On the Propagation of Gravitation from a Pulsating Source

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According to an idea underlying the classical relativity, a pulsating (or simply expanding or simply contracting) spherical source does not generate an external dynamical (i.e. non-stationary) gravitational field. The relativists believe that this idea is well based on account of the so-called Birkhoff's theorem, which, contrary to the fundamental principles of general relativity, states that the external gravitational field of a non-stationary spherical mass is necessarily static. However, as shown in several papers [2, 3, 4, 7, 8], Birkhoff's theorem is, in fact, a vicious circle arising from the introduction of inadmissible implicit transformations which eliminate in advance the boundary conditions defining the radial motion of the sphere bounding the matter, namely the boundary conditions inducing the non-stationary states of the gravitational field. In the present paper we deal with the rigorous mathematical theory of the subject and put forward the corresponding form of the spacetime metric in order to prepare a thorough study of the equations of gravitation related to the dynamical states of the gravitational field.

1 S$\Theta(4)$-invariant metrics and gravitational disturbances

Let us first consider a general spacetime metric

$$\sum_{i,j=0}^{3} g_{ij}dx_i dx_j \tag{1.1}$$

namely a form of signature $(+1,-1,-1,-1)$ on a open set $U \subset \mathbb{R}\times\mathbb{R}^3$. In order that the local time and the proper time of the observes be definable, the timelike character of x_0 must be clearly indicared together with its distinction from the spacelike character of the coordinates x_1, x_2, x_3. This is why, according to Levi-Civita [l], the components g_{00}, g_{11}, g_{22}, g_{33} of the metric tensor must satisfy the conditions $g_{00}>0$, $g_{11}<0$, $g_{22}<0$, $g_{33}<0$.

Our investigation of an $S\Theta(4)$-invariant (or $\Theta(4)$-invariant) metric follows Levi-Civita's point of view by allowing at the same time a slight generalization which will be fully justified. More precisely, an allowable $S\Theta(4)$-invariant (or $\Theta(4)$-invariant) metric will satisfy the conditions $g_{00}>0$, $g_{11}\leqslant 0$, $g_{22}\leqslant 0$, $g_{33}\leqslant 0$. We recall [9] the explicit form of such a metric

$$ds^2 = \left(f dx_0 + f_1(xdx)\right)^2 - \ell_1^2 dx^2 - \frac{\ell^2-\ell_1^2}{\rho^2}(xdx)^2,$$
$$x_0 = t, \ell(t,0) = \ell_1(t,0),$$

which is invariant by the action of the group $S\Theta(4)$ consisting of the matrices of the form

$$\begin{pmatrix} 1 & O_H \\ O_V & A \end{pmatrix}, \quad O_H = (0,0,0), \quad O_V = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$
$$A \in SO(3)$$

as well as by the action of the group $\Theta(4)$ consisting of the matrices of the same form for which $A \in O(3)$. Note that the given form of the metric does not contain the important functions

$$h = \rho f_1 = \rho f_1(t,\rho), \qquad g = \rho\ell_1 = \rho\ell_1(t,\rho),$$

because they are not C^∞ on the subspace $\mathbb{R}\times\{(0,0,0)\}$. However, as already noted [9], on account of their geometrical and physical significance, it is very convenient to insert them into the metric, thus obtaining

$$ds^2 = \left(f dx_0 + \frac{h}{\rho}(xdx)\right)^2 - \left(\frac{g}{\rho}\right)^2 dx^2 - \frac{1}{\rho^2}\left(\ell^2 - \left(\frac{g}{\rho}\right)^2\right)(xdx)^2 \tag{1.2}$$

and then

$$g_{00} = f^2, \quad g_{ii} = \left(h^2-\ell^2\right)\frac{x_i^2}{\rho^2} - \left(\frac{g}{\rho}\right)^2\left(1-\frac{x_i^2}{\rho^2}\right),$$
$$(i=1,2,3).$$

We contend that $g_{ii}\leqslant 0$, $(i=1,2,3)$, if and only if $|h|\leqslant\ell$. In fact, if $|h|\leqslant\ell$, we have obviously $g_{ii}\leqslant 0$, $(i=1,2,3)$. On the other hand, if $|h|>\ell$, by choosing $x_1=\rho$, $x_2=x_3=0$, we have $g_{11}=h^2-\ell^2>0$.

The $S\Theta(4)$ -invariant metric (1.2), considered with the condition $|h|\leqslant\ell$, is assumed to represent the gravitational field generated by a spherical isotropic non-rotating, in general pulsating, distribution of matter. This field is related intuitively to a radial uniform propagation of spherical gravitational (and possibly electromagnetic) disturbances issuing from the matter and governed by the time according to the following rule:

> The emission of a disturbance takes place at a given instant from the entirety of the sphere bounding the matter (namely from the totality of the points of this sphere) and reaches the totality of any other sphere $S_\rho : \| x \| = \rho > 0$ outside the matter at another instant.

The assignment of a given instant t to every point of the sphere S_ρ means that we consider an infinity of simultaneous events $\{(t, x)\,|x \in S_\rho\}$ related to S_ρ. This conception of simultaneity is restricted to the considered sphere S_ρ and cannot be extended radially (for greater or less values of ρ). So the present situation differs radically from that encountered in special relativity. In particular, the synchronization of clocks in S_ρ cannot be carried out by the standard method put forward by Einstein, because there are no null geodesies of the metric associated with curves lying on S_ρ. The idea of synchronization in $S_\rho : \| x \| = \rho > 0$ is closely related to the very definition of the $S\Theta(4)$-invariant field: For any fixed value of time t, the group $S\Theta(4)$ sends the subspace $\{t\} \times S_\rho$ of $\mathbb{R} \times \mathbb{R}^3$ onto itself, so that the group $S\Theta(4)$ assigns the value of time t to every point of the sphere S_ρ. Specifically, given any two distinct points x and y of S_ρ, there exists an operation of $S\Theta(4)$ sending (t, x) onto (t, y). This operation appears as an abstract mathematical mapping and must be clearly distinguished from a rotation in $\mathbb{R}^3$ in the sense of classical mechanics. Such a rotation in $\mathbb{R}^3$ is a motion defined with respect to a pre-existing definition of time, whereas the assignment of the value of time t to every point of S_ρ, is an "abstract operation" introducing the time in the metric.

Let S_m be the sphere bounding the matter. As will be shown later on, the "synchronization" in S_m induces the synchronization in any other sphere S_ρ outside the matter by means of the propagation process of gravitation. In a stationary state, the radius of S_m reduces to a constant, say σ, and every point of S_m can be written as $x = \alpha\sigma$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in S_1$, S_1 being the unit sphere:

$$S_1 = \Big\{\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 | \\ \| \alpha \| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} = 1\Big\}.$$

Now, in a non-stationary state, the radius of S_m will be a function of time, say $\sigma(t)$, and the equation of S_m can be written as $x = \alpha\sigma(t)$ with $\alpha \in S_1$. So each value of time t defines both the radius $\sigma(t)$ and the "simultaneous events" $\{(t, \alpha\sigma(t))\,|\,\alpha \in S_1\}$. This simultaneity is also closely related to the definition of the $S\Theta(4)$ invariant field: $\{(t, \alpha\sigma(t))\,|\,\alpha \in S_1\}$ remains invariant by the action of $S\Theta(4)$. From these considerations it follows that the first principles related to the notion of time must be introduced axiomatically on the basis of the very definition of the $S\Theta(4)$-invariance. Their physical justification is to be sought a posteriori by taking into account the results provided by the theory itself.

This being said, according to our assumptions, it makes sense to consider as a function of time the curvature radius $g(t, \rho) = \rho\ell_1(t, \rho)$ of a sphere $\|x\| = \rho = \text{const} > 0$ outside the matter. The same assumptions allow to define, as functions of time, the radius $\sigma(t)$ and the curvature radius, denoted by $\zeta(t)$, of the sphere bounding the matter. These positive functions, $\sigma(t)$ and $\zeta(t)$, constitute the boundary conditions at finite distance for the non-stationary field outside the pulsating source. They are assumed to be C^∞, but they cannot be analytic, because the vanishing of $|\sigma'(t)| + |\zeta'(t)|$ on certain compact time intervals does not imply its vanishing on $\mathbb{R}$.

Since the internal field extends to the external one through the sphere $\|x\| = \sigma(t)$, the non-stationary (dynamical) states of the gravitational field outside the pulsating source are induced by the radial motion of this sphere, namely by the motion defined mathematically by the boundary conditions $\sigma(t)$ and $\zeta(t)$. So, it is reasonable to assume that, if $\sigma'(t) = \zeta'(t) = 0$ on a compact interval of time $[t_1, t_2]$, no propagation of gravitational disturbances takes place in the external space during $[t_1, t_2]$ (at least if there is no diffusion of disturbances). It follows that the gravitational radiation in the external space depends on the derivatives $\sigma'(t)$ and $\zeta'(t)$, so that we may identify their pair with the gravitational disturbance inducing the dynamical states outside the matter. More precisely, the non-stationary-states are generated by the propagation of the gravitational disturbance in the exterior space, so that we have first to clarify the propagation process. Our intuition suggests that the propagation of gravitation is closely related to the radial propagation of light, and this is why we begin by defining the function governing the radial propagation of light from the sphere bounding the matter.

2 Radial null geodesics

We recall that a curve $x(v) = (x_0(v), x_1(v), x_2(v), x_3(v))$ is a geodesic line with respect to (1.1) if

$$\frac{D}{dv}\frac{dx(v)}{dv} = q(v)\frac{dx(v)}{dv}\,.$$

So we are led to introduce the vector

$$Y^j = \frac{d^2 x_j}{dv^2} + \sum_{k,\ell=0}^{3} \Gamma^j_{k\ell}\frac{dx_k}{dv}\frac{dx_\ell}{dv} - q(v)\frac{dx_j}{dv}, \quad (j = 0, 1, 2, 3),$$

which allows to write the equations of a geodesic in their general form

$$Y^0 = 0\,, \qquad Y^1 = 0\,, \qquad Y^2 = 0\,, \qquad Y^3 = 0\,.$$

On the other hand, a null line (not necessarily geodesic) is defined by the condition

$$\sum_{i,j=0}^{3} g_{ij}\frac{dx_i}{dv}\frac{dx_j}{dv} = 0\,, \qquad (v \neq s),$$

which implies

$$\sum g_{ij}\frac{dx_i}{dv}\frac{d^2x_j}{dv^2}+\sum\Gamma_{i,k\ell}\frac{dx_i}{dv}\frac{dx_k}{dv}\frac{dx_\ell}{dv}=0$$

so that by setting

$$X_j=\sum g_{ij}\frac{dx_i}{dv}$$

we deduce by an easy computation the relation

$$\sum_{j=0}^{3}X_jY^j=0$$

which is valid for every null line.

Now, let

$$d_t=\{x_1=\alpha_1\rho,\;\; x_2=\alpha_2\rho,\;\; x_3=\alpha_3\rho,\;\; \alpha_1^2+\alpha_2^2+\alpha_3^2=1,\;\; \rho\geqslant\sigma(t)\}$$

be a half-line issuing from a point of the sphere $\|x\|=\sigma(t)$. The vanishing of (1.2) on d_t gives rise to two radial null lines defined respectively by the equations

$$\frac{dt}{d\rho}=\frac{-h(t,\rho)+\ell(t,\rho)}{f(t,\rho)} \tag{2.1}$$

$$\frac{dt}{d\rho}=\frac{-h(t,\rho)-\ell(t,\rho)}{f(t,\rho)} \tag{2.2}$$

Proposition 2.1. *The above defined null lines are null geodesics.*

Proof. By using a transformation defined by an element of the group $S\Theta(4)$, we may assume, without restriction of generality, that d_t is defined by the equations $x_1=\rho$, $x_2=0$, $x_3=0$, where $\rho\geqslant\sigma(t)$. Then taking into account the expressions of the Christoffel symbols [9], we see that

$$\Gamma^2_{00}=\Gamma^2_{01}=\Gamma^2_{11}=0,\qquad \Gamma^3_{00}=\Gamma^3_{01}=\Gamma^3_{11}=0,$$

so that the equations $Y^2=0$, $Y^3=0$ are identically verified. Moreover $x_2=x_3=0$ imply

$$Y^0=\frac{d^2t}{dv^2}+\Gamma^0_{00}\left(\frac{dt}{dv}\right)^2+\Gamma^0_{11}\left(\frac{d\rho}{dv}\right)^2+ +2\Gamma^0_{01}\frac{dt}{dv}\frac{d\rho}{dv}-q(v)\frac{dt}{dv},$$

$$Y^1=\frac{d^2\rho}{dv^2}+\Gamma^1_{00}\left(\frac{dt}{dv}\right)^2+\Gamma^1_{11}\left(\frac{d\rho}{dv}\right)^2+ +2\Gamma^1_{01}\frac{dt}{dv}\frac{d\rho}{dv}-q(v)\frac{d\rho}{dv}.$$

Now, let $t=\xi(\rho)$ be a solution of (2.1) and take $v=\rho$. Then the equation $Y^1=0$ gives

$$\Gamma^1_{00}\big(\xi(\rho),\rho\big)\big(\xi'(\rho)\big)^2+\Gamma^1_{11}\big(\xi(\rho),\rho\big)+ +2\Gamma^1_{01}\big(\xi(\rho),\rho\big)\xi'(\rho)=q(\rho)$$

so that it defines the function $q(\rho)$. Next, since the equations $Y^1=0$, $Y^2=0$, $Y^3=0$ are fulfilled, the condition $\sum_{j=0}^{3}X_jY^j=0$ reduces to $X_0Y^0=0$, and since

$$X_0=g_{00}\frac{dt}{dv}+g_{01}\frac{dx_1}{dv}=f^2\frac{dt}{d\rho}+fh= =f^2\left(\frac{-h+\ell}{f}\right)+fh=f\ell>0\,,$$

it follows also that $Y^0=0$. In the same way taking into account that $-f\ell<0$, we prove the assertion regarding (2.2).

Corollary 2.1. *The equation (2.1), resp. (2.2), defines the radial motion of the photons issuing from (resp. approaching to) the pulsating spherical mass.*

In fact, since $|h|\leqslant\ell$, we have $-h+\ell\geqslant 0$, which impies $dt/d\rho\geqslant 0$, and $-h-\ell\leqslant 0$ which implies $dt/d\rho\leqslant 0$.

Remark 2.1. The condition $|h|\leqslant\ell$ has been introduced in order to ensure the physical validity of the spacetime metric. Now we see that it is absolutely indispensable in order to define the radial motion of light. In fact, if $h>\ell$ (resp. $-h>\ell$), the photons issuing from (resp. approaching to) the spherical mass would be inexistent for the metric. A detailed discussion of the inconsistencies resulting from the negation of the condition $|h|\leqslant\ell$ is given in the paper [6].

Remark 2.2. As already remarked, the propagation of the gravitation from the pulsating source is closely related to the radial propagation of te outgoing light which is defined by (2.1). Regarding the equation (2.2), which defines the radial propagation of the incoming light, it is not involved in our study, because there are no gravitational disturbances coming from the "infinity".

3 On the solutions of (2.1)

Let us consider a photon emitted radially at an instant u from the sphere bounding the matter. Its velocity at this instant, namely

$$\frac{d\rho}{dt}=\frac{f\big(u,\sigma(u)\big)}{\ell\big(u,\sigma(u)\big)-h\big(u,\sigma(u)\big)}$$

is greater than the radial velocity $|\sigma'(u)|$ of this sphere, whence the condition

$$\frac{\ell\big(u,\sigma(u)\big)-h\big(u,\sigma(u)\big)}{f\big(u,\sigma(u)\big)}\,|\sigma'(u)|<1$$

which implies in particular the validity of the condition

$$\frac{\ell\big(u,\sigma(u)\big)-h\big(u,\sigma(u)\big)}{f\big(u,\sigma(u)\big)}\,\sigma'(u)<1 \tag{3.1}$$

which is trivially valid if $\sigma'(u)\leqslant 0$.

This being said, let us consider the open set

$$U=\big\{(t,\rho)\in\mathbb{R}^2\mid\rho>\sigma(t)\big\}$$

and denote by F its frontier:

$$F=\big\{(t,\rho)\in\mathbb{R}^2\mid\rho=\sigma(t)\big\}\,.$$

Since the equation (2.1) is conceived on the closed set $\overline{U} = U \cup F$, the functions f, h, ℓ are defined on $\overline{U}$. However, since we have to define the solutions of (2.1) by using initial conditions in F, we are led to extend the function

$$\alpha(t, \rho) = \frac{-h(t, \rho) + \ell(t, \rho)}{f(t, \rho)}$$

to a C^∞ function $\hat{\alpha}(t, \rho) \geqslant 0$ on an open set W containing $\overline{U}$. It is not necessary to indicate a precise extension on W because its values on $W - \overline{U}$ play an auxiliary part and are not involved in the final result.

This remark applies also to the derivatives of the functions f, h, ℓ at the points of F. In fact, although the definition of these derivatives takes into account the extension $\hat{\alpha}(t, \rho)$, their values on F, on account of the continuity, are defined uniquely by their values on U.

This being said, for each fixed point $(u, \sigma(u)) \in F$, the differential equation

$$\frac{dt}{d\rho} = \hat{\alpha}(t, \rho)$$

possesses a unique local solution $t = \hat{\xi}(u, \rho)$ taking the value u for $\rho = \sigma(u)$. Let $]\, \rho_1(u), \rho_2(u)\, [$ be the maximal interval of validity of this local solution $(\rho_1(u) < \sigma(u) < \rho_2(u))$.

Lemma 3.1. *There exists a real number* $\epsilon > 0$ *such that* $\sigma(u) < \epsilon < \rho_2(u)$ *and* $(\hat{\xi}(u, \rho), \rho) \in U$ *for every* $\rho \in\,]\sigma(u), \epsilon]$.

Proof. Assume that such a number does not exist. Then we can find a sequence of values $\rho_n > \sigma(u)$ converging to $\sigma(u)$ and such that $(\hat{\xi}(u, \rho_n), \rho_n) \notin U$, which means that $\sigma(\hat{\xi}(u, \rho_n)) \geqslant \rho_n$, and implies, in particular $\hat{\xi}(u, \rho_n) \neq u$. It follows that

$$\frac{\hat{\xi}(u, \rho_n) - u}{\rho_n - \sigma(u)} \cdot \frac{\sigma(\hat{\xi}(u, \rho_n)) - \sigma(u)}{\hat{\xi}(u, \rho_n) - u} = \frac{\sigma(\hat{\xi}(u, \rho_n)) - \sigma(u)}{\rho_n - \sigma(u)} \geqslant 1$$

and since $\hat{\xi}(u, \sigma(u)) = u$, $\rho_n \to \sigma(u)$, we obtain

$$\frac{\partial \hat{\xi}(u, \sigma(u))}{\partial \rho}\, \sigma'(u) \geqslant 1\,,$$

or

$$\frac{-h(u, \sigma(u)) + \ell(u, \sigma(u))}{f(u, \sigma(u))}\, \sigma'(u) \geqslant 1$$

which contradicts (3.1). This contradiction proves our assertion.

Lemma 3.2. *We also have* $(\hat{\xi}(u, \rho), \rho) \in U$ *for every* $\rho \in\,]\, \epsilon, \rho_2(u)\, [$.

Proof. If not, the set of values $\rho \in\,]\, \epsilon, \rho_2(u)\, [$ for which $\sigma(\hat{\xi}(u, \rho)) = \rho$ is not empty. Let ρ_0 be the greatest lower bound of this set. Then $\sigma(\hat{\xi}(u, \rho_0)) = \rho_0$. Let $\hat{\xi}(u, \rho_0) = t_0$ and let $\psi(t_0, \rho)$ be the local solution of the differential equation

$$\frac{dt}{d\rho} = \hat{\alpha}(t, \rho)$$

for which $\psi(t_0, \rho_0) = t_0$. The uniqueness of the solution implies obviously that $\psi(t_0, \rho) = \hat{\xi}(u, \rho)$. On the other hand, for every $\rho \in\,]\sigma(u), \rho_0[$, we have $\sigma(\hat{\xi}(u, \rho)) < \rho$. Moreover $\hat{\xi}(u, \rho_0) \neq \hat{\xi}(u, \rho)$ because the equality $\hat{\xi}(u, \rho_0) = \hat{\xi}(u, \rho)$ would imply

$$\rho_0 = \sigma(\hat{\xi}(u, \rho_0)) = \sigma(\hat{\xi}(u, \rho)) < \rho$$

contradicting the choice of ρ. On the other hand

$$\sigma(\hat{\xi}(u, \rho_0)) - \sigma(\hat{\xi}(u, \rho)) = \rho_0 - \sigma(\hat{\xi}(u, \rho)) > \rho_0 - \rho > 0$$

so that we can write

$$\frac{\hat{\xi}(u, \rho_0) - \hat{\xi}(u, \rho)}{\rho_0 - \rho} \cdot \frac{\sigma(\hat{\xi}(u, \rho_0)) - \sigma(\hat{\xi}(u, \rho))}{\hat{\xi}(u, \rho_0) - \hat{\xi}(u, \rho)} = \frac{\sigma(\hat{\xi}(u, \rho_0)) - \sigma(\hat{\xi}(u, \rho))}{\rho_0 - \rho} \geqslant 1$$

or

$$\frac{\psi(t_0, \rho_0) - \psi(t_0, \rho)}{\rho_0 - \rho} \cdot \frac{\sigma(t_0) - \sigma(\psi(t_0, \rho))}{t_0 - \psi(t_0, \rho)} \geqslant 1$$

and for $\rho \to \rho_0$ we find

$$\frac{\partial \psi(t_0, \rho_0)}{\partial \rho}\, \sigma'(t_0) \geqslant 1$$

or

$$\frac{-h(t_0, \sigma(t_0)) + \ell(t_0, \sigma(t_0))}{f(t_0, \sigma(t_0))}\, \sigma'(t_0) \geqslant 1$$

which contradicts (3.1). This contradiction proves our assertion.

Proposition 3.1. *Let* $\xi(u, \rho)$ *be the restriction of the solution* $\hat{\xi}(u, \rho)$ *to the interval* $[\sigma(u), \rho_2(u)\, [$ *. Then* $\xi(u, \rho)$ *does not depend on the extension* $\hat{\alpha}(t, \rho)$ *of* $\alpha(t, \rho)$*, so that it is the unique local solution of (2.1) in* $\overline{U}$ *satisfying the condition* $\xi(u, \sigma(u)) = u$.

In fact, since $\hat{\xi}(u, \sigma(u)) = u$ and $(\hat{\xi}(u, \rho), \rho) \in U$ for $\rho > \sigma(u)$, the definition of $\xi(u, \rho)$ on $[\sigma(u), \rho_2(u)[$ depends uniquely on the function $\alpha(t, \rho)$ which is defined on $\overline{U}$

In general, the obtained solution $\xi(u, \rho)$ is defined on a bounded interval $[\sigma(u), \rho_2(u)\, [$. However the physical conditions of the problem require that the emitted photon travel to infinity. In fact, the pulsating source (whenever it is expanding) can not overtake the photon emitted radially at the instant u. Consequently the functions f, h, ℓ involved in the metric must be such that, for each value of $u \in \mathbb{R}$, the solution $\xi(u, \rho)$ of (2.1) be defined on the half-line $[\sigma(u), +\infty\, [$, so that $\rho_2(u) = +\infty$ and $(\xi(u, \rho), \rho) \in U$ for every $\rho \in\,]\, \sigma(u), +\infty\, [$. Then the corresponding curves $(\xi(u, \rho), \rho)$ issuing from the points of F are the leaves of a foliation of $\overline{U}$ representing the paths of the photons emitted radially from the sphere bounding the matter (see Figure 1 shown in Page 79).

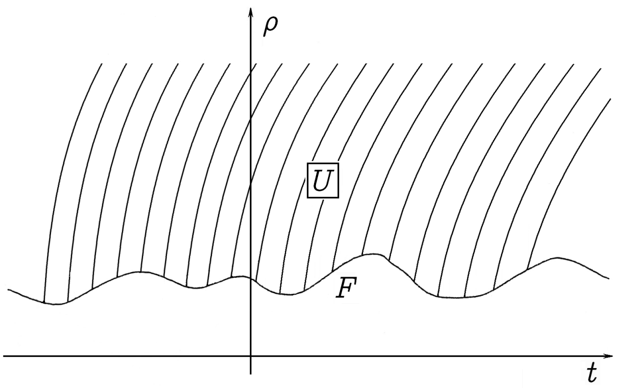

Fig. 1: Foliation representing the paths of the photons emitted radially from the sphere bounding the matter.

4 Propagation function of light and canonical metric

The solution $\xi(u,\rho)$ appears as a function of two variables: On the one hand the time $u \in \mathbb{R}$ on the sphere bounding the matter, and on the other hand the radial coordinate $\rho \in [\sigma(u), +\infty[$.

Proposition 4.1. *The function $\xi(u,\rho)$, $(u,\rho) \in \overline{U}$, fulfils the conditions*

$$\frac{\partial\xi(u,\rho)}{\partial u} > 0, \quad \frac{\partial\xi(u,\rho)}{\partial\rho} \geqslant 0$$

the first of which allows to solve with respect to u the equation $t = \xi(u,\rho)$, where $\xi(u,\sigma(u)) = u$, and obtain thus the instant u of the radial emission of a photon as a function of (t,ρ): $u = \pi(t,\rho)$. The so obtained function $\pi(t,\rho)$ on $\overline{U}$ satisfies the conditions

$$\frac{\partial\pi(t,\rho)}{\partial t} > 0\,, \qquad \frac{\partial\pi(t,\rho)}{\partial\rho} \leqslant 0\,, \qquad \pi\big(t,\sigma(t)\big) = t\,.$$

Proof. Since $-h + \ell \geqslant 0$, the condition $\xi(u,\rho)/\partial\rho \geqslant 0$ is obvious on account of (2.1). On the other hand, taking the derivatives of both sides of the identity $\xi(u,\sigma(u)) = u$ we obtain

$$\frac{\partial\xi\big(u,\sigma(u)\big)}{\partial u} + \frac{\partial\xi\big(u,\sigma(u)\big)}{\partial\rho}\,\sigma'(u) = 1$$

or

$$\frac{\partial\xi\big(u,\sigma(u)\big)}{\partial u} + \frac{-h\big(u,\sigma(u)\big) + \ell\big(u,\sigma(u)\big)}{f\big(u,\sigma(u)\big)}\,\sigma'(u) = 1$$

whence, on account of (3.1),

$$\frac{\partial\xi\big(u,\sigma(u)\big)}{\partial u} > 0$$

for every $u \in \mathbb{R}$. It remains to prove that, for each fixed value $u_0 \in \mathbb{R}$, and for each fixed value $\rho_0 > \sigma(u_0)$, we have $\partial\xi(u_0,\rho_0)/\partial u > 0$.

Now, $\rho_0 > \sigma(u_0)$ implies that there exists a straight line segment

$$\big[-\epsilon_1 + \xi(u_0,\rho_0), \epsilon_1 + \xi(u_0,\rho_0)\big] \times \{\rho_0\}\,, \epsilon_1 > 0,$$

contained in U. Let us denote by L_1, L_0, L_2 respectively the leaves containing the points

$$\big(-\epsilon_1 + \xi(u_0,\rho_0), \rho_0\big),\ \big(\xi(u_0,\rho_0), \rho_0\big),\ \big(\epsilon_1 + \xi(u_0,\rho_0), \rho_0\big).$$

L_0 is defined by the solution $\xi(u_0,\rho)$ of (2.1), whereas L_1 and L_2 are defined respectively by two solutions $\xi(u_1,\rho)$ and $\xi(u_2,\rho)$ with convenient values u_1 and u_2. Since $L_1 \cap L_0 = \varnothing$, $L_0 \cap L_2 = \varnothing$, it follows obviously that $u_1 < u_0$ and $u_0 < u_2$. The same reasoning shows that, if $u_1 < u' < u_0 < u'' < u_2$, then

$$\xi(u_1,\rho_0) < \xi(u',\rho_0) < \xi(u_0,\rho_0) < \xi(u'',\rho_0) < \xi(u_2,\rho_0),$$

so that $\xi(u,\rho_0)$ is a strictly increasing function of u on the interval $[u_1,u_2]$. It follows that $\partial\xi(u_0,\rho_0)/\partial u > 0$ as asserted. Regarding the last assertion, it results trivially from the identity $\xi(\pi(t,\rho),\rho) = t$, which implies

$$\frac{\partial\xi}{\partial u}\cdot\frac{\partial\pi}{\partial t} = 1\,, \qquad \frac{\partial\xi}{\partial u}\cdot\frac{\partial\pi}{\partial\rho} + \frac{\partial\xi}{\partial\rho} = 0\,.$$

Remark. Let u_1 and u_2 be two instants such that $u_1 < u_2$, and let ρ be a positive length. If the values $\xi(u_1,\rho)$ and $\xi(u_2,\rho)$ are both definable, which implies, in particular, $\xi(u_1,\rho) \geqslant u_1$ and $\xi(u_2,\rho) \geqslant u_2$, then $\xi(u_1,\rho) < \xi(u_2,\rho)$.

The function $\pi(t,\rho)$ characterizes the radial propagation of light and will be called *propagation function*. Its physical significance is the following : If a photon reaches the sphere $\|x\| = \rho$ at the instant t, then $\pi(t,\rho)$ is the instant of its radial emission from the sphere bounding the matter.

Proposition 4.2 *If a photon emitted radially from the sphere bounding the matter reaches the sphere $\|x\| = \rho$ at the instant t, then its radial velocity at this instant equals*

$$-\frac{\partial\pi(t,\rho)/\partial t}{\partial\pi(t,\rho)/\partial\rho}\,.$$

In fact, since

$$\frac{dt}{d\rho} = \frac{-h+\ell}{f} = \frac{\partial\xi(u,\rho)}{\partial\rho}\,,$$

the velocity in question equals

$$\frac{d\rho}{dt} = \left(\frac{\partial\xi(u,\rho)}{\partial\rho}\right)^{-1} = -\left(\frac{\partial\xi(u,\rho)}{\partial u}\frac{\partial\pi(t,\rho)}{\partial\rho}\right)^{-1} =$$
$$= -\frac{\partial\pi(t,\rho)/\partial t}{\partial\pi(t,\rho)/\partial\rho}\,.$$

Remark. The preceding formula applied to the classical propagation function $t - \frac{\rho}{c}$, gives the value c.

Since the parameter u appearing in the solution $\xi(u,\rho)$ represents the time on the sphere bounding the matter and describes the real line, we are led to define a mapping $\Gamma : \overline{U} \to \overline{U}$, by setting $\Gamma(t,\rho) = (\pi(t,\rho),\rho) = (u,\rho)$.

Proposition 4.3. *The mapping Γ is a diffeomorphism which reduces to the identity on the frontier F of U. Moreover it transforms the leaf $\{(t,\rho)\in\overline{U}\mid t=\xi(u,\rho)\}$ issuing from a point $(u,\sigma(u))\in F$ into a half-line issuing from the same point and parallel to the ρ-axis. Finally it transforms the general $\Theta(4)$ invariant metric (1.2) into another $\Theta(4)$-invariant metric for which $h=\ell$, so that the new propagation function is identical with the new time coordinate.*

Proof. The mapping Γ is one-to-one and its jacobian determinant $\partial\pi(t,\rho)/\partial t$ is strictly positive everywhere. Consequently Γ is a diffeomorphism. Moreover, since each leaf is defined by a fixed value of u, its transform in the new coordinates (u,ρ) is actually a half-line parallel to the ρ-axis. Finally, since $t=\xi(u,\rho)$ and $\partial\xi/\partial\rho=(-h+\ell)/f$, it follows that

$$\begin{aligned} f dt + \frac{h}{\rho}(xdx) &= \left(f\frac{\partial\xi}{\partial u}\right)du + \left(f\frac{\partial\xi}{\partial\rho}\right)d\rho + hd\rho \\ &= \left(f\frac{\partial\xi}{\partial u}\right)du + \left(f\left(\frac{-h+\ell}{f}\right)+h\right)d\rho \\ &= \left(f\frac{\partial\xi}{\partial u}\right)du + \ell d\rho \\ &= \left(f\frac{\partial\xi}{\partial u}\right)du + \ell\frac{(xdx)}{\rho} \end{aligned}$$

with

$$f=f\big(\xi(u,\rho),\rho\big),\quad h=h\big(\xi(u,\rho),\rho\big),\quad \ell=\ell\big(\xi(u,\rho),\rho\big).$$

So the remarkable fact is that, in the transformed $\Theta(4)$-invariant metric, the function h equals ℓ. The corresponding equation (2.1) reads

$$\frac{du}{d\rho}=0$$

whence $u=\text{const}$, so that the new propagation function is identified with the time coordinate u. (This property follows also from the fact that the transform of $\pi(t,\rho)$ is the function $\pi(\xi(u,\rho),\rho)=u$.)

The Canonical Metric. In order to simplify the notations, we write $f(u,\rho)$, $\ell(u,\rho)$, $g(u,\rho)$ respectively instead of

$$f\big(\xi(u,\rho),\rho\big)\frac{\partial\xi(u,\rho)}{\partial u}\,,\quad \ell\big(\xi(u,\rho),\rho\big)\,,\quad g\big(\xi(u,\rho),\rho\big)$$

so that the transformed metric takes the form

$$\begin{aligned} ds^2 &= \left(f(u,\rho)du+\ell(u,\rho)\frac{(xdx)}{\rho}\right)^2 - \\ &-\left[\left(\frac{g(u,\rho)}{\rho}\right)^2 dx^2+\left(\big(\ell(u,\rho)\big)^2-\left(\frac{g(u,\rho)}{\rho}\right)^2\right)\frac{(xdx)^2}{\rho^2}\right] \end{aligned} \tag{4.1}$$

which will be termed Canonical.

The equality $h=\ell$ implies important simplifications: Since the propagation function of light is identified with the new time coordinate u, it does not depend either on the unknown functions f, ℓ, g involved in the metric or on the boundary conditions at finite distance $\sigma(u)$, $\zeta(u)$. The radial motion of a photon emitted radially at an instant u_0 from the sphere $\|x\|=\sigma(u)$ will be defined by the equation $u=u_0$, which, when u_0 describes $\mathbb{R}$, gives rise to a foliation of $\overline{U}$ by half-lines issuing from the points of F and parallel to the ρ-axis (Figure 2). This property makes clear the physical significance of the new time coordinate u. Imagine that the photon emitted radially at the instant u_0 is labelled with the indication u_0. Then, as it travels to infinity, it assigns the value of time u_0 to every point of the corresponding ray. This conception of time differs radically from that introduced by special relativity. In this last theory, the equality of values of time at distinct points is defined by means of the process of synchronization. In the present situation the equality of values of time along a radial half-line is associated with the radial motion of a single photon. The following proposition is obvious (although surprising at first sight).

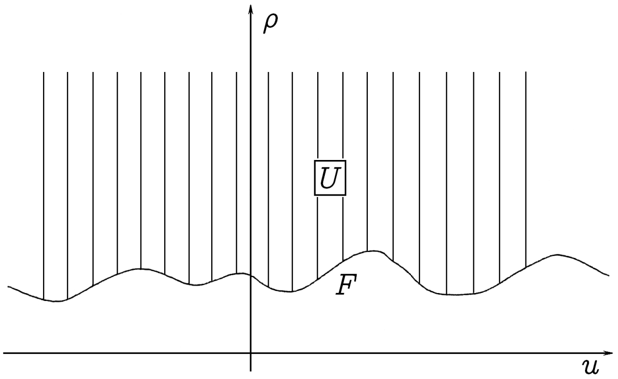

Fig. 2: The rise to a foliation of $\overline{U}$ by half-lines issuing from the points of F and parallel to the ρ-axis.

Proposition 4.4. *With respect to the canonical metric, the radial velocity of propagation of light is infinite.*

Note that the classical velocity of propagation of light, namely c, makes sense only with respect to the time defined by synchronized clocks in an inertial system.

We emphasize that the canonical metric is conceived on the closed set $\{(u,x)\in\mathbb{R}\times\mathbb{R}^3\mid\|x\|\geqslant\sigma(u)\}$ namely on the exterior of the matter, and it is not possible to assign to it a universal validity on $\mathbb{R}\times\mathbb{R}^3$. In fact, ℓ is everywhere strictly positive, whereas h vanishes for $\rho=0$, so that the equality $h(t,\|x\|)=\ell(t,\|x\|)$ cannot hold on a neighbourhood of the origin. It follows that the canonical metric is incompatible with the idea of a punctual source.

5 Propagation function of gravitational disturbances

We recall that, $\sigma(u)$ and $\zeta(u)$ being respectively the radius and the curvature radius of the sphere bounding the matter, we are led to identify the pair of derivatives $(\sigma'(u),\zeta'(u))$ with the gravitational disturbance produced at the instant u on the entirety of the sphere in question. This local disturbance induces a radial propagation process with propagation paths identified with the radial geodesies and wave fronts covering successively the spheres $\|x\| = \rho = \text{const}$. This process modifies step by step the field outside the matter and thus gives rise to a non-stationary (dynamical) state of the gravitational field. It follows that apart from any theory aimed at determining the gravitational field, we have first to elucidate the propagation process of the gravitational disturbance. In order to carry out this investigation, we refer constantly to the canonical metric (4.1) which, without restriction of generality, gives rise to significant simplifications. This being said, the gravitational disturbance being produced at the instant u on the sphere bounding the matter, let us consider the instant $t = \psi(u,\rho)$ at which it reaches the sphere $\|x\| = \rho$, so that we have, in particular, $\psi(u,\sigma(u)) = u$. We assume naturally that the pulsating source does not hinder the propagation of gravitation outside the matter. In other words, every time that the sphere bounding the matter is expanding, it can not overtake the advancing gravitational disturbance. This is the case if and only if $(\psi(u,\rho),\rho) \in U$ for every $\rho > \sigma(u)$. On the other hand, on account of the physical conditions of the problem, the derivative $\dfrac{\partial\psi(u,\rho)}{\partial\rho}$ cannot be negative, so that the equation $t = \psi(u,\rho)$ defines non decreasing functions of ρ giving rise to a foliation of $\overline{U}$ by curves issuing from the points of F. Because of this foliation, we have the condition

$$\frac{\partial\psi(u,\rho)}{\partial u} > 0 \tag{5.1}$$

which allows to solve the equation $t = \psi(u,\rho)$ with respect to u, thus obtaining the propagation function of the gravitational disturbance $u = e(t,\rho)$ relative to the canonical metric (4.1). Note that, on account of (5.1), by setting $\Delta(u,\rho) = (\psi(u,\rho),\rho) = (t,\rho)$, we define a diffeomorphism $\Delta : \overline{U} \to \overline{U}$, the restriction of which to F is the identity.

Proposition 5.1 *If the gravitational disturbance emitted at the instant u reaches the sphere $\|x\| = \rho$ at the instant t, then its radial velocity at this instant equals*

$$-\frac{\partial e(t,\rho)/\partial t}{\partial e(t,\rho)/\partial\rho}\,.$$

Proof. The velocity in question equals

$$\frac{d\rho}{dt} = \frac{1}{dt/d\rho} = \frac{1}{\partial\psi(u,\rho)/\partial\rho}$$

and since the derivation of the identity

$$e\big(\psi(u,\rho),\rho\big) = u$$

with respect to ρ gives

$$\frac{\partial e}{\partial t}\frac{\partial\psi}{\partial\rho} + \frac{\partial e}{\partial\rho} = 0\,,$$

we obtain

$$\frac{1}{\partial\psi(u,\rho)/\partial\rho} = -\frac{\partial e(t,\rho)/\partial t}{\partial e(t,\rho)/\partial\rho}$$

as asserted.

Remark. Since the radial velocity of propagation of light is infinite with respect to the canonical metric (4.1), the velocity of radial propagation of the gravitational disturbance is necessarily less than (or possibly equal to) that of light. In fact, we can establish the identity of the two propagation functions on the basis of a hypothesis which suggests itself quite naturally.

Proposition 5.2. *If the diffeomorphism Δ transforms the canonical metric (4.1) into another physically admissible $\Theta(4)$-invariant metric on $\overline{U}$, then the propagation function of the gravitational disturbance is identical with the propagation function of light.*

Proof. In order to transform (4.1) by means of Δ, we have simply to replace u in (4.1) by $e(t,\rho)$ thus obtaining the $\Theta(4)$-invariant metric

$$\begin{aligned} ds^2 &= \left(F dt + \frac{H}{\rho}(x dx)\right)^2 - \\ &- \left[\left(\frac{G}{\rho}\right)^2 dx^2 + \left(L^2 - \left(\frac{G}{\rho}\right)^2\right)\frac{(x dx)^2}{\rho^2}\right] \end{aligned} \tag{5.2}$$

where

$$F = F(t,\rho) = f\big(e(t,\rho),\rho\big)\frac{\partial e(t,\rho)}{\partial t}\,, \tag{5.3}$$

$$\begin{aligned} H = H(t,\rho) &= f\big(e(t,\rho),\rho\big)\frac{\partial e(t,\rho)}{\partial\rho} + \ell\big(e(t,\rho),\rho\big), \\ G = G(t,\rho) &= g\big(e(t,\rho),\rho\big), \\ L = L(t,\rho) &= \ell\big(e(t,\rho),\rho\big). \end{aligned} \tag{5.4}$$

In the new metric (5.2), each value of $t = \psi(u,\rho)$ is the instant at which the disturbance emitted at the instant u reaches the sphere $\|x\| = \rho$. Consequently $e(t,\rho)$ is also the propagation function of the gravitational disturbance with respect to (5.2).

We now prove that the derivative $\partial e(t,\rho)/\partial\rho$ vanishes identically on $\overline{U}$.

We argue by contradiction. If this derivative does not vanish, the propagation function $e(t,\rho)$ of the gravitational disturbance is distinct from the propagation function of light with respect to (4.1), hence also with respect to the transformed metric (5.2). This last being admissible, according to our assumption, it satisfies the condition

$$|H(t,\rho)| \leqslant L(t,\rho)\,,$$

so that the radial motion of the photons issuing from the matter is defined by the equation

$$\frac{dt}{d\rho} = \frac{-H(t,\rho) + L(t,\rho)}{F(t,\rho)}\,.$$

On account of (5.3) and (5.4), we have

$$\frac{-H(t,\rho) + L(t,\rho)}{F(t,\rho)} = -\frac{\partial e(t,\rho)/\partial\rho}{\partial e(t,\rho)/\partial t}$$

so that the preceding equation reads

$$\frac{\partial e(t,\rho)}{\partial t}\,dt + \frac{\partial e(t,\rho)}{\partial \rho}\,d\rho = 0$$

whence $e(t,\rho) = \text{const}$ and since $e(t,\sigma(t)) = t$, $e(t,\rho)$ is the propagation function of light with respect to (5.2) contrary to our assumptions. This contradiction proves our assertion, namely that $\partial e(t,\rho)/\partial\rho = 0$ on $\overline{U}$.

This being proved, since the condition $\psi(e(t,\rho),\rho) = t$ implies

$$\frac{\partial \psi}{\partial u}\frac{\partial e}{\partial \rho} + \frac{\partial \psi}{\partial \rho} = 0\,,$$

the derivative $\partial\psi/\partial\rho$ vanishes identically on $\overline{U}$. In other words, $\psi(t,\rho)$ does not depend on ρ, so that

$$\psi(u,\rho) = \psi\big(u,\sigma(u)\big) = u$$

for every $\rho \geqslant \sigma(u)$. It follows that the propagation function of the gravitational disturbance is the same as that of light with respect to (4.1), hence also with respect to any admissible transformation of (4.1).

From now on we will not distinguish the propagation function of gravitational disturbances from that of light. So we can begin by the consideration of the canonical metric (4.1) for the study of the equations of gravitation related to a spherical pulsating source. This investigation will be carried out in another paper.

Submitted on February 09, 2007
Accepted on February 14, 2007

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