Geometrical Derivation of the Lepton PMNS Matrix Values

Franklin Potter

Sciencegems.com, 8642 Marvale Drive, Huntington Beach, CA, USA. E-mail: frank11hb@yahoo.com

The linear superposition of generators of the 3 discrete binary rotational subgroups [332], [432], [532] of the Standard Model determine the PMNS matrix elements. The 6 leptons are 3-D entities representing these 3 groups, one group for each lepton family.

1 Introduction

Numerous attempts to derive the neutrino PMNS matrix from various discrete group horizontal symmetries have led to partial success. Herein I determine the true source of the PMNS matrix elements by using the linear superposition of the generators for 3 discrete binary rotational subgroups of the Standard Model (SM) electroweak gauge group $SU(2)_L$ x $U(1)_Y$.

In a series of articles [1–4] I have proposed 3 discrete binary rotational subgroups of the SM gauge group for 3 lepton families in R^3 and the related 4 discrete binary rotational subgroups in R^4 for 4 quark families, one binary group for each family. The generators for these 7 binary groups are quaternions operating in R^3, in R^4, and in C^2. I use these binary group quaternion generators to calculate the matrix elements for the PMNS mixing matrix for the leptons.

In another article under preparation I use the same approach, with an important modification, to calculate the standard CKM mixing matrix for the quarks as well as a proposed CKM4 mixing matrix for four quark families.

The SM local gauge group $SU(2)_L$ x $U(1)_Y$ x $SU(3)_C$ defines an electroweak(EW) interaction part and a color interaction part. The EW isospin states define the flavor of the fundamental lepton and quark states. However, experiments have determined that these left-handed flavor states are linear superpositions of mass eigenstates.

For the 3 lepton families, one has the neutrino flavor states ν_e, ν_μ, ν_τ and the mass states ν_1, ν_2, ν_3 related by the PMNS matrix U_{ij}

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}$$

From experiments [5], the PMNS angles have been estimated to be

$$\theta_{12} = 32.6° - 34.8°, \quad \theta_{13} = 8.5° - 9.4°,$$

$$\theta_{23} = 37.2° - 39.8°, \quad \delta = (0.77 - 1.36)\pi.$$

Consequently, for the normal hierarchy of neutrino masses, one has the empirically determined PMNS matrix

$$\begin{bmatrix} 0.822 & 0.547 & -0.150 + 0.038i \\ -0.356 + 0.0198i & 0.704 + 0.0131i & 0.614 \\ 0.442 + 0.0248i & -0.452 + 0.0166i & 0.774 \end{bmatrix}$$

which can be compared to my resultant derived PMNS matrix in the standard parametrization

$$\begin{bmatrix} 0.817 & 0.557 & -0.149e^{-i\delta} \\ -0.413 - 0.084e^{i\delta} & 0.605 - 0.057e^{i\delta} & -0.673 \\ -0.383 + 0.090e^{i\delta} & 0.562 + 0.061e^{i\delta} & 0.725 \end{bmatrix}$$

In the SM the EW isospin symmetry group that defines the lepton and quark flavor states is assumed to be the Lie group SU(2) with its two flavor eigenstates per family. In this context there is no fundamental reason for Nature to have more than one fermion family, and certainly no reason for having 3 lepton families and at least 3 quark families. As far as I know, this normal interpretation of the SM provides no answer that dictates the actual number of families, although the upper limit of 3 lepton families with low mass neutrinos is well established via Z^0 decays and via analysis of the CMB background. There are claims also that one cannot have more than 15 fundamental fermions (plus 15 antifermions) without violating certain cosmological constraints.

My geometrical approach makes a different choice, for I utilize discrete binary rotational subgroups of SU(2) instead, a different subgroup for each family. Each discrete binary group has two eigenstates and three group generators, just like SU(2). Whereas the three generators for the SU(2) Lie group are essentially the 2 x 2 Pauli matrices, the three generators for each of the 3 lepton discrete binary groups [332], [432], [532], (also labeled 2T, 2O, 2I) in R^3 and the 4 quark discrete groups [333], [433], [343], [533], (also labeled 5-cell, 16-cell, 24-cell, 600-cell) in R^4 are not exactly the Pauli matrices.

I propose that this difference between the discrete subgroup generators and the Pauli matrices is the fundamental source of the lepton and the quark mixing matrices, and the calculated results verify this conjecture. In other words, one requires the mixing of the different family discrete groups in order to have a complete set of three generators equivalent to the three SU(2) generators, separately for the leptons and for the quarks. The mixing matrices, PMNS and CKM4, express this linear superposition of the discrete group generators.

2 The PMNS calculation

In order to calculate the PMNS values one can use either unit quaternions or unitary 2x2 complex matrices. The unit quaternion generators are equivalent to the SU(2) generators.

The unit quaternion q = a + b**i** + c**j** + d**k**, where the coefficients a, b, c, d are real numbers for the one real and three imaginary axes. The unit quaternion spans the space R^4 while the imaginary prime part spans the subspace R^3. With $i^2 = j^2 = k^2 = -1$, the quaternion can be expressed as an SU(2) matrix

$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$$

Both the quaternions and the SU(2) matrices operate in the unitary plane C^2 with its two orthogonal complex axes, so the quaternion can be written also as q = u + v**j**, with u = a + b**i** and v = c + d**i**. The three Pauli matrices σ_x, σ_y, σ_z, are the simple quaternions k, j, and i, respectively.

For the three lepton families, each family representing its own binary rotational group, [332], [432], and [532], two of the three generators R_i, i = 1, 2, 3, in each group are equivalent to two of the three Pauli matrices. Therefore, only the remaining generator for each lepton family contributes to the mixing that produces the PMNS matrix. That is, in the notation of H.M.S. Coxeter [6], R_1 = j, R_3 = i, and

$$R_2 = -i\cos\frac{\pi}{q} - j\cos\frac{\pi}{p} + k\sin\frac{\pi}{h} \tag{1}$$

for the three binary groups [p q r] and the h values 4, 6, and 10, respectively.

Defining the golden ratio $\phi = (\sqrt{5}+1)/2$, the appropriate generators R_2 are listed in the table. The sum of all three R_2 generators should be k, so one has three equations for three unknowns, thereby determining the listed multiplicative factor for each R_2 generator's contribution to k after overall normalization.

Table 1: Lepton Family Discrete Group Assignments

Family	Group	R_2	Factor	Angle°
ν_e, e	[332]	$-\frac{1}{2}i - \frac{1}{2}j + \frac{1}{\sqrt{2}}k$	-0.2645	105.337
ν_μ, μ	[432]	$-\frac{1}{2}i - \frac{1}{\sqrt{2}}j + \frac{1}{2}k$	0.8012	36.755
ν_τ, τ	[532]	$-\frac{1}{2}i - \frac{\phi}{2}j + \frac{\phi^{-1}}{2}k$	-0.5367	122.459

The resulting angles in the table are determined by the arccosines of the factors, but they are twice the rotation angles required in R^3, a property of quaternion rotations. Using one-half these angles produces

$$\theta_1 = 52.67°, \quad \theta_2 = 18.38°, \quad \theta_3 = 61.23°, \tag{2}$$

resulting in

$$\theta_{12} = 34.29°, \quad \theta_{13} = -8.56°, \quad \theta_{23} = -42.85°. \tag{3}$$

Note that $|\ \theta_{12} - \theta_{13}\ | = |\ \theta_{23}\ |$ because of normalization.

Products of the sines and cosines of these angles in the standard parameterization are the PMNS entries, producing matrix values which compare favorably with the empirical estimates, as shown earlier. One has $\sin^2\theta_{12} = 0.3176$ and $\sin^2\theta_{13} = 0.0221$, both within 1σ of the empirically determined values from the neutrino experiments, according to the Particle Data Group in 2012. However, $\sin^2\theta_{23} = 0.4625$ is outside the PDG 1σ range but agrees with the recent T2K [7] estimate $\sin^2 2\theta_{23} = 1.0$, making $|\ \theta_{23}\ | = 45°$ with $\delta \simeq 0$.

3 Conclusions

This fit of the PMNS mixing matrix derived from the three separate R_2 generators indicates that the lepton families faithfully represent the discrete binary rotational groups [332], [432], and [532] in R^3 that were introduced first in my geometrical approach back in 1986 and expanded in detail over the past two decades. In particular, the 6 lepton states are linear superpositions of the two degenerate basis states in each of the 3 groups. My approach within the realm of the Standard Model local gauge group makes the ultimate *unique* connection to the discrete group Weyl E_8 x Weyl E_8 in 10-D spacetime and to the Golay-24 code in information theory [1].

One can conclude that leptons are 3-dimensional objects, geometrically different from the quarks which require a 4-dimensional space for their existence. Their mass ratios derive from a mathematical syzygy relation to the j-invariant of elliptic modular functions associated with these specific binary groups. In addition, one can predict that no more lepton families exist because the appropriate binary rotational symmetry groups in 3-D space have been exhausted. However, sterile neutrinos remain viable [1, 4].

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