures $n = 3/2$ and $n = 1$ respectively.

The observed motion of the line of apsides in numerous eclipse binaries can be explained, in numerous cases, by their elliptic form. Because matter is more strongly concentrated in a binary system than in regular stars, the binary components interact like two point-masses, so there should be no motion of the line of apsides. Therefore the velocity of the line of apsides should be proportional (in the first order approximation) to $\alpha - 1/2$, where α is sensitive to changes in the structure of a star (as we showed above). Many theoretical studies on this theme give contradictory formulae for the velocity, depending on hypotheses about the properties of rotation in the pair. Russell, in his initial studies of this problem, supposed the rotating components solid bodies. This theory, being applied to the system Y Cygni by Russell and Dugan [23], gave $\alpha - 1/2 = 0.034$, which is the polytropic structure $1/2 < n < 2$. Other researchers, having made other suppositions, obtained larger n: $n \simeq 3$. It is probable that we can be most sure only that, because we observe motion of the line of apsides in binaries, the stars have no strong concentration of matter at the centre.

Blackett supposed a law according to which the ratio between the magnetic momentum P_H and angular momentum U is constant for all rotating space bodies. If this law is correct, we could have a possibility of determining the structures of stars in an independent way. We denote by k the ratio between the moments of the inertia of an arbitrary structured star rotating with the angular velocity ω and of the same star if it would be homogeneous throughout. Then

$$U = \frac{2}{5}\,k\omega M R^2, \qquad k = \frac{5}{3}\,\frac{I_{x_0}}{x^2 M_{x_0}}\,,$$

where I_{x_0} is the dimensionless moment of inertia. Using Blanchett's formula [24]

$$\frac{P_H}{U} = \beta\,\frac{G^{1/2}}{2c} \tag{3.27}$$

(β is a dimensionless multiplier, equal to about 1), and having the magnetic magnitude at the pole $H = 2P_H/R^3$, we can calculate k. For the Earth ($k = 0.88$), we obtain $\beta = 0.3$. Supposing $k = 0.16$ for stars, Blackett has found: $\beta = 1.14$ for the Sun and $\beta = 1.16$ for 78 Virginis (its magnetic field has been measured by Babcock).

If Blackett's law (3.27) is valid throughout the Universe and $\beta = 0.3$ for all space bodies, not just for the Earth, then $k = 0.60$ should be accepted for stars. Comparing $k = 0.60$ with Table 6, we come to the same conclusion that we have obtained by completely different methods: that stars have polytropic structures of class $n = 3/2$.

Table 6

n	k
0	1.00
1	0.65
3/2	0.52
2	0.40
2.5	0.28
3	0.20

For the convective model of a star (calculated in §2.8) we obtain $k = 0.26$. This is much less than required. The same convective model with a variable absorption coefficient (Cowling's model) gives even less: $k = 0.19$.

The agreement of our value $n = 3/2$ with other data, obtained by very different methods, verifies Blackett's law. It is possible his formula (3.27) should be written without β, but with the denominator $2\pi c$.

3.5 Conclusions about the internal constitution of stars

The most certain conclusions about the structure of stars are derived from the theory of pulsation of Cepheids. We have concluded that Cepheids have structures close to the polytropic one of class $n = 3/2$, for which $\rho_c = 6\,\bar{\rho}$. This conclusion is verified by other data, whereas each of them could be doubtful when being considered in isolation. At the same time all the data, characterizing stars of different classes, lead to the same result. It is probable that stars are really close to being homogeneous, having a low concentration of matter at the centre like the bulky planets, Jupiter and Saturn. Such a distribution of matter, as we saw in the ultimate case of the convective model, cannot be explained by a strong concentration of an energy source at the centre, or by a special kind of absorption coefficient. The real reason is that the radiant pressure B is included in the mechanical equilibrium equation through the gaseous pressure in the exponent $1/4$. Therefore the structural characteristics M_{x_0} and X_0, determined by the function ρ_1, have small changes even in very different models. Hence, in order to obtain the observable low concentration of matter at the centre of stars, we can search for the reason only in the heat equilibrium equation. The polytropic model $n = 3/2$ differs from other polytropic models by a smaller value of x_0. In order to make x_0 smaller, the gaseous pressure should decrease more strongly in the upper layers of a star. Such a rapid decrease in the pressure is possible only if the surface layers are heavy. In other words, in the case of the strong increase of the molecular weight in the surface layers of a star. Such an explanation is in complete agreement with our conclusion about the high concentration of hydrogen in the internal regions of stars. If the average molecular weight changes from $\mu = 1/2$ at the centre to $\mu = 2$ at the surface of a star, such a change of the molecular weight can be sufficient.

What is the goal of introducing the variable μ? Let us assume that μ depends on the temperature as

$$\mu_1 = \frac{1}{T_1^s}\,, \tag{3.28}$$

where s is a positive determined exponent. Increase of the molecular weight at the surface should result in an increase of the absorption coefficient κ (transition from $\kappa = \kappa_{\rm T}$ to $\kappa = \kappa_{\rm Ch}$). At the same time, under energy sources concentrated at the centre, the quantity $\kappa_1 L_x/M_x$ can remain almost the same. If $\kappa_1 L_x/M_x = \text{const} = 1$, equation (2.22a) leads to

$$p_1 = B_1 = T_1^4\,, \qquad \lambda = 1\,. \tag{3.29}$$

Instead of T_1 we introduce the characteristics

$$u_1 = \frac{T_1}{\mu_1} = T_1^{1+s} = \mu_1^{-\frac{1+s}{s}}\,, \tag{3.30}$$

which keeps the ideal gas equation in the regular form $p_1 = u_1\,\rho_1$. According to (3.29), we have

$$p_1 = u_1^{\frac{4}{1+s}}. \tag{3.31}$$

where we should equate the exponent $4/(1+s)$ to $n+1$ according to formula (2.7a).

Thus we have

$$\rho_1 = u_1^n\,, \qquad n = \frac{3-s}{1+s}\,, \tag{3.32}$$

so the function u_1 is determined by Emden's equation of class n. Hence, in order to obtain the structure $n = 3/2$, there should be $s = 3/5$ — the very low increase of the molecular weight: for instance, under such s the molecular weight μ increases 4 times at the distance x_1 where

$$\mu_1 = \left(\frac{1}{4}\right)^{\frac{8}{3}} = 0.025\,, \qquad T_1 = \left(\frac{1}{4}\right)^{\frac{5}{3}} = 0.10\,. \tag{3.33}$$

At $x > x_1$ the molecular weight remains unchanged, the equilibrium of a star is determined by the regular system of the equilibrium equations. However at the numerical values (3.33) almost the whole mass of a star is accounted for (see Table 4, for instance), so we obtain small corrections to the polytropic structure $n = 3/2$. Naturally, tables of Emden's function taken under $n = 3/2$ show that $x_1 = 5.6$ and $M_{x_1} = 11.0$. Applying formula (2.27a), we obtain $x_0 = 7.0$ instead of $x_0 = 6$, as expected for such a polytropic structure. These calculations show that the observed structures of stars* verify our result about the high content of hydrogen in the internal regions of a star, obtained from the "mass-luminosity" relation. At the same time, it should be taken into account that the hydrogen content in the surface layers of stars is also substantial. Therefore on the average we have $\mu < 2$ inside a star, so the problem about homogeneity of the molecular weight of stars is not completely solved with the above.

We saw that the dimensionless mass M_{x_0} is almost the same in completely different models of stars. For polytropic structures of the classes $n = 3/2$ and $n = 2$, convective models, and models described in Table 2, we obtained approximately the same numerical values of M_{x_0}. Therefore we can surely accept $M_{x_0} = 11$. What about x_0? According to observed structures of stars, we accept $x_0 = 6$. Hence $\bar{\rho}_c = 6.5\,\bar{\rho}$. In order to obtain $\kappa = \kappa_1$ from the observed "mass-luminosity" relation, we should have $\lambda L_{x_0}/M_{x_0}^3 = 1.0\times10^{-3}$. Thus we obtain $\lambda L_{x_0} = 1.5$. As a result, using these numerical values in formulae (2.28), (2.30), and (2.31), we have a way of calculating the physical conditions at the centre of any star. We now make this calculation for the Sun. Assuming $\mu_{c\odot} = 1/2$, we obtain

$$\begin{gathered} \rho_{c\odot} = 9.2\,, \qquad p_{c\odot} = 9.5\times10^{15}\ \text{dynes/cm}^2, \\ \gamma_{c\odot} = 0.4\times10^{-3}, \qquad B_{c\odot} = 3.8\times10^{12}\ \text{dynes/cm}^2, \\ T_{c\odot} = 6.3\times10^6\ \text{degrees}. \end{gathered} \tag{3.34}$$

Of the data the most soundly calculated is $\gamma_{c\odot}$, because it is dependent only on M_{x_0}. Thus a low temperature at the centre of the Sun, about 6 million degrees, is obtained because of low numerical values of $\mu_{c\odot}$ and x_0. Having such low temperatures, it is scarcely possible to explain the origin of stellar energy by thermonuclear reactions.

The results indicate possible ways to continue our research into the internal constitution of stars. They open a way for a physical interpretation of the Russell-Hertzsprung diagram, which is directly linked to the origin of stellar energy.

PART II

Chapter 1

The Russell-Hertzsprung Diagram and the Origin of Stellar Energy

1.1 An explanation of the Russell-Hertzsprung diagram by the theory of the internal constitution of stars

The Russell-Hertzsprung diagram connects the luminosity L of a star to its spectral class or, in other words, the reduced temperature $T_{\rm eff}$. The theory of the internal constitution of stars uses the radius R of a star instead of the effective temperature $T_{\rm eff}$. It follows from the Stephan-Boltzmann law

$$L = 4\pi R^2 \sigma T_{\rm eff}^4\,, \qquad \sigma = \frac{1}{4}\,\alpha c\,,$$

where c is the velocity of light, α is the radiant energy density constant. Thus, the Russell-Hertzsprung diagram is

*The fact that the molecular weight is variable does not change the formulas, determining the pulsation period of Cepheids. The variability of μ can include a goal only if the whole structure of star has been changed.

the same for the correlation $L(R)$ or $M(R)$, if we use the "mass-luminosity" relation. Due to the existence of numerous sequences in the Russell-Hertzsprung diagram (the main sequence, the sequences of giants, dwarfs, etc.) the correlations $L(R)$ and $M(R)$ are not sufficiently clear. In this paragraph we show that for most stars the correlations $L(R)$ and $M(R)$ are directly connected to the mechanism generating stellar energy. The essence of the correlation $L(R)$ becomes clear, as soon as we replace the observable characteristics of stars (the masses M, the luminosities L, and the radii R) with the parameters which determine the physical conditions inside stars. The method of such calculations and the precision of the obtained results were discussed in detail in Part I of this research.

First we calculate the average density of a star

$$\rho = \frac{3M}{4\pi R^3}\,. \tag{1.1}$$

Then, having the mechanical equilibrium of a star, we calculate the average pressure within. This internal pressure is in equilibrium with the weight of the column whose aperture is one square centimeter and whose length is the radius of the star. The pressure is $p = g\rho R$. Because of $g = GM/R^2$,

$$p = \frac{3G}{4\pi}\frac{M^2}{R^4}\,. \tag{1.2}$$

What can be said about the temperature of a star? It should be naturally calculated by the energy flow of excess radiation F_R

$$F_R = \frac{L}{4\pi R^2}\,, \tag{1.3}$$

because the flow is connected to the gradient of the temperatures. If we know what mechanism transfers energy inside a star, we can calculate the temperature T by formula (1.1) or (1.2)

$$T = f(L, M, R)\,. \tag{1.4}$$

For instance, if energy is dragged by radiations, according to §1.2, we have

$$F_R = -\frac{c}{\kappa\rho}\frac{dB}{dr}\,, \tag{1.5}$$

where κ is the absorption coefficient per unit mass, B is the radiant pressure

$$B = \frac{1}{3}\,\alpha T^4. \tag{1.6}$$

We often use the radiant pressure B instead of the temperature. By formula (1.3) we can write

$$B \simeq \frac{\kappa F_R}{c}\,\rho R\,,$$

which, by using (1.1) and (1.3), gives

$$B \simeq \frac{3LM}{(4\pi)^2 cR^4}\,\kappa\,. \tag{1.4a}$$

If we know how κ depends on B and ρ, formula (1.4a) leads to equation (1.4). So formulae (1.1), (1.2), and (1.4a) permit calculation of the average numerical values of the density, the pressure, and the temperature for any star. Exact numerical values of the physical parameters at a given point inside a star (at the centre, for instance) can be obtained, if we multiply the formulae by dimensionless "structural" coefficients. We studied the structural coefficients in detail in Part I of this research. We studied them by both mathematical methods (solving the system of the dimensionless differential equations and mechanical equilibrium and heat equilibrium of a star) and empirical methods (the analysis of observable properties of stars).

Values of ρ, p, and T, calculated by formulae (1.1), (1.2), and (1.4), should be connected by the equation of the phase state of matter. Hence, we obtain the first theoretical correlation

$$F_1\,(L, M, R) = 0\,, \tag{1.7}$$

which almost does not depend on the kind of energy generation in stars.

For instance, a star built on an ideal gas has

$$p = \frac{\Re T}{\mu}\,\rho\,.$$

Dividing (1.2) by (1.1), we obtain

$$T \simeq \frac{G}{\Re}\,\mu\,\frac{M}{R}\,, \qquad B \simeq \frac{\alpha G^4}{3\Re^4}\,\mu\,\frac{M^4}{R^4}\,, \tag{1.8}$$

$$\gamma = \frac{B}{p} \simeq M^2\mu^4. \tag{1.9}$$

Comparing (1.8) to formula (1.4a), obtained for the energy transfer by radiation, we obtain the correlation (1.7) in clear form

$$L \simeq M^3\,\frac{\mu^4}{\kappa}\,. \tag{1.7a}$$

Another instance — a star built on a degenerate gas

$$p \simeq \rho^{\frac{5}{3}}\,,$$

then formulae (1.1) and (1.2) lead to

$$RM^{1/3} = \text{const}\,, \tag{7.b}$$

so in this case we just obtain the correlation like (1.7), where there is no L.

Formula (1.7a), which is true for an ideal gas, can include R only through κ. Therefore this formula is actually the "mass-luminosity" relation, which is in good agreement with observational data $L \sim M^3$, if $\mu^4/\kappa = \text{const} = 0.08$. The calculations are valid under the low radiant pressure $\gamma < 1$. As we see from formula (1.9), inside extremely bulky stars the value of γ can be more than 1. In such cases formula (1.2) will determine the radiant pressure

$$B \simeq \frac{M^2}{R^4},$$

not the gaseous one. Comparing to formula (1.4a), we have

$$L \simeq \frac{M}{\kappa}. \tag{1.7b}$$

Astronomical observations show that super-giants do not have the huge variations of M which are predicted by this formula. Therefore, in Part I, we came to the conclusion that $\gamma \leqslant 1$ for stars of regular masses $M \leqslant 100 M_{\odot}$, so formula (1.9) gives for them: $\mu = 1/2$. Hence, $\kappa = 0.8$, which is approximately equal to Thomson's absorption coefficient. This is very interesting, for we have obtained that the radiant pressure places a barrier to the existence of extremely large masses for stars, although there is no such barrier in the theory based on the equilibrium equations of stars.

Until now, we hardly used the heat equilibrium equation, which requires that the energy produced inside a star should be equal to its radiation into space. According to the heat equilibrium equation, the average productivity of energy by one gram of stellar matter can be calculated by the formula

$$\varepsilon = \frac{L}{M}. \tag{1.10}$$

On the other hand, if the productivity of energy is determined by some other reactions, ε would be a function of ρ and T. This function would also be dependent on the kinetics of the supposed reaction. Thus formulae (1.10), (1.1), (1.4), and the equation of the reaction demand the existence of the second correlation

$$F_2\left(L, M, R\right) = 0, \tag{1.11}$$

which is fully determined by the mechanism that generates energy in the reaction. For an ideal gas, R disappears from the first correlation $F_1 = 0$ (1.7). For this reason formula (1.11) transforms into the relation $L(R)$ or $M(R)$, which become directly dependent on the kind of energy sources in stars. For a degenerate gas we obtain another picture: as we saw above, in this case $M(R)$ is independent of energy sources, and then M and L are connected by equation (1.11).

1.2 Transforming the Russell-Hertzsprung diagram to the physical characteristics specific to the central regions of stars

Our task is to find those processes which generate energy in stars. In order to solve this problem, we must know physical conditions inside stars. In other words, we should proceed from the observed characteristics L, M, R to physical parameters.

We denote by a bar all the quantities expressed in terms of their numerical values in the Sun. Assuming, according to our conclusion in Part I, that stars have the same structure, we can, by formulae (1.1), (1.2), and (1.10), strictly calculate the central characteristics of stars

$$\bar{p}_c = \frac{\bar{M}^2}{\bar{R}^4}, \qquad \bar{\rho}_c = \frac{\bar{M}}{\bar{R}^3}, \qquad \bar{\varepsilon}_c = \frac{\bar{L}}{\bar{M}}. \tag{1.12}$$

Even for very different structures of stars, it is impossible to obtain distorted results by the formulae. As we saw in the previous paragraph, we can calculate the temperature (or, which is equivalent, the radiant pressure) in two ways, either way being connected to suppositions. First, the radiant pressure can be obtained through the flow of energy, i. e. through ε by formula (1.4a). The exact formula of that relation, by equations (1.27) in §1.3 (Part I), is

$$B_c = \frac{\varepsilon_c \kappa_c}{4\pi G c\,\lambda}\, p_c, \tag{1.13}$$

where λ is the structural parameter of the main system of the dimensionless equations of equilibrium: its numerical value is about 1. Second, for an ideal gas, the radiant pressure can be calculated directly from formulae (1.12)

$$\frac{\bar{B}_c}{\bar{\mu}^4} = \left(\frac{\bar{p}_c}{\bar{\rho}_c}\right)^4 = \frac{\bar{M}^4}{\bar{R}^4}. \tag{1.14}$$

Formulae (1.13) and (1.14) must lead to the same result. This requirement leads to the "mass-luminosity" relation. Our conclusion that all stars (except for while dwarfs) are built on an ideal gas is so well grounded that it is fair to use formula (1.14) in order to calculate the temperature or the radiant pressure in stars. Naturally, Eddington [21] showed: under temperatures of about a few million degrees, because of the ionization of matter, the atoms of even heavy elements take up so little space (about one millionth of their normal sizes) that van der Waals' corrections are negligible if the density is even much more than 1. However, because of plasma, there could be substantial electrostatic interactions between particles, making the pressure negative, and the gas approaches properties of a super-ideal one. The approximate theory of such phenomena in strong electrolytes has been developed by Debye and Hückell. Eddington and Rosseland applied the theory to a gas inside stars. They came to the conclusion that the electric pressure cannot substantially change the internal constitution of stars. Giving no details of that theory, we can show directly that the electric pressure is negligible in stars built on hydrogen. We compare the kinetic energy of particles to the energy of Coulomb interaction

$$kT > \frac{z^2 e^2}{r}.$$

As soon as the formula becomes true in a gas, the gas becomes ideal. Cubing the equation we obtain

$$\frac{(kT)^3}{n} = \frac{(kT)^4}{p} > z^6 e^6,$$

where n is the number of particles in a unit volume. Because the radiant pressure is given by the formula

$$B = \frac{\pi^2}{45}\frac{(kT)^4}{(\hbar c)^3}\,, \tag{1.6a}$$

a gas becomes ideal as soon as the ratio between the radiant pressure and the gaseous pressure becomes

$$\gamma > \frac{\pi^2 z^6}{45}\left(\frac{e^2}{\hbar c}\right)^3.$$

Because of formula (1.9), this ratio is determined by the mass of a star. Because $\gamma = 1$ under $\bar{M} = 100$, we obtain $\gamma^{\frac{1}{2}} \approx \bar{M}/100$. So, for an ideal gas, we obtain the condition

$$100 M_\odot > M > \frac{100\pi}{\sqrt{45}}\, z^3 \left(\frac{e^2}{\hbar c}\right)^{3/2} M_\odot\,. \tag{1.15}$$

which is dependent only on the mass of a star.

For hydrogen or singly ionized elements, we have $z = 1$. Hence, for hydrogen contents of stars, the electric pressure can play a substantial rôle only in stars with masses less than 0.01–0.02 of the mass of the Sun.

It is amazing that of all possible states of matter in stars there are realized those states which are the most simple from the theoretical point of view.

Now, if we know $\bar{M}$ and $\bar{R}$ for a star, assuming the same molecular weight $\bar{\mu} = 1$ for all stars (by our previous conclusions), we can calculate its central characteristics $\bar{\rho}_c$ and $\bar{T}_c$ by formulae (1.12) and (1.14). The range, within which the calculated physical parameters are located, is so large ($10^{-8} < \bar{\rho}_c < 10^6$, $10^{-2} < \bar{T}_c < 10^2$, $10^{-3} < \bar{\varepsilon}_c < 10^4$), that we use logarithmic scales. We use the abscissa for $\log \bar{\rho}_c$, while the ordinate is used for $\log \bar{B}_c$ (or equivalently, $4 \log \bar{T}_c$). If an energy generation law like $\varepsilon_c = f(\rho_c, T_c)$ exists in Nature, the points $\log \bar{\varepsilon}_c$ plotted along the z-coordinate axis will build a surface. On the other hand, the equilibrium condition requires formula (1.13), so the equilibrium states of stars should be possible only at the transection of the above surfaces*. Hence, stars should be located in the plane ($\log \bar{\rho}_c$, $\log \bar{B}_c$) along a line which is actually the relation $M(R)$ transformed to the physical characteristics inside stars. There in the diagram, we draw the numerical values of $\log \bar{\varepsilon}_c$ in order to picture the whole volume.

1.3 The arc of nuclear reactions

The equation for the generation of energy by thermonuclear reactions is

$$\varepsilon = A\,\rho\tau^2\varepsilon^{-\tau}, \qquad \tau = \frac{a}{T_{\mathrm{m}}^{1/3}}\,, \tag{1.16}$$

*The energy generation surface, drawn from the energy generation law $\varepsilon_c = f(\rho_c, T_c)$, and the energy drainage surface, drawn from formula (1.13).

where T_{m} is temperature expressed in millions of degrees. For instance, for the proton-proton reaction, the constants a and A take the values

$$a = 33.8, \qquad A = 4{\times}10^3. \tag{1.17}$$

In order to find the arc of the relation between ρ_c and B_c, on which stars should be located if nuclear reactions are the sources of their energy, we eliminate ε_c from formula (1.16) by formula (1.13)

$$\lambda 4\pi G c B_c = A \kappa_c\, p_c\, \rho_c \tau_c^2 e^{-\tau_c}. \tag{1.18}$$

As the exponent indicates (see formula (1.16)), ε is very sensitive to temperature. Therefore, inside such stars, a core of free convection should exist, as was shown in detail in Part I, §2.8. We showed there that λ cannot be calculated separately for stars within which there is a convective core: the equilibrium equations determine only λL_{x_0}, where L_{x_0} is the dimensionless luminosity

$$L_{x_0} = \int_0^{x_0} \varepsilon_1 \rho_1 x^2 dx\,. \tag{1.19}$$

In this formula x_0 is the dimensionless radius (see Part I). The subscript 1 on ε and ρ means that the quantities are taken in terms of their numerical values at the centre of a star. In the case under consideration (stars inside which thermonuclear reactions occur).

$$L_{x_0} = \int_0^{x_0} \rho_1^2 \tau_1^2 e^{-(\tau_1 - \tau_c)} x^2 dx\,.$$

Because this integral includes the convective core (where $\rho_1 = T_1^{3/2}$),

$$L_{x_0}(\tau_c) = \int_0^{x_0} T_1^{1/3} x^2 e^{-\tau_c\left(T_1^{-1/3} - 1\right)} dx\,. \tag{1.20}$$

The integral $L_{x_0}(\tau_c)$ can be easily taken by numerical methods, if we use Emden's solution $T_1(x)$ for stars of the polytropic structure 3/2. The calculations show that numerical values of the integral taken under very different τ_c are very little different from 1. For instance,

$$L_{x_0}(33.8) = 0.67, \qquad L_{x_0}(7.3) = 1.15.$$

For the proton-proton reactions formula (1.17), the first value of L_{x_0} is 1 million degrees at the centre of a star, the second value is one hundred million degrees. Assuming $L_{x_0} \approx 1$ (according to our conclusions in Part I), Table 3 gives $\lambda \approx 3$ in stars where the absorption coefficient is constant.

In Part I of this research we found the average molecular weight 1/2 for all stars. We also found that all stars have structures very close to the polytropic structure of the class 3/2. Under these conditions, the central temperature of the Sun should be $6{\times}10^6$ degrees. Therefore Bethe's carbon-nitrogen cycle is improbable as the source of stellar energy. As an example, we consider proton-proton reactions. Because of the numerical values obtained for the constants a and A (1.17), formula (1.18) gives

$$\log \rho_c = 0.217\,\tau_c - 5.5 \log \tau_c + 5.26 - \frac{1}{2} \log \frac{\kappa_c}{\mu} . \quad (1.21)$$

Taking κ_c/μ constant in this formula, we see that ρ_c has the very slanting minimum (independent of the temperature) at $\tau_c = 11$ that is $T_c = 30{\times}10^6$ degrees. In a hydrogen star where the absorption coefficient is Thomson, the last term of (1.21) is zero and the minimal value of ρ_c is 100. Hence, stars undergoing proton-proton reactions internally should be located along the line $\rho_c \approx 100$ in the diagram for (ρ_c, B_c). It appears that stars of the main sequence satisfy the requirement (in a rude approximation). Therefore, it also appears that the energy produced by thermonuclear reactions could explain the luminosity of most of stars. But this is only an illusion. This illusion disappears completely as soon as we construct the diagram for ($\log \bar{\rho}_c$, $\log \bar{B}_c$) using the data of observational astronomy.

1.4 Distribution of stars on the physical conditions diagram

Currently we know all three parameters (the mass, the bolometric absolute stellar magnitude, and the spectral class) for approximately two hundred stars. In our research we should use only independent measurements of the quantities. For this reason, we cannot use the stellar magnitudes obtained by the spectroscopic parallax method, because the basis of this method is the "mass-luminosity" relation.

For stars of the main sequence we used the observational data collection published in 1948 by Lohmann [26], who generalized data by Parenago and Kuiper. For eclipse variable stars we used data collections mainly by Martynov [27], Gaposchkin [28], and others. Finally, we took particularly interesting data about super-giants from collections by Parenago [29], Kuiper [7], and Struve [30]. Some important data about the masses of sub-dwarfs were given to the writer by Prof. Parenago in person, and I'm very grateful to him for his help, and critical discussion of the whole research. Consequently, we used the complete data of about 150 stars.

The stellar magnitudes were obtained by the above mentioned astronomers by the trigonometric parallaxes method and the empirically obtained bolometric corrections (Petit, Nickolson, Kuiper). In order to go from the spectral class to the effective temperature, we used Kuiper's temperature scale. Then we calculated the radius of a star by the formula

$$5 \log \bar{R} = 4.62 - m_b - 10 \log \bar{T}_{\rm eff} , \qquad (1.22)$$

where m_b is the bolometric stellar magnitude of the star. Then, by formulae (1.12) and (1.14), we calculated $\log \bar{\rho}_c$, $\log \bar{B}_c$, $\log \bar{\varepsilon}_c$. We calculated the characteristics for every star on our list. The results are given in Fig. 2*. There the abscissa takes the logarithm of the matter density, $\log \bar{\rho}_c$, while the ordinate takes the logarithm of the radiant energy density, $\log \bar{B}_c$, where both values are taken at the centre of a star†. Each star is plotted as a point in the numerical value of $\log \bar{\varepsilon}_c$ — the energy productivity per second from one gramme of matter at the centre of a star with respect to the energy productivity per second at the centre of the Sun. In order to make exploration of the diagram easier, we have drawn the net values of the fixed masses and radii. Bold lines at the left side and the right side are the boundaries of that area where the ideal gas law is true (stars land in exactly this area). The left bold line is the boundary of the ultimately large radiant pressure ($\gamma = 1$). The bold line in the lower part of the diagram is the boundary of the ultimately large electric pressure, drawn for hydrogen by formula (1.15). This line leads to the right side bold lines, which are the boundaries of the degeneration of gas calculated for hydrogen (the first line) and heavy elements (the second line).

We built the right boundary lines in the following way. We denote by n_e the number of free electrons inside one cubic centimetre, and μ_e the molecular weight per electron. Then

$$\rho = \mu_e m_{\rm H} n_e ,$$

so Sommerfeld's condition of degeneration

$$\frac{n_e \hbar^3}{2} \frac{1}{(2\pi m_e k T)^{3/2}} > 1 \qquad (1.23)$$

can be re-written as

$$\rho > 10^{-8} \mu_e T^{3/2}. \qquad (1.24)$$

For the variables p and ρ, we obtain the degeneration boundary equation‡

$$\begin{aligned} p &= k \mu_e^{5/3} \rho^{5/3}, \\ \bar{p} &= k \frac{\rho_\odot^{5/3}}{p_\odot} \bar{\rho}^{5/3} \mu_e^{5/3} , \end{aligned} \qquad (1.25)$$

which coincides with the Fermi gas state equation $p = K\rho^{5/3} = K_{\rm H} \mu_e^{5/3} \rho^{5/3}$ (formula 1.9 in Part I), if

$$K \approx K_{\rm H} = 9.89{\times}10^{12}.$$

*Of course not all the stars are shown in the diagram, because that would produce a very dense concentration of points. At the same time, the plotted points show real concentrations of stars in its different parts. — Editor's remark.

†The bar means that both values are expressed in multiples of the corresponding values at the centre of the Sun. — Editor's remark.

‡The degeneration boundary equation is represented here in two forms: expressed in absolute values of p and in multiples of the pressure in the Sun. — Editor's remark.

At the centre of the Sun, as obtained in Part I of this research (see formula 3.34),

$$\rho_{c\odot} = 9.2\,, \qquad p_{c\odot} = 9.5{\times}10^{15}, \qquad \gamma_{c\odot} = 0.4{\times}10^{-3}, \qquad B_{c\odot} = 3.8{\times}10^{12}, \qquad T_{c\odot} = 6.3{\times}10^{6}, \tag{1.26}$$

then we obtain

$$\bar{p} = 4{\times}10^{-2}\,\bar{\rho}^{\,5/3}\mu_e^{5/3}.$$

The right side boundaries drawn in the diagram are constructed for $\mu_e = 1$ and $\mu_e = 2$. At the same time these are lines along which stars built on a degenerate gas (the lines of Chandrasekhar's "mass-radius" relation) should be located. In this case the ordinate axis has the meaning $\log\,(\bar{p}/\bar{\rho})^4$ that becomes the logarithm of the radiant energy density $\log \bar{B}$ for ideal gases only. In this sense we have drawn white dwarfs and Jupiter on the diagram. Under low pressure, near the boundary of strong electric interactions, the degeneration lines bend. Then the lines become constant density lines, because of the lowering of the ionization level and the appearance of normal atoms. The lines were constructed according to Kothari's "pressure-ionization" theory [31]. Here we see a wonderful consequence of Kothari's theory: the maximum radius which can be attained by a cold body is about the radius of Jupiter.

Finally, this diagram contains the arc along which should be located stars whose energy is generated by proton-proton reactions. The arc is built by formula (1.21), where we used the central characteristics of the Sun (1.26) obtained in Part I.

The values $\log \bar{\varepsilon}_c$ plotted for every star builds the system of isoergs — the lines of the same productivity of energy. The lines were drawn through the interval of ten changes of $\bar{\varepsilon}_c$. If a "mass-luminosity" relation for stars does not contain their radii, $\bar{\varepsilon}_c$ should be a function of only the masses of stars. Hence, the isoergs should be parallel to the constant mass lines. In general, we can suppose the "mass-luminosity" relation as the function

$$L \sim M^{\alpha}, \tag{1.27}$$

then the interval between the neighbouring isoergs should decrease with increasing α according to the picture drawn in the upper left part of the diagram. We see that the real picture does not correspond to formula (1.27) absolutely. Only for giants, and the central region of the main sequence (at the centre of the diagram) do the isoergs trace a path approximately parallel to the constant mass lines at the interval $\alpha = 3.8$. In all other regions of the diagram the isoergs $\bar{\varepsilon}_c$ are wonderfully curved, especially in the regions of supergiants (the lower left part of the diagram) and hot subdwarfs (the upper right part). As we will soon see, the curvilinearity can be explained. In the central concentration of stars we see two opposite tendencies of the isoergs to be curved. We have a large dataset here, so the isoergs were drawn very accurately. The twists are in exact agreement with the breaks, discovered by Lohmann [26], in the "mass-luminosity" relation for stars of the main sequence. It is wonderful that this tendency, intensifying at the bottom, gives the anomalously large luminosities for sub-giants (the satellites of Algol) — the circumstance, considered by Struve [30]. For instance, the luminosity of the satellite of XZ Sagittarii, according to Struve, is ten thousand times more than that calculated by the regular "mass-luminosity" relation. There we obtain also the anomalously large luminosity, discovered by Parenago [29], for sub-dwarfs of small masses. The increase of the opposite tendency at the top verifies the low luminosity of extremely hot stars, an increase which leads to Trumpler stars. It is very doubtful that masses of Trumpler stars measured through their Einstein red shift are valid. For this reason, the diagram contains only Trumpler stars of "intermediate" masses. Looking at the region of sub-giants and sub-dwarfs (of large masses and of small ones) we see that ε is almost constant there, and independent of the masses of the stars. Only by considering altogether the stars located in the diagram we can arrive at the result obtained in Part I of this research: $L \sim M^3$.

So, the first conclusion that can be drawn from our consideration of the diagram is: deviations from the "mass-luminosity" relation are real, they cannot be related to systematic errors in the observational data. The possibility of drawing the exact lines of constant $\bar{\varepsilon}_c$ itself is wonderful: it shows that ε is a simple function of ρ and B. Hence, the luminosity L is a simple function of M and R. Some doubts can arise from the region located below and a little left of the central region of the diagram, where the isoergs do not coincide with L for sub-dwarfs of spectral class F–G and L of normal dwarfs of class M. It is most probable that the inconsistency is only a visual effect, derived from errors in experimental measurements of the masses and radii of the sub-dwarfs.

As a whole our diagram shows the plane image of the surface $\varepsilon\,(\rho, B)$. We obtained much more than expected: we should obtain only one section of the surface, but we obtained the whole surface, beautifully seen in the central region of the diagram. Actually, we see no tendency for stars to be distributed along a sequence $\varepsilon = \text{const}$. Thus, of the two equations determining ε, there remains only one: *the energy productivity in stars is determined by the energy drainage (radiation) only*. This conclusion is very important. Thus the mechanism that generates energy in stars is not of any kid of reactions, but is like the generation of energy in the process of its drainage. The crude example is the energy production when a star, radiating energy into space, is cooling down: the star compresses, so the energy of its gravitational field becomes free, cooling the star (the well-known Helmholtz-Kelvin mechanism). Naturally, in a cooling down (compress-

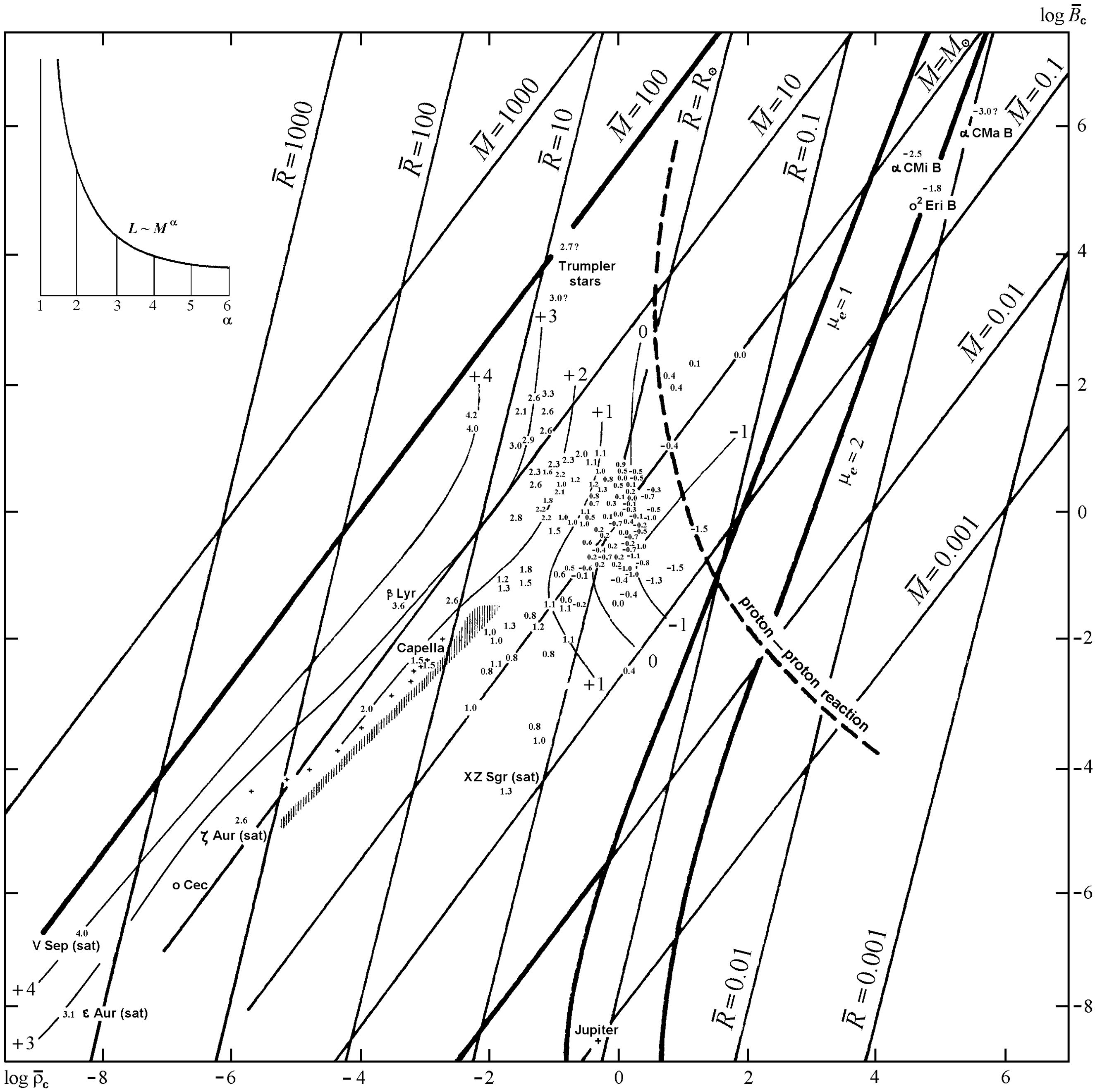


Fig. 2: The diagram of physical conditions inside stars (the stellar energy diagram): the productivity of stellar energy sources independence of the physical conditions in the central regions of stars. The abscissa is the logarithm of the density of matter, the ordinate is the logarithm of the radiant energy density (both are taken at the centre of stars in multiples of the corresponding values at the centre of the Sun). The small diagram at the upper left depicts the intervals between the neighbouring isoergs.

ing) star the quantity of energy generated is determined by the speed of this process. At the same time the speed is regulated by the heat drainage. Of course, the Helmholtz-Kelvin mechanism is only a crude example, because of the inapplicable short period of the cooling (a few million years). At the same time the mechanism that really generates energy in stars should also be self-regulating by the radiation. In contrast to reaction, such a mechanism should be called a *machine*.

It should be noted that despite many classes of stars in the diagram, the filling of the diagram has some limitations.

First there is the main direction along which stars are concentrated under a huge range of physical conditions — from the sequence of giants, then the central concentration in our diagram (the so-called main sequence of the Hertzsprung-Russel diagram), to sub-dwarfs of class A and white dwarfs. In order to amplify the importance of this direction, we indicated the main location of normal giants by a hatched strip. The main direction wonderfully traces an angle of exactly 45°. Hence, all stars are concentrated along the line, determined by the equation*

$$B \sim \rho\mu^4. \tag{1.28}$$

Because stars built on a degenerate gas satisfy this direction, a more accurate formula is

$$p \sim \rho^{5/5} \tag{1.28a}$$

Second, there is in the main direction (1.28) a special point — the centre of the main sequence†, around which stars are distributed at greater distances, and in especially large numbers.

Thus, there must exist two fundamental constants which determine the generation of energy in stars:

1. The coefficient of proportionality of equation (1.28);
2. One of the coordinates of the "main point", because its second coordinate is determined by the eq. (1.28).

The above mentioned symmetry of the surface $\varepsilon\,(\rho, B)$ is connected to the same two constants.

Concluding the general description of the diagram, we note: this diagram can also give a practical profit in calculations of the mass of a star by its luminosity and the spectral class. Naturally, having the radius calculated, we follow the line $R = \text{const}$ to that point where $\log\bar{\varepsilon} + \log\bar{M}$ gives the observed value of $\log\bar{L}$.

1.5 Inconsistency of the explanation of stellar energy by Bethe's thermonuclear reactions

It is seemingly possible that the existence of the uncovered main direction along which stars are concentrated in our diagram support a stellar energy mechanism like reactions. In the real situation the equation of the main direction (1.28) contradicts the kinetics of any reaction. Naturally, equation (1.28) can be derived from the condition of energy drainage (1.13) only if

$$\varepsilon \sim \frac{1}{T}\,, \quad \text{under} \quad \rho \sim T^4, \tag{1.29}$$

i. e. only if the energy productivity increases with decrease in temperature and hence the density. The directions of all the isoergs in the diagram, and also the numerical values $\varepsilon = 10^3$–10^4 in giants and super-giants under the low temperatures inside them (about a hundred thousands degrees) cannot be explained by nuclear reactions. It is evident therefore, that the possibility for nuclear reactions is just limited by the main sequence of the Russell-Hertzsprung diagram (the central concentration of stars in our diagram).

The proton-proton reaction arc is outside the main sequence of stars. If we move the arc to the left, into the region of the main sequence stars, we should change the constant A in the reaction equation (1.16) or change the physical characteristics at the centre of the Sun (1.26) as we found in Part I. Equation (1.18) shows that the shift of the proton-proton reaction arc along the density axis is proportional to the square of the change of the reaction constant A. Hence, in order to build the proton-proton reaction arc through the main concentration of stars we should take at least $A = 10^5$–10^6 instead of the well-known value $A = 4{\times}10^3$. This seems very improbable, for then we should ignore the central characteristics of the Sun that we have obtained, and hence all conclusions in Part I of this research which are in fine agreement with observational data. Only in a such case could we arrive at a temperature of about 20 million degrees at the centre of the Sun; enough for proton-proton reactions and also Bethe's carbon-nitrogen cycle.

All theoretical studies to date on the internal constitution of stars follow this approach. The sole reason adduced as proof of the high concentration of matter in stars, is the slow motion of the lines of apsides in compact binaries. However the collection published by Luyten, Struve, and Morgan [32] shows no relation between the velocity of such motion and the ratio of the star radius to the orbit semi-axis. At the same time, such a relation would be necessary if the motion of the lines of apsides in a binary system is connected to the deformations of the stars. Therefore we completely agree with the conclusion of those astronomers, that no theory correctly explains the observed motions of apsides. Even if we accept that the arc of nuclear reactions could intersect the central concentration of stars in our diagram (the stars of the main sequence in the Russell-Hertzsprung diagram), we should explain why the stars are distributed not along this arc, but fill some region around it. One could explain this circumstance by a "dispersion" of the parameters included in the main equations. For instance, one relates this dispersion

*See formula (1.14). — Editor's remark.

†The main sequence in the sense of the Russell-Hertzsprung diagram, is here the central concentration of stars. — Editor's remark.

to possible differences in the chemical composition of stars, their structure etc. Here we consider the probability of such explanations.

The idea that stars can have different chemical compositions had been introduced into the theory in 1932 by Strömgren [16], before Bethe's hypothesis about nuclear sources for stellar energy. He used only the heat drainage condition (1.13), which leads to the "mass-luminosity" relation (1.7a) for ideal gases. In chapter 2 of Part I we showed in detail that the theoretical relation (1.7a) is in good agreement (to within the accuracy of Strömgren's data) with the observed correlation for hydrogen stars (where we have Thomson's absorption coefficient, which is independent of physical conditions). Introducing some a priori suppositions (see §2.7, Part I), Eddington, Strömgren and other researchers followed another path; they attempted to explain non-transparency of stellar matter by high content of heavy elements, which build the so-called Russell mix. At the same time the absorption theory gives such a correlation $\kappa(\rho, B)$ for this mix which, being substituted into formula (1.7a), leads to incompatibility with observational data. Strömgren showed that such a "difficulty" can be removed if we suppose different percentages of heavy elements in stars, which substantially changes the resulting absorption coefficient κ. Light element percentages X can be considered as the hydrogen percentage. Comparing the theoretical formula to the observable "mass-luminosity" relation gives the function $X(\rho, B)$ or $X(M, R)$. Looking at the Strömgren surface from the physical viewpoint we can interpret it as follows. As we know, the heat drainage equation imposes a condition on the energy generation in stars. This is condition (1.13), according to which κ and μ depend on the chemical composition of a star. Let us suppose that the chemical composition is determined by one parameter X. Then

$$\varepsilon = f_1(\rho, B, X)\,. \qquad \text{(I)}$$

For processes like a reaction, the energy productivity ε is dependent on the same variables by the equation of this reaction

$$\varepsilon = f_2(\rho, B, X)\,. \qquad \text{(II)}$$

So we obtain the condition $f_1 = f_2$, which will be true only if a specific relation $X(\rho, B)$ is true in the star. The parameter X undergoes changes within the narrow range $0 \leqslant X \leqslant 1$, so stars should fill a region in the plane (ρ, B). Some details of the Russell-Hertzsprung diagram can be obtained as a result of an additional condition, imposed on $X(\rho, B)$: Strömgren showed that arcs of $X = \text{const}$ can be aligned with the distribution of stars in the Russell-Hertzsprung diagram. Kuiper's research [33] is especially interesting in this relation. He discovered that stars collected in open clusters are located along one of Strömgren's arcs $X = \text{const}$ and that the numerical values of X are different for different clusters. Looking at this result, showing that stellar clusters are different according to their hydrogen percentage, one can perceive an evolutionary meaning — the proof of the nuclear transformations of elements in stars.

Strömgren's research prepared the ground for checking the whole nuclear hypothesis of stellar energy: substituting the obtained correlation $X(\rho, B)$ into the reaction equation (II), we must come to the well-known relation (I). The nuclear reaction equation (1.16), where X is included through A, had not passed that examination. Therefore they introduced the second parameter Y into the theory — the percentage of helium. As a result, every function f_1 and f_2 can be separately equated to the function $\varepsilon(\rho, B)$ known from observations. Making the calculations for many stars, it is possible to obtain two surfaces: $X(\rho, B)$ and $Y(\rho, B)$. However, both surfaces are not a consequence of the equilibrium conditions of stars. It remains unknown as to why such surfaces exist, i. e. why the observed ε is a simple function of ρ and B? It is very difficult to explain this result by evolutionary transformations of X and Y, if the transformation of elements procedes in only one direction. Of course, taking a very small part of the plane (ρ, B), the evolution of elements can explain changes of X and Y. For instance, calculations made by Masevich [34] gave a monotone decrease of hydrogen for numerous stars located between the spectral classes B and G. To the contrary, from the class G to the class M, the hydrogen percentage increases again (see the work of Lohmann work [26] cited above). As a result we should be forced to think that stars evolve in two different ways. In such a case the result that the chemical composition of stars is completely determined by the physical conditions inside them can only be real if there is a balanced transformation of elements. Then the mechanism that generates energy in stars becomes the Helmholtz-Kelvin mechanism, not reactions. Nuclear transformations of elements only become an auxiliary circumstance which changes the thermal capacity of the gas. At the same time, the balanced transformation of elements is excluded from consideration, because it is possible only if the temperature becomes tens of billions of degrees, which is absolutely absent in stars.

All the above considerations show that the surfaces $X(\rho, B)$ and $Y(\rho, B)$ obtained by the aforementioned researchers are only a result of the trimming of formulae (I) and (II) to the observed relation $\varepsilon(\rho, B)$. Following this approach, we cannot arrive at a solution to the stellar energy problem and the problem of the evolution of stars. This conclusion is related not only to nuclear reactions; it also shows the impossibility of any sources of energy whose productivity is not regulated by the heat drainage condition. Naturally, the coincidence of the surfaces (I) and (II) manifests their identity. In a real situation the second condition is not present*.

*For reactions, the energy productivity increases with the increase of the density. In the heat drainage condition we see the opposite: equation (1.13). Therefore the surfaces (I) and (II), located over the plane (ρ, B), should be oppositely inclined — their transection should be very sharp.

So we get back to our conclusion of the previous paragraph: there are special physical conditions, the main direction (1.28) and the main point in the plane (ρ, B), about which stars generate exactly as much energy as they radiate into space. In other words, stars are *machines* which generate radiant energy. The heat drainage is the power regulation mechanism in the machines.

1.6 The "mass-luminosity" relation in connection with the Russell-Hertzsprung diagram

The luminosity of stars built on an ideal gas, radiant transfer of energy and low radiant pressure, is determined by formula (1.7a). This formula is given in its exact form by (2.38) in Part I. We re-write formula (2.38) as

$$\bar{\varepsilon} = \frac{\bar{L}}{\bar{M}} = 1.04{\times}10^4 \,\frac{\mu^4}{\kappa_c}\left(\frac{\lambda L_{x_0}}{M_{x_0}^3}\right)\bar{M}^2, \tag{1.30}$$

where M_{x_0} is the dimensionless mass of a star, κ_c is the absorption coefficient at its centre. It has already been shown that the structural multiplier of this formula has approximately the same numerical value

$$\frac{\lambda L_{x_0}}{M_{x_0}^3} \simeq 2{\times}10^{-3} \tag{1.31}$$

for all physically reasonable models of stars. The true "mass-luminosity" relation is shown in Fig. 2 by the system of isoergs $\bar{\varepsilon} = \bar{L}/\bar{M} = \text{const}$. If we do not take the radius of a star into account, we obtain the correlation shown in Fig. 1, Part I. There L is approximately proportional to the cube of M, although we saw a dispersion of points near this direction $L \sim M^3$. As we mentioned before, in Part I, the comparison of this result to formula (1.30) indicates that: (1) the radiant pressure plays no substantial rôle in stars, (2) stars are built on hydrogen.

Now we know that the dispersion of points near the average direction $L \sim M^3$ is not stochastic. So we could compare the exact correlation to the formula (1.30), and also check our previous conclusions.

Our first conclusion about the negligible rôle of the radiant pressure is confirmed absolutely, because of the mechanical equilibrium of giants. Naturally, comparing formula (1.7b) to (1.7a), we see that the greater the rôle of the radiant pressure, the less ε is dependent on M, so the interval between the neighbouring isotherms should increase for large masses. Such a tendency is completely absent for bulky stars (see the stellar energy diagram, Fig. 2). This result, in combination with formula (1.9) (its exact form is formula 2.47, Part I), leads to the conclusion that giants are built mainly on hydrogen (the molecular weight ½). Thus we calculate the absorption coefficient for giants. We see in the diagram that red giants of masses $\approx 20 M_\odot$ have $\log \bar{\varepsilon} = 3$. By formulae (1.30) and (1.31), we obtain

$$\frac{\kappa_c}{\mu^4} = 8\,. \tag{1.32}$$

If $\mu = 1/2$, we obtain $\kappa_c = 0.5$. This result implies that the non-transparency of giants is derived from Thomson's dispersion of light in free electrons ($\kappa_{\text{T}} = 0.40$), as it should be in a pure hydrogen star.

The main peculiarity of the "mass-luminosity" relation is the systematic curvilinearity of the isoergs in the plane (ρ, B). Let us show that this curvilinearity cannot be explained by the changes of the coefficient in formula (1.30). First we consider the multiplier containing the molecular weight and the absorption coefficient.

The curvilinearity of the isoergs shows that for the same mass the diagram contains anomalous low luminosity stars at the top and anomalous bright stars at the bottom. Hence, the left part of (1.32) should increase under higher temperatures, and should decrease with lower temperatures. Looking from the viewpoint of today's physics, such changes of the absorption coefficient are impossible. Moreover, for the ultimate inclinations of the isoergs, we obtain absolutely impossible numerical values of the coefficient (1.32). For instance, in the case of super-giants, the lower temperature stars, this coefficient is 100 times less than that in giants. Even if we imagine a star built on heavy elements, we obtain that κ is about 1. In hot super-giants (the direction of Trumpler stars) the coefficient (1.32) becomes 200. Because of high temperatures in such stars, the absorption coefficient cannot be so large.

In order to explain the curvilinearity by the structural multiplier (1.31), we should propose that it be anomalously large in stars of high luminosity (sub-giants) and anomalously small in stars like Trumpler stars. We note that the dimensionless mass M_{x_0} included in (1.31) cannot be substantially changed, as shown in Part I. So the structural multiplier (1.31) can be changed by only λL_{x_0}. Employing the main system of the dimensionless equations of equilibrium of stars, we easily obtain the equation

$$\frac{dB_1}{dp_1} = \frac{\lambda L_x}{M_x}\,, \tag{1.33}$$

which is equation (2.22) of Part I, where B_1 and p_1 are the radiant pressure and the gaseous pressure expressed in multiples of their values at the centre of a star. Here the absorption coefficient κ is assumed constant from the centre to the surface, i. e. $\kappa_1 = 1$. Applying this equation to the surface layers of a star, we deduce that the structural coefficient is

$$\frac{\lambda L_{x_0}}{M_{x_0}} = \frac{B_1}{p_1}\,. \tag{1.34}$$

We denote the numerical values of the functions at the boundary between the surface layer and the "internal" layers of a star by the subscript 0. We consider two ultimate cases of the temperature gradients within the "internal" layers:

1. The "internal" zone of a star is isothermal:
$$\frac{\lambda L_{x_0}}{M_{x_0}} = \frac{1}{p_{1_0}}\,, \tag{1.34a}$$
2. The "internal" zone of a star is convective ($B_1 = p_1^{8/5}$):
$$\frac{\lambda L_{x_0}}{M_{x_0}} = p_{1_0}\,. \tag{1.34b}$$

In the first theoretical case, spreading the isothermal zone to almost the surface of a star, we can make the structural coefficient as large as we please. This case is attributed to sub-giants and anomalous bright stars in general. The second theoretical case can explain stars of anomalously low luminosity. Following this way, i. e. spreading the convective zone inside stars, Tuominen [35] attempted to explain the low luminosity of Trumpler stars.

The isothermy can appear if energy is generated mainly in the upper layers of a star. The spreading of the convective zone outside the Schwarzschild boundary can occur if energy is generated in moved masses of stellar gas, i. e. under forced convection. A real explanation by physics should connect the above peculiarities of the energy generation to the physical conditions inside stars or their general characteristics L, M, R. Before attempting to study the theoretical possibility of such relations, it is necessary to determine them first from observational data. Dividing $\bar{\varepsilon}$ by $\bar{M}^2$ for every star, we obtain the relation of the structural coefficient of formula (1.30) for ρ and B. But, at the same time, the determination of this relation in this way is somewhat unclear. There are no clear sequences or laws, so we do not show it here. Generally speaking, a reason should be simpler than its consequences. Therefore, it is most probable that the structural coefficient is not the reason. It is most probable that the reason for the incompatibility of the observed "mass-luminosity" relation with formula (1.30) is that equation (1.30) itself is built incorrectly. This implies that the main equations of equilibrium of stars are also built incorrectly. This conclusion is in accordance with our conclusion in the previous paragraph: energy is generated in stars like in machines — their workings are incompatible with the standard principles of today's mechanics and thermodynamics.

1.7 Calculation of the main constants of the stellar energy state

The theoretical "mass-luminosity" relation (1.30) is obtained as a result of comparing the radiant energy B calculated by the excess energy flow (formula 1.4a or 1.13) to the same B calculated by the phase state equation of matter (through p and ρ by formula 1.14). Therefore the incompatibility of the theoretical correlation (1.30) to observational data can be considered as the incompatibility of both the values of B. So we denote by B^* the radiant pressure calculated by the ideal gas equation. For the radiant transport of energy in a star, formulae (1.4a) and (1.13) lead to
$$\frac{\bar{B}^*}{\bar{\kappa}} = \bar{\varepsilon}\bar{p}\,. \tag{1.35}$$

By this formula we can calculate $\bar{B}^*/\bar{\kappa}$ for every star of the stellar energy diagram (Fig. 2). As a result we can find the correlation of the quantity $\bar{B}^*/\bar{\kappa}$ to $\bar{p}$ and $\bar{\rho}$. Fig. 3 shows the stellar energy diagram transformed in this fashion. Here the abscissa is $\log\bar{\rho}$, while the ordinate is $\log\bar{p}$. In order to make the diagram readable, we have not plotted all stars. We have plotted only the Sun and a few giants. At the same time we drawn the lines of constant $\bar{B}^*/\bar{\kappa}$ through ten intervals. The lines show the surface $\log\bar{B}^*/\bar{\kappa}\,(\log\bar{\rho},\log\bar{p})$. For the constant absorption coefficient κ, the lines show the system of isotherms. If $B^* = B$, there should be a system of parallel straight lines, inclined at 45° to the $\log\bar{p}$ axis and following through the interval 0.25. As we see, the real picture is different in principle. There is in it a wonderful symmetry of the surface $\log\bar{B}^*/\bar{\kappa}$. Here the origin of the coordinates coincides with the central point of symmetry of the isoergs. At the same time it is the main point mentioned in relation in the stellar energy diagram. The coordinates of the point with respect to the Sun are
$$\begin{aligned} &\log\bar{\rho}_0 = -0.58, \quad \log\bar{p}_0 = -0.53, \\ &\log\bar{B}_0 = +0.22, \quad \log\bar{B}_0^* = +0.50. \end{aligned} \tag{1.36}$$

Using the data, we deduce that the main point is attributed to a star of the Russell-Hertzsprung main sequence, which has spectral class A4. Rotating the whole diagram around the main point by 180°, we obtain almost the same diagram, only the logarithms of the isotherms change their signs. Hence, if
$$\frac{\frac{B^*}{\kappa}}{\frac{B_0^*}{\kappa}} = f\left(\frac{p}{p_0}, \frac{\rho}{\rho_0}\right),$$
we have
$$f\left(\frac{p}{p_0}, \frac{\rho}{\rho_0}\right) f\left(\frac{p_0}{p}, \frac{\rho_0}{\rho}\right) = 1\,. \tag{1.37}$$

The relation (1.37) is valid in the central region of the diagram. An exception is white dwarfs, in which B^*/κ is 100 times less than that required by formula (1.37), i. e. 100 times less than that required for the correspondence to giants after the 180° rotation of the diagram. It is probable that this circumstance is connected to the fact that white dwarfs are located close to the boundary of degenerate gas.

Besides the isotherms, we have drawn the main direction along which stars are distributed. Now the equation of the direction (1.28) can be written in the more precise form
$$\log\frac{\bar{B}}{\bar{\rho}} = +0.80\,. \tag{1.38}$$

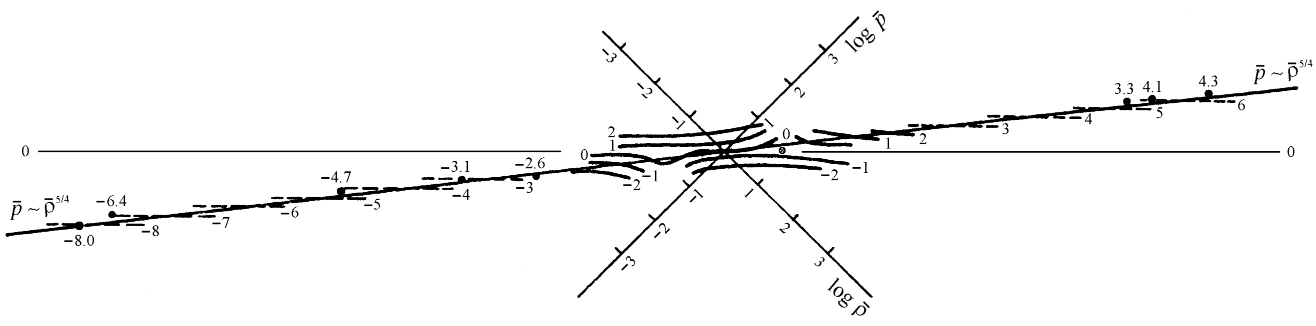


Fig. 3: Isotherms of stellar matter. The coordinate axes are the logarithms of the matter density and the gaseous pressure. Dashed lines show isotherms of an ideal gas.

Because of the very large range of the physical states in the diagram, the main direction is drawn very precisely (to within 5%). It should be noted that, despite their peculiarities, white dwarfs satisfy the main direction like all regular stars.

A theory of the internal constitution of stars, which could explain observational data (the relation 1.37, for instance), should be built on equations containing the coordinates of the main point. This circumstance is very interesting: it shows that there is an absolute system of "physical coordinates", where physical quantities of absolutely different dimensions can be combined. Such combinations can lead to a completely unexpected source of stellar energy. Therefore it is very important to calculate the absolute numerical values of the constants (1.36). Assuming in (1.36) a mostly hydrogen content for stars $\mu = 1/2$, and using the above calculated physical characteristics at the centre of the Sun (1.26), we obtain

$$\rho_0 = 2.4, \quad p_0 = 2.8\times10^{15}, \quad B_0 = 6.3\times10^{12}. \qquad (1.39)$$

We calculate B_0^* by formula (1.13). Introducing the average productivity of energy ε

$$B_c^* = \frac{\varepsilon \kappa_c p_c}{4\pi G c} \frac{M_{x_0}}{\lambda L_{x_0}}, \qquad (1.40)$$

assuming κ_c equal to Thomson's absorption coefficient, $\varepsilon_\odot = 1.9$, $M_{x_0} = 11$, and the structural multiplier according to (1.31). We then obtain for the Sun, $B_{c\odot}^* = 1.1\times10^{12}$ instead of $B_{c\odot} = 3.8\times10^{12}$. Hence,

$$B_0^* = 4.1\times10^{12} \approx B_0\,. \qquad (1.41)$$

We introduce the average number of electrons in one cubic centimetre n_e instead of the density of matter: $\rho = 1.66\times10^{-24} n_e$. Then the equation of the main direction becomes

$$\frac{3B}{n_e} = 1.4\times10^{-11} = 8.7 \text{ eV}, \qquad (1.42)$$

which is close to the hydrogen ionization potential, i. e. $\chi_0 = 13.5$ eV. Thus the average radiant energy per particle in stars (calculated by the ideal gas formula) is constant and is about the ionization energy of the hydrogen atom. Fig. 3 shows that, besides the main direction, the axis $\rho = \rho_0$ is also important. Its equation can be formulated through the average distance between particles in a star

$$r = 0.55\,(n_e)^{-1/3}$$

as follows

$$r = 0.51\times10^{-8} = r_{\rm H} = \frac{e^2}{2\chi_0}, \qquad (1.43)$$

where $r_{\rm H}$ is the radius of the hydrogen atom, e is the charge of the electron. As a result we obtain the very simple correlation between the constants of the lines (1.42) and (1.43), which bears a substantial physical meaning.

In the previous paragraph we showed that the peculiarities of the "mass-luminosity" relation* cannot be explained by changes of the absorption coefficient κ. Therefore the lines $B^*/\kappa = \text{const}$ should bear the properties of the isotherms. The isotherms drawn in Fig. 3 are like the isotherms of the van der Waals gas. The meaning of this analogy is that there is a boundary near which the isotherms become distorted, at which the regular laws of thermodynamics are violated. The asymptotes of the boundary line (the boundary between two different phases in the theory of van der Waals) are axes (1.42) and (1.43). The distortion of the isotherms increases with approach to the axis $\rho = \rho_0$ or $r = r_{\rm H}$. That region is filled by stars of the Russell-Hertzsprung main sequence. The wonderful difference from van der Waals' formula is the fact that there are two systems of the distortions, equation (1.37), which become smoothed with the distance from the axis $\rho = \rho_0$ (for both small densities and large densities).

Stars can radiate energy for a long time only under conditions close to the boundaries (1.42) and (1.43). This most probably happens because the mechanism generating energy in stars works only if the standard laws of classical physics are broken.

The results are completely unexpected from the viewpoint of contemporary theoretical physics. The results show

*The dispersion of showing-stars points around the theoretically calculated direction "mass-luminosity". — Editor's remark.

that in stars the classical laws of mechanics and thermodynamics are broken much earlier than predicted by Einstein's theory of relativity, and it occurs under entirely different circumstances. The main direction constants (1.42) and (1.43) show that the source of stellar energy is not Einstein's conversion of mass and energy (his mass-energy equivalence principle), but by a completely different combination of physical quantities.

Here we limit ourselves only to conclusions which follow from the observational data. A generalization of the results and subsequent theoretical consequences will be dealt with in the third part of this research. In the next chapter we only consider some specific details of the Russell-Hertzsprung diagram, not previously discussed.

Chapter 2

Properties of Some Sequences in the Russell-Hertzsprung Diagram

2.1 The sequence of giants

The stellar energy diagram (see Fig. 2) shows that the "mass-luminosity" relation has the most simple form for stars of the Russell-Hertzsprung main sequence

$$L \sim M^{\alpha}, \qquad \alpha = 3.8. \tag{2.1}$$

Cepheids, denoted by crosses in the diagram, also satisfy the relation (2.1). Using the pulsation equation $P\sqrt{\rho} = c_1$ we obtained (see formula 3.25 of Part I)

$$\left(0.30 - \frac{1}{5\alpha}\right)\left(m_b - 4.62\right) + \log P + 3\log \bar{T}_{\text{eff}} = \log c_1\,, \tag{2.2}$$

where $\bar{T}_{\text{eff}}$ is the reduced temperature of a star, expressed in multiples of the reduced temperature of the Sun, m_b is the absolute stellar magnitude, P is the pulsation period (days). We plot stars in a diagram where the abscissa is $m_b - 4.62$, while the ordinate is $\log P + \log \bar{T}_{\text{eff}}$. As a result we should obtain a straight line, which gives both the constant c_1 (see §3.3 of Part I) and the angular coefficient $0.30 - 1/5\alpha$. Fig. 4 shows this diagram, built using the collected data of Becker [36], who directly calculated $\bar{T}_{\text{eff}}$ and m_b by the radiant velocities arc (independently of the distances). As a result the average straight line satisfying all the stars has the angular coefficient 0.25 and $c_1 = 0.075$. Hence, $\alpha = 4$, which is in fine accordance with the expected result ($\alpha = 3.8$). Such a coincidence makes Melnikov's conclusion unreasonable: that Cepheids have the same masses ($\alpha = \infty$), as shown by the dashed line in Fig. 4.

In §1.5 of Part I we showed that the "mass-luminosity" relation for giants is explained by the fact that the structural coefficient $\lambda L_{x_0}/M_{x_0}^3$ has the same value $\simeq 2{\times}10^{-3}$ (1.31) for all stars. In order to obviate difficulties which appear if one attempts to explain the luminosity of giants by nuclear reactions, one attributes to them an exotic internal constitution (the large shell which covers a normal star). Therefore, the simple structure of giants we have obtained gives an additional argument for the inconsistency of the nuclear sources of stellar energy. At the same time, because of their simple structure, giants and super-giants are quite wonderful. For instance, for a giant like the satellite of ε Aurigae we obtain its central density at 10^{-4} of the density of air, and the pressure at about 1 atmosphere. Therefore, it is quite possible that in moving forward along the main direction we can encounter nebulae satisfying the condition (1.42). Such nebulae can generate their own energy, just like stars.

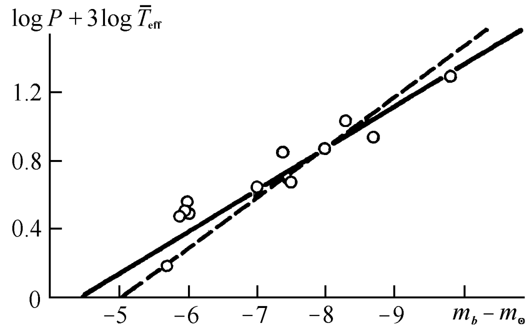


Fig. 4: Finding the exponent index α in the $L \sim M^{\alpha}$ relation for Cepheids.

Because of the physical conditions in giants, obtained above, the huge amounts of energy radiating from them cannot be explained by nuclear reactions. Even if this were true, their life-span would be very short. For reactions, the upper limit of the life-span of a star (the full transformation of its mass into radiant energy) can be obtained as the ratio of $\bar{\varepsilon}$ to c^2. So, by formula (1.40), we obtain

$$t = \frac{t_0}{4\gamma_c}\left(\frac{M_{x_0}}{\lambda L_{x_0}}\right), \tag{2.3}$$

where

$$t_0 = \frac{\kappa_{\text{T}}\, c}{\pi G} = 6{\times}10^{16}\ \text{sec} = 2{\times}10^{9}\ \text{years} \tag{2.4}$$

and $\gamma_c = B_c/p_c$ is the ratio of the radiant pressure to the gaseous one. As obtained, the structural multiplier here is about 4. Therefore

$$t = \frac{t_0}{\gamma_c}\,. \tag{2.5}$$

In giants $\gamma_c \approx 1$, so we obtain that t is almost the same as t_0. At the same time, as we know, the percentage of energy which could be set free in nuclear reactions is no more than 0.008. Hence, the maximum life-span of a giant is about $1.6{\times}10^{7}$ years, which is absolutely inapplicable. This gives additional support for our conclusion that the mechanism of stellar energy is not like reactions.

It is very interesting that the constant (2.4) has a numerical value similar to the time constant in Hubble's relation (the red shift of nebulae). It is probable that the exact form of the Hubble equation should be

$$\nu = \nu_0 e^{-t/t_0}, \tag{2.6}$$

where ν is the observed frequency of a line in a nebula spectrum when it is located at t light years from us, ν_0 is its normal frequency. According to the General Theory of Relativity the theoretical correlation between the constant t_0 and the average density $\bar{\rho}$ of matter in the visible part of the Universe

$$t_0 \simeq \frac{1}{\sqrt{\pi G \bar{\rho}}}, \tag{2.7}$$

which, independently of its theoretical origin, is also the very interesting empirical correlation. Because of (2.4) and (2.7), we re-write equation (2.6) as follows

$$\nu = \nu_0 e^{-\kappa_{\mathrm{T}} \bar{\rho} x}, \tag{2.8}$$

where $x = ct$ is the path of a photon. Formula (2.8) is like the formula of absorption, and so may give additional support to the explanation of the nebula red shift by unusual processes which occur in photons during their journey towards us. It is possible that in this formula $\bar{\rho}$ is the average density of the intergalactic gas.

2.2 The main sequence

The contemporary data of observational astronomy has sufficiently filled the Russell-Hertzsprug diagram, i. e. the "luminosity — spectral class" plane. As a result we see that there are no strong arcs $L(\bar{T}_{\mathrm{eff}})$ and $L(R)$, but regions filled by stars. In the previous chapter we showed that such a dispersion of points implies that the energy productivity in stars is regulated exclusively by the energy drainage (the radiation). So the mechanism generating stellar energy is not like any reactions. It is possible that only the main sequence of the Russell-Hertzsprung diagram can be considered a line along which stars are located. According to Parenago [38], this direction is

$$m_b = m_\odot - 1.62x, \qquad x = 10 \log \bar{T}_{\mathrm{eff}}\,. \tag{2.9}$$

An analogous relation had been found by Kuiper [8] as the $M(R)$ relation

$$\log \bar{R} = 0.7 \log \bar{M}\,. \tag{2.10}$$

Using formulae (1.12) and (1.14), we could transform formula (2.10) to a correlation $B(\rho)$. At the same time, looking at the stellar energy diagram (Fig. 2), we see that the stars of the Russell-Hertzsprung "main sequence" have no $B(\rho)$ correlation, but fill instead a ring at the centre of the diagram. This incompatibility should be considered in detail.

In the stellar energy diagram, the Russell-Hertzsprung main sequence is the ring of radius c filled by stars. The boundary equation of this region is

$$\log^2 \bar{B} + \log^2 \bar{\rho} = c^2. \tag{2.11}$$

We transform this equation to the variables $\bar{M}$ and $\bar{R}$ by formulae (1.12) and (1.14). We obtain

$$17 \log^2 \bar{M} - 38 \log \bar{M} \log \bar{R} + 25 \log^2 \bar{R} = c^2. \tag{2.12}$$

As we have found, for stars located in this central region (the Russell-Hertzsprung main sequence), the exponent of the "mass-luminosity" relation is about 4. Therefore, using formulae

$$\log \bar{M} = -0.1 m_b\,, \qquad 5 \log \bar{R} = -m_b - x\,,$$

we transform (2.12) to the form

$$m_b^2 + 2{\times}1.51 m_b x + 2.44 x^2 = c_1^2\,. \tag{2.13}$$

The left side of this equation is almost a perfect square, hence we have the equation of a very eccentric ellipse, with an angular coefficient close to 1.51. The exact solution can be found by transforming (2.13) to the main axes using the secular equation. As a result we obtain

$$\frac{a}{b} = 8.9, \qquad \alpha = -1.58\,, \tag{2.14}$$

where a and b are the main axis and the secondary axis of the ellipse respectively, α is the angle of inclination of its main axis to the abscissa's axis. Because of the large eccentricity, there is in the Russell-Hertzsprung diagram the illusion that stars are concentrated along the line a, the main axis of the ellipse. The calculated angular coefficient $\alpha = -1.58$ (2.14) is in close agreement with the empirically determined $\alpha = -1.62$ (2.9).

Thus the Russell-Hertzsprung main sequence has no physical meaning: it is the result of the scale stretching used in observational astrophysics. In contrast, the reality of the scale used in our stellar energy diagram (Fig. 2) is confirmed by the homogeneous distribution of the isoergs.

As obtained in Part I of this research, from the viewpoint of the internal constitution of stars, stars located at the opposite ends of the main sequence (the spectral classes O and M) differ from each other no more than stars of the same spectral class, but of different luminosity. Therefore the "evolution of a star along the main sequence" is a senseless term.

The results show that the term "sequence" was applied very unfortunately to groups of stars in the Russell-Hertzsprung diagram. It is quite reasonable to change this terminology, using the term "region" instead of "sequence": the region of giants, the main region, etc.

2.3 White dwarfs

There is very little observational data related to white dwarfs. Only for the satellite of Sirius and for o^2 Eridani do we know values of all three quantities L, M, and R. For Sirius' satellite we obtain

$$\bar{M} = 0.95, \quad \bar{R} = 0.030, \quad \varepsilon = 1.1\times10^{-2}, \quad \rho = 10^4, \quad \rho_c = 3\times10^5, \quad p_c = 1\times10^{22}. \tag{2.15}$$

For an ideal gas and an average molecular weight $\mu = 1/2$, we obtain $T_c = 2\times10^8$ degrees. The calculations show that white dwarfs generate energy hundreds of times smaller than regular stars. Looking at the isoergs in Fig. 2 and the isotherms in Fig. 3, we see that the deviation of white dwarfs from the "mass-luminosity" relation is of a special kind; not the same as that for regular stars. At the same time white dwarfs satisfy the main direction in the stellar energy diagram: they lie in the line following giants. Therefore it would be natural to start our brief research into the internal constitution of white dwarfs by proceeding from the general supposition that they are hot stars whose gas is at the boundary of degeneration

$$\rho = A\,T^{3/2}, \qquad A = 10^{-8}\,\mu_e\,. \tag{2.16}$$

We now show that, because of high density of matter in white dwarfs, the radiant transport of energy F_R is less than the transport of energy by the electron conductivity F_T

$$F_R = -\frac{1}{3}\,\bar{v}_e\,\bar{\lambda}\,\bar{c}_v\,n_e\,\frac{dT}{dr}\,,$$

where $\bar{\lambda}$ is the mean free path of electrons moved at the average velocity $\bar{v}_e$, $\bar{c}_v$ is the average heat capacity per particle. Also

$$\lambda = \frac{1}{n_i\,\sigma}\,, \quad n_i = \frac{n_e}{z}\,, \quad \sigma = \pi r^2, \quad c_v = \frac{3}{2}\,k\,, \tag{2.18}$$

where n_i is the number of ions deviating the electrons, σ is the ion section determined by the 90° deviation condition

$$m_e v_e^2 = \frac{ze^2}{r}\,, \tag{2.19}$$

i. e. the condition to move along a hyperbola.

Substituting (2.19) and (2.18) into formula (2.17) and eliminating $\bar{v}$ by the formula

$$\bar{v}^5 = \frac{12}{\sqrt{\pi}}\left(\frac{2\,kT}{m_e}\right)^{5/2},$$

we obtain

$$F_T = -\frac{24}{ze^4}\left(\frac{2\,k^7T^5}{\pi^3 m_e}\right)^{1/2}\frac{dT}{dr}\,. \tag{2.20}$$

The radiant flow can be written as

$$F_R = -\frac{4}{3}\,\frac{c\alpha T^3}{\kappa\rho}\,\frac{dT}{dr}\,, \tag{2.21}$$

hence

$$\frac{F_R}{F_T} = \frac{zT^{1/2}}{\kappa\rho}\left(\frac{\alpha ce^4\pi^{3/2}m_e^{1/2}}{k^{7/2}18\sqrt{2}}\right) = \frac{2.6\,zT^{1/2}}{\kappa\rho}\,. \tag{2.22}$$

Using (2.15) it is easily seen that even if $\kappa \simeq 1$, $F_R < F_T$ in the internal regions of white dwarfs. We can apply the formulae obtained to the case of the conductive transport of energy, if we eliminate κ with the effective absorption coefficient κ^*

$$\kappa^* = \frac{2.6\,zT^{1/2}}{\rho}\,. \tag{2.23}$$

Thus, if white dwarfs are built on an ideal gas whose state is about the degeneration boundary, their luminosity should be more than that calculated by the "mass-luminosity" formula (the heat equilibrium condition).

We consider the regular explanation for white dwarfs, according to which they are stars built on a fully degenerate gas. For the full degeneration, we use Chandrasekhar's "mass-radius" formula (see formula 2.32, Part I). With $\bar{M} = 1$ we obtain

$$\bar{R} = 0.042 \quad (\mu_e = 1)\,, \qquad \bar{R} = 0.013 \quad (\mu_e = 2)\,.$$

The observable radius (2.15) cannot be twice as small, so we should take Sirius' satellite as being composed of at least 50% hydrogen. From here we come to a serious difficulty: because of the high density of white dwarfs, even for a few million degrees internally, they should produce much more energy than they can radiate.We now show that such temperatures are necessary for white dwarfs.

Applying the main equations of equilibrium to the surface layer of a star, we obtain

$$\frac{B}{p} = \frac{L\kappa}{4\pi GcM} = \frac{\varepsilon\kappa}{4\pi Gc}\,, \tag{2.24}$$

where κ is the absorption coefficient in the surface layer. At the boundary of degeneration we can transform the left side by (2.16)

$$\rho_0 = \frac{3\varepsilon\kappa}{4\pi Gc}\,\frac{A^2\Re}{\mu\alpha}$$

so that

$$\rho_0 = 125\,\varepsilon\kappa\left(\frac{\mu_e^2}{\mu}\right), \qquad T_0^{3/2} = 1.25\times10^{10}\,\varepsilon\kappa\left(\frac{\mu_e}{\mu}\right). \tag{2.25}$$

We see from formula (2.22) that even in the surface layer the quantity F_T can be greater than F_R. Substituting κ^* (2.23) into (2.25), we obtain

$$T_0 = 2.5\times10^7\,\varepsilon^{2/5}\left(\frac{z}{\mu}\right)^{2/5}. \tag{2.26}$$

For $\varepsilon = 10^{-2}$, $\mu = 1$, and $z = 1$, we obtain

$$T_0 = 4\times10^6, \quad \rho_0 = 80, \quad \kappa_0^* = 65,$$

thus for such conditions, $\kappa > \kappa^*$.

We know that in the surface layer the temperature is linked to the depth h as follows

$$T = \frac{g\mu}{4\Re}\,h\,. \tag{2.27}$$

In the surface of Sirius' satellite we have $g = 3\times10^7$. Hence $h_0 = 3\times10^7$. Therefore the surface layer is about 2% of the radius of the white dwarf, so we can take the radius at the observed radius of the white dwarf.

It should be isothermal in the degenerated core, because the absorption coefficient rapidly decreases with increasing density. For a degenerate gas we can transform formula (2.23) in a simple way, if we suppose the heat capacity proportional to the temperature. Then, in the formula for F_R (2.20), the temperature remains in the first power, while $T_0^{3/2}$ should be eliminated with the density by (2.16). As a result we obtain $F_R \sim \rho T$ and also

$$\kappa_1^* \simeq 2.6\times10^{-8}\left(\frac{T}{\rho}\right)^2 z\mu_e\,. \tag{2.28}$$

Even for 4×10^6 degrees throughout a white dwarf, the average productivity of energy calculated by the proton-proton reaction formula (1.16) is $\varepsilon = 10^2$ erg/sec, which is much more than that observed. In order to remove the contradiction, we must propose a very low percentage of hydrogen, which contradicts the calculation above,* which gives hydrogen as at least 50% of its contents. So the large observed value of the radius of Sirius' satellite remains unexplained.

So we should return to our initial point of view, according to which white dwarfs are hot stars at the boundary of degeneration, but built on heavy elements. The low luminosity of such stars is probably derived from the presence of endothermic phenomena inside them. That is, besides energy generating processes, there are also processes where ε is negative. This consideration shows again that the luminosity of stars is unexplained within the framework of today's thermodynamics.

*By Chandrasekhar's formula for a fully degenerate gas. — Editor's remark.

References

1. Bethe H. A. Energy production in stars. *Physical Review*, 1939, v. 55, No. 5, 434–456.
2. Schwarzschild K. Über das Gleichgewicht der Sonnenatmosphäre. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Mathematisch — physicalische Klasse*. 1906, H. 6, 41–53.
3. Schmidt W. Der Massenaustausch in freier Luft und verwandte Erscheinungen. Hamburg, 1925 (*Probleme der Kosmischen Physik*, Bd. 7).
4. Milne E. A. The analysis of stellar structure. *Monthly Notices of the Royal Astronomical Society*, 1930, v. 91, No. 1, 4–55.
5. Kuiper G. P. Note on Hall's measures of ε Aurigae. *Astrophys. Journal*, 1938, v. 87, No. 2, 213–215.
6. Parenago P P. Physical characteristics of sub-dwarfs. *Astron. Journal — USSR*, 1946, v. 23, No. 1, 37.
7. Chandrasekhar S. The highly collapsed configurations of a stellar mass. Second paper. *Monthly Notices of the Royal Astronomical Society*, 1935, v. 88, No. 4, 472–507.
8. Kuiper G. P. The empirical mass-luminosity relation. *Astrophys. Journal*, 1938, v. 88, No. 4, 472–507.
9. Russell H. N., Moore Ch. E. The masses of the stars with a general catalogue of dynamical parallaxes. Astrophys. Monographs, Chicago, 1946.
10. Trumpler R.J. Observational evidence of a relativity red shift in class O stars. *Publications of the Astron. Society of the Pacific*, 1935, v. 47, No. 279, 254.
11. Parenago P. P. The mass-luminosity relation. *Astron. Journal — USSR*, 1937, v. 14, No. 1, 46.
12. Baize P. Les masses des étoiles doubles visuelles et la relation empirique masse-luminosité. *Astronomique — Paris*, 1947, t. 13, fasc. 2, 123–152.
13. Eddington A. S. The internal constitution of the stars. Cambridge, 1926, 153.
14. Chandrasekhar S. An introduction to the study of stellar structure. Astrophys. Monographs, Chicago, 1939, 412.
15. Eddington A. S. The internal constitution of the stars. Cambridge, 1926, 135.
16. Strömgren B. The opacity of stellar matter and the hydrogen content of the stars. *Zeitschrift für Astrophysik*, 1932, Bd. 4, H. 2, 118–152; On the interpretation of the Hertzsprung-Russell-Diagram. *Zeitschrift für Astrophysik*, 1933, Bd. 7, H. 3, 222–248.
17. Cowling T. G. The stability of gaseous stars. Second paper. *Monthly Notices of the Royal Astronomical Society*, 1935, v. 96, No. 1, 57.
18. Ledoux P. On the radial pulsation of gaseous stars. *Astrophys. Journal*, 1945, v. 102, No. 2, 143–153.
19. Parenago P. P. Stellar astronomy. Moscow, 1938, 200.
20. Becker W. Spektralphotometrische Untersuchungen an δ Cephei-Sternen X. *Zeitschrift für Astrophysik*, 1940, Bd. 19, H. 4/5, 297.
21. Eddington A. S. The internal constitution of the stars. Cambridge, 1926, 191.
22. Luyten W. J. On the ellipticity of close binaries. *Monthly Notices of the Royal Astronomical Society*, 1938, v. 98, No. 6, 459–466.
23. Russell H. N., Dugan R. S. Apsidal motion in Y Cygni and other stars. *Monthly Notices of the Royal Astronomical Society*, 1930, v. 91, No. 2, 212–215.

24. Blackett P. M. S. The magnetic field of massive rotating bodies. *Nature*, 1947, v. 159, No. 4046, 658–666.
25. Eddington A. S. The internal constitution of the stars. Cambridge, 1926, 163.
26. Lohmann W. Die innere Struktur der Masse-Leuchtkraftfunktion und die chemische Zusammensetzung der Sterne der Hauptreihe. *Zeitschrift für Astrophysik*, 1948, Bd. 25, H. 1/2, 104.
27. Martynov D. Ya. Eclipse variable stars. Moscow, 1939.
28. Gaposchkin S. Die Bedeckungsveränderlichen. *Veröffentlichungen der Universitätssternwarte zu Berlin-Babelsberg*, 1932, Bd. 9, H. 5, 1–141.
29. Parenago P. P. Physical characteristics of sub-dwarfs. *Astron. Journal — USSR*, 1946, v. 23, No. 1, 31–39.
30. Struve O. The masses and mass-ratios of close binary systems. *Annales d'Astrophys.*, 1948, t. 11, Nom. 2, 117–123.
31. Kothari D. S. The theory of pressure-ionization and its applications. *Proceedings of the Royal Society of London*, Ser. A, 1938, v. 165, No. A923, 486–500.
32. Luyten W. J., Struve O., Morgan W. W. Reobservation of the orbits of ten spectroscopic binaries with a discussion of apsidal motions. *Publications of Yerkes Observatory*, Univ. of Chicago, 1939, v. 7, part. 4, 251–300.
33. Kuiper G. P. On the hydrogen contents of clusters. *Astrophys. Journal*, 1937, v. 86, No. 2, 176–197.
34. Masevich A. G. Stellar evolution where is corpuscular radiations, considered from the viewpoint of the internal constitution of stars. *Astron. Journal — USSR*, 1949, v. 26, No. 4, 207–218.
35. Tuominen J. Über die inneren Aufbau der TRÜMPLERschen Sterne. *Zeitschrift für Astrophysik*, 1943, Bd. 22, H. 2, 90–110.
36. Becker W. Spektralphotometrische Untersuchungen an δ Cephei-Sternen X. *Zeitschrift für Astrophysik*, 1940, Bd. 19, H. 4/5, 297; Spektralphotometrische Untersuchungen an δ Cephei-Sternen. *Zeitschrift für Astrophysik*, 1941, Bd. 20, H. 3, 229.
37. Melnikov O. A. On the stability of masses of ling-periodical Cepheids. *Proceedings of Pulkovo Astron. Observatory*, 1948, v. 17, No. 2 (137), 47–62.
38. Parenago P. P. The generalized relation mass-luminosity. *Astron. Journal — USSR*, 1939, v. 16, No. 6, 13.

Progress in Physics is an American scientific journal on advanced studies in physics, registered with the Library of Congress (DC, USA): ISSN 1555-5534 (print version) and ISSN 1555-5615 (online version). The journal is peer reviewed and listed in the abstracting and indexing coverage of: Mathematical Reviews of the AMS (USA), DOAJ of Lund University (Sweden), Zentralblatt MATH (Germany), Scientific Commons of the University of St. Gallen (Switzerland), Open-J-Gate (India), Referential Journal of VINITI (Russia), etc. *Progress in Physics* is an open-access journal published and distributed in accordance with the Budapest Open Initiative: this means that the electronic copies of both full-size version of the journal and the individual papers published therein will always be acessed for reading, download, and copying for any user free of charge. The journal is issued quarterly (four volumes per year).

Electronic version of this journal:
http://www.ptep-online.com

Postal address for correspondence:

Department of Mathematics and Science
University of New Mexico
200 College Road, Gallup, NM 87301, USA

Printed in the United States of America

ISSN 1555-5534

9 771555 553907 >