A Derivation of Planck's Constant from the Principles of Electrodynamics

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A formula for Planck's constant is derived from the Bohr model and Larmor formula, leading to its expression as a function of the proton-to-electron mass ratio, the elementary charge of an electron, and variables as the speed of light and vacuum permittivity. While Planck's constant obtained from its theoretical formula deviates from the Committee on Data of the International Science Council (CODATA) value by a tiny epsilon due to modelling assumptions or geometrical aspects, 98.6% of this deviation is explained by the relativistic effect of electron mass and the mass gap due to the binding energy of electron. As such the relative error of Planck's constant adjusted for the aforementioned factors remains about 22.2 parts per million.

1 Introduction

The Planck's constant known as quantity h, is a fundamental constant in physics of importance in quantum mechanics, statistical mechanics, electronics and metrology. The constant h appears in Max Planck's work on black-body radiation and its spectrum [11–13], a collaborative effort on Kirchhoff's law. In 1905, Einstein publishes the photoelectric effect for the measurement of quantized energies of photons $E = h\nu$, where ν is the frequency of electromagnetic waves [3]. The photoelectric experiment is conducted inside a vacuum chamber exposed to light at different frequencies, causing electrons to be ejected from a metal plate. Einstein's photoelectric relation expresses the kinetic energy of ejected electrons by the relation $eV = h\nu - w$, where w is the work function of the metal, representing the energy level that electromagnetic waves must exceed to eject electrons from the plate. Early photoelectric experiments by Hughes [14] and Richardson and Compton [4], yield estimates of h/e with uncertainties of about 10%. As Millikan refined the experiment, he obtained a value of $h = 6.57 \times 10^{-34}$ J s [9].

The Kibble balance, formerly called a watt balance, is a metrological instrument to measure the weight of a tiny object very precisely by the electric current and voltage powering the balance. This instrument, developed in 1975 by Bryan Kibble, is used to measure Planck's constant on the basis of the Josephson and quantum Hall effect. The Josephson effect, is described by the set of equations $I(t) = I_c \sin(\varphi(t))$ and

$$\frac{\partial \varphi}{\partial t} = \frac{2\,e}{\hbar} V(t)\,,$$

where $V(t)$ and $I(t)$ are the voltage and current flowing through the Josephson junction, I_c the critical current, $\hbar$ the reduced Planck's constant, and e the elementary charge. A Josephson junction is a superconducting tunnel junction made of a thin film of a few micrometers separating superconducting wires [5, 6], whereas the Hall effect is produced by a current flowing through a conductor exposed to a magnetic field perpendicular to the current. The method exploits discretised jumps in the resistivity computed as

$$R = \frac{V_{Hall}}{I_{ch}} = \frac{h}{e^2\,\nu},$$

where V_{Hall} is the Hall voltage and I_{ch} the channel current, e the elementary charge, and h Planck's constant. The divisor ν can be an integer $\nu = 1, 2, 3, ..$ or a fractional number $\nu = 1/3, 2/5, 3/7, ..$ producing jumps as the density of electrons varies. An example of such quantization are Landau levels representing discretised energies as a proposed solution to the Schrödinger's equation [7].

A Planck's constant of $h = 6.62607034(12)\times10^{-34}$ J s was obtained in recent work by a team of researchers using a watt balance to demonstrate its capability [15]. The joule balance is an enhanced watt balance where dynamic measurements are replaced by a static measurement for convenience purpose. The performance of the joule balance was demonstrated by measuring Planck's constant, $h = 6.626104(59) \times 10^{-34}$ J s with an 8.9 ppm uncertainty [18]. A detailed view of the historical development of Planck's constant measurements is provided in Reiner [16].

In the present work, a formula for Planck's constant was obtained from the Bohr model and Larmor formula, see Section 2. The coupling between both models into a single expression for quantity h involves a membrane representation of the electron as a surface covering the Bohr sphere, where the flux of energies radiating across the membrane is determined by the mass of the proton.

Table 1: Fundamental constants from Committee on Data for Science and Technology (CODATA), 2014 [10].

Constant	Symbol	Value	Unc. u *
Planck constant	h	$6.626070040(81) \times 10^{-34}$ J s	8.7×10^{-8}
Electron mass	m_e	$9.10938356(11) \times 10^{-31}$ kg	8.8×10^{-8}
Proton mass	m_p	$1.672621898(21) \times 10^{-27}$ kg	8.9×10^{-8}
Elementary charge	q, e	$1.602176620(89) \times 10^{-19}$ C	4.4×10^{-8}
Vacuum permittivity	ε_0	$8.8541878128(13) \times 10^{-12}$ F / m	–
Speed of light	c	299 792 458 m/s	–

SI units, Intern. Committee for Weights and Measures.
* u, means relative standard uncertainty, source [17].

Planck's constant predicted by the present model and its deviation from CODATA (see Table 1) are provided in Section 3. The attribution of errors by modeling assumptions is described at the end. Of the deviation, 98.6% is explained by the relativistic effect of electron mass and mass gap due to the binding energy of an electron in its orbital, which is a fairly promising result.

2 Method

2.1 Larmor formula

The Larmor formula expresses the power radiated by a non-relativistic charged particle as a result of acceleration [8]. The Larmor formula in its current form appears in more recent works, see the Bremsstrahlung effect and the study of electromagnetic radiation emitted in cyclotrons. The electromagnetic wave as a bimodal function is often represented as a tuple of two undulatory waves moving in the same direction, where functions in Hilbert space $\mathcal{L}^2$ are orthogonal by the inner product. The magnetic cardioid or lemiscate are geometric representations, involving the interaction between an electron and an electric field. These basics of currents and electromagnetism are useful wave representation of the electron. The magnetic field, commonly denoted by the letter B, is represented by an E-field in the current context. Such an E-field is denoted as E_θ, where θ is the angle between the radial electric field E_r and the orientation of E_θ itself.

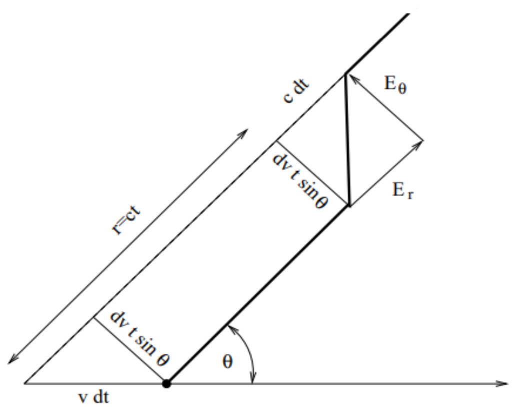

Fig. 1: E-field in the region of an electromagnetic pulse in polar coordinates.

As we suppose the E-field is proportional to the inverse of the wave frequency, where the ratio of wave frequencies is equal to the ratio of velocities, we have $\frac{E_\theta}{E_r} = \frac{v_r}{v_\theta}$. By the Pythagorean theorem, we get:

$$\frac{E_\theta}{E_r} = \frac{\Delta v\, t\, \sin(\theta)}{c\, \Delta t}\,, \tag{1}$$

where E_θ and E_r are the tangential and radial components of the E-field respectively, c the speed of light, and t the time of a pulse Δt. From equality $v/t = dv/dt$, we get $v\Delta t = t\Delta v$. By definition, t is the time to accelerate a charged particle q from rest to velocity v.

The radial component of an E-field as in Coulomb's law, is expressed as follows:

$$E_r = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\,, \tag{2}$$

where r is the radius, q the charge of the particle and ε_0 the vacuum permittivity.

Given the acceleration term $a = \frac{\Delta v}{\Delta t}$ and joint relation $r = c\,t$, (1) and (2) lead to:

$$E_\theta = \frac{q\,a}{4\pi\varepsilon_0\, c^2 r}\,\sin(\theta)\,. \tag{3}$$

By Poynting's theorem, i.e. $S = c\varepsilon_0\, E^2$, the flux is expressed as:

$$S = \frac{1}{16\,\pi^2\, c^3\, \varepsilon_0\, r^2}\, q^2\, a^2\, \sin^2\theta\,. \tag{4}$$

The angular element in spherical coordinates is

$$d\Omega = r^2 \sin\theta\, d\theta\, d\varphi\,,$$

leading to the below expression for the power radiated by an electron:

$$P = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} S\, r^2 \sin\theta\, d\theta\, d\varphi\,. \tag{5}$$

As

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \sin^3\theta\, d\theta\, d\varphi = \frac{8\pi}{3}\,,$$

we obtain:

$$P = \frac{8\pi}{3}\frac{q^2 a^2}{16\pi^2 c^3 \varepsilon_0}\,, \tag{6}$$

which is the Larmor formula for the power radiated by a particle of charge q under acceleration a, in say Watt per squared steradians where the variables in the argument are expressed in the International System of Units (SI).

2.2 Thomson cross section to Planck formula

Considering an E-field where the field lines are collinear and pulsed in the direction orthogonal to the electron orbital, the energy flux over a cross section σ_e transverse to the power inflow, is given by:

$$P_{in} = c\varepsilon_0 E_r^2 \sigma_e\,, \tag{7}$$

where the energy flux is the speed of light times the energy density as given by Poynting's theorem.

The power radiated by a ground state electron revolving around a nucleus is given by the Larmor formula, which can be expressed as follows:

$$P_{out} = \frac{8\pi}{3}\frac{q^2\,(q\,E_r/m_e)^2}{16\pi^2\, c^3\, \varepsilon_0}\,. \tag{8}$$

As $P_{in} = P_{out}$, (7) and (8) lead to the well-known Thomson cross section for a free electron in its orbital:

$$\sigma_e = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_0\, m_e\, c^2}\right)^2 . \tag{9}$$

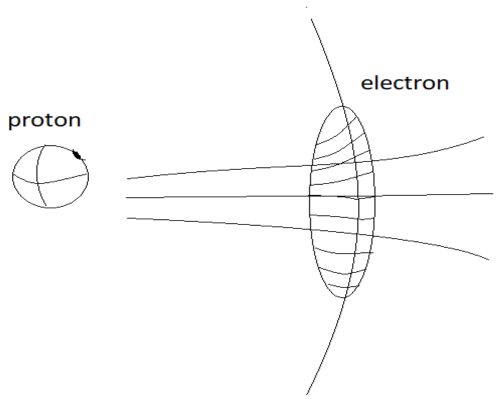

Fig. 2: Membrane representation of the electron where the electron is represented as a surface covering the Bohr sphere. A more natural shape for the atom of hydrogen would be a Horn Torus, or "apple shape" having field lines connecting its poles. The flux of energy crossing the membrane is determined by the mass of the proton as used in the scaling of the Thomson cross section.

By the squared-mass scaling rule, we multiply (9) by $\left(m_p/m_e\right)^2$, a scaling of the Thomson cross section to the Bohr sphere, yielding:

$$\sigma_0 = \frac{8\pi}{3}\left(\frac{q^2 m_p}{4\pi\varepsilon_0\, m_e^2\, c^2}\right)^2 . \tag{10}$$

The scaled Bohr radius, expressed as

$$r_1 = \frac{\varepsilon_0 h^2}{4\pi^2\, m_e\, q^2},$$

is a non-standard Bohr radius of electron orbital obtained by rescaling in a way that E in Poynting's theorem $S = c\varepsilon_0 E^2$ is the standard wave of an electric field, for consistency with the Thomson cross section. By the scaled Bohr radius, the surface of the Bohr sphere $4\pi r_1^2$ is expressed as follows:

$$\sigma_s = \frac{\varepsilon_0^2\, h^4}{4\pi^3\, m_e^2\, e^4} . \tag{11}$$

The standard Bohr radius

$$r_0 = \frac{\varepsilon_0\, h^2}{\pi\, m_e\, e^2}$$

representing the radius of an electron orbital in the Bohr model [1, 2], is based on the electron identity $n\frac{h}{2\pi r} = m_e\, v$, where h is a quantity defined as the product of electron momentum by one circumference of the ring, n the number of electrons, m_e the mass of an electron, and v its velocity.

As the electron from the Thomson cross section rescaled by the squared-mass scaling rule covers the whole surface of the Bohr sphere, we can match σ_0 with σ_s, i.e. (10) and (11). As such the Bohr sphere stands as a membrane of the electron, as seen in Fig. 2. As a result, the one circumference momentum of the electron, also known as the Planck's constant, is expressed as follows:

$$h = \frac{e^2}{c\,\varepsilon_0}\sqrt{\pi\sqrt{\frac{2}{3}\frac{m_p}{m_e}}}\,, \tag{12}$$

where e is the elementary charge of an electron, m_e the mass of an electron, m_p the mass of a proton, ε_0 the vacuum permittivity, and c the speed of light.

3 Results

The Planck's constant computed from (12) with values in Table 1, yields $h = 6.6368 \times 10^{-34}$ J s, deviating from its CODATA value by 1.62 parts per thousand. Of this deviation, 87.4% is explained by the non-relativistic approximation of electron mass, 11.2% by the binding energy of the electron orbital, and 1.37% remains unexplained (see Fig. 3).

By introducing the relativistic mass of the electron $m_{el} = \frac{1}{\sqrt{1-(v_e/c)^2}}\, m_e$ into (12), with the electron velocity

$$v_e = \frac{e}{\sqrt{4\pi\varepsilon_0\, r_0\, m_e}}$$

resulting from the equilibrium between centripetal and Coulomb's force, where e is the elementary charge of the electron, r_0 the standard Bohr radius, m_e the mass of an electron, and ε_0 the vacuum permittivity, leads to the new value $h = 6.6247 \times 10^{-34}$ J s.

The binding energy of the electron in its orbital, as given by the potential energy using the rescaled Bohr radius r_e,

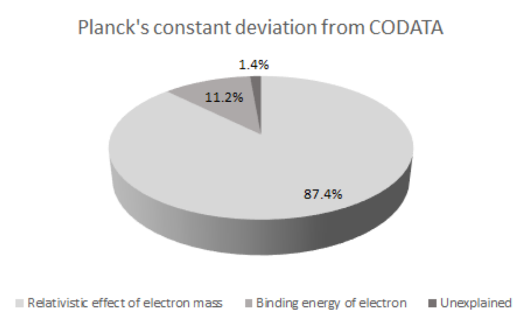

Fig. 3: Attribution of Planck's constant deviation from its CODATA value. Of a relative error of 1.62 parts per thousand, 87.4% is explained by the non-relativistic approximation of electron mass, 11.2% by the binding energy of the electron orbital, and 1.37% remains unexplained.

gives $K = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0\, r_e}$. By substracting the mass gap $\Delta m_e = K/c^2 \simeq 3.059\times10^{-34}$ kg from the mass of the electron and applying relativistic adjustment (multiplying electron mass by the inverse of the Lorentz-FitzGerald contraction), yields a new Planck's value $h = 6.62592\times10^{-34}$ J s, of an accuracy of about 22.2 parts per million with respect to actual measurements (as explained by modelling assumptions or geometrical aspects, e.g. shape of atom departing from a perfect sphere).

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