Analytical Study of the Van der Pol Equation in the Autonomous Regime

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The Van der Pol differential equation was constructed for an autonomous regime using link's law. The Van der Pol equation was studied analytically to determine fixed points, stability criteria, existence of limit cycles and solved numerically. The graphs of the equation are drawn for different values of damping coefficient μ.

1 Introduction

Balthazar Van der Pol (1899-1959) was a Dutch electrical engineer who initiated experimental dynamics in the laboratory during the 1920's and 1930's. He first introduced his (now famous) equation in order to describe triode oscillations in electric circuits, in 1927.

Van der Pol found stable oscillations, now known as limit cycles, in electrical circuits employing vacuum tubes. When these circuits are driven near the limit cycle they become entrained, i.e. the driving signal pulls the current along with it. The mathematical model for the system is a well known second order ordinary differential equation with cubic non linearity: the Van der Pol equation. The Van der Pol equation has a long history of being used in both the physical and biological sciences. For instance, Fitzhugh [1] and Nagumo [2] used the equation in a planer field as a model for action potential of neurons. The equation has also been utilized in seismology to model the plates in a geological fault [3].

During the first half of the twentieth century, Balthazar Van der Pol pioneered the field of radio telecommunication [4–9]. The Van der pol equation with large value of nonlinearity parameter has been studied by Cartwright and Littlewood in 1945 [10]; they showed that the singular solution exists. Also analytically, Lavinson [11] in 1949, analyzed the Van der Pol equation by substituting the cubic non linearity for piecewise linear version and showed that the equation has singular solution also. Also, the Van der Pol Equation for Nonlinear Plasma Oscillations has been studied by Hafeez and Chifu in 2014 [12]; they showed that the Van der pol equation depends on the damping co-efficient μ which has varying behaviour. In this article, the analytical study of the Van der Pol equation in the autonomous regime is studied.

2 Description of the Van der Pol oscillator

The Van der Pol oscillator is a self-maintained electrical circuit made up of an Inductor (L), a capacitor initially charged with a capacitance (C) and a non-linear resistance (R); all of them connected in series as indicated in Fig. 1 below. This oscillator was invented by Van der Pol while he was trying to find out a new way to model the oscillations of a self-maintained electrical circuit. The characteristic intensity-tension U_R of the nonlinear resistance (R) is given as:

$$U_R = -R_0\, i_0 \left[\frac{i}{i_0} - \frac{1}{3}\left(\frac{i}{i_0}\right)^3\right] \tag{1}$$

where i_0 and R_0 are the current and the resistance of the normalization respectively. This non linear resistance can be obtained by using the operational amplifier (op-amp). By applying the link's law to Fig. 1 below,

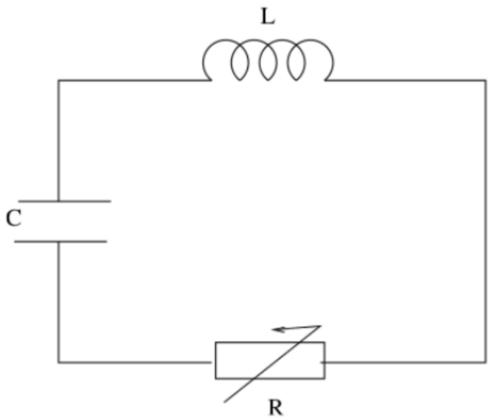

Fig. 1: Electric circuit modelizing the Van der Pol oscillator in an autonomous regime.

we have:

$$U_L + U_R + U_C = 0 \tag{2}$$

where U_L and U_C are the tension to the limits of the inductor and capacitor respectively and are defined as

$$U_L = L\frac{di}{d\tau} \tag{3}$$

$$U_C = \frac{1}{C}\int i d\tau. \tag{4}$$

Substituting (1), (3) and (4) in (2), we have:

$$L\frac{di}{d\tau} - R_0\, i_0 \left[\frac{i}{i_0} - \frac{1}{3}\left(\frac{i}{i_0}\right)^3\right] + \frac{1}{C}\int i d\tau = 0. \tag{5}$$

Differentiating (5) with respect to τ, we have

$$L\frac{d^2i}{d\tau^2} - R_0\left[1 - \frac{i^2}{i_0^2}\right]\frac{di}{d\tau} + \frac{i}{C} = 0. \quad (6)$$

Setting

$$x = \frac{i}{i_0} \quad (7)$$

and

$$t = \omega_e \tau \quad (8)$$

where $\omega_e = \frac{1}{\sqrt{LC}}$ is an electric pulsation, we have:

$$\frac{d}{d\tau} = \omega_e \frac{d}{dt} \quad (9)$$

$$\frac{d^2}{d\tau^2} = \omega_e^2 \frac{d^2}{dt^2}. \quad (10)$$

Substituting (9) and (10) in (6), yields

$$\frac{d^2x}{dt^2} - R_0\sqrt{\frac{C}{L}}\left(1 - x^2\right)\frac{dx}{dt} + x = 0. \quad (11)$$

By setting $\mu = R_0\sqrt{\frac{C}{L}}$ Eq. (11) takes dimensional form as follows

$$\ddot{x} - \mu\,(1 - x^2)\,\dot{x} + x = 0 \quad (12)$$

where μ is the scalar parameter indicating the strength of the nonlinear damping, and (12) is called the Van der Pol equation in the autonomous regime.

3 Analytical study

3.1 Fixed points and stability

Transforming the higher order ODE (12) into a system of simultaneous ODE's i.e. let $x_1 = x$ and $x_2 = \dot{x}$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \mu\,(1 - x_1^2)\,x_2 \end{bmatrix}. \quad (13)$$

Introducing the standard transformation

$$y = x \quad (14)$$

$$z = \dot{x} - \mu\left(x - \frac{x^3}{3}\right) \quad (15)$$

and letting

$$F(x) = \mu\left(\frac{x^3}{3} - x\right), \quad (16)$$

now

$$\dot{y} = \dot{x}. \quad (17)$$

Using (15) we have,

$$\dot{y} = z + \mu\left(y - \frac{y^3}{3}\right) \quad (18)$$

and

$$\dot{z} = \ddot{x} - \mu\dot{x}\,(1 - x^2)$$

$$\dot{z} = -\mu\,(y^2 - 1)\,\dot{x} - x - \mu\,(1 - y^2)\,\dot{x} = -x = -y. \quad (19)$$

This transformation puts the equation into the form:

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} z - \mu\left(\frac{y^3}{3} - y\right) \\ -y \end{bmatrix}. \quad (20)$$

Eq. (20) is the particular case of Lienard's Equation

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} z - f(y) \\ -y \end{bmatrix} \quad (21)$$

where $f(y) = \mu\left(\frac{y^3}{3} - y\right)$. Linearizing (20) around the origin i.e. fixed point (0,0), we have

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \mu & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}. \quad (22)$$

The characteristic equation of (22) is given as

$$\lambda^2 - \mu\,\lambda + 1 = 0 \quad (23)$$

with eigenvalues of

$$\lambda_\pm = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} \quad (24)$$

and eigenvectors of

$$\vec{e}_+ = \begin{bmatrix} \frac{-2}{\mu - \sqrt{(\mu^2 - 4)}} \\ 1 \end{bmatrix}, \quad \vec{e}_- = \begin{bmatrix} \frac{-2}{\mu + \sqrt{(\mu^2 - 4)}} \\ 1 \end{bmatrix}. \quad (25)$$

The stability of this fixed point depends on the signs of the eigenvalues of the Jacobian matrix (22).

3.2 Existence of the limit cycles

Let us now analytically study the amplitude of the limit cycle by using the average method [13]. Considering the following transformations

$$x(t) = A(t)\cos\left(t + \varphi(t)\right) = A\cos\psi \quad (26)$$

$$\dot{x}(t) = -A(t)\sin\left(t + \varphi(t)\right) = -A\sin\psi \quad (27)$$

where $A(t)$ is the amplitude, $\varphi(t)$ being the phase and with $\psi(t) = \varphi(t) + t$. Supposing the amplitude and phase feebly vary during a period $T = 2\pi$, we have the fundamental equations of the average method as follows:

$$\dot{A}(t) = -\frac{\mu}{2\pi}\int_0^{2\pi} f(A\cos\psi, -A\sin\psi)\sin\psi\,d\psi \quad (28)$$

$$\dot{\varphi}(t) = -\frac{\mu}{2\pi}\int_0^{2\pi} f(A\cos\psi, -A\sin\psi)\sin\psi\,d\psi \quad (29)$$

Eqs. (28) and (29) help to determine the amplitude $A(t)$ and phase $\varphi(t)$ of the oscillator. Applying this method to (12) for which

$$f(x, \dot{x}, t) = (1 - x^2)\,\dot{x}$$

then, we have

$$f(A, \psi) = -A \sin\psi + A^3 \sin\psi \cos^2\psi. \quad (30)$$

Substituting (30) into (28) and (29), we get

$$\dot{A}(t) = -\frac{\mu}{2\pi}\int_0^{2\pi}\left(-A\sin^2\psi + A^3\sin^2\psi\cos^2\psi\right)\mathrm{d}\psi \quad (31)$$

$$\dot{\varphi}(t) = -\frac{\mu}{2\pi}\int_0^{2\pi}\left(-A\sin\psi\cos\psi + A^3\sin\psi\cos^3\psi\right)\mathrm{d}\psi. \quad (32)$$

Integration of (31) and (32) gives the evolution equation of the amplitude $A(t)$ and the phase $\varphi(t)$:

$$\dot{A}(t) = -\frac{\mu A(t)}{2}\left(1 - \frac{A^2(t)}{4}\right) \quad (33)$$

$$\dot{\varphi}(t) = 0. \quad (34)$$

The average method states that the amplitude and the phase feebly vary during a period. Therefore $\dot{A}(t) = 0$, and the amplitude is eventually $A(t) = 2$.

4 Numerical solution

The numerical solution to the Van der Pol equation for various values of μ are presented in Figs. 2 to 4.

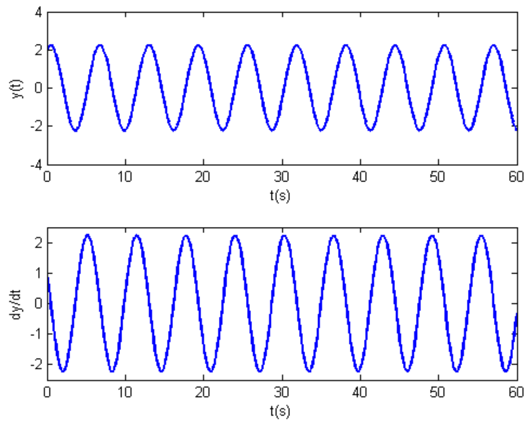

Fig. 2: Plot of $y(t)$ and dy/dt against $t(s)$ for $\mu = 0$.

5 Discussion

The classical Van der Pol equation (12) depends on the damping coefficient μ and the following varying behaviors were obtained. When $\mu < 0$, the system will be damped and the limit $\lim_{t\to\infty} \to 0$. From Fig. 2, where $\mu = 0$, there is no damping and we have a simple harmonic oscillator. From Figs. 3 and 4, where $\mu \geq 0$, the system will enter a limit cycle, with continuous energy to be conserved. The wave generated by this oscillator is periodic with sinusoidal form for $\mu \ll 1$ and relaxation for large value of μ [14] with fix amplitude equal to 2. Also when $-\infty < \mu \leq 0$ and $\lambda_\pm$ is $\mathrm{Re}(\lambda_\pm) < 0$, the point is stable; if $\mu = 0$ and $\lambda_\pm = \pm i$, the point is marginally stable and unstable; if $0 \leq \mu < \infty$ and $\lambda_\pm$ is $\mathrm{Re}(\lambda_\pm) > 0$, the origin is unstable. If $0 \leq \mu \leq 2$ and $\lambda_\pm$ is $\mathrm{Im}(\lambda_\pm) \neq 0$, then the fixed point (0,0) is an unstable center. If $2 < \mu < \infty$ and $\lambda_\pm$ is $\mathrm{Im}(\lambda_\pm) = 0$, then the fixed point (0,0) is still unstable.

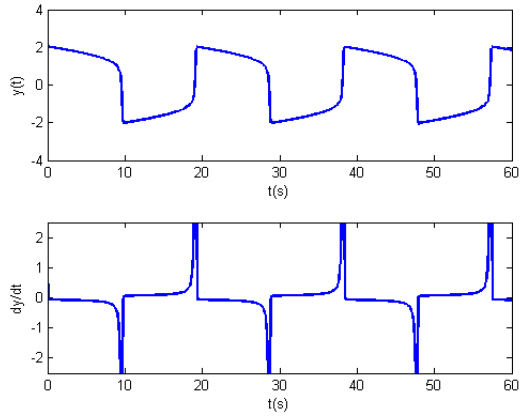

Fig. 3: Plot of $y(t)$ and dy/dt against $t(s)$ for $\mu = 10$.

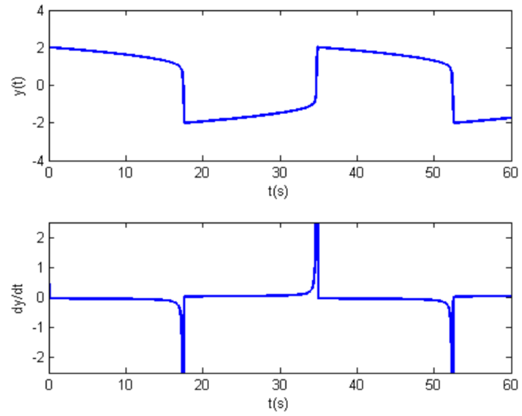

Fig. 4: Plot of $y(t)$ and dy/dt against $t(s)$ for $\mu = 20$.

6 Conclusion

In the above analysis, a class of analytical study of the Van der Pol equation in the autonomous regime is presented. Analytically, we conclude that the fixed point (0,0) is unstable whatever the value of the damping coefficient μ and the system enters a limit cycle with amplitude $A(t)$ of the Van der Pol oscillator limit cycle equal to 2. We showed that there exists a unique limit cycle.

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