On the Geometry of Background Currents in General Relativity

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In this preliminary work, we shall reveal the intrinsic geometry of background currents, possibly of electromagnetic origin, in the space-time of General Relativity. Drawing a close analogy between the object of our present study and electromagnetism, we shall show that there exists an inherent, fully non-linear, conservative third-rank radiation current which is responsible for the irregularity in the curvature of the background space(-time), whose potential (generator) is of purely geometric origin.

1 Introduction

Herein we attempt to study, in a way that has never been fully explored before, the nature of background radiation fields from a purely geometric point of view. One may always expect that empty (matter-free) regions in a space(-time) of non-constant sectional curvature are necessarily filled with some kind of pure radiation field that may be associated with a class of null electromagnetic fields. As is common in practice, their description must therefore be attributed to the Weyl tensor alone, as the only remaining geometric object in emptiness (with the cosmological constant neglected). An in-depth detailed elaboration on the nature of the physical vacuum and emptiness, considering space(-time) anisotropy, can be seen in [6, 7].

Our present task is to explore the geometric nature of the radiation fields permeating the background space(-time). As we will see, the thrilling new aspect of this work is that our main stuff of this study (a third-rank background current and its associates) is geometrically non-linear and, as such, it cannot be gleaned in the study of gravitational radiation in weak-field limits alone. Thus, it must be regarded as an essential part of Einstein's theory of gravity.

Due to the intended concise nature of this preliminary work, we shall leave aside the more descriptive aspects of the subject.

2 A third-rank geometric background current in a general metric-compatible manifold

At first, let us consider a general, metric-compatible manifold $\mathbb{M}_D$ of arbitrary dimension D and coordinates x^α. We may extract a third-rank background current from the curvature as follows:

$$J_{\mu\nu\rho} = J_{\mu[\nu\rho]} = \nabla_\lambda R^\lambda{}_{\mu\nu\rho}\,,$$

where square brackets on a group of indices indicate anti-symmetrization (similarly, round brackets will be used to indicate symmetrization). Of course, ∇ is the covariant derivative, and, with $\partial_\mu = \frac{\partial}{\partial x^\mu}$,

$$R^\lambda{}_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \Gamma^\sigma_{\mu\rho}\Gamma^\lambda_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\sigma\rho}$$

are the usual components of the curvature tensor R of the metric-compatible connection Γ whose components are given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\, g^{\lambda\sigma}\left(\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu}\right) + \Gamma^\lambda_{[\mu\nu]} - \\ - g^{\lambda\alpha}\left(g_{\mu\beta}\Gamma^\beta_{[\alpha\nu]} + g_{\nu\beta}\Gamma^\beta_{[\alpha\mu]}\right).$$

Here $g_{\mu\nu}$ are the components of the fundamental symmetric metric tensor g and $\Gamma^\lambda_{[\mu\nu]}$ are the components of the torsion tensor. The (generalized) Ricci tensor and scalar are then given, as usual, by the contractions $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and $R = R^\mu{}_\mu$, respectively.

We may introduce the traceless Weyl curvature tensor W through the decomposition

$$R^\mu{}_{\alpha\beta\gamma} = K^\mu{}_{\alpha\beta\gamma} + \\ + \frac{1}{D-2}\left(\delta^\mu_\beta R_{\alpha\gamma} + g_{\alpha\gamma}R^\mu{}_\beta - \delta^\mu_\gamma R_{\alpha\beta} - g_{\alpha\beta}R^\mu{}_\gamma\right),$$

$$K^\mu{}_{\alpha\beta\gamma} = W^\mu{}_{\alpha\beta\gamma} + \\ + \frac{1}{(D-1)(D-2)}\left(\delta^\mu_\gamma g_{\alpha\beta} - \delta^\mu_\beta g_{\alpha\gamma}\right) R\,,$$

$$K_{\alpha\beta} = K_{(\alpha\beta)} = K^\mu{}_{\alpha\mu\beta} = -\frac{1}{D-2}\, g_{\alpha\beta}R\,,$$

for which $D > 2$. In particular, we shall take into account the following useful relation:

$$R^\mu{}_{\alpha\beta\gamma}R^{\alpha\beta\gamma\nu} = W^\mu{}_{\alpha\beta\gamma}W^{\alpha\beta\gamma\nu} + \\ + \frac{1}{D-2}\left(K^\mu{}_\alpha{}^\nu{}_\beta R^{\alpha\beta} + K^{\alpha\mu\beta\nu}R_{\alpha\beta} - K^{\alpha\beta\mu\nu}R_{[\alpha\beta]}\right) + \\ + \frac{1}{D-2}\left(2RR^{\mu\nu} + g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - R^\mu{}_\alpha R^{\nu\alpha} - R_\alpha{}^\mu R^{\alpha\nu}\right) + \\ + \frac{2}{D-2}\left(R^{(\mu\alpha)}R^{[\nu}{}_{\alpha]} + R^{(\mu\alpha)}R_\alpha{}^\nu - R^\mu{}_\alpha R^{(\nu\alpha)}\right) - \\ - \frac{2}{(D-2)^2}\, RR^{\mu\nu}\,.$$

Now, for an arbitrary tensor field T, we have, as usual,

$$(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta)\, T^{\mu\nu\cdots}_{\rho\sigma\cdots} =$$
$$= R^{\lambda}{}_{\rho\alpha\beta} T^{\mu\nu\cdots}_{\lambda\sigma\cdots} + R^{\lambda}{}_{\sigma\alpha\beta} T^{\mu\nu\cdots}_{\rho\lambda\cdots} + \cdots - R^{\mu}{}_{\lambda\alpha\beta} T^{\lambda\nu\cdots}_{\rho\sigma\cdots} -$$
$$- R^{\nu}{}_{\lambda\alpha\beta} T^{\mu\lambda\cdots}_{\rho\sigma\cdots} - 2\Gamma^{\lambda}_{[\alpha\beta]} \nabla_\lambda T^{\mu\nu\cdots}_{\rho\sigma\cdots}\,.$$

For a complete set of general identities involving the curvature tensor R and their relevant physical applications in Unified Field Theory, see [1–5].

At this point, we can define a second-rank background current density (field strength) f through

$$f^{\nu\rho} = f^{[\nu\rho]} = \nabla_\mu J^{\mu\nu\rho} = \nabla_\mu \nabla_\lambda R^{\lambda\mu\nu\rho} = -\nabla_{[\lambda}\, \nabla_{\mu]} R^{\lambda\mu\nu\rho}\,.$$

An easy calculation gives, in general,

$$f^{\mu\nu} = -\frac{1}{2}\left(R^{\mu}{}_{\alpha\beta\gamma} R^{\alpha\beta\gamma\nu} - R^{\nu}{}_{\alpha\beta\gamma} R^{\alpha\beta\gamma\mu}\right) -$$
$$- R_{[\alpha\beta]} R^{\alpha\beta\mu\nu} - \Gamma^{\alpha}_{[\rho\sigma]} \nabla_\alpha R^{\rho\sigma\mu\nu}\,.$$

In analogy to the electromagnetic source, we may define a first-rank current density through

$$j^\mu = \nabla_\nu f^{\mu\nu}\,.$$

Then, a somewhat lengthy but straightforward calculation shows that

$$\nabla_\mu\, j^\mu = R_{[\mu\nu]} f^{\mu\nu} + \Gamma^{\sigma}_{[\mu\nu]} \nabla_\sigma f^{\mu\nu}\,.$$

We may also define the field strength f through a sixth-rank curvature tensor F whose components are given by

$$F_{\mu\nu\rho\sigma\lambda\delta} = F_{[\mu\nu][\rho\sigma]\lambda\delta} =$$
$$= \frac{1}{2}\left(R_{\mu\delta\lambda\alpha} \overline{R}^{\alpha}{}_{\nu\rho\sigma} - R_{\nu\delta\lambda\alpha} \overline{R}^{\alpha}{}_{\mu\rho\sigma}\right) +$$
$$+ \frac{1}{2}\left(R_{\mu\rho\lambda\alpha} \overline{R}^{\alpha}{}_{\nu\delta\sigma} - R_{\nu\rho\lambda\alpha} \overline{R}^{\alpha}{}_{\mu\delta\sigma}\right) +$$
$$+ \frac{1}{2}\left(R_{\nu\sigma\lambda\alpha} \overline{R}^{\alpha}{}_{\mu\delta\rho} - R_{\mu\sigma\lambda\alpha} \overline{R}^{\alpha}{}_{\nu\delta\rho}\right) +$$
$$+ \frac{1}{2}\left(R_{\mu\lambda\delta\alpha} \overline{R}^{\alpha}{}_{\nu\rho\sigma} - R_{\nu\lambda\delta\alpha} \overline{R}^{\alpha}{}_{\mu\rho\sigma}\right) - R_{\mu\nu\delta\alpha} \overline{R}^{\alpha}{}_{\lambda\rho\sigma}\,.$$

where $\overline{R}_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$.

If we define a second-rank anti-symmetric tensor B by

$$B_{\mu\nu} = F_{\mu\nu\rho\sigma}{}^{\sigma\rho} =$$
$$= \frac{1}{2}\left(R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma}{}_{\nu} - R_{\nu\alpha\beta\gamma} R^{\alpha\beta\gamma}{}_{\mu}\right) - R_{\mu\nu\alpha\beta} R^{[\alpha\beta]}\,,$$

we then obtain

$$f_{\mu\nu} = B_{\mu\nu} + \Gamma^{\alpha}_{[\rho\sigma]} \nabla_\alpha R^{\rho\sigma}{}_{\mu\nu}\,,$$

such that in the case of vanishing torsion, the quantities f and B are completely equivalent.

3 A third-rank radiation current relevant to General Relativity

Having defined the basic geometric objects of our theory, let us adhere to the standard Riemannian geometry of General Relativity in which the torsion vanishes, that is $\Gamma^{\lambda}_{[\mu\nu]} = 0$, and so the connection is the symmetric Levi-Civita connection. However, let us also take into account discontinuities in the first derivatives of the components of the metric tensor in order to take into account discontinuity surfaces corresponding to any existing background energy field. As we will see, we shall obtain a physically meaningful background current which is strictly conservative.

Now, in connection with the results of the preceding section, if we employ the simplified relation (which is true in the absence of torsion)

$$R^{\mu}{}_{\alpha\beta\gamma} R^{\alpha\beta\gamma\nu} = W^{\mu}{}_{\alpha\beta\gamma} W^{\alpha\beta\gamma\nu} +$$
$$+ \frac{1}{D-2}\left(K^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\, R^{\alpha\beta} + K^{\alpha\mu\beta\nu} R_{\alpha\beta}\right) +$$
$$+ \frac{1}{D-2}\left(2RR^{\mu\nu} + g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - 4R^{\mu}{}_{\alpha} R^{\nu\alpha}\right) -$$
$$- \frac{2}{(D-2)^2}\, RR^{\mu\nu}\,,$$

as well as the relations

$$K^{\mu}{}_{\alpha\beta\gamma} K^{\alpha\beta\gamma\nu} = W^{\mu}{}_{\alpha\beta\gamma} W^{\alpha\beta\gamma\nu} + \frac{1}{(D{-}1)\,(D{-}2)^2}\, g^{\mu\nu} R^2,$$

$$K^{\mu\alpha\nu\beta} R_{\alpha\beta} = W^{\mu\alpha\nu\beta} R_{\alpha\beta} +$$
$$+ \frac{1}{(D-1)\,(D-2)}\left(R^{\mu\nu} - g^{\mu\nu} R\right) R\,,$$

we obtain the desired relation

$$f^{\mu\nu} = -\frac{1}{2}\left(W^{\mu}{}_{\alpha\beta\gamma} W^{\alpha\beta\gamma\nu} - W^{\nu}{}_{\alpha\beta\gamma} W^{\alpha\beta\gamma\mu}\right) -$$
$$- \frac{1}{D-2}\left(W^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} - W^{\nu}{}_{\alpha}{}^{\mu}{}_{\beta}\right) R^{\alpha\beta}.$$

If the metric tensor is perfectly continuous, it is obvious that

$$f^{\mu\nu} = 0\,.$$

In deriving this relation we have used the symmetry $W_{\mu\nu\rho\sigma} = W_{\rho\sigma\mu\nu}$. This shows that, in the presence of metric discontinuity, the field strength f depends on the Weyl curvature alone which is intrinsic to the background space(-time) only when matter and non-null electromagnetic fields are absent. We see that, in spaces of constant sectional curvature, we will strictly have $J^{\lambda\mu\nu} = 0$ and $f^{\mu\nu} = 0$ since the Weyl curvature vanishes therein. In other words, in the sense of General Relativity, the presence of background currents is responsible for the irregularity (anisotropy) in the curvature of the background space(-time). Matter, if not elementary particles, in this sense, can indeed be regarded as a form of perturbation with respect to the background space(-time).

Furthermore, it is now apparent that

$$J_{\mu\nu\rho} + J_{\nu\rho\mu} + J_{\rho\mu\nu} = 0\,.$$

This relation is, of course, reminiscent of the usual Bianchi identity satisfied by the components of the Maxwellian electromagnetic field tensor.

Also, we obtain the conservation law

$$\nabla_\mu j^\mu = 0\,.$$

which becomes trivial when the metric is perfectly continuous.

Hence, the formal correspondence between our present theory and the ordinary theory of electromagnetism may be completed, in the simplest way, through the relation

$$J_{\mu\nu\rho} = \nabla_\lambda R^\lambda{}_{\mu\nu\rho} = \nabla_\mu \Phi_{\nu\rho}\,,$$

where the anti-symmetric field tensor Φ given by

$$\Phi_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

plays a role similar to that of the electromagnetic field strength. However, it should be emphasized that it exists in General Relativity's fully non-linear regime. In addition, it vanishes identically in the absence of curvature anisotropy. Interestingly, if one is willing to regard electromagnetism as a kind of non-linear gravity, one may alternatively regard Φ as being the complete equivalent of Maxwell's electromagnetic field strength. However, we shall not further pursue this interest here.

Furthermore, we obtain the relation

$$f_{\mu\nu} = \Box\, \Phi_{\mu\nu}\,,$$

where $\Box = \nabla_\mu \nabla^\mu$. That is, the wave equation

$$\Box\,\Phi_{\mu\nu} = -\frac{1}{2}\left(W_{\mu\alpha\beta\gamma}W^{\alpha\beta\gamma}{}_\nu - W_{\nu\alpha\beta\gamma}W^{\alpha\beta\gamma}{}_\mu\right) - \\ - \frac{1}{D-2}\left(W_{\mu\alpha\nu\beta} - W_{\nu\alpha\mu\beta}\right) R^{\alpha\beta}.$$

In the absence of metric discontinuity, we obtain

$$\Box\,\Phi_{\mu\nu} = 0\,.$$

Let us now introduce a vector potential ϕ such that the curl of which gives us the field strength f. Instead of writing $f_{\mu\nu} = \partial_\nu\phi_\mu - \partial_\mu\phi_\nu$ and instead of expressing the field strength f in terms of the Weyl tensor, let us write its components in the following equivalent form:

$$f_{\mu\nu} = -\frac{1}{2}\left(R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma}{}_\nu - R_{\nu\alpha\beta\gamma}R^{\alpha\beta\gamma}{}_\mu\right) = \\ = \nabla_\nu\phi_\mu - \nabla_\mu\phi_\nu\,.$$

In order for the potential ϕ to be purely geometric, we shall have

$$\nabla_\nu\phi_\mu = -\frac{1}{2}\,R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma}{}_\nu\,,$$

from which an "equation of motion" follows somewhat effortlessly:

$$\frac{D\phi_\mu}{Ds} = -\frac{1}{2}\,R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma}{}_\nu\,\frac{dx^\nu}{ds}\,,$$

where $\frac{D\phi_\mu}{Ds} = \frac{dx^\nu}{ds}\nabla_\nu\phi_\mu$.

Note that, in the absence of metric discontinuity, the vector potential ϕ is a mere gradient of a smooth scalar field Θ: $\phi_\mu = \nabla_\mu\Theta$.

Now, it remains to integrate the equation

$$\partial_\nu\phi_\mu = -\frac{1}{2}\,R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma}{}_\nu + \Gamma^\lambda_{\mu\nu}\phi_\lambda$$

by taking a closed contour P associated with the surface area dS spanned by infinitesimal displacements in two different directions, that is,

$$dS^{\mu\nu} = d_1x^\mu d_2x^\nu - d_1x^\nu d_2x^\mu\,.$$

An immediate effect of this closed-loop integration is that, by using the generally covariant version of Stokes' theorem and by explicitly assuming that the integration factor Z given by

$$Z^\rho{}_\mu = \frac{1}{2}\oiint_S\left(\nabla_\lambda\Gamma^\rho_{\mu\nu} - \nabla_\nu\Gamma^\rho_{\mu\lambda}\right)dS^{\lambda\nu} = \\ = \frac{1}{2}\oiint_S\left(R^\rho{}_{\mu\lambda\nu} + \Gamma^\rho_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\sigma\nu}\right)dS^{\lambda\nu} = \\ = \frac{1}{2}\oiint_S\left(R^\rho{}_{\mu\lambda\nu} + 2\Gamma^\rho_{\sigma\lambda}\Gamma^\sigma_{\mu\nu}\right)dS^{\lambda\nu}$$

does not depend on ϕ, the integral $\oint_P \Gamma^\lambda_{\mu\nu}\phi_\lambda dx^\nu$ shall indeed vanish identically.

Hence, we are left with the expression

$$\Delta\phi_\mu = -\frac{1}{2}\oint_P R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma}{}_\nu dx^\nu\,.$$

By introducing a new integration factor X satisfying $X^{\alpha\beta\gamma} + X^{\beta\gamma\alpha} + X^{\gamma\alpha\beta} = 0$ as follows:

$$X^{\alpha\beta\gamma} = X^{[\alpha\beta]\gamma} = \oint_P R^{\alpha\beta\gamma}{}_\nu dx^\nu = \\ = \frac{1}{2}\oiint_S\left(\nabla_\mu R^{\alpha\beta\gamma}{}_\nu - \nabla_\nu R^{\alpha\beta\gamma}{}_\mu\right)dS^{\mu\nu}\,,$$

we obtain, through direct partial integration,

$$\Delta\phi_\mu = -\frac{1}{2}\left(R_{\mu\alpha\beta\gamma}X^{\alpha\beta\gamma} - \int X^{\alpha\beta\gamma}dR_{\mu\alpha\beta\gamma}\right).$$

Simplifying, by keeping in mind that $X = X\,(R, dR)$, we finally obtain

$$\Delta\phi_\mu = \frac{1}{2}\int R_{\mu\alpha\beta\gamma}\,dX^{\alpha\beta\gamma}\,.$$

The simplest desired result of this is none other than

$$\Delta\phi_\mu = \frac{1}{2}\,R_{\mu\alpha\beta\gamma}X^{\alpha\beta\gamma}\,,$$

which, expressed in terms of the Weyl tensor, the Ricci tensor, and the Ricci scalar, is

$$\Delta\phi_\mu = \frac{1}{2}\,W_{\mu\alpha\beta\gamma}X^{\alpha\beta\gamma} + \frac{1}{D-2}\left(X^{\alpha}{}_{\mu}{}^{\beta}R_{\alpha\beta} - X^{\alpha}{}_{\beta}{}^{\beta}R_{\alpha\mu}\right) + + \frac{1}{(D-1)\,(D-2)}\,X_{\mu}{}^{\alpha}{}_{\alpha}R\,.$$

Hence, through Einstein's field equation (i.e. through the energy-momentum tensor T)

$$R_{\mu\nu} = \pm\,\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}T\right),$$

where G is Newton's gravitational constant and c is the speed of light, we may see how the presence of (distributed) matter affects the potential ϕ.

4 Final remarks

At this point, having outlined our study in brief, it remains to be seen whether our fully geometric background current may be associated with any type of conserved material current which is already known in the literature. It is also tempting to ponder, from a purely physical point of view, on the possibility that the intrinsic curvature of space(-time) owes its existence to null (light-like) electromagnetic fields or simply pure radiation fields.

In this case, let the null electromagnetic (pure radiation) field of the background space(-time) be denoted by φ, for which

$$\varphi_{\mu\nu}\varphi^{\mu\nu} = 0\,.$$

Then we may express the components of the Weyl tensor as

$$W_{\mu\nu\rho\sigma} = \varphi_{\mu\nu}\varphi_{\rho\sigma} - \varphi_{\mu\rho}\varphi_{\nu\sigma} + \varphi_{\mu\sigma}\varphi_{\nu\rho}\,,$$

such that the relation $W^{\rho}{}_{\mu\rho\nu} = 0$ is satisfied.

If this indeed is the case, then we shall have a chance to better understand how matter actually originates from such a pure radiation field in General Relativity. This will hopefully also open a new way towards the full geometrization of matter in physics.

Finally, as a pure theory of gravitation, the results in the present work may be compared to those given in [8] and [9], wherein, based on the theory of chronometric invariants [7], a new geometric formulation of gravity (which is fully equivalent to the standard form of General Relativity) is presented in a way very similar to that of the electromagnetic field, based solely on a second-rank anti-symmetric field tensor.

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