Type III Spacetime with Closed Timelike Curves

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We present a symmetric spacetime, admitting closed timelike curves (CTCs) which appear after a certain instant of time, *i.e.*, a time-machine spacetime. These closed timelike curves evolve from an initial spacelike hypersurface on the planes $z = constant$ in a causally well-behaved manner. The spacetime discussed here is free from curvature singularities and a 4D generalization of the Misner space in curved spacetime. The matter field is of pure radiation with cosmological constant.

1 Introduction

One of the most intriguing aspects of Einstein's theory of gravitation is that solutions of field equations admit closed timelike curves (CTC). Presence of CTC in a spacetime leads to time-travel which violates the causality condition. The first one being Gödel's spacetime [1] which admits closed timelike curves (CTC) everywhere and an eternal time-machine spacetime. There are a considerable number of spacetimes in literature that admitting closed timelike curves have been constructed. A small sample would be [1–21]. One way of classifying such causality violating spacetimes would be to categorize the metrics as either eternal time-machine in which CTC always exist (in this class would be [1, 2]), or as time-machine spacetimes in which CTC appear after a certain instant of time. In the latter category would be the ones discussed in [18–20]. Many of the models, however, suffer from one or more severe drawbacks. For instance, in some of these solutions, for example [13, 14, 20], the weak energy condition (WEC) is violated indicating unrealistic matter-energy content and some other solutions have singularities.

Among the time-machine spacetimes, we mention two: the first being Ori's compact core [17] which is represented by a vacuum metric locally isometric to *pp* waves and second, which is more relevant to the present work, the Misner space [22] in 2D. This is essentially a two dimensional metric (hence flat) with peculiar identifications. The Misner space is interesting in the context of CTC as it is a prime example of a spacetime where CTC evolve from causally well-behaved initial conditions.

The metric for the Misner space [22]

$$ds^2_{Misn} = -2\,dt\,dx - t\,dx^2 \tag{1}$$

where $-\infty < t < \infty$ but the co-ordinate x is periodic. The metric (1) is regular everwhere as det $g = -1$ including at $t = 0$. The curves $t = t_0$, where t_0 is a constant, are closed since x is periodic. The curves $t < 0$ are spacelike, but $t > 0$ are timelike and the null curves $t = t_0 = 0$ form the chronology horizon. The second type of curves, namely, $t = t_0 > 0$ are closed timelike curves (CTC). This metric has been the subject of intense study and quite recently, Levanony and Ori [23], have studied the motion of extended bodies in the 2D Misner space and its flat 4D generalizations. A non-flat 4D spacetime, satisfying all the energy conditions, but with causality violating properties of the Misner space, primarily that CTC evolve smoothly from an initially causally well-behaved stage, would be physically more acceptable as a time-machine spacetime.

In this paper, we shall attempt to show that causality violating curves appear in non-vacuum spacetime with comparatively simple structure. In section 2, we analyze the spacetime; in section 3, the matter distribution and energy condition; in section 4, the spacetime is classified and its kinematical properties discussed; and concluding in section 5.

2 Analysis of the spacetime

Consider the following metric

$$\begin{aligned} ds^2 = 4\,r^2\,dr^2 + e^{2\,\alpha\,r^2}\left(dz^2 - t\,d\phi^2 - 2\,dt\,d\phi\right) + \\ + 4\,\beta\,z\,r\,e^{-\alpha\,r^2}\,dr\,d\phi \end{aligned} \tag{2}$$

where ϕ coordinate is assumed periodic $0 \leqslant \phi \leqslant \phi_0$, where α is an integer and $\beta > 0$ is a real number. We have used co-ordinates $x^1 = r$, $x^2 = \phi$, $x^3 = z$ and $x^4 = t$. The ranges of the other co-ordinates are $t, z \in (-\infty, \infty)$ and $0 \leqslant r < \infty$. The metric has signature $(+, +, +, -)$ and the determinant of the corresponding metric tensor $g_{\mu\nu}$, det $g = -4\,r^2\,e^{6\,\alpha\,r^2}$. The non-zero components of the Einstein tensor are

$$\begin{aligned} G^{\mu}_{\mu} &= 3\,\alpha^2, \\ G^{t}_{\phi} &= -\frac{1}{2}\,e^{-6\,\alpha\,r^2}\,\beta^2\,. \end{aligned} \tag{3}$$

Consider an azimuthal curve γ defined by $r = r_0$, $z = z_0$ and $t = t_0$, where r_0, z_0, t_0 are constants, then we have from the metric (2)

$$ds^2 = -\,t\,e^{2\,\alpha\,r^2}\,d\phi^2. \tag{4}$$

These curves are null for $t = 0$, spacelike throughout for $t = t_0 < 0$, but become timelike for $t = t_0 > 0$, which indicates the presence of closed timelike curves (CTC). Hence CTC form at a definite instant of time satisfy $t = t_0 > 0$.

It is crucial to have analysis that the above CTC evolve from a spacelike $t = constant$ hypersurface (and thus t is a time coordinate) [17]. This can be ascertained by calculating the norm of the vector $\nabla_\mu t$ (or by determining the sign of the component g^{tt} in the inverse metric tensor $g^{\mu\nu}$ [17]). We find from (2) that

$$g^{tt} = t\, e^{-2\alpha r^2} + \beta^2 z^2 e^{-6\alpha r^2} . \tag{5}$$

A hypersurface $t = constant$ is spacelike provided $g^{tt} < 0$ for $t = t_0 < 0$, but becomes timelike provided $g^{tt} > 0$ for $t = t_0 > 0$. Here we choose the z-planes defined by $z = z_0$, (z_0, a constant equal to zero) such that the above condition is satisfied. Thus the spacelike $t = constant < 0$ hypersurface can be chosen as initial conditions over which the initial may be specified. There is a Cauchy horizon for $t = t_0 = 0$ called Chronology horizon which separates the causal and non-causal parts of the spacetime. Hence the spacetime evolves from a partial Cauchy hypersurface (initial spacelike hypersurface) in a causally well-behaved manner, up to a moment, *i.e.*, a null hypersurface $t = 0$ and CTC form at a definite instant of time on $z = constant$ plane.

Consider the Killing vector $\eta = \partial_\phi$ for metric (2) which has the normal form

$$\eta^\mu = (0, 1, 0, 0) . \tag{6}$$

Its co-vector is

$$\eta_\mu = \left(2\beta z\, r\, e^{-\alpha r^2},\ -t\, e^{2\alpha r^2},\ 0,\ -e^{2\alpha r^2}\right) . \tag{7}$$

The (6) satisfies the Killing equation $\eta_{\mu;\nu} + \eta_{\nu;\mu} = 0$. For cyclicly symmetric metric, the norm $\eta_\mu \eta^\mu$ of the Killing vector is spacelike, closed orbits [24–28]. We note that

$$\eta^\mu \eta_\mu = -t\, e^{2\alpha r^2} \tag{8}$$

which is spacelike for $t < 0$, closed orbits (ϕ co-ordinate being periodic).

An important note is that the Riemann tensor $R_{\mu\nu\rho\sigma}$ can be expressed in terms of the metric tensor $g_{\mu\nu}$ as

$$R_{\mu\nu\rho\sigma} = k\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right) \tag{9}$$

where $k = -\alpha^2$ for the spacetime (2).

Another important note is that if we take $\beta = 0$, then the spacetime represented by (2) is maximally symmetric vacuum spacetime and locally isometric anti-de Sitter space in four-dimension. One can easily show by a number of transformations the standard form of locally isometric AdS_4 metric [29]

$$ds^2 = \frac{3}{(-\Lambda)\, x^2}\left(-dt^2 + dx^2 + d\phi^2 + dz^2\right) \tag{10}$$

where one of the co-ordinate ϕ being periodic.

3 Matter distribution of the spacetime and energy condition

Einstein's field equations taking into account the cosmological constant

$$G^{\mu\nu} + \Lambda\, g^{\mu\nu} = T^{\mu\nu} , \quad \mu, \nu = 1, 2, 3, 4 . \tag{11}$$

Consider the energy-momentum tensor of pure radiation field [30]

$$T^{\mu\nu} = \rho\, n^\mu\, n^\nu \tag{12}$$

where n^μ is the null vector defined by

$$n^\mu = (0, 0, 0, 1) . \tag{13}$$

The non-zero component of the energy-momentum tensor

$$T^t_{\ \phi} = -\rho\, e^{2\alpha r^2} . \tag{14}$$

Equating field equations (11) using (3) and (14), we get

$$\begin{aligned} \Lambda &= -3\,\alpha^2, \\ \rho &= \frac{1}{2}\,\beta^2\, e^{-8\alpha r^2}, \quad 0 \leqslant r < \infty . \end{aligned} \tag{15}$$

The energy-density of pure radiation or null dust decreases exponentially with r and vanish at $r \to \pm\infty$. The matter field pure radiation satisfy the energy condition and the energy density ρ is always positive.

4 Classification and kinematical properties of the spacetime

For classification of the spacetime (2), we can construct the following set of null tetrads $(k, l, m, \bar{m})$ as

$$k_\mu = (0, 1, 0, 0) , \tag{16}$$

$$l_\mu = \left(-2\beta z\, r\, e^{-\alpha r^2},\ \frac{t}{2}\, e^{2\alpha r^2},\ 0,\ e^{2\alpha r^2}\right) , \tag{17}$$

$$m_\mu = \frac{1}{\sqrt{2}}\,(2\,r, 0, i\, e^{\alpha r}, 0) , \tag{18}$$

$$\bar{m}_\mu = \frac{1}{\sqrt{2}}\left(2\,r, 0, -i\, e^{\alpha r^2}, 0\right) , \tag{19}$$

where $i = \sqrt{-1}$. The set of null tetrads above are such that the metric tensor for the line element (2) can be expressed as

$$g_{\mu\nu} = -k_\mu\, l_\nu - l_\mu\, k_\nu + m_\mu\, \bar{m}_\nu + \bar{m}_\mu\, m_\nu . \tag{20}$$

The vectors (16)–(19) are null vectors and are orthogonal except for $k_\mu\, l^\mu = -1$ and $m_\mu\, \bar{m}^\mu = 1$. Using this null tetrad above, we have calculated the five Weyl scalars

$$\begin{aligned} \Psi_3 &= -\frac{i\,\alpha\beta\, e^{-2\alpha r^2}}{2\sqrt{2}} , \\ \Psi_4 &= -\frac{1}{4}\,\beta\, e^{-2\alpha r^2}\left(i + 2\,\alpha\, z\, e^{\alpha r^2}\right) \end{aligned} \tag{21}$$

are non-vanishing, while $\Psi_0 = \Psi_1 = \Psi_3 = 0$. The spacetime represented by (2) is of type III in the Petrov classification scheme. Note that the non-zero Weyl scalars Ψ_3 and Ψ_4 are finite at $r \to 0$ and vanish as $r \to \pm\infty$ indicating asymptotic flatness of the spacetime (2). The metric (2) is free from curvature singularities. The curvature invariant known as Kretchsmann scalar is given by

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 24\,\alpha^4 \tag{22}$$

and the curvature scalar

$$R = -12\,\alpha^2 \tag{23}$$

are constant being non-zero.

Using the null tetrad (16) we have calculated the *Optical* scalars [30] the *expansion*, the *twist* and the *shear* and they are

$$\begin{aligned} \Theta &= \frac{1}{2}\, k^{\mu}_{\,;\mu} = 0\,, \\ \omega^2 &= \frac{1}{2}\, k_{[\mu\,;\,\nu]}\, k^{\mu\,;\,\nu} = 0\,, \\ \sigma\,\bar{\sigma} &= \frac{1}{2}\, k_{(\mu\,;\,\nu)}\, k^{\mu\,;\,\nu} - \Theta^2 = 0 \end{aligned} \tag{24}$$

and the null vector (16) satisfy the geodesics equation

$$k_{\mu\,;\,\nu}\, k^{\nu} = 0\,. \tag{25}$$

Thus the spacetime represented by (2) is non-diverging, has shear-free null geodesics congruence. One can easily show that for constant r and z, the metric (2) reduces to conformal Misner space in 2D

$$ds^2_{confo} = \Omega\, ds^2_{Misn} \tag{26}$$

where $\Omega = e^{2\,\alpha\, r^2}$ is a constant.

5 Conclusion

Our primary motivation in this paper is to write down a metric for a spacetime that incorporates the Misner space and its causality violating properties and to classify it. The solution presented here is non-vacuum, cyclicly symmetric metric (2) and serves as a model of time-machine spacetime in the sense that CTC appear at a definite instant of time on the z-plane. Most of the CTC spacetimes violate one or more energy conditions or unrealistic matter source and are unphysical. The model discussed here is free from all these problems and matter distribution is of pure radiation field with negative cosmological constant satisfying the energy condition.

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