The Curse of Dimensionality in Physics

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The curse of dimensionality is a well-discussed issue in mathematics. Physicists also require n-dimensional space but because of the phase space choice, there is no need to worry about its consequences. This issue connected with dimensionality and related problems are discussed in this paper.

1 Introduction

We live in a 3-dimensional world and any dimension beyond this is called hyper-dimension. In the early decades of the 19th century, many articles were published listing works on "hyper-volume and surface" n-dimensional geometry. Swiss mathematician Ludwig Schlafli wrote a treatise on the subject in the early 1850's [1]. In 1858, a short description of this was translated into English by Arthur Cayley which gives the volume formula for an n-ball, commenting that it was determined long ago. In this paper, there were footnotes citing papers published in 1839 and 1841 [2] by the mathematician Eugene Catalan regarding descriptive geometry, number theory, *etc*. Though the earliest works encountered problems in computations, it was William Kingdon Clifford who published a solution in 1866 [3]. In the 1897 thesis, Heyl derived formulas for both volume and surface area and gives a clear idea of multidimensional geometry [4]. In 1911, Duncan Sommerville published a bibliography of non-Euclidean and n-dimensional geometry [5] giving the details on the works on hyper-sphere volumes. A book *An Introduction to the Geometry of N Dimensions* [6], by Duncan Sommerville, was published in 1929 which explains the n-ball formula and has a table of values for dimensions 1 to 7. In this paper, in the first section, we will give the formula for hyper-volume whose derivation is available in many statistical mechanics textbooks [7, 8]. In the other sections, we will discuss the so called "curse of dimensionality" and its consequences.

2 Hyper-volume

The n-dimensional volume of an Euclidean ball of radius R in n-dimensional Euclidean space [9] is

$$V_n(R) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} R^n \,, \tag{1}$$

where Γ is Euler's gamma function. The gamma function extends the factorial function to non-integer arguments. The volume of an n-dimensional sphere depends on the radius of the sphere (if we are considering the momentum space, the radius will be momentum) and the number of degrees of freedom. Now we want to know how the variation in n and R affects volume. For that, in the next section, we will numerically evaluate the variation of hyper-volume with increasing n for different radius.

3 The curse of dimensionality

We are all accustomed to live in low dimension spaces, mostly up to three dimensions. But relativity says we live in four dimensions [10] where the fourth dimension is time. String theory uses about ten dimensions [11, 12]. Our intuition about space can be misleading in high dimensions, rather more surprises awaits us there. Consider the case of an n dimensional sphere, and let us evaluate the volume for different dimensions for radius $R = 1$ and $R = 1.5$ which are given in Table 1. Initially an increase in volume is observed but later, volume decreases dramatically and almost approaches to zero at higher and higher dimensions. This effect is called the "curse of dimensionality" [13], often described as a phenomenon that arises when studying and using high-dimensional spaces. For $R = 1$, we can see that after reaching 5.26 the volume decreases, whereas for $R = 1.5$, after reaching 177.22, the volume decreases. These numbers depend on how the ratio $\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}+1)$ changes with n. Richard Bellman was the one who coined the term in 1957 [14, 15] when considering problems in dynamic programming.

In Fig. 1, we plot a graph with n along the x-axis and volume along the y-axis for (R = 1, 1.05, 1.10, 1.15, 1.20). In the graph, we can see that the volume first increases with n, reaches a maximum value for a particular value of n, called n_{max}. If we increase n further, the volume decreases. We can see that n_{max} shifts towards the right when R increases. All plots show that the volume of the n-ball vanishes to nothing as n approaches infinity.

4 What is really happening to the volume for large n?

First, we will check how the dimension will be influenced by the radius R. Taking the logarithm of the expression for the n-dimensional volume and applying Stirling's approximation in (1), we get

$$\ln V_n(R) \simeq \frac{n}{2}\,\ln\pi + n\ln R - \frac{n}{2}\ln\frac{n}{2} + \frac{n}{2}\,. \tag{2}$$

Dimension n	V_n for $R = 1$	V_n for $R = 1.5$
0	1	1
1	2	3
2	3.14	7.06
3	4.19	14.13
4	4.94	24.98
5	5.26	39.97
6	5.17	58.86
7	4.73	80.72
8	4.06	104.02
9	3.30	126.80
10	2.55	147.05
11	1.88	162, 97
12	1.33	173.24
13	0.91	177.22
14	0.59	174.94
15	0.38	167.03

Table 1: Values of hyper-volumes for $R = 1$ and $R = 1.5$.

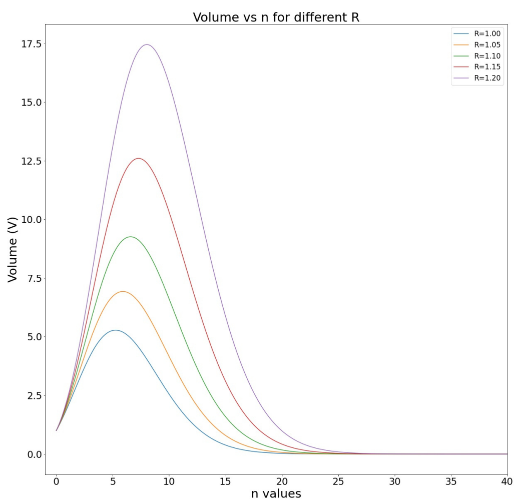


Fig. 1: A graph between n and volume for $R = 1, 1.05, 1.10, 1.15, 1.20$.

To find when the volume will decrease for different R, we take the derivative with respect to n of the above equation which gives

$$\frac{1}{V_n(R)}\frac{dV_n(R)}{dn} \simeq \frac{1}{2}\ln\pi + \ln R - \frac{1}{2}\ln\frac{n}{2}\,. \tag{3}$$

In order for the volume to be a maximum, $\frac{dV_n(R)}{dn}$ must be zero for a particular n. Hence we obtain

$$n_{max} \simeq 2\pi R^2\,. \tag{4}$$

This relation of n_{max} for various R has a parabolic-type dependence which means the radius has no role in the decrease of volume. Next, we will find out what is happening to the volume for large n. There are arguments to show that data confined in the volume will be spreading to an outer shell for large n [16, 17]. Let us check whether this is true or not. For a sphere with radius ΔR less than R, the volume will be

$$V_n(R) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}(R-\Delta R)^n\,. \tag{5}$$

The volume of the shell will be given by subtracting (1)–(5). We calculated the volumes of n-dimensional spheres and shells for different n which is given in Table 2. A graph is also plotted with n along the x-axis and volumes of n-dimensional sphere and shell along the y-axis as in Fig. 2.

Dimension n	$V_n(R)$	$V_n(\Delta R)$
4	4.93	1.69
5	5.26	2.15
10	2.55	1.66
15	0.38	0.30
20	0.02	0.02
99	9.47×10^{-40}	9.47×10^{-40}
100	2.36×10^{-40}	2.36×10^{-40}

Table 2: Values of volumes of n-dimensional sphere and shell for different n.

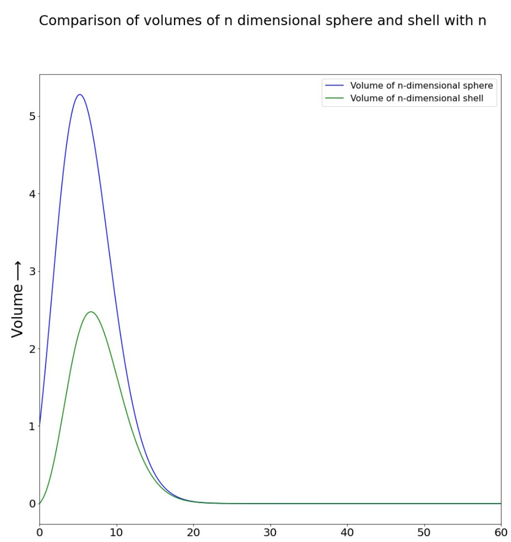


Fig. 2: A graph between n and volumes of n-dimensional sphere and shell.

Initially the volume of the shell is much less than the volume of the sphere. As n increases, both volumes decrease and become equal. We also found the percentage change in volume of the sphere to shell. The fraction of volume contained in the shell with thickness ΔR will be equal to

$$\text{Fractional volume} = \frac{\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}R^n - \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}(R-\Delta R)^n}{\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}R^n}.$$

On simplification,

$$\text{Fractional volume} = 1 - \left(1 - \frac{\Delta R}{R}\right)^n . \tag{6}$$

For $R = 1, \Delta R = 0.1$ and $R = 2, \Delta R = 0.1$, the fractional volumes in percentage are given in Table 3. All these show that the popular concept that the volume content is spreading into the surface area is not correct. Hence the curse of dimensionality remains unchanged.

Dimension	$R = 1$	$R = 2$
1	10	5
5	40.95	22.62
10	65.13	40.12
15	79.41	53.67
20	87.84	64.15
299	100	99.99
300	100	99.99

Table 3: Values of percentage of fractional volume for different n.

5 How Physicists overcome the curse using Statistical Mechanics

Statistical Mechanics (SM) provides the basis for many important branches of physics, including atomic and molecular physics, solid state physics, biophysics, astrophysics, environmental and socioeconomic physics. In statistical mechanics, we are interested in finding the thermodynamic properties of a system using n-dimensional space [18, 19]. It involves number of particles of the order of 10^{23} which are in continuous movement and hence have a continuous transformation in their position and momenta. So in order to predict the properties, we need to have information about all the possible values of position and momentum. For this, we construct a new space called a "phase space" which is a fusion of momentum and position spaces which is a six-dimensional space for N particles. In this space, the bridging equation to find the properties was given by Boltzmann [20, 21] as

$$S = k \ln \Omega \tag{7}$$

where S is the entropy, k is the Boltzmann constant and Ω is the number of available states in phase space which is given by [7, 22]

$$\Omega = \frac{V^{\frac{n}{3}} \pi^{\frac{n}{2}}}{h^n \, \Gamma\left(\frac{n}{2}+1\right)} R^n, \tag{8}$$

where V is the spatial volume and h is Planck's constant. Momentum volume of $3N$-dimension is [7, 22]

$$V_n = V_{3N} = \frac{\pi^{\frac{3N}{2}} R^{3N}}{\Gamma\left(\frac{3N}{2}+1\right)} = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma\left(\frac{n}{2}+1\right)} . \tag{9}$$

In SM, we never have V_{3N} alone. We have both spatial volume V and momentum volume V_{3N} such that the total volume

$$V_{Total} = V^N V_{3N} . \tag{10}$$

But SM requires only Ω, the number of micro-states. Substituting for V_{Total}, we find the number of micro-states as

$$\Omega = \frac{V_{Total}}{h^{3N} N!} = \frac{V^N V_{3N}}{h^{3N} N!} = \frac{V^N \pi^{\frac{3N}{2}} R^{3N}}{h^{3N} N! \Gamma\left(\frac{3N}{2}+1\right)} \tag{11}$$

where $N!$ is used to avoid Gibbs paradox [7]. Simple calculations show that the number of micro-states (Ω) goes to infinity even for N just above 3 (Ω is of the order of 10^{1000} for $N = 100$). But because of the bridging equation, we require only $\ln \Omega$ and for that we carry out the following steps. Let us consider non-relativistic classical particles with energy $E = p^2/2m$. Then we have the radius of the momentum sphere $R = p = \sqrt{2mE}$ and we get

$$\Omega = \frac{\left(\frac{V}{h^3}\right)^N (2\pi m E)^{\frac{3N}{2}}}{N! \, \Gamma\left(\frac{3N}{2}+1\right)} . \tag{12}$$

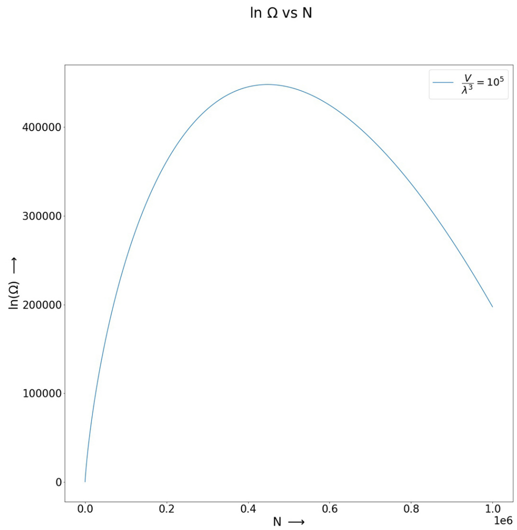
Applying Stirling's approximation and carrying out suitable simplifications we arrive at

$$\ln \Omega \simeq N \ln \frac{V}{\lambda^3} - N \ln N + \frac{5}{2} N \tag{13}$$

where λ is the de Broglie wavelength. So we plot a graph between $\ln \Omega$ and N as in Fig. 3. The first graph shows a nonlinear variation because our choice of V/λ^3 is not realistic. In practice V/λ^3 will be always greater than 10^{25} and hence $\ln \Omega$ will be always proportional to N. This shows that in SM there is no need to worry about the decrease in volume of the n-dimensional space and we are not affected by the curse of dimensionality.

6 Conclusion

In statistical mechanics, in micro-canonical ensembles, we use the hyper-dimensional space to find the thermodynamic properties of a system. There are much literature [16, 17, 23, 24] showing that hyper-dimensional volume vanishes at large dimension or for large N. But this does not affect the properties of a system, which remains a paradox for physicists. This paradox is resolved in this paper. In SM, the classical particles are always in motion and hence to specify them we require both position and momentum simultaneously, which results in the phase space. We showed that because of the choice of the phase space, the curse of hyper-dimension is not affecting the properties and calculations in SM.



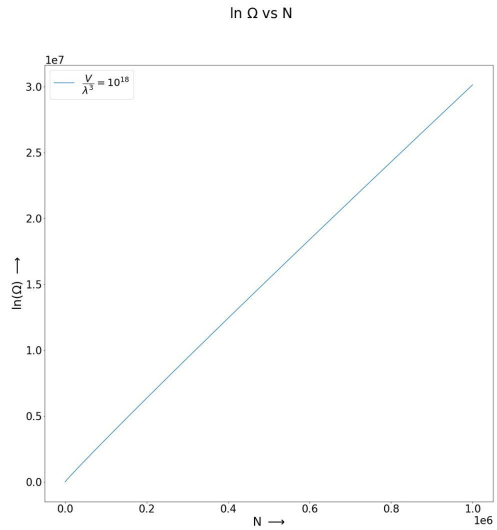


Fig. 3: A graph between $\ln\Omega$ and N for different V/λ^3.

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