

# Theoretical Study on Polarized Photon

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A solution of electromagnetic four-potential for polarized photon is obtained by solving its wave equations in elliptic cylindrical coordinates. An explicit energy wave function for the photon is presented in the form of a linear combination of the electric field and magnetic field from the solution. This wave function is used to calculate the angular momentum value of the photon. The elliptic coordinate parameter,  $a$ , for the photon is considered to be equal to a quarter of its wavelength.

## 1 Introduction

Photon as a quantum of light has attracted many researchers to develop explanations on its behaviors and to experiment to determine its properties. The photon as a fundamental wave-particle which moves at the speed of light serves like a messenger traveling from one place to another, which is necessary for the physical world to work properly. The classical view on light is provided by Maxwell's theory of electromagnetism [1], hence light is considered as a bundle of electromagnetic transverse waves. The particle view of light in modern physics may be provided by Einstein [2], so a photon has not only energy but also momentum. Work has been done to unify these two views. An expression for photon wave function is introduced by using the Riemann-Silberstein vector which is a linear combination of the electric field and magnetic field of the photon. An overview of the work on photon wave function is available in [3].

A photon has wave-particle duality which may be explained by a single entity as a joint wave-particle [4]. A more specific view on the electromagnetic structure for the photon is presented in [5], which is for circularly polarized photons. Hence the photon in circular polarization may be viewed as a charged moving electric capacitor with electric charge distributed circularly on its cylindrical surface of radius  $\lambda/2\pi$ , where  $\lambda$  is the wavelength of the photon.

In this article, we present our theoretical study on polarized photons. It is well known that polarized light has the property of certain orientation which may be generated by an optical polarizer. Recent experiment [6] shows that the transmission intensity of polarized light strongly correlates to the orientation of elliptically-shaped holes on the transmission plate. This as an example indicates that the transverse field strength of photons in the polarized light is not circularly distributed evenly as different from that of circularly polarized photons. The novelty of this article is on: the wave equation for the photon is solved within the elliptic cylindrical coordinates; an explicit photon energy wave function is presented based on the expression of Riemann-Silberstein vector wave function (in the next section); quantum expressions of the energy density, energy current density and the angular momentum or spin density for the photon are derived from the wave

function. We are not aware of such work in the literature.

This article is divided into the following sections: Introduction, Method, Results and Discussions, and Conclusion. The Introduction section provides a brief overview on our current understanding of the photon.

In the Method section, we will use similar method as in [5]. First we obtain a solution for the electromagnetic four-potential by solving the wave equations in elliptic cylindrical coordinates. The electromagnetic four-potential generally includes a scalar potential, which is an electric potential divided by the speed of light, and a vector potential. Then show to get the electric field and magnetic field from the solution of the four-potential; an explicit energy wave function for the photon is presented as a linear combination of the electric field and magnetic field; other expressions such as photon energy density, energy current density and angular momentum density are derived based on quantum mechanics.

In the Results and Discussions section we show the results for the photon expressions developed in the previous section, such as the four-potential, electromagnetic fields, the wave function, energy and energy current densities, and angular momentum for the photon; fairly detailed work is presented in evaluating the angular momentum value for the photon; some particularities are discussed. The Conclusion section provides a brief summary of the work presented in this article. We use MKS units in this work.

## 2 Method

In the space region where there are no other free electric charge and electric current, the electric potential  $\psi$  and the vector potential  $\mathbf{A}$  satisfy the following wave equations, respectively,

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0, \quad (1)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = 0, \quad (2)$$

where  $c$  is the speed of light,  $t$  is time,  $\nabla^2$  is the Laplacian operator, and  $\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$  is D'Alembert's operator which is also written as  $\square$ . In obtaining these equations the set of Maxwell equations with Lorenz gauge is employed. The Lorenz gauge

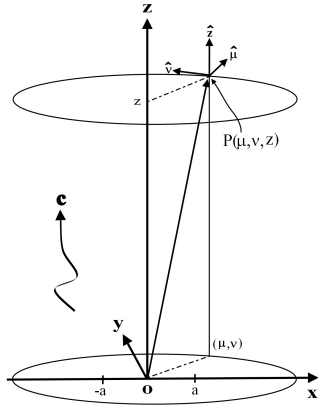


Fig. 1: A drawing of the elliptic cylindrical coordinate system together with the cartesian coordinates, where  $\hat{\mu}$ ,  $\hat{\nu}$ , and  $\hat{z}$  are unit vectors for the coordinate system and  $a$  is a length parameter that marks the focal points on  $x$  of the ellipse. The major axis of the ellipse is  $x$ . The wave symbol represents a photon moving in the direction of the positive  $z$  axis at the speed of light  $c$ .

is given by

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \psi}{\partial t} = 0. \tag{3}$$

Eqs. (1) and (2) are satisfied with solutions for traveling waves.

For the polarized photon, we solve (1) and (2) in elliptic cylindrical coordinates as shown in Fig. 1. Where the relationships between the cartesian and elliptic cylindrical coordinates are

$$\begin{aligned} x &= a \cosh \mu \cos \nu, \\ y &= a \sinh \mu \sin \nu, \\ z &= z, \end{aligned} \tag{4}$$

where  $x, y, z$  are cartesian coordinate values and  $\mu, \nu, z$  are elliptic cylindrical coordinate values,  $a$  is a length parameter which specifies the focal points of the ellipse,  $\mu \in (0, \infty)$  and  $\nu \in (0, 2\pi)$ . The value of  $a$  will be considered later to be proportional to the wavelength of the photon. The scale factors are

$$\begin{aligned} h_\mu &= h_\nu = a\gamma, \\ h_z &= 1, \end{aligned} \tag{5}$$

where  $\gamma = \sqrt{\sinh^2 \mu + \sin^2 \nu}$ .

We find for this particular case that the vector potential  $\mathbf{A}$  has a  $z$  component only so  $\mathbf{A} = \hat{z}A_z$  and  $\nabla^2 \mathbf{A} = \hat{z}\nabla^2 A_z$ , where  $\hat{z}$  is the unit vector for the  $z$  axis. The Laplacian operator  $\nabla^2$

for the elliptic cylindrical coordinates is expressed as

$$\nabla^2 = \frac{1}{a^2\gamma^2} \left( \frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2} \right) + \frac{\partial^2}{\partial z^2}. \tag{6}$$

Hence (1) and (2) in the elliptic cylindrical coordinates are satisfied with the following general solution:

$$f = f_0 e^{-\mu} \sin(\phi), \tag{7}$$

where  $f$  is a general quantity that may represent either  $\psi$  or  $A_z$  here,  $f_0$  is the corresponding constant,  $\phi = kz + \nu - \omega t$ , and  $k = \omega/c$ , and  $\omega$  is the angular frequency of the photon. We choose the “-” sign in the exponential function to make the solution to be limited in space. Here we let the photon travel in the  $z$  direction. And we arbitrarily choose the sine function here, one may choose cosine function as well but the results should be similar. By using the Lorenz gauge we have the following relationship for the electric potential constant,  $\psi_0$ , and the vector potential constant,  $A_0$ , as

$$A_0 = \psi_0/c. \tag{8}$$

Once we have the solution of the four-potential we can calculate [7] the electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$  using the following equations,

$$\mathbf{E} = -\nabla\psi - \frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{a\gamma} \left( \hat{\mu} \frac{\partial}{\partial \mu} + \hat{\nu} \frac{\partial}{\partial \nu} \right) \psi, \tag{9}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{a\gamma} \left( \hat{\mu} \frac{\partial}{\partial \nu} - \hat{\nu} \frac{\partial}{\partial \mu} \right) A_z, \tag{10}$$

where  $\hat{\mu}, \hat{\nu}$  are unit vectors for  $\mu$  and  $\nu$ , respectively, and “ $\times$ ” represents the vector cross operator. In deriving (9) for the electric field, we used this case relationship:

$$\frac{\partial \psi}{\partial z} + \frac{\partial A_z}{\partial t} = 0.$$

Both the electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$  are vectors with  $\mu$  and  $\nu$  components, which are perpendicular to the direction of the wave propagation. They represent transverse waves.

As we know, a photon is a packet of energy in electromagnetic field form and moves at the speed of light. This means that the electric field  $\mathbf{E}$  or the magnetic field  $\mathbf{B}$  of the photon can not exist alone and they are both like two faces of one body. We have the following expression of the electromagnetic field  $\mathbf{F}$  suit for the photon

$$\mathbf{F} = \frac{1}{\sqrt{2}} \left( \sqrt{\epsilon} \mathbf{E} + i \frac{\mathbf{B}}{\sqrt{u}} \right), \tag{11}$$

where  $\epsilon$  is the permittivity and  $u$  is the permeability in the space region where photon absorption is negligible, and  $i$  is

the imaginary unit. This expression is known as the Riemann-Silberstein vector and was introduced as a photon wave function in [8]. Here the choice of “+” sign for the imaginary part is arbitrary, one may choose “-” for similar results. Like  $\mathbf{E}$  or  $\mathbf{B}$ ,  $\mathbf{F}$  is also a vector which satisfies the wave equation and also represents a transverse traveling wave. The field  $\mathbf{F}$  is a complex vector in general and is characterized as a quantum vector wave function. Hence methods developed in quantum mechanics may be employed here [9]. By the dimensional analysis we know that  $\mathbf{F}$  represents an energy density wave function. In the following we use  $\mathbf{F}$  to derive expressions for energy and current densities and then the angular momentum for the photon. For clarity, the cartesian coordinates are used in the following work. We start from the wave equations:

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{F}}{\partial t^2} - \nabla^2 \mathbf{F} = 0, \quad (12)$$

and

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{F}^*}{\partial t^2} - \nabla^2 \mathbf{F}^* = 0, \quad (13)$$

where  $\mathbf{F}^*$  is the conjugate of  $\mathbf{F}$ . And

$$\mathbf{F} = \hat{\mathbf{x}}F_x + \hat{\mathbf{y}}F_y, \quad (14)$$

where  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  are unit vectors and  $F_x, F_y$  are the field components for  $x$  and  $y$  axes, respectively. As a transverse wave,  $\mathbf{F}$  has  $x$  and  $y$  components only and the  $z$  component,  $F_z$  is zero. Since our original solution for  $\mathbf{F}$  is in elliptic coordinates with components of  $\mu$  and  $\nu$ , we may convert those to  $x$  and  $y$  components using the following matrix multiplication,

$$\begin{bmatrix} \hat{\mu} \\ \hat{\nu} \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} \sinh \mu \cos \nu & \cosh \mu \sin \nu \\ -\cosh \mu \sin \nu & \sinh \mu \cos \nu \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{bmatrix}. \quad (15)$$

Since  $F_x$  and  $F_y$  are explicit functions of  $\mu$  and  $\nu$ , in order to do their derivatives with respect to  $x$  and  $y$  we need partial derivatives of  $\mu$  and  $\nu$  to  $x$  and  $y$  by using the following matrix form:

$$\begin{bmatrix} \delta\mu \\ \delta\nu \end{bmatrix} = \frac{1}{a\gamma^2} \begin{bmatrix} \sinh \mu \cos \nu & \cosh \mu \sin \nu \\ -\cosh \mu \sin \nu & \sinh \mu \cos \nu \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}, \quad (16)$$

where  $\delta$  is a tiny increment. In obtaining (16), we first do the tiny variations of (4) for  $x$  and  $y$  to  $\mu$  and  $\nu$  to get a conversion matrix between the two coordinate systems. And then find the inverse matrix as in (16). Eq. (15) is equivalent to (16) if we replace each variation together with its scale factor such as  $a\gamma$  in the latter equation by the corresponding unit vector.

As is common in quantum mechanics to find the energy density and the energy current density for the photon, we do this operation:

$$\mathbf{F}^* \cdot (12) - \mathbf{F} \cdot (13), \quad (17)$$

where “ $\cdot$ ” represents the dot product operator and “ $*$ ” is the complex conjugate symbol, and

$$\mathbf{F}^* \cdot \nabla^2 \mathbf{F} = \sum_i^{x,y,z} F_i^* \nabla^2 F_i, \quad (18)$$

where the summation is over the three cartesian components. By a few mathematical operations, we have the following form of energy current and density continuity equation:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \quad (19)$$

with

$$\mathbf{j} = \frac{c^2}{2i\omega} \sum_i^{x,y,z} (F_i \nabla F_i^* - F_i^* \nabla F_i) \quad (20)$$

and

$$\rho = \frac{1}{2i\omega} \left( \mathbf{F}^* \cdot \frac{\partial \mathbf{F}}{\partial t} - \mathbf{F} \cdot \frac{\partial \mathbf{F}^*}{\partial t} \right) = \mathbf{F} \cdot \mathbf{F}^*, \quad (21)$$

where  $\mathbf{j}$  is the energy current density and  $\rho$  is the energy density for the photon,  $\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$ . The photon propagation phase factor is  $e^{-i\phi}$  in this case (see next section) and  $\frac{\partial \mathbf{F}}{\partial t} = i\omega \mathbf{F}$ . The energy density  $\rho$  is positive.

Now the angular momentum increment for the photon is

$$d\mathbf{S} = \hat{\mathbf{z}} (xj_y - yj_x) \frac{dV}{c^2}, \quad (22)$$

where  $\mathbf{S}$  is the angular momentum vector or spin for the photon,  $j_x/c^2$  and  $j_y/c^2$  are the momentum densities in the  $x$  and  $y$  directions, respectively, and  $dV$  is the tiny volume in space. Notice that  $\mathbf{j}$  needs to be divided by  $c^2$  to be converted to the momentum density. The angular momentum for the photon in the present case has only the  $z$  component and zero  $x$  and  $y$  components. Eq. (22) may be rewritten in the form of spin momentum density as

$$\frac{d\mathbf{S}}{dV} = \frac{1}{c^2} (xj_y - yj_x). \quad (23)$$

In the next section we present results using relationships developed here and also provide discussions on the results.

### 3 Results and Discussions

To start this section we first present the mathematical solution of the four-potential for the polarized photon, which are two traveling wave functions, one for the electric potential  $\psi$ , which is a scalar, and the other for the vector potential  $\mathbf{A}$ . These functions are desirable since they are limited in space and show wave-particle duality with a limited length. These basic representations are important since, from which we may derive other physical quantities for the photon, such as electromagnetic fields and the spin angular momentum.

Now the solution for the four-potential in elliptic cylindrical coordinates is

$$\psi = \psi_0 e^{-\mu} \sin(\phi) \quad (24)$$

and

$$\mathbf{A} = \hat{\mathbf{z}}A_0 e^{-\mu} \sin(\phi), \quad (25)$$

where we assume that the photon travels in the  $z$  direction.

The vector potential in this case has only a  $z$  component. The choice of the sine function here is arbitrary, one may use the cosine function but the result should be similar since they only have a phase difference of  $\pi/2$ . Notice that (24) and (25) are in the same form with corresponding magnitude, and with the same phase change in both space and time. Since the physical meaning of the electric potential  $\psi$  is clear,  $c^2\mathbf{A}$  may be interpreted as an electric potential current or the total-electric-potential current density flowing in the same direction as the photon, which satisfy the continuity equation given by the Lorenz gauge condition (3). Hence the Lorenz gauge may be considered as the conservation of the total-electric-potential, a physical quantity of the integration of electric potential in the whole space. With the Lorenz gauge, we can get the relationship between the two constants as in (8)

Comparing with that of circularly polarized photons [5], the strength of the four-potential for the elliptically polarized photon decreases exponentially with  $\mu$  in the single space region, while the other is divided into two regions by a parameter  $r_0$  and decreases with  $1/r$  for  $r > r_0$ , where  $r$  is the radial value in polar cylindrical coordinates. As a result, the potential strength for the polarized photon with certain energy decreases quicker with distance from its center than that for circularly polarized photon, and hence the polarized photon may occupy less space.

Now we present expressions for the electric and magnetic fields using (9) and (10):

$$\mathbf{E} = \frac{\psi_0 e^{-\mu}}{a\gamma} [\hat{\mu} \sin(\phi) - \hat{\nu} \cos(\phi)], \quad (26)$$

and

$$\mathbf{B} = \frac{A_0 e^{-\mu}}{a\gamma} [\hat{\mu} \cos(\phi) + \hat{\nu} \sin(\phi)]. \quad (27)$$

These results of  $\mathbf{E}$  and  $\mathbf{B}$  show that they are transverse waves and are perpendicular to each other. The energy density in classical theory for the photon is

$$\rho = \frac{1}{2} \left( \epsilon \mathbf{E}^2 + \frac{\mathbf{B}^2}{u} \right) = \frac{\epsilon \psi_0^2}{a^2 \gamma^2} e^{-2\mu} \quad (28)$$

and the Poynting vector is

$$\mathbf{P} = \frac{\mathbf{E} \times \mathbf{B}}{u} = \hat{\mathbf{z}} \frac{c \epsilon \psi_0^2}{a^2 \gamma^2} e^{-2\mu}, \quad (29)$$

where, in converting  $A_0$ , we used (8). These quantities are finite in space and are physically meaningful. The magnitudes of these quantities decrease exponentially with  $2\mu$ . Since the factor  $a^2 \gamma^2$  is equal to the combination of scale factors for both  $\mu$  and  $\nu$ , it can be canceled in each space integration by the same volume factor as shown later. With the Poynting

vector, the photon may be viewed as a packet of energy moving at the speed of light along its propagation direction.

Since a photon is actually a quantum entity in modern physics view, we need an integral expression as (11). This is a linear combination of both the electric field and magnetic field for the elliptically polarized photon. Therefore we have a photon wave function. There are at least two advantages to have the wave function. First it can be used to calculate the value of the angular momentum for the photon; secondly it may be used to calculate the penetration probability for the photon in a sub-wavelength hole since in the view of quantum mechanics it represents the photon probability distribution. But in this article, we aim at the angular momentum value for the photon with the wave function.

In the following, we first obtain an explicit wave function using the developed expression in last section, (11), secondly derive the component expressions for energy current densities, and finally calculate the angular momentum value for the photon. This procedure has been first applied successfully to the circularly polarized photon. In this article, we report results on elliptically polarized photon.

By inserting results from (26) and (27) into (11), we have a photon wave function:

$$\mathbf{F} = \frac{\sqrt{\epsilon} \psi_0 e^{-i\phi}}{\sqrt{2} a \gamma} e^{-\mu} (i \hat{\mu} - \hat{\nu}). \quad (30)$$

Using the unit vector conversion (15), we have the cartesian components of  $\mathbf{F}$  as

$$F_x = \frac{\sqrt{\epsilon} \psi_0 e^{-i\phi}}{\sqrt{2} a \gamma^2} e^{-\mu} (i \sinh \mu \cos \nu + \cosh \mu \sin \nu), \quad (31)$$

$$F_x^* = \frac{\sqrt{\epsilon} \psi_0 e^{i\phi}}{\sqrt{2} a \gamma^2} e^{-\mu} (-i \sinh \mu \cos \nu + \cosh \mu \sin \nu), \quad (32)$$

$$F_y = i F_x, \quad (33)$$

and  $F_z$  is zero.

Due to the simple relationship between  $F_y$  and  $F_x$ , we have

$$F_y^* \nabla F_y = F_x^* \nabla F_x \quad (34)$$

and

$$F_y \nabla F_y^* = F_x \nabla F_x^*. \quad (35)$$

Hence in this case, (20) becomes

$$\mathbf{j} = \frac{c^2}{i\omega} (F_x \nabla F_x^* - F_x^* \nabla F_x) \quad (36)$$

and the work is reduced to one component. Furthermore since

$$F_x \nabla F_x^* = (F_x^* \nabla F_x)^*, \quad (37)$$

we have

$$\mathbf{j} = -\frac{2c^2}{\omega} \text{Im}(F_x^* \nabla F_x), \quad (38)$$

where “Im” means taking the real value of the imaginary part. And similarly, (21) becomes

$$\rho = \mathbf{F} \cdot \mathbf{F}^* = 2F_x \cdot F_x^* \tag{39}$$

Now insertimg (31) and (32) into (39), we have

$$\rho = \frac{\epsilon\psi_0^2}{a^2\gamma^2} e^{-2\mu}, \tag{40}$$

which is the same as that of (28) for photon energy density. Now we do integration of (40) in space with the tiny volume,  $dV = a^2\gamma^2 d\mu dv dz$ . Assuming the photon length is  $n\lambda$ , where  $\lambda$  is the wavelength of the photon and  $n$  may be a positive integer, but is not exactly determined in the present work. The result should be equal to the photon energy  $\hbar\omega$ , where  $\hbar$  is the reduced Planck constant. By doing that, we determine the electric potential constant to be

$$\psi_0 = \sqrt{\frac{2\hbar c}{\epsilon n}} \frac{1}{\lambda}. \tag{41}$$

Now we evaluate the energy current densities for the photon.  $F_x$  contains  $z$  explicitly in  $\phi$  of the exponential function, therefore the derivative with  $z$  is simple. We have  $\frac{\partial F_x}{\partial z} = -ikF_x$  and

$$j_z = \frac{c\epsilon\psi_0^2}{a^2\gamma^2} e^{-2\mu}, \tag{42}$$

which is consistent with the Poynting vector (29).

And from (38), we have

$$j_x = -\frac{2c^2}{\omega} \text{Im} \left( F_x^* \frac{\partial F_x}{\partial x} \right) \tag{43}$$

and

$$j_y = -\frac{2c^2}{\omega} \text{Im} \left( F_x^* \frac{\partial F_x}{\partial y} \right). \tag{44}$$

The work is now turned to calculate  $\frac{\partial F_x}{\partial x}$  and  $\frac{\partial F_x}{\partial y}$ . Because  $F_x$  contains explicit variables of  $\mu$  and  $\nu$ , we need the following equations to calculate the cartesian derivatives,

$$\frac{\partial F_x}{\partial x} = \frac{\partial F_x}{\partial \mu} \frac{\partial \mu}{\partial x} + \frac{\partial F_x}{\partial \nu} \frac{\partial \nu}{\partial x} \tag{45}$$

and

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_x}{\partial \mu} \frac{\partial \mu}{\partial y} + \frac{\partial F_x}{\partial \nu} \frac{\partial \nu}{\partial y}, \tag{46}$$

where  $\frac{\partial \mu}{\partial x}, \frac{\partial \nu}{\partial x}, \frac{\partial \mu}{\partial y}, \frac{\partial \nu}{\partial y}$  may be obtained from (16). We find that

$$\frac{\partial F_x}{\partial \mu} = \beta \left[ i \left( \cosh \mu - \sinh \mu - 2 \frac{\sinh^2 \mu \cosh \mu}{\gamma^2} \right) \cos \nu + \left( \sinh \mu - \cosh \mu - 2 \frac{\sinh \mu \cosh^2 \mu}{\gamma^2} \right) \sin \nu \right], \tag{47}$$

$$\frac{\partial F_x}{\partial \nu} = \beta \left[ -i \left( \cosh \mu + \sinh \mu + 2 \frac{\sinh \mu \cos^2 \nu}{\gamma^2} \right) \sin \nu + \left( \cosh \mu + \sinh \mu - 2 \frac{\cosh \mu \sin^2 \nu}{\gamma^2} \right) \cos \nu \right], \tag{48}$$

where  $\beta = \sqrt{\epsilon}\psi_0 e^{-\mu} e^{-i\phi} / \sqrt{2}a\gamma^2$ .

Now the cartesian derivatives are

$$\begin{aligned} \frac{\partial F_x}{\partial x} = \beta' & \left[ i \left( \cosh^2 \mu \sin^2 \nu - \sinh^2 \mu \cos^2 \nu + \sinh \mu \cosh \mu - \right. \right. \\ & \left. \left. - 2 \sinh \mu \cosh \mu \cos^2 \nu \frac{\sinh^2 \mu - \sin^2 \nu}{\gamma^2} \right) - \right. \\ & \left. - \sin \nu \cos \nu \left( 1 + 2 \sinh \mu \cosh \mu + \right. \right. \\ & \left. \left. + 2 \cosh^2 \mu \frac{\sinh^2 \mu - \sin^2 \nu}{\gamma^2} \right) \right], \tag{49} \end{aligned}$$

$$\begin{aligned} \frac{\partial F_x}{\partial y} = \beta' & \left[ i \sin \nu \cos \nu \left( 1 - 2 \sinh \mu \cosh \mu - \right. \right. \\ & \left. \left. - 2 \sinh^2 \mu \frac{\cosh^2 \mu + \cos^2 \nu}{\gamma^2} \right) + \right. \\ & \left. + \sinh^2 \mu \cos^2 \nu - \cosh^2 \mu \sin^2 \nu + \sinh \mu \cosh \mu - \right. \\ & \left. - 2 \sinh \mu \cosh \mu \sin^2 \nu \frac{\cosh^2 \mu + \cos^2 \nu}{\gamma^2} \right], \tag{50} \end{aligned}$$

where  $\beta' = \sqrt{\epsilon}\psi_0 e^{-\mu} e^{-i\phi} / \sqrt{2}a^2\gamma^4$ . These expressions are a little bit long but manageable. The purpose here is to serve as check points to guide the reader to the final correct results.

Using (43) and (44), we have

$$j_x = -\beta'' \sin \nu \left( \cosh \mu + \sinh \mu \frac{\cosh^2 \mu + \cos^2 \nu}{\gamma^2} \right), \tag{51}$$

$$j_y = \beta'' \cos \nu \left( \sinh \mu + \cosh \mu \frac{\sinh^2 \mu - \sin^2 \nu}{\gamma^2} \right), \tag{52}$$

where  $\beta'' = c^2\epsilon\psi_0^2 e^{-2\mu} / \omega a^3 \gamma^4$ .

Now using (23), we have

$$\begin{aligned} \frac{dS}{dV} = \alpha e^{-2\mu} & \left( \frac{\sinh \mu \cosh \mu}{\gamma^2} + \right. \\ & \left. + \frac{\sinh^2 \mu \cosh^2 \mu - \sin^2 \nu \cos^2 \nu}{\gamma^4} \right), \tag{53} \end{aligned}$$

where  $\alpha = \epsilon\psi_0^2 / \omega a^2 \gamma^2$ .

To calculate the spin value, we integrate (53) in the whole space. There are two parts to be integrated on the right hand side of the equation. This integration is a bit challenging since each integration part is divergent at  $\mu = 0$  and  $\nu = 0, \pi$ . To avoid this problem we work around by first doing the integration of the second part which fortunately produces an exact

term to cancel the first part and the remaining is finite and manageable. We now show the integration of the second part:

$$\begin{aligned}
 I &= \int_0^{n\lambda} \int_0^\infty \int_0^{2\pi} dz d\mu dv \\
 &\quad e^{-2\mu} \frac{\sinh^2 \mu \cosh^2 \mu - \sin^2 v \cos^2 v}{\gamma^4} \\
 &= n\lambda \int_0^\infty \int_0^{2\pi} d\mu dv \\
 &\quad e^{-2\mu} \frac{\sinh^2 \mu \cosh^2 \mu - \sin^2 v \cos^2 v}{\gamma^4},
 \end{aligned} \tag{54}$$

where the scale factors in the integration volume are canceled within the  $\alpha$  factor and we omit the rest of the constants here for simplicity. This integration may be further separated into sub-integration as

$$I_1 = \int_0^\infty \int_0^{2\pi} e^{-2\mu} \frac{\sinh^2 \mu \cosh^2 \mu}{(\sinh^2 \mu + \sin^2 v)^2} d\mu dv \tag{55}$$

and

$$I_2 = \int_0^\infty \int_0^{2\pi} e^{-2\mu} \frac{\sin^2 v \cos^2 v}{(\sinh^2 \mu + \sin^2 v)^2} d\mu dv. \tag{56}$$

These may be done by the partial integration method: for (55) first integrate with  $\mu$  and for (56) first integrate with  $v$ . Hence we have

$$\begin{aligned}
 I_1 &= - \int_0^\infty \int_0^{2\pi} e^{-2\mu} \frac{\sinh \mu \cosh \mu}{\sinh^2 \mu + \sin^2 v} d\mu dv + \\
 &\quad + \frac{1}{2} \int_0^\infty \int_0^{2\pi} e^{-2\mu} \frac{\sinh^2 \mu + \cosh^2 \mu}{\sinh^2 \mu + \sin^2 v} d\mu dv - \\
 &\quad - \frac{1}{2} \left[ e^{-2\mu} \sinh \mu \cosh \mu \int_0^{2\pi} \frac{dv}{\sinh^2 \mu + \sin^2 v} \right]_0^\infty
 \end{aligned} \tag{57}$$

and

$$I_2 = \frac{1}{2} \int_0^\infty \int_0^{2\pi} e^{-2\mu} \frac{\cos^2 v - \sin^2 v}{\sinh^2 \mu + \sin^2 v} d\mu dv. \tag{58}$$

Now the last integration term in (57) is zero at both  $\mu = 0$  and  $\mu \rightarrow \infty$ . Hence (54) becomes

$$\begin{aligned}
 I &= -n\lambda \int_0^\infty \int_0^{2\pi} e^{-2\mu} \frac{\sinh \mu \cosh \mu}{\sinh^2 \mu + \sin^2 v} d\mu dv + \\
 &\quad + n\lambda \int_0^\infty \int_0^{2\pi} e^{-2\mu} d\mu dv,
 \end{aligned} \tag{59}$$

where the second integration term is the second integration term of (57) minus that of (58). And finally by finishing the second integration we have

$$I = - \int_0^{n\lambda} \int_0^\infty \int_0^{2\pi} e^{-2\mu} \frac{\sinh \mu \cosh \mu}{\gamma^2} dz d\mu dv + n\lambda\pi. \tag{60}$$

The first integration term in (60) cancels exactly the integration of the first part in (53) so the angular momentum for the photon is

$$S = \frac{\epsilon\psi_0^2}{\omega} n\lambda\pi = \hbar, \tag{61}$$

where we used (41). The value of spin or the angular momentum calculated here for the elliptically polarized photon is indeed  $\hbar$ .

Before concluding this section we consider the elliptic coordinate parameter  $a$  for the photon. The divergence of the electric field (26) is zero everywhere except at the two focal points ( $x = \pm a$ ). This leads us to believe that electricity may only exist in these two focal points formed traveling lines. To further consider the value of  $a$  we take a look at that for circularly polarized photon [5]. In that case the electromagnetic field occupies two space regions divided by  $r_0$  with the center core region carrying zero angular momentum for spin one. The elliptically polarized photon may be understood as transformed from the circularly polarized photon with its core region collapsed by its energy popped out without change in its length of circumference. If that is the case then  $a = \lambda/4$ .

#### 4 Conclusion

To conclude this article we summarize what has been presented here. First, we have solved the wave equations for the electromagnetic four-potential in the elliptic cylindrical coordinates for the polarized photon. The solution for each potential is an electromagnetic traveling wave and its transverse strength decreases exponentially with  $\mu$ . These expressions for the four-potential are simple but essential representations since they may be used to obtain other physical quantities for the polarized photon.

We first obtained the electric field and magnetic field for the photon from the four-potential solution. Then we have presented the energy wave function explicitly, which is a linear combination of the electric field and magnetic field. Using concepts from quantum mechanics, we first derived expressions then evaluated for photon energy, energy current, and angular momentum densities. Work is shown particularly in calculating the value of the angular momentum or spin for the photon. Considerations are given about the value of the elliptic coordinate parameter  $a$  which may be equal to a quarter of the photon wavelength.

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