Gödel's Universe Revisited

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This paper investigates the Gödel's exact solution of the Einstein equations which describes a stationary homogeneous cosmological Universe inducing closed timelike curves CTCs). This model is generally dismissed because it exhibits a rotational symmetry and it requires a non zero cosmological constant in contradiction with the current astronomical observations. If the cosmological term is assumed to be slightly variable, we show that this metric can be compatible with the Hubble expansion, which makes the Gödel model a viable representation of our Universe.

Introduction

In his original paper [1], Kurt Gödel has derived an exact solution to Einstein's field equations in which the matter takes the form of a pressure-free perfect fluid (dust solution). This $\Re^4$ manifold is homogeneous but non-isotropic and it exhibits a specific rotational symmetry which allows for the existence of *closed time like curves* since the light cone opens up and tips over as the Gödel radial coordinate increases. In addition, it implies a *non zero cosmological term* and a constant scalar curvature, therefore it doesnot admit a *Hubble* expansion *in the whole*, which tends to contradict all current observations.

We suggest here to stick to the Gödel model which we consider as the *true* Universe, and we state that the Hubble expansion can yet be maintained in a particular location with specific coordinates transformations, where the Gödel rotation is *unobservable*.

In this distinguished location, our derivations lead to an *open Universe* without cosmological term and as a result, no future singularity will ever appear in this *local* World.

Our model however, is bound to a main restriction: for physical reasons, it provides a solution which holds only for the existence of the cosmic scale factor, *within* the Gödel metric.

This improved Gödel Universe which we present here, has nevertheless the advantage of agreeably coping with the observational facts.

Some notations

Space-time indices: 0, 1, 2, 3.
Newton's gravitation constant: G.
The velocity of light is $c = 1$.
Space-time signature: -2.

1 Homogeneous space-times

1.1 Roberston-Walker space

Our actual observed Universe is spatially *homogeneous*: if we can see these observations identically in different directions, the model is said *isotropic*. The Robertson-Walker metric is an exact spherically symmetric solution. This property would imply that the Universe admits a six-parameter group of isometries whose surfaces of transitivity are spacelike three-surfaces of constant curvatures. (An action of a group is transitive on the manifold $\mathfrak{M}$, if it can map any point of $\mathfrak{M}$ into any other point of $\mathfrak{M}$.) The spatial metric is expressed by

$$dl^2 = \frac{dr^2}{1 + r^2/F^2} + r^2 \left(\sin^2\theta\, d\varphi^2 + d\theta^2\right). \tag{1.1}$$

In the full RW model $F(t)$ is called the *cosmic scale factor* which varies with the (cosmic) proper time t of the whole space.

For an *open* (*infinite*) Universe, with *negative* curvature

$$K(t) = \frac{k}{F^2}, \qquad \text{where } k = -1. \tag{1.2}$$

and the three-spaces are *diffeomorphic* to $\Re_3$.

The standard formulation is given by

$$(ds^2)_{\rm RW} = F^2 \left(d\eta^2 – d\chi^2 - \sinh^2\chi\,(\sin^2\theta\, d\varphi^2 + d\theta^2)\right) \tag{1.3}$$

with the usual parametrizations

$$dt = F\, d\eta \quad \text{and} \quad r = F \sinh\chi\,. \tag{1.4}$$

In the RW Universe, the matter with mean density ρ is non interacting (dust) and the energy-momentum tensor is that of a *pressure free perfect fluid*:

$$T_{ab} = \rho\, u_a u_b\,. \tag{1.5}$$

From the corresponding field equations we arrive at the temporal coordinate [2]

$$\eta = \pm \int \frac{dF}{F\sqrt{\left[\frac{8\pi G}{3}\rho F^2 + 1\right]}}\,, \tag{1.6}$$

$$F = F_0 (\cosh\eta - 1)\,, \tag{1.7}$$

with

$$F_0 = \frac{4\pi G\rho F^3}{3}\,, \tag{1.8}$$

Where the ± sign depends on the light emitted either from the coordinates origin or reaching this origin.

1.2 The Gödel metric

The Gödel line element is generically given by

$$(ds^2)_{\rm G} = B^2\left[dx_0^2 - dx_1^2 + \frac{e^{2x_1}}{2}\,dx_2^2 - dx_3^2 + {}\right.$$
$$\left. + 2e^{2x_1}(dx_0 + dx_2)\right], \qquad (1.9)$$

where $B > 0$ is a constant in the original formulation.

This space-time has a five dimensional group of isometries which is transitive. It admits a five dimensional *Lie algebra of Killing vector fields* generated by a time translation ∂_{x_0}, two spatial translations ∂_{x_1}, ∂_{x_2} plus two further Killing vector fields:

$$\partial_{x_3} - x_2\partial_{x_3} \quad \text{and} \quad 2e^{x_1}\partial_{x_0} + x_2\partial_{x_3} + \left[e^{2x_1} - \frac{x_2^2}{2}\,\partial_{x_2}\right].$$

In all current papers, the Gödel metric is always described as the direct sum of the metric

$$(ds^2)_{\rm G_1} = B^2\left[dx_0^2 - dx_1^2 + dx_2^2\,\frac{e^{2x_1}}{2} + {}\right.$$
$$\left. + 2e^{x_1}(dx_0 + dx_2)\right] \qquad (1.10)$$

on the manifold $\mathfrak{M}_1 = \Re_3$ and

$$(ds^2)_{\rm G_2} = B^2(-dx_3^2) \qquad (1.11)$$

on the manifold $\mathfrak{M}_2 = \Re_1$.

This means that in the usual treatments, in order to analyze the properties of the Gödel solution it is always sufficient to consider only $\mathfrak{M}_1$. The coordinate dx_3 is deemed irrelevant and is thus simply suppressed in the classical representation, which in our opinion reveals a certain lack of completeness. In what follows, we consider the complete solution, where we assign a specific meaning to dx_3.

Let us remark that the Gödel space is homogeneous but not isotropic.

1.3 Classical features of Gödel's metric

Computing the connection coefficients $\Gamma^{\,c}_{ab}$ from the g_{ab} given in (1.9) eventually yield

$$R_{00} = 1, \quad R_{22} = e^{2x_1}, \quad R_{02} = R_{20} = e^{x_1}. \qquad (1.12)$$

All other R_{ab} vanish.

Hence:

$$R = \frac{1}{B^2}\,. \qquad (1.13)$$

The unit vector (world velocity) following the x_0-lines is shown to have the following contravariant components

$$\frac{1}{B},\ 0,\ 0,\ 0$$

and the covariant components

$$B,\ 0,\ Be^{x_1},\ 0$$

so we obtain

$$R_{ab} = \frac{1}{B^2}\,u_a u_b\,. \qquad (1.14)$$

Since the curvature scalar is a constant, the Gödel field equations read

$$(G_{ab})_{\rm G} = R_{ab} - \frac{1}{2}\,g_{ab}R = 8\pi G\rho\, u_a u_b + \Lambda g_{ab}\,, \qquad (1.15)$$

where Λ is the cosmological term which is here inferred as $-4\pi G\rho$, i.e.:

$$\frac{1}{B^2} = 8\pi G\rho\,, \qquad (1.16)$$

$$\Lambda = -\frac{R}{2} = -\frac{1}{2B^2}\,. \qquad (1.17)$$

We next define new coordinates (t, w, ϕ) on $\mathfrak{M}_1$ by

$$E^{x_1} = \cosh 2w + \cos\phi\,\sinh 2w\,, \qquad (1.18)$$

$$x_2 e^{x_1} = \sqrt{2}\,\sin\phi\,\sinh 2w\,, \qquad (1.19)$$

$$\tan\frac{1}{2}\left(\phi + \frac{x_0 - 2t}{\sqrt{2}}\right) = e^{-2w}\tan\frac{\phi}{2}\,. \qquad (1.20)$$

This leads to the new line element

$$(ds^2)_{\rm G} = 4B^2\Big(\big(dt^2 - dw^2 - dy^2 + \sinh^4 w - \sinh^2 w\big)\,d\phi^2 + {}$$
$$+ 2\sqrt{2}\,\sinh^2 w\,d\phi\,dt\Big) \qquad (1.21)$$

which exhibits the rotational symmetry of the solution about the axis $w = 0$, since we clearly see that the g_{ab} do not depend on θ. Gödel inferred that matter everywhere rotates with the angular velocity $2\sqrt{4\pi G\rho}$.

Let us consider the *reduced* Gödel metric

$$(ds^2)_{\rm G_1} = 4B^2\Big(\big(dt^2 - dw^2 + \sinh^4 w - \sinh^2 w\big)\,d\phi^2 + {}$$
$$+ 2\sqrt{2}\,\sinh^2 w\,d\phi\,dt\Big).$$

All light rays emitted from an event on the symmetry axis reconverge at a later event on this axis, with the null geodesics forming a circular *cusp* [3].

If a curve c is defined by $\sinh^4 w = 1$, that is

$$c = \ln(1 + \sqrt{2}), \qquad (1.22)$$

hence, any circle $w > \ln(1+\sqrt{2})$ in the plane $t = 0$, is a *closed timelike curve*.

2 The modified Gödel metric

2.1 Conformal transformation

Now we will assume that the Λ-term is slightly varying with the time t, so B is also variable through the dust density. See (1.16) for detail.

By setting

$$y = r \cosh w, \tag{2.1}$$

where r is another radial parameter, we choose:

$$B = \frac{1}{2}\left(1 - \frac{L_0}{2\sqrt{t^2 - y^2}}\right)^2 \tag{2.2}$$

where L_0 is a constant whose meaning will become apparent in the next sub-section. B is now identified with a conformal factor.

Note: one of the *Kretschmann scalar* is no longer an invariant

$$R_{abcd} R^{abcd} = \frac{6}{B^4} \tag{2.3}$$

which reflects the fact that the Gödel space-time may be not fully homogeneous.

Anticipating on our postulate, we will state that the variation of B is only localized in a certain region of the Gödel model. The Λ-term remains constant throughout the complete metric as initially derived, thus preserving its homogeneity.

2.2 The postulate

Our fundamental assumption will now consist of considering our *observed* Universe as being *local*. By local we mean that the rotation ϕ is *unobservable* since we assume that our world is situated at

$$w = 0.$$

Our (local) Universe is now becoming isotropic.

In this case, the Gödel metric reduces to a standard *conformal solution* where the light cone is centered about the t-axis:

$$(ds^2)_{\rm G} = \left[1 - \frac{L_0}{2\sqrt{t^2 - r^2}}\right]^4 \left(dt^2 - dr^2\right). \tag{2.4}$$

We now make the following transformations

$$L_0 = F_0 \tag{2.5}$$

with F_0 defined in (1.8)

$$r = \frac{F_0}{2} e^{\eta} \sinh\chi, \qquad t = \frac{F_0}{2} e^{\eta} \cosh\chi, \tag{2.6}$$

$$\frac{F_0}{2} e^{\eta} = \sqrt{t^2 - r^2}, \tag{2.7}$$

$$\tanh\chi = \frac{r}{t}, \tag{2.8}$$

and we retrieve the *Roberston-Walker* metric for an *open* Universe with the sole radial coordinate r:

$$(ds^2)_{\rm RW} = F^2(\eta)\left[d\eta^2 - d\chi^2\right]. \tag{2.9}$$

Remark: The Weyl tensor of the Gödel solution

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} + \frac{R}{3}\,\delta^{a}{}_{[c}\,\delta_{d]}{}^{b} + 2\delta^{[a}{}_{[c} R_{d]}{}^{b]} \tag{2.10}$$

which has Petrov type D, vanishes for (2.9). Indeed, the equivalent metric (2.4) implies that $C^{ab}{}_{cd} = 0$ for this conformally flat space-time.

Our observed Universe would then be devoid of the Weyl curvature which explains why it is purely described in terms of the Ricci tensor alone. In this view, Einstein was perhaps an even more exceptional visionary mind than is yet currently admitted.

2.3 Hubble expansion

In our local world, the null geodesics are obviously given by $(ds^2)_{\rm RW} = 0$, that is

$$d\eta = \pm\, d\chi \tag{2.11}$$

and integrating

$$\chi = \pm\eta + const. \tag{2.12}$$

Let us place ourselves at $t(\eta)$, where we observe a light ray emitted at χ where its frequency is ν_0. In virtue of (2.12), the emission time will be $t(\eta-\chi)$, and we observe an apparent frequency given by:

$$\nu = \frac{\nu_0 F(\eta - \chi)}{F(\eta)}. \tag{2.13}$$

As $F(\eta)$ increases monotonically, we have $\nu < \nu_0$ which is the expression of a red shifted light. Most observed red shifts are rather small, so that $F(\eta-\chi)$ can be expanded as a Taylor series about $t(\eta-\chi) = t(\eta)$ and we finally get, limiting to the first two terms

$$F(\eta - \chi) = F(\eta) + \left[t(\eta-\chi) - t(\eta)\right] F'(\eta) \tag{2.14}$$

$$= F(\eta)\left\{1 + H_0\left[t(\eta-\chi) - t(\eta)\right]\right\} \tag{2.15}$$

where F' denotes differentiation with respect to η

$$H_0 = \frac{F'(\eta)}{F(\eta)} \tag{2.16}$$

is the present numerical value of *Hubble constant*.

The Gödel solution has a non-zero cosmological term, but **not** the local RW metric.

This agrees with the fact that our open local Universe has a singularity in the past and no singularity in the future [4], in accordance with astronomical observations.

Concluding remarks

Closed timelike curves turn out to exist in many other exact solutions to Einstein's field equations.

It would seem that the first model exhibiting this property was pioneered by C. Lanczos in 1924 [5], and later rediscovered under another form by W. J. Van Stockum in 1937 [6].

However, unlike the Gödel solution, the dust particles of these Universes are rotating about a geometrically distinguished axis.

Even worse, the matter density is shown to increase with radius w, a feature which seriously contradicts all current observations.

In this sense, the Gödel metric appears as a more plausible model characterizing a broaden Universe which is compatible with our astronomical data, provided one is prepared to accept the fact that our observed world is purely local.

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