Introducing the Space Metric of a Rotating Massive Body and Four New Effects of General Relativity

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This paper introduces and proves the space metric of a rotating spherical body (approximated by a mass-point). This is a new metric to General Relativity, which is an extension and replacement of Schwarzschild's mass-point metric (since all cosmic bodies rotate). Physically observable characteristics of such a space are calculated, including the curvature of space and others. It is shown that the curvature of such a space has two components: a component created by the gravitational field (it decreases with distance from the body) and a constant curvature component created by the rotation of space (it does not depend on distance). Using Einstein's equations, the Riemannian conditions are calculated under which the introduced metric is valid (with the conditions, the Einstein equations vanish). Four new effects of General Relativity are calculated: the deflection of light rays and mass-bearing particles near a rotating body, a length-stretching effect along the geographical longitudes, a time-loss effect in the clocks co-moving with the Earth's rotation (to the East) and a time increment when moving to the West.

1 Introduction

This is the fourth paper in the series of papers on the effects of the space curvature, caused by the rotation of space.

The first [1] of these studies, besides many other scientific results obtained in it, showed that the rotation of space makes it curved. Then, two subsequent studies [2, 3] predicted four new effects of General Relativity, the origin of which is the space curvature caused by the rotation of space.

The first two effects are the deflection of light rays and mass-bearing particles in the field of a rotating body [2].

When a body rotates, the space around it curves towards the direction of its rotation and the centre of the body (around which it rotates), thereby creating a "slope of the hill" descending "down" along the equator in the direction, in which the body rotates, and also to the centre of the body. Therefore, when a particle travels freely to a rotating body, it "rolls down" the slope of the space curvature along the equator in the direction, in which the body rotates, as well as to the centre of the body. As a result, the following two effects should occur in the field of a rotating body:

1. A particle travelling freely to a rotating body should be deflected slightly from its radial trajectory in the equatorial direction, in which the body rotates, i.e., along the geographical longitudes;
2. The particle should gain a small increase of its velocity, and its path should become physically "stretched" for a little, causing the particle to reach the body with a delay in time compared to if the body did not rotate.

That is, light rays and mass-bearing particles should be deflected near a rotating body due to the curvature of space caused by its rotation. These two effects should take place both for mass-bearing particles and for light rays (massless light-like particles such as photons).

The other two effects are the length stretching and time loss/gain, expected in the field of a rotating body due to the curvature of its space, caused by its rotation [3]:

3. Since the diurnal rotation of the Earth around its axis curves the Earth's space making it "stretched" along the geographical longitudes, then the measured length of a standard rod should be greater when the rod is installed in the longitudinal direction;
4. Due to the same reason, there should be a time loss on board an airplane flying to the East (the direction in which the Earth's space rotates), and also a time increment when flying in the opposite direction, to the West.

Both of the effects are maximum at the equator (where the curvature of the Earth's space caused by its rotation is maximum and, therefore, space is maximally "stretched") and decrease towards the North and South Poles.

The above four effects, namely — the deflection of light rays and mass-bearing particles in the field of a rotating body, and also the length stretching and time loss/gain in the field of a rotating body — are new fundamental effects of the General Theory of Relativity, which were predicted "au bout d'un stylo". These four effects can be considered as an addition to the well-known Einstein effect of the deflection of light rays in the field of a gravitating body (which does not take the rotation of space into account).

2 Problem statement

When calculating the mentioned four new effects in the field of a rotating body, our task was to deduce the effects in their "pure form", i.e., without any other factors taken into account. To do this, the simplest metric was used, which described the four-dimensional space (space-time) of a rotating body, the mass of which is so small that the gravitational field it creates can be neglected.

This space metric is easy to deduce. Consider the metric of an empty space, which does not rotate or deform

$$ds^2 = c^2dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (1)$$

where and below, in terms of the spherical coordinates, r is the radial coordinate, dr is the elementary segment length along the radial r-axis, θ is the polar coordinate angle measured from the North Pole to the equator, $r\,d\theta$ is the elementary arc length along the θ-axis (along the geographical latitudes), φ is the geographical longitude (equatorial coordinate axis), and $r\sin\theta\, d\varphi$ is the elementary arc length along the equatorial φ-axis.

Assume that the space rotates along the equatorial axis φ, i.e., along the geographical longitudes, with the linear velocity $v_3 = \omega r^2 \sin^2\theta$, where $\omega = const$ is the angular velocity of this rotation. Since by definition of v_i (13)

$$v_3 = \omega r^2 \sin^2\theta = -\frac{c\, g_{03}}{\sqrt{g_{00}}} \qquad (2)$$

then we have

$$g_{03} = -\frac{1}{c}\, v_3 \sqrt{g_{00}} = -\frac{\omega r^2 \sin^2\theta}{c}\,, \qquad (3)$$

and the metric of such a rotating empty space has the form

$$ds^2 = c^2dt^2 - 2\,\omega r^2\sin^2\theta\, dt\, d\varphi - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \qquad (4)$$

As you can see, the non-zero components of the fundamental metric tensor $g_{\alpha\beta}$ of this metric are

$$\left.\begin{array}{lll} g_{00} = 1\,, & g_{03} = -\dfrac{\omega r^2\sin^2\theta}{c} & \\ g_{11} = -1\,, & g_{22} = -r^2, & g_{33} = -r^2\sin^2\theta \end{array}\right\}, \qquad (5)$$

where $g_{00} = 1$ means that the space is free of gravitational fields or such fields can be neglected: with $g_{00} = 1$ the gravitational field potential w, the general formula of which for any space metric is $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ (12), is either equal to zero $\mathrm{w} = 0$ or approaches zero $\mathrm{w} \to 0$.

The deflection of light rays and mass-bearing particles in the field of a rotating body [2], and also the length stretching and time loss/gain in the field of a rotating body [3] were obtained in the space of the above metric (4). Thanks to the above approximation, expressed with the simplest metric (4) describing a rotating empty space, it was possible to obtain the mentioned effects of the space curvature created by the rotation of space in their "pure form", without adding any other geometric or physical factors.

But real experiments conducted in an Earth-bound laboratory must take the gravitational field of the Earth into account. From this follows the problem statement for this paper:

Problem statement

Our task now is to re-calculate the space curvature effects caused by the rotation of space — the deflection of light rays and mass-bearing particles, and also the length stretching and time loss/gain in the field of a rotating body — for the case, where the gravitational field of the rotating body is taken into account.

To do this, we need the metric of such a space. We deduce it from Schwarzschild's mass-point metric, which describes a spherically symmetric space filled with the gravitational field created in emptiness by a spherical massive island of substance (approximated by a mass-point)

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (6)$$

where r is the radial distance from the centre of the massive island, $r_g = 2GM/c^2$ is its gravitational radius, calculated for its mass M, and the non-zero components of the fundamental metric tensor $g_{\alpha\beta}$ are

$$\left.\begin{array}{ll} g_{00} = 1 - \dfrac{r_g}{r}\,, & g_{11} = -\dfrac{1}{1 - \frac{r_g}{r}} \\ g_{22} = -r^2, & g_{33} = -r^2\sin^2\theta \end{array}\right\}. \qquad (7)$$

As before, we assume that the space rotates along the equatorial axis φ (along the geographical longitudes) with the linear velocity $v_3 = \omega r^2\sin^2\theta$, where $\omega = const$ is the angular velocity of this rotation. Since by definition of v_i (13)

$$v_3 = \omega r^2\sin^2\theta = -\frac{c\, g_{03}}{\sqrt{g_{00}}}\,, \qquad (8)$$

and, hence,

$$g_{03} = -\frac{1}{c}\, v_3\sqrt{g_{00}} = -\frac{\omega r^2\sin^2\theta}{c}\sqrt{1 - \frac{r_g}{r}} \neq 0\,, \qquad (9)$$

then we obtain the desired metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - 2\,\omega r^2\sin^2\theta\sqrt{1 - \frac{r_g}{r}}\, dt\, d\varphi - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (10)$$

which describes a spherically symmetric space, which is filled with the gravitational field created in emptiness by a rotating spherical island of matter (approximated by a mass-point) and rotates together with this body.

It is the metric (10), in the space of which we are going to re-calculate the space curvature effects, created due to the rotation of space.

We will do this in the following steps. First, we need to give a short description of the mathematical formalism we are using — the mathematical apparatus of chronometric invariants, which are physically observable quantities in the space-time of General Relativity.

Second, we calculate the physically observable chr.inv.-characteristics of the space of a rotating mass-point, which is the space of the metric (10).

Third, it is not a fact that the space described by the introduced metric of a rotating mass-point (10) is Riemannian. By definition, a Riemannian space is such one, the metric of which has the Riemannian square form $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$, determined by the Riemann fundamental metric tensor $g_{\alpha\beta}$, is invariant $ds^2 = inv$ everywhere in the space, and also satisfies Einstein's field equations, which are the specific relation between the Ricci curvature tensor, the fundamental metric tensor multiplied by the curvature scalar, and the energy-momentum tensor of the "space filler" (the latter targets non-empty Riemannian spaces filled with distributed matter). The above three requirements are specific to the family of Riemannian spaces.

Finding a metric that satisfies the first two conditions is easy, but satisfying the third condition (Einstein's field equations) is problematic. This is why, until now, only a small number of space metrics have been proven to be Riemannian and used in the General Theory of Relativity.

A space metric satisfies the field equations, if the components of the fundamental metric tensor $g_{\alpha\beta}$ (specific to this metric) and the components of the energy-momentum tensor of the medium (that fills the space), substituted into the field equations, make the left-hand and right-hand sides of the equations identical (the field equations vanish). In an empty Riemannian space, the left-hand side of the field equations itself after the above substitution must become zero (since in this case the energy-momentum tensor of distributed matter on the right-hand side is zero).

Most likely, the introduced metric of the space of a rotating mass-point (10) does not satisfy the field equations. For this reason, at our third step, we will substitute the $g_{\alpha\beta}$ components from the introduced metric (10) into the left-hand terms of the field equations (the right-hand side of the equations is zero, since the space of a rotating mass-point we are considering is not filled with distributed matter). The relations (particular conditions) that vanish the resulting field equations are *Riemannian conditions*, under which the introduced metric (10) is Riemannian and, therefore, can be used in the framework of General Relativity.

At our fourth step, we will deduce formulae for the space curvature effects in the field of a rotating massive body, i.e., in the space of the metric (10), which is the final task of this research.

3 Chronometrically invariant quantities

We use the mathematical apparatus of chronometric invariants, which uniquely determines physically observable quantities in the four-dimensional pseudo-Riemannian space (space-time of General Relativity). This mathematical formalism was created in 1944 by Abraham Zelmanov.

In addition to the publications by Zelmanov [4–6], which were very concise, an extended review of the chronometrically invariant formalism was given in each of our three research monographs (together with L. Borissova), originally published in 2001 [7, 8] and 2013 [9]. In 2023 we published the most comprehensive survey of the Zelmanov formalism [10], where we collected almost everything that we know in this field personally from Zelmanov and based on our own research studies. The most complete list of the research studies performed using the chronometrically invariant formalism as of January 2023 can be found in the Bibliography to our survey [10].

In short, Zelmanov unambiguously determined physically observable quantities in the space-time of General Relativity as the projections of four-dimensional tensor quantities onto the time line and the three-dimensional spatial section, associated with an observer. Such projections remain invariant throughout the observer's three-dimensional spatial section (his observable three-dimensional physical reference space), i.e., they are "chrono-metric invariants" in his reference frame and depend on the properties of his physical reference space, such as the gravitational potential, rotation, deformation, curvature, etc.

The chronometrically invariant projections of any four-dimensional tensor quantity are calculated using operators of projection, which take the physical properties and geometric structure of the observer's space into account. For detail, see the References to chronometric invariants, e.g., the most detailed survey [10].

Below you can find only the necessary minimum of this mathematical formalism, which is necessary for understanding and reproducing the results obtained in this study.

Projecting the four-dimensional displacement vector dx^α ($\alpha = 0, 1, 2, 3$) onto the time line of an observer gives the physically observable chr.inv.-time interval $d\tau$

$$d\tau = \sqrt{g_{00}}\,dt - \frac{1}{c^2}\,v_i\,dx^i, \quad i = 1, 2, 3, \tag{11}$$

where g_{00} is expressed with the chr.inv.-potential w (physically observable potential) of the gravitational field that fills the space of the observer as

$$\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right), \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{12}$$

and v_i is the three-dimensional vector of the linear velocity of rotation of the observer's space

$$v_i = -\frac{c\,g_{0i}}{\sqrt{g_{00}}}\,, \quad v^i = -c\,g^{0i}\sqrt{g_{00}}\,, \quad v_i = h_{ik}\,v^k. \tag{13}$$

Projecting dx^α onto the observer's three-dimensional spatial section gives the three-dimensional chr.inv.-displacement vector dx^i (which coincides with the three-dimensional coordinate displacement vector). As a result, $d\tau$ distinguishes the chr.inv.-velocity vector $\mathrm{v}^i = dx^i/d\tau$ (physically observable three-dimensional velocity) from the three-dimensional coordinate velocity vector $u^i = dx^i/dt$.

The three-dimensional chr.inv.-spatial interval $d\sigma$ (physically observable three-dimensional interval) is determined

$$d\sigma^2 = h_{ik}\, dx^i dx^k, \tag{14}$$

using the three-dimensional chr.inv.-metric tensor h_{ik}

$$h_{ik} = -g_{ik} + \frac{1}{c^2}\, v_i v_k\,, \quad h^{ik} = -g^{ik}, \quad h^i_k = \delta^i_k\,, \tag{15}$$

which is the chr.inv.-projection of the fundamental metric tensor $g_{\alpha\beta}$ onto the observer's spatial section and possesses all properties of $g_{\alpha\beta}$ throughout the spatial section (the observer's three-dimensional space).

The square of the four-dimensional (space-time) interval $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ is therefore expressed with chronometrically invariant (physically observable) quantities as

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{16}$$

Thanks to the splitting of space-time into three-dimensional spatial sections pierced by time lines, which is specific to the chronometrically invariant formalism, we can reveal the true nature of three-dimensional rotations. When $v_i \neq 0$, i.e., the reference body of an observer rotates (together with his reference space), then this rotation cannot be vanished by a coordinate transformation (by moving the observer to another reference frame within his three-dimensional spatial section). This happens because the rotation speed v_i (13) is determined by the mixed (space-time) components g_{0i} of the fundamental metric tensor $g_{\alpha\beta}$, and not by its three-dimensional spatial components g_{ik} dependent on time (as it is considered in classical mechanics, where time is just a parameter, and not the fourth coordinate). Since the components of $g_{\alpha\beta}$ are cosines of the angles between the respective coordinate lines, then three-dimensional rotations are due to the *non-holonomity* of space-time, which means that time lines are not orthogonal to three-dimensional spatial sections.

If all g_{0i} are zero, then such space-time is *holonomic*. In this case the three-dimensional spatial section is everywhere orthogonal to the time lines that pierce it. If at least one of the components g_{0i} is different from zero, then such space-time is *non-holonomic*, and the spatial section $x^0 = const$ is inclined to the time lines (at different points it can be inclined to the time lines at different angles depending on the local geometric structure of the particular four-dimensional space-time).

In general, the physical reference space of a real observer can be filled with a gravitational field, rotate, deform, be inhomogeneous and curved.

The chr.inv.-vector of the gravitational inertial force F_i, where the first (gravitational) term is created by the gradient of the gravitational potential w and the second (inertial) term is created by the centrifugal force of inertia, is

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,. \tag{17}$$

The antisymmetric chr.inv.-tensor A_{ik} of the angular velocity of rotation of space is

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \tag{18}$$

which is related to F_i by two identities

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{19}$$

$$\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0\,, \tag{20}$$

where asterisk denotes the chr.inv.-derivation operators

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\, v_i \frac{{}^*\partial}{\partial t}\,. \tag{21}$$

Antisymmetric chr.inv.-tensors can be used to create the corresponding chr.inv.-pseudovectors (marked with an asterisk) using the antisymmetric chr.inv.-discriminant tensor

$$\varepsilon^{ikm} = \frac{e^{ikm}}{\sqrt{h}}\,, \qquad \varepsilon_{ikm} = e_{ikm}\sqrt{h}\,, \tag{22}$$

where $h = \det\| h_{ik}\|$. This tensor is the chr.inv.-analogy of the Levi-Civita antisymmetric unit tensor e^{ikm} (the components of e^{ikm} are either $+1$ or -1 depending on the transposition of its indices).* For example, the antisymmetric chr.inv.-tensor A_{ik} of the angular velocity of rotation of space has the corresponding chr.inv.-pseudovector Ω^{*i} of this rotation

$$\left.\begin{aligned} &\Omega^{*i} = \frac{1}{2}\,\varepsilon^{ikm}A_{km}\,, \qquad \Omega_{*i} = \frac{1}{2}\,\varepsilon_{imn}A^{mn} \\ &\varepsilon^{ipq}\,\Omega_{*i} = \frac{1}{2}\,\varepsilon^{ipq}\varepsilon_{imn}A^{mn} = \\ &\qquad = \frac{1}{2}\left(\delta^p_m\delta^q_n - \delta^q_m\delta^p_n\right)A^{mn} = A^{pq} \end{aligned}\right\}. \tag{23}$$

The symmetric chr.inv.-tensor D_{ik} of the deformation rate of space is formulated as

$$\left.\begin{aligned} &D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \qquad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t} \\ &D = h^{ik}D_{ik} = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \quad h = \det\| h_{ik}\| \end{aligned}\right\}. \tag{24}$$

*For detail, see pages 14–16 in our comprehensive survey of the Zelmanov chronometric invariants [10], or §2.3 in our monograph [8].

The chr.inv.-Christoffel symbols of the 1st rank $\Delta_{jk,m}$ and the 2nd rank Δ^i_{nk} (their physical sense is the coefficients of inhomogeneity of space) are

$$\Delta^i_{nk} = h^{im}\Delta_{nk,m} = \frac{1}{2}\, h^{im}\left(\frac{^*\partial h_{nm}}{\partial x^k} + \frac{^*\partial h_{km}}{\partial x^n} - \frac{^*\partial h_{nk}}{\partial x^m}\right). \quad (25)$$

The physically observable curvature of space is expressed with the chr.inv.-curvature tensor C_{lkij} that possesses all properties of the Riemann-Christoffel curvature tensor throughout the three-dimensional spatial section associated with the observer. Its subsequent contractions give the chr.inv.-Ricci curvature tensor C_{ik} and the chr.inv.-scalar curvature C

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right) = H_{lkij} - \frac{1}{2}\left(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + A_{kl}D_{ij} + A_{li}D_{jk}\right), \quad (26)$$

$$C_{lk} = C_{lki\cdot}^{\cdots i} = H_{lk} - \frac{1}{2}\left(A_{kj}D^j_l + A_{lj}D^j_k + A_{kl}D\right), \quad (27)$$

$$C = h^{lk}C_{lk} = h^{lk}H_{lk}\,, \quad (28)$$

where, for a better association with the Riemann-Christoffel curvature tensor, we denote

$$H_{lki\cdot}^{\cdots j} = \frac{^*\partial\Delta^j_{il}}{\partial x^k} - \frac{^*\partial\Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}\,. \quad (29)$$

From the above definitions we see that the physically observable curvature of space depends on not only the gravitational inertial force (hidden in the second chr.inv.-derivatives of the chr.inv.-metric tensor), but also the rotation, deformation and inhomogeneity of space and, therefore, does not vanish in the absence of the gravitational field.

By analogy with absolute (general covariant) derivatives, the corresponding chr.inv.-derivatives are introduced

$$^*\nabla_i\, Q_k = \frac{^*\partial Q_k}{dx^i} - \Delta^l_{ik}\, Q_l\,, \quad (30)$$

$$^*\nabla_i\, Q^k = \frac{^*\partial Q^k}{dx^i} + \Delta^k_{il}\, Q^l, \quad (31)$$

$$^*\nabla_i\, Q_{jk} = \frac{^*\partial Q_{jk}}{dx^i} - \Delta^l_{ij}\, Q_{lk} - \Delta^l_{ik}\, Q_{jl}\,, \quad (32)$$

$$^*\nabla_i\, Q^k_j = \frac{^*\partial Q^k_j}{dx^i} - \Delta^l_{ij}\, Q^k_l + \Delta^k_{il}\, Q^l_j\,, \quad (33)$$

$$^*\nabla_i\, Q^{jk} = \frac{^*\partial Q^{jk}}{dx^i} + \Delta^j_{il}\, Q^{lk} + \Delta^k_{il}\, Q^{jl}, \quad (34)$$

$$^*\nabla_i\, Q^i = \frac{^*\partial Q^i}{\partial x^i} + \Delta^j_{ji}\, Q^i, \qquad \Delta^j_{ji} = \frac{^*\partial \ln\sqrt{h}}{\partial x^i}, \quad (35)$$

$$^*\nabla_i\, Q^{ji} = \frac{^*\partial Q^{ji}}{\partial x^i} + \Delta^j_{il}\, Q^{il} + \Delta^l_{li}\, Q^{ji}, \qquad \Delta^l_{li} = \frac{^*\partial \ln\sqrt{h}}{\partial x^i}, \quad (36)$$

which, in particular, exhibit some properties of the chr.inv.-metric tensor h_{ik} and the chr.inv.-discriminant tensor ε_{ijk} (used further in calculations)

$$^*\nabla_i\, h_{jk} = 0\,, \quad {}^*\nabla_i\, h^k_j = 0\,, \quad {}^*\nabla_i\, h^{jk} = 0\,, \quad (37)$$

$$^*\nabla_l\, \varepsilon_{ijk} = 0\,, \qquad {}^*\nabla_l\, \varepsilon^{ijk} = 0\,, \quad (38)$$

Einstein's field equations, having the well-known general covariant (four-dimensional) form

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa\, T_{\alpha\beta} + \lambda\, g_{\alpha\beta} \quad (39)$$

can also be presented in chr.inv.-form, i.e., in the form of their physically observable chr.inv.-projections.

Note, that the Zelmanov formalism uses $\varkappa = \frac{8\pi G}{c^2}$, but not $\varkappa = \frac{8\pi G}{c^4}$ as Landau and Lifshitz did in their *The Classical Theory of Fields* [11]. This is because, since Ricci's tensor $R_{\alpha\beta}$ has the dimension [cm^{-2}] and the energy-momentum tensor $T_{\alpha\beta}$ has the dimension of mass density [gram/cm^3], if we used $\varkappa = \frac{8\pi G}{c^4}$ on the right-hand side of the field equations, then we would not use the energy-momentum tensor $T_{\alpha\beta}$ itself, but $c^2 T_{\alpha\beta}$ as Landau and Lifshitz did (which is not correct at all from the point of view of physical sense and physically observable quantities).

To understand the chr.inv.-Einstein equations that below, we should note that any tensor or tensor equation of the 2nd rank has three chr.inv.-projections: the time projection, the space-time (mixed) projection and the spatial projection; for detail, see [10]. So, the energy-momentum tensor $T_{\alpha\beta}$ of a distributed matter has the following chr.inv.-projections

$$\varrho = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\, T^i_0}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik}, \quad (40)$$

where ϱ is the observable mass density of the distributed matter, J^i is its observable momentum density, and U^{ik} is the observable stress-tensor of the matter field.

The general covariant Einstein field equations (39) also have three chr.inv.-projections, which are called the chr.inv.-Einstein equations

$$\frac{^*\partial D}{\partial t} + D_{jl}D^{jl} + A_{jl}A^{lj} + {}^*\nabla_j F^j - \frac{1}{c^2}\, F_j F^j = -\frac{\varkappa}{2}\left(\varrho c^2 + U\right) + \lambda c^2, \quad (41)$$

$$^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}\, F_j A^{ij} = \varkappa\, J^i, \quad (42)$$

$$\frac{^*\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D^j_k + A^{\cdot j}_{k\cdot}\right) + D D_{ik} + 3A_{ij}A^{\cdot j}_{k\cdot} - \frac{1}{c^2}\, F_i F_k + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - c^2 C_{ik} = \frac{\varkappa}{2}\left(\varrho c^2 h_{ik} + 2\,U_{ik} - U h_{ik}\right) + \lambda c^2 h_{ik}\,. \quad (43)$$

With the above mathematical tools, we now have everything we need to consider the space of a rotating massive body using the chronometrically invariant formalism.

4 Physically observable characteristics of the space of a rotating massive body

Consider a space of the rotating Schwarzschild metric, which we have introduced (10). It has the form

$$ds^2 = \left(1-\frac{r_g}{r}\right)c^2dt^2 - 2\,\omega r^2\sin^2\theta\,\sqrt{1-\frac{r_g}{r}}\,dt\,d\varphi - \frac{dr^2}{1-\frac{r_g}{r}} - r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right). \quad (44)$$

Such a space rotates in the equatorial plane along the geographical longitudes φ with an angular velocity $\omega = const$. The linear velocity of this rotation is $v_3 = \omega r^2\sin^2\theta$

$$v_3 = \omega r^2\sin^2\theta = -\frac{c\,g_{03}}{\sqrt{g_{00}}}\,, \qquad v_1 = v_2 = 0\,, \quad (45)$$

hence, non-zero components of the fundamental metric tensor of the above space metric are

$$\left.\begin{aligned} g_{00} &= 1-\frac{r_g}{r}\,, \qquad g_{03} = -\frac{\omega r^2\sin^2\theta}{c}\sqrt{1-\frac{r_g}{r}} \\ g_{11} &= -\frac{1}{1-\frac{r_g}{r}}\,, \qquad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta \end{aligned}\right\}. \quad (46)$$

Respectively, the chr.inv.-metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2}v_i v_k$ (15) of a rotating Schwarzschild space has only the following non-zero components

$$\left.\begin{aligned} h_{11} &= \frac{1}{1-\frac{r_g}{r}}\,, \qquad h_{22} = r^2 \\ h_{33} &= r^2\sin^2\theta\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right) \end{aligned}\right\}, \quad (47)$$

and, respectively, calculating the determinant of the chr.inv.-metric tensor h_{ik}, we obtain

$$h = \det\|\,h_{ik}\,\| = h_{11}\,h_{22}\,h_{33} = \frac{r^4\sin^2\theta}{1-\frac{r_g}{r}}\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right), \quad (48)$$

$$\sqrt{h} = \frac{r^2\sin\theta}{\sqrt{1-\frac{r_g}{r}}}\sqrt{1+\frac{\omega^2r^2\sin^2\theta}{c^2}}\,. \quad (49)$$

As is seen from the above formulae, the matrix h_{ik} is strict diagonal: all of its non-diagonal components h_{ik} $(i \neq k)$ are zero. Therefore, the upper-index components of h_{ik} are obtained just like the invertible matrix components to any diagonal matrix as $h^{ik} = (h_{ik})^{-1}$. They are

$$\left.\begin{aligned} h^{11} &= 1-\frac{r_g}{r}\,, \qquad h^{22} = \frac{1}{r^2} \\ h^{33} &= \frac{1}{r^2\sin^2\theta\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)} \end{aligned}\right\}. \quad (50)$$

In particular, as a result, the square of the linear velocity, with which the space rotates $v^2 = v_i v^i = v_i h^{ik} v_k$ (13) is

$$v^2 = v_3\,h^{33}\,v_3 = \frac{\omega^2r^2\sin^2\theta}{1+\frac{\omega^2r^2\sin^2\theta}{c^2}}\,. \quad (51)$$

As is seen from (47), the obtained chr.inv.-metric tensor h_{ik} does not depend on time. This means that the chr.inv.-tensor of the deformation rate of space D_{ik} (24) is zero

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} = 0\,, \quad (52)$$

i.e., a rotating Schwarzschild space does not deform.

Taking into account that the linear velocity $v_3 = \omega r^2\sin^2\theta$ with which the space rotates does not depend on time

$$\frac{\partial v_3}{\partial t} = 0 \quad (53)$$

and also that the gravitational field potential $\mathrm{w} = c^2(1-\sqrt{g_{00}})$ in the present case is

$$\mathrm{w} = c^2\left(1-\sqrt{1-\frac{r_g}{r}}\right), \quad (54)$$

we obtain the components of the chr.inv.-vector of the gravitational inertial force F_i (17). They are

$$F_1 = \frac{1}{\sqrt{g_{00}}}\frac{\partial \mathrm{w}}{\partial r} = -\frac{c^2r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\,, \qquad F_2 = F_3 = 0\,, \quad (55)$$

$$F^1 = h^{11}F_1 = -\frac{c^2r_g}{2r^2}\,, \qquad F^2 = F^3 = 0\,. \quad (56)$$

Since the gravitational inertial force in the present case is a radially acting force F_1 that depends only on $x^1 = r$, i.e.

$$\frac{{}^*\partial F_k}{\partial x^i} = 0\,, \qquad i \neq k\,, \quad (57)$$

then according to the 1st Zelmanov identity (19) we have

$$\frac{{}^*\partial A_{ik}}{\partial t} = 0\,, \quad (58)$$

i.e., the rotation of the space of the rotating Schwarzschild metric is stationary.

According to the definition of the chr.inv.-tensor of the angular velocity of rotation of space A_{ik} (18), only the following components of it are non-zero in the space of the rotating Schwarzschild metric: $A_{13} \neq 0$, $A_{31} \neq 0$, $A^{13} \neq 0$, $A^{31} \neq 0$,

$A_{23} \neq 0$, $A_{32} \neq 0$, $A^{23} \neq 0$, $A^{32} \neq 0$. Using the definition of A_{ik} (18), after some algebra we obtain

$$A_{13} = \frac{1}{2}\frac{\partial v_3}{\partial r} + \frac{1}{2c^2}F_1 v_3 = \omega r \sin^2\theta - \frac{\omega r_g \sin^2\theta}{4\left(1-\frac{r_g}{r}\right)}\,, \tag{59}$$

$$A_{31} = -A_{13} = -\,\omega r \sin^2\theta + \frac{\omega r_g \sin^2\theta}{4\left(1-\frac{r_g}{r}\right)}\,, \tag{60}$$

$$A^{13} = h^{11}h^{33}A_{13} = = \frac{\left(1-\frac{r_g}{r}\right)\omega}{r\left(1+\frac{\omega^2 r^2\sin^2\theta}{c^2}\right)} - \frac{\omega r_g}{4r^2\left(1+\frac{\omega^2 r^2\sin^2\theta}{c^2}\right)}\,, \tag{61}$$

$$A^{31} = -A^{13} = = -\frac{\left(1-\frac{r_g}{r}\right)\omega}{r\left(1+\frac{\omega^2 r^2\sin^2\theta}{c^2}\right)} + \frac{\omega r_g}{4r^2\left(1+\frac{\omega^2 r^2\sin^2\theta}{c^2}\right)}\,, \tag{62}$$

$$A_{23} = \frac{1}{2}\frac{\partial v_3}{\partial \theta} = \omega r^2 \sin\theta\cos\theta\,, \tag{63}$$

$$A_{32} = -A_{23} = -\,\omega r^2 \sin\theta\cos\theta\,, \tag{64}$$

$$A^{23} = h^{22}h^{33}A_{23} = \frac{\omega\cot\theta}{r^2\left(1+\frac{\omega^2 r^2\sin^2\theta}{c^2}\right)}\,, \tag{65}$$

$$A^{32} = -A^{23} = -\frac{\omega\cot\theta}{r^2\left(1+\frac{\omega^2 r^2\sin^2\theta}{c^2}\right)}\,. \tag{66}$$

Find the physically observable scalar angular velocity Ω, with which the space rotates. Its square is calculated as

$$\Omega^2 = \Omega_{*i}\Omega^{*i} = \Omega_{*1}\Omega^{*1} + \Omega_{*2}\Omega^{*2} = = h_{11}\Omega^{*1}\Omega^{*1} + h_{22}\Omega^{*2}\Omega^{*2}. \tag{67}$$

In the space of the rotating Schwarzschild metric, which we are considering, we have

$$\Omega^{*1} = \frac{1}{2}\varepsilon^{1km}A_{km} = \frac{e^{1km}}{2\sqrt{h}}A_{km} = \frac{e^{123}}{2\sqrt{h}}A_{23} + \frac{e^{132}}{2\sqrt{h}}A_{32} \tag{68}$$

and, taking into account that $e^{123} = +1$ and $e^{132} = -1$, and also $A_{32} = -A_{23}$, we obtain

$$\Omega^{*1} = \frac{e^{123}}{2\sqrt{h}}A_{23} + \frac{e^{123}}{2\sqrt{h}}A_{23} = \frac{e^{123}}{\sqrt{h}}A_{23} = \frac{A_{23}}{\sqrt{h}}\,. \tag{69}$$

In the same way, we obtain

$$\Omega^{*2} = \frac{1}{2}\varepsilon^{2km}A_{km} = \frac{e^{2km}}{2\sqrt{h}}A_{km} = = \frac{e^{213}}{2\sqrt{h}}A_{13} + \frac{e^{231}}{2\sqrt{h}}A_{31} = \frac{e^{213}}{\sqrt{h}}A_{13} = -\frac{A_{13}}{\sqrt{h}}\,. \tag{70}$$

Finally, substituting A_{13} (59), A_{23} (63), $h = \det\| h_{ik}\|$ (48), h_{11} and h_{22} (47) into Ω^2 (67), we obtain the physically observable scalar angular velocity Ω of the rotation of space

$$\Omega = \sqrt{\Omega_{*i}\Omega^{*i}} = \frac{\omega}{\sqrt{1+\frac{\omega^2 r^2\sin^2\theta}{c^2}}} \times \times \sqrt{1 - \frac{3r_g\sin^2\theta}{2r} + \frac{r_g^2\sin^2\theta}{16r^2\left(1-\frac{r_g}{r}\right)}}\,. \tag{71}$$

If there is no mass ($M = 0$), then the gravitational radius is $r_g = 2GM/c^2 = 0$. In this case, $g_{00} = 1 - \frac{r_g}{r} = 1$ and the formulae for h_{ik} (47–50), A_{ik} (59–66) and Ω (71) we have obtained in the space of the rotating Schwarzschild metric transform into the corresponding formulae in the spherically symmetric rotating space without the gravitational field, which we have obtained earlier; see page 43 in the previous paper [1].

To calculate the chr.inv.-Einstein equations in the space of the rotating Schwarzschild metric, we need the chr.inv.-Ricci curvature tensor C_{ik} containing in the third, tensor chr.inv.-Einstein equation (43). The chr.inv.-Ricci tensor C_{ik} (27) consists of the chr.inv.-derivatives of the chr.inv.-Christoffel symbols Δ^i_{nk} and the products of Δ^i_{nk} with each other. In turn, Δ^i_{nk} (25) are the re-combination of the chr.inv.-derivatives of the chr.inv.-metric tensor h_{ik} (47). Therefore, at first we calculate the non-zero chr.inv.-derivatives of h_{ik}

$$\frac{^*\partial h_{11}}{\partial r} = -\frac{r_g}{\left(1-\frac{r_g}{r}\right)^2 r^2}\,, \tag{72}$$

$$\frac{^*\partial h_{22}}{\partial r} = 2r\,, \tag{73}$$

$$\frac{^*\partial h_{33}}{\partial r} = 2r\sin^2\theta\left(1+\frac{2\omega^2 r^2\sin^2\theta}{c^2}\right), \tag{74}$$

$$\frac{^*\partial h_{33}}{\partial \theta} = 2r^2\sin\theta\cos\theta\left(1+\frac{2\omega^2 r^2\sin^2\theta}{c^2}\right). \tag{75}$$

The chr.inv.-Christoffel symbols Δ^i_{nk} (25) in the rotating Schwarzschild metric space have the non-zero components

$$\left.\begin{array}{ll} \Delta^1_{11} = \frac{1}{2}h^{11}\frac{^*\partial h_{11}}{\partial r}\,, & \Delta^1_{22} = \frac{1}{2}h^{11}\frac{^*\partial h_{22}}{\partial r} \\ \Delta^1_{33} = -\frac{1}{2}h^{11}\frac{^*\partial h_{33}}{\partial r}\,, & \Delta^2_{12} = \frac{1}{2}h^{22}\frac{^*\partial h_{22}}{\partial r} \\ \Delta^2_{21} = \frac{1}{2}h^{22}\frac{^*\partial h_{22}}{\partial r}\,, & \Delta^2_{33} = -\frac{1}{2}h^{22}\frac{^*\partial h_{33}}{\partial \theta} \\ \Delta^3_{13} = \frac{1}{2}h^{33}\frac{^*\partial h_{33}}{\partial r}\,, & \Delta^3_{23} = \frac{1}{2}h^{33}\frac{^*\partial h_{33}}{\partial \theta} \\ \Delta^3_{31} = \frac{1}{2}h^{33}\frac{^*\partial h_{33}}{\partial r}\,, & \Delta^3_{32} = \frac{1}{2}h^{33}\frac{^*\partial h_{33}}{\partial \theta} \end{array}\right\}. \tag{76}$$

After some algebra using the obtained formulae for the non-zero components of h^{ik} (50) and the chr.inv.-derivatives of the non-zero components of h_{ik} (72–75), we obtain

$$\Delta^1_{11} = -\frac{r_g}{2r^2\left(1-\frac{r_g}{r}\right)}\,, \tag{77}$$

$$\Delta^1_{22} = -r\,, \tag{78}$$

$$\Delta^1_{33} = -r\sin^2\theta\left(1+\frac{2\,\omega^2r^2\sin^2\theta}{c^2}\right), \tag{79}$$

$$\Delta^2_{12} = \Delta^2_{21} = \frac{1}{r}\,, \tag{80}$$

$$\Delta^2_{33} = -\sin\theta\cos\theta\left(1+\frac{2\,\omega^2r^2\sin^2\theta}{c^2}\right), \tag{81}$$

$$\Delta^3_{13} = \Delta^3_{31} = \frac{1}{r\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)}\left(1+\frac{2\,\omega^2r^2\sin^2\theta}{c^2}\right), \tag{82}$$

$$\Delta^3_{23} = \Delta^3_{32} = \frac{\cot\theta}{1+\frac{\omega^2r^2\sin^2\theta}{c^2}}\left(1+\frac{2\,\omega^2r^2\sin^2\theta}{c^2}\right). \tag{83}$$

The non-zero contracted chr.inv.-Christoffel symbols Δ^i_{i1} and Δ^i_{i2} are calculated from their definition based on the determinant $h = \det\|\,h_{ik}\,\|$; see (35) or (36). Using the formulae for h (48) and its square root (49) obtained in the space of the rotating Schwarzschild metric, we obtain

$$\Delta^i_{i1} = \frac{{}^*\partial\ln\sqrt{h}}{\partial r} = \frac{2}{r\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)}\left(1+\frac{3\,\omega^2r^2\sin^2\theta}{2\,c^2}\right) - \frac{r_g}{2r^2\left(1-\frac{r_g}{r}\right)}\,, \tag{84}$$

$$\Delta^i_{i2} = \frac{{}^*\partial\ln\sqrt{h}}{\partial\theta} = \frac{\cot\theta}{1+\frac{\omega^2r^2\sin^2\theta}{c^2}}\left(1+\frac{2\,\omega^2r^2\sin^2\theta}{c^2}\right). \tag{85}$$

Based on the above formulae, we calculate the non-zero chr.inv.-derivatives of the contracted chr.inv.-Christoffel symbols Δ^i_{i1} and Δ^i_{i2}. After some algebra, we obtain

$$\frac{{}^*\partial\Delta^i_{i1}}{\partial r} = -\frac{2}{r^2\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)^2} - \frac{3\,\omega^2\sin^2\theta}{c^2\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)^2} - \frac{3\,\omega^4r^2\sin^4\theta}{c^4\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)^2} + \frac{r_g}{r^3\left(1-\frac{r_g}{r}\right)^2}\left(1-\frac{r_g}{2r}\right), \tag{86}$$

$$\frac{{}^*\partial\Delta^i_{i1}}{\partial\theta} = \frac{2\,\omega^2r\sin\theta\cos\theta}{c^2\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)^2}\,, \tag{87}$$

$$\frac{{}^*\partial\Delta^i_{i2}}{\partial r} = \frac{2\,\omega^2r\sin\theta\cos\theta}{c^2\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)^2} = \frac{{}^*\partial\Delta^i_{i1}}{\partial\theta}\,, \tag{88}$$

$$\frac{{}^*\partial\Delta^i_{i2}}{\partial\theta} = -\frac{1}{\sin^2\theta\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)} - \frac{2\,\omega^2r^2\sin^2\theta}{c^2\left(1+\frac{\omega^2r^2\sin^2\theta}{c^2}\right)^2} - \frac{2\,\omega^4r^4\sin^2\theta}{c^4\left(1+\frac{\omega^2r^2\sin^4\theta}{c^2}\right)^2}\,. \tag{89}$$

Now, using the quantities calculated above, we calculate the chr.inv.-Ricci curvature tensor C_{ik} in the space of the rotating Schwarzschild metric. Since the space we are considering does not deform ($D_{ik} = 0$), then in this case the general formula for $C_{lk} = C_{lki\cdot}^{\cdots i}$ (27) is simplified to

$$C_{lk} = H_{lk} = H_{lki\cdot}^{\cdots i} = \frac{{}^*\partial\Delta^i_{il}}{\partial x^k} - \frac{{}^*\partial\Delta^i_{kl}}{\partial x^i} + \Delta^m_{il}\,\Delta^i_{km} - \Delta^m_{kl}\,\Delta^i_{im}\,, \tag{90}$$

which, according to the non-zero chr.inv.-Christoffel symbols calculated in the space of the rotating Schwarzschild metric (see above), has the following non-zero components

$$C_{11} = \frac{{}^*\partial\Delta^i_{i1}}{\partial r} + \Delta^2_{21}\Delta^2_{12} + \Delta^3_{31}\Delta^3_{13} - \frac{{}^*\partial\Delta^1_{11}}{\partial r} + \Delta^1_{11}\Delta^1_{11} - \Delta^1_{11}\Delta^i_{i1}\,, \tag{91}$$

$$C_{12} = \frac{{}^*\partial\Delta^i_{i1}}{\partial\theta} + \Delta^3_{31}\Delta^3_{23} - \Delta^2_{21}\Delta^i_{i2}\,, \tag{92}$$

$$C_{21} = \frac{{}^*\partial\Delta^i_{i2}}{\partial r} + \Delta^3_{32}\Delta^3_{13} - \Delta^2_{12}\Delta^i_{i2}\,, \tag{93}$$

$$C_{22} = \frac{{}^*\partial\Delta^i_{i2}}{\partial\theta} - \frac{{}^*\partial\Delta^1_{22}}{\partial r} + 2\,\Delta^2_{12}\Delta^1_{22} + \Delta^3_{32}\Delta^3_{23} - \Delta^1_{22}\Delta^i_{i1}\,, \tag{94}$$

$$C_{33} = -\frac{{}^*\partial\Delta^1_{33}}{\partial r} - \frac{{}^*\partial\Delta^2_{33}}{\partial\theta} + 2\,\Delta^3_{13}\Delta^1_{33} + 2\,\Delta^3_{23}\Delta^2_{33} - \Delta^1_{33}\Delta^i_{i1} - \Delta^2_{33}\Delta^i_{i2}\,. \tag{95}$$

To calculate these components, we calculate the unknown derivatives contained in them. We obtain

$$\frac{{}^*\partial\Delta^1_{11}}{\partial r} = \frac{r_g}{r^3\left(1-\frac{r_g}{r}\right)^2}\left(1-\frac{r_g}{2r}\right), \tag{96}$$

$$\frac{{}^*\partial\Delta^1_{33}}{\partial r} = -\sin^2\theta\left(1+\frac{6\,\omega^2r^2\sin^2\theta}{c^2}\right), \tag{97}$$

$$\frac{{}^*\partial\Delta^2_{33}}{\partial\theta} = \sin^2\theta + \frac{2\,\omega^2r^2\sin^4\theta}{c^2} - \cos^2\theta - \frac{6\,\omega^2r^2\sin^2\theta\cos^2\theta}{c^2}\,. \tag{98}$$

Substituting the non-zero necessary chr.inv.-Christoffel symbols and their chr.inv.-derivatives into these general formulae (91–95), after some algebra and non-trivial transformations we obtain formulae for the non-zero components of the chr.inv.-Ricci tensor in the space of the rotating Schwarzschild metric. They have the form

$$C_{11} = \frac{3\,\omega^2 \sin^2\theta}{c^2\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)} - \frac{\omega^4 r^2 \sin^4\theta}{c^4\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)^2} + \frac{r_g}{r^3\left(1-\frac{r_g}{r}\right)\left(1+\frac{\omega^2 r^2 \sin^4\theta}{c^2}\right)}\left(1+\frac{3\,\omega^2 r^2 \sin^2\theta}{2c^2}\right), \tag{99}$$

$$C_{12} = \frac{3\,\omega^2 r \sin\theta\cos\theta}{c^2\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)} - \frac{\omega^4 r^3 \sin^3\theta\cos\theta}{c^4\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)^2}, \tag{100}$$

$$C_{21} = \frac{3\,\omega^2 r \sin\theta\cos\theta}{c^2\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)} - \frac{\omega^4 r^3 \sin^3\theta\cos\theta}{c^4\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)^2}, \tag{101}$$

$$C_{22} = \frac{3\,\omega^2 r^2 \cos^2\theta}{c^2\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)} - \frac{\omega^4 r^4 \sin^2\theta\cos^2\theta}{c^4\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)^2}, \tag{102}$$

$$C_{33} = \frac{3\,\omega^2 r^2 \sin^2\theta}{c^2} - \frac{\omega^4 r^4 \sin^4\theta}{c^4\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)}, \tag{103}$$

where $C_{12} = C_{21}$ means that the space of the rotating Schwarzschild metric has a certain curvature symmetry.

Using the obtained components of the chr.inv.-Ricci tensor C_{ik} (99–103) and the upper-index components h^{ik} (50) of the chr.inv.-metric tensor, we calculate the physically observable chr.inv.-scalar curvature $C = h^{ik}C_{ik}$ (28) of the space of the rotating Schwarzschild metric. Since only h^{11}, h^{22}, h^{33} are non-zero in such a space, then $C = h^{11}C_{11} + h^{22}C_{22} + h^{33}C_{33}$. After some algebra, we obtain

$$C = \frac{6\,\omega^2}{c^2\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)} - \frac{2\,\omega^4 r^2 \sin^2\theta}{c^4\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)^2} + \frac{r_g}{r^3\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)} \times \left(1-\frac{3\,\omega^2 r^2 \sin^2\theta}{2c^2}+\frac{\omega^4 r^4 \sin^4\theta}{c^4}\right), \tag{104}$$

where the first two terms are due only to the rotation of space, and the third term (in the second and third lines of the formula) is due to the combined action of the gravitational field and the rotation of space.

This is the *physically observable chr.inv.-scalar curvature* of the three-dimensional space of a rotating massive body. It is this curvature of space that is registered in astronomical observations and laboratory measurements in the space near such rotating massive bodies as stars and planets.

In the absence of a massive island of substance producing the gravitational field ($M = 0$, $r_g = 2GM/c^2 = 0$), the obtained formula (104) transforms into the formula

$$C = \frac{6\,\omega^2}{c^2\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)} - \frac{2\,\omega^4 r^2 \sin^2\theta}{c^4\left(1+\frac{\omega^2 r^2 \sin^2\theta}{c^2}\right)^2}, \tag{105}$$

obtained recently in a rotating spherically symmetric space without a gravitational field; see page 45 in the first paper [1] of this series of papers.

At small speeds of rotation, the obtained formula for the chr.inv.-scalar curvature (104) takes the simplified form

$$C = \frac{6\,\omega^2}{c^2} + \frac{r_g}{r^3}. \tag{106}$$

From the obtained simplified formula for C (106), we see that the rotation of a massive body at slow rotations creates a constant curvature field that does not depend on distance from its source (the rotating body), whereas the gravitational field of the body creates a curvature that decreases inversely proportional to r^3 from it.

If the massive body approximated by a mass-point does not rotate ($\omega = 0$), then the space metric of a rotating massive body (10), which we have introduced and considered here, transforms into the Schwarzschild mass-point metric (6). In this case the obtained formula for the physically observable chr.inv.-scalar curvature (104) transforms into

$$C = \frac{r_g}{r^3}, \tag{107}$$

which is the same as the three-dimensional scalar curvature of a spherically symmetric gravitational field, which Landau and Lifshitz give in their *The Classical Theory of Fields* [11]; see page 325 of §100 in the 4th final English edition, or pages 378–379 of §97 in the 3rd French edition. The only difference is that their curvature has a negative sign. This is because in the years, when they wrote their book (the 1st edition was issued in 1939), Zelmanov's chronometrically invariant formalism had not yet been created. Therefore, Landau and Lifshitz believed that the three-dimensional components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$ create an observable metric tensor. On the contrary, the chronometrically invariant formalism clearly proves that the physically observable metric tensor that possesses all properties of the fundamental metric tensor throughout the three-dimensional spatial section associated with an observer (his observable three-dimensional space) is $h_{ik} = -g_{ik} + \frac{1}{c^2}v_i v_k$ (15). This is why their curvature of a non-rotating centrally symmetric gravitational field is negative, and the truly physically observable chr.inv.-curvature (107), which we have just deduced using the chronometrically invariant formalism, has a positive sign, as it should be according to the physical sense of this quantity.

Consider a few typical numerical examples of the curvature of space caused by rotating cosmic bodies.

The first typical example is the Sun: $r_\odot \simeq 7.0\times10^{10}$ cm, $M_\odot \simeq 2.0\times10^{33}$ gram, $r_{g\odot} = 2GM_\odot/c^2 \simeq 3.0\times10^5$ cm, $\omega_\odot \simeq \simeq 2.87\times10^{-6}$ sec^{-1} (we are considering the Carrington rotation of the Sun at the equator with a sidereal period of 25.38 days). According to the obtained formula (106), the expected constant curvature of space due to the proper rotation of the Sun is $C = 6\omega_\odot^2/c^2 \simeq 5.6\times10^{-32}$ cm^{-2}, while the variable curvature of space due to the gravitational field of the Sun at a distance of one solar radius $r_\odot$ from its centre (i.e., on the Sun's surface) is 4 orders of magnitude greater: $C = r_{g\odot}/r_\odot^3 \simeq \simeq 8.8\times10^{-28}$ cm^{-2}.

Since the curvature of space due to the Sun's rotation is constant, and the curvature due to its gravitational field decreases inversely proportional to r^3 from it, then there is a spherical surface in the cosmos on which these curvatures are equal to each other: $C = r_g/r^3 = 6\omega^2/c^2$. For the Sun, this is a spherical surface surrounding the Sun at a distance of

$$r = \sqrt[3]{\frac{c^2 r_{g\odot}}{6\omega_\odot^2}} \simeq 1.8\times10^{12}\ \text{cm} \simeq 25\, r_\odot\,. \qquad (108)$$

Starting from the distance $r \simeq 1.8\times10^{12}$ cm $\simeq 25\, r_\odot$ from the centre of the Sun, the contribution of the Sun's rotation to the observable curvature of space (it remains constant with distance) exceeds the contribution of the Sun's gravitational field (since it decreases inversely proportional to r^3). For comparison: Mercury, the closest planet to the Sun, orbits the Sun at a distance of $r = 57.9$ mln km $= 82.7\, r_\odot$.

For the Earth ($r_\oplus = 6.37\times10^8$ cm, $M_\oplus = 5.97\times10^{27}$ gram, $r_{g\oplus} = 0.884$ cm, $\omega_\oplus = 7.27\times10^{-5}$ sec^{-1}), the constant curvature of space caused by the Earth's rotation is $C = 6\omega_\oplus^2/c^2 \simeq \simeq 3.5\times10^{-29}$ cm^{-2} that is 3 orders of magnitude greater than the constant curvature $C = 6\omega_\odot^2/c^2 \simeq 5.6\times10^{-32}$ cm^{-2} caused by the rotation of the Sun. The curvature of space caused by the Earth's gravitational field on the Earth's surface ($r = r_\oplus$) is $C = r_{g\oplus}/r_\oplus^3 \simeq 3.4\times10^{-27}$ cm^{-2}.

At a distance of

$$r = \sqrt[3]{\frac{c^2 r_{g\oplus}}{6\omega_\oplus^2}} \simeq 2.93\times10^9\ \text{cm} \simeq 29\,300\ \text{km} \simeq 4.6\, r_\oplus \qquad (109)$$

from the centre of the Earth (or at an altitude of $h = r - r_\oplus \simeq \simeq 23\,000$ km $\simeq 3.6\, r_\oplus$ above the Earth's surface) the contributions of the Earth's rotation and its gravitational field to the curvature of space become equal to each other. At higher altitudes, the contribution of the Earth's rotation to the curvature of space, since it remains constant with altitude, is greater than the contribution of the Earth's gravitational field (the latter becomes comparatively negligible, since it decreases inversely proportional to r^3).

For our Galaxy ($r \simeq 30\,000$ pc $\simeq 10^{23}$ cm, $M \simeq 2\times10^{11} M_\odot$, $r_g \simeq 6\times10^{16}$ cm, $T \simeq 200$ mln years, $\omega = 2\pi/T \simeq 10^{-15}$ sec^{-1}), the constant curvature of space caused by its rotation is $C = = 6\omega^2/c^2 \simeq 7\times10^{-51}$ cm^{-2}, while the curvature caused by its gravitational field at its edge ($r \simeq 30\,000$ pc $\simeq 10^{23}$ cm) is 2 orders of magnitude weaker: $C = r_g/r^3 \simeq 6\times10^{-53}$ cm^{-2}. The distance from the Galactic centre, at which the contribution of the rotation of the Galaxy to the curvature of space becomes equal to the contribution of its gravitational field is

$$r = \sqrt[3]{\frac{c^2 r_g}{6\omega^2}} \simeq 2.1\times10^{22}\ \text{cm} \simeq 7\,000\ \text{parsec}. \qquad (110)$$

	$^*C = \frac{r_g}{r^3}$, cm^{-2}	$^\dagger C = \frac{6\omega^2}{c^2}$, cm^{-2}	$^\ddagger r = \sqrt[3]{\frac{c^2 r_g}{6\omega^2}}$
Galaxy	6×10^{-53}	7×10^{-51}	7 000 pc
Sun	8.8×10^{-28}	5.6×10^{-32}	$25\, r_\odot$
Earth	3.4×10^{-27}	3.5×10^{-29}	$4.6\, r_\oplus$
Pulsars (min)		1.9×10^{-21}	
Pulsars (max)		1.4×10^{-13}	

*The variable (decreasing) curvature of space caused by the gravitational field of the cosmic body at a distance equal to its radius from its centre.

†The constant curvature of space caused by the rotation of the cosmic body.

‡The distance from the centre of the cosmic body at which the contribution of its rotation to the curvature of space becomes equal to the contribution of its gravitational field.

The observed frequencies of radio-pulsars are in the range from $\omega_{\min} = 0.53$ to $\omega_{\max} = 4501$ sec^{-1}. Therefore, the constant curvature of space caused by pulsars is in the range of magnitudes from $C \simeq 1.9\times10^{-21}$ to $C \simeq 1.4\times10^{-13}$ cm^{-2}.

As a result of the above calculation, we arrive at the following conclusion:

Conclusion on the background curvature of space

> The curvature of space caused by the gravitational field of rotating massive bodies decreases inversely proportional to r^3 and, therefore, becomes negligibly small already in the immediate vicinity of these bodies, at a distance of a few of their radii from them. However, the rotation of these bodies creates a constant curvature field, which is much weaker than the curvature caused by their gravitational fields near these bodies, but does not depend on the distance to them. Moreover, such rapidly rotating cosmic objects as pulsars create strong fields of a constant curvature, the magnitude of which is many orders greater than the constant curvature fields caused by other rotating stars and Galaxies.
>
> It seems that the space of the entire Universe is filled with a constant curvature field that is the superposition of the constant curvature fields caused by rapidly rotating cosmic bodies such as pulsars. This is the basis for considering the background space of our Universe as a *constant curvature space*.

This is a very interesting theoretical discovery that requires further study and analysis by astronomers.

5 Einstein's field equations in the space of a rotating massive body

As mentioned on page 81, Einstein's equations are one of the necessary conditions for a space to be Riemannian. Therefore, the considered space metric of a rotating massive body (10) is Riemannian under some particular conditions (*Riemannian conditions*) by which the Einstein equations for this space metric vanish. Now our task is to find out the Riemannian conditions for the space metric (10).

As we showed above (52), the space of a rotating massive body, which we are considering, does not deform ($D_{ik}=0$), and is not filled with any distributed matter such as gas, dust, electromagnetic fields, etc. The latter means that the energy-momentum tensor of distributed matter is zero ($T_{\alpha\beta}=0$) and, hence, the entire right-hand side of the Einstein field equations is zero. With taking the above into account, the chr.inv.-Einstein equations (41–43) take the simplified form

$$A_{jl}A^{lj} + {}^*\nabla_j F^j - \frac{1}{c^2}F_j F^j = 0\,, \tag{111}$$

$$^*\nabla_j A^{ij} - \frac{2}{c^2}F_j A^{ij} = 0\,, \tag{112}$$

$$2A_{ij}A_{k\cdot}^{\cdot j} - \frac{1}{c^2}F_i F_k + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - c^2 C_{ik} = 0\,. \tag{113}$$

The *1st Riemannian condition* for the space metric of a rotating massive body (10), which we are considering, follows from the obtained scalar chr.inv.-Einstein equation (111). Since $A_{jl}A^{lj} = -A_{jl}A^{jl}$ is the square of the chr.inv.-tensor A_{jl} of the angular velocity of the rotation of space, taken with the opposite sign, and the Zelmanov operator of the chr.inv.-physical divergence (marked with a tilde)

$$^*\widetilde{\nabla}_j = {}^*\nabla_j - \frac{1}{c^2}F_j\,, \tag{114}$$

gives a divergence that is physically registered by the observer, for instance, $^*\widetilde{\nabla}_j F^j$ according to (31) is

$$^*\widetilde{\nabla}_j F^j = \frac{^*\partial F^j}{\partial x^j} + \Delta^j_{jl}F^l - \frac{1}{c^2}F_j F^j = {}^*\nabla_j F^j - \frac{1}{c^2}F_j F^j, \tag{115}$$

then the scalar chr.inv.-Einstein equation (111) gives:

The 1st Riemannian condition

> In the space of a rotating massive body, the physically observable rotation of space is always balanced by the physically observable divergence of the acting gravitational inertial force:
>
> $$A_{jl}A^{jl} = {}^*\widetilde{\nabla}_j F^j, \tag{116}$$
>
> or, which is the same,
>
> $$2\Omega^2 = {}^*\widetilde{\nabla}_j F^j. \tag{117}$$

P.S. The alternative form (117) of the 1st Riemannian condition (116) is obtained using the components of A_{jl} (59–66) that we have calculated earlier in the space of a rotating massive body, after which we have

$$A_{jl}A^{jl} = \frac{2\omega^2}{1+\frac{\omega^2 r^2\sin^2\theta}{c^2}} - \frac{3\omega^2 r_g \sin^2\theta}{r\left(1+\frac{\omega^2 r^2\sin^2\theta}{c^2}\right)} + \frac{\omega^2 r_g^2\sin^2\theta}{8r^2\left(1+\frac{\omega^2 r^2\sin^2\theta}{c^2}\right)\left(1-\frac{r_g}{r}\right)} = 2\Omega^2, \tag{118}$$

where Ω^2 is the square of the physically observable scalar angular velocity Ω (71) with which the space rotates.

The *2nd Riemannian condition* for the space metric of a rotating massive body follows from the obtained vector chr.inv.-Einstein equation (112):

The 2nd Riemannian condition

> In the space of a rotating massive body, the physically observable divergence of the rotation of space is always and everywhere equal to zero:
>
> $$^*\widetilde{\nabla}_j A^{ij} = 0\,, \tag{119}$$
>
> which means that the physically observable rotation of such a space is *homogeneous* (i.e., such a space rotates always and everywhere homogeneously).

P.S. And here is why. Using the definition of the operator of the chr.inv.-physical divergence $^*\widetilde{\nabla}_j$ (114) that is physically registered by the observer, we calculate the chr.inv.-physical divergence of the contravariant tensor of the angular velocity of rotation of space A^{ij}. According to the general formula for the chr.inv.-derivative $^*\nabla_j$ of an arbitrary contravariant tensor of the 2nd rank (36), we obtain

$$^*\widetilde{\nabla}_j A^{ij} = \frac{^*\partial A^{ij}}{\partial x^j} + \Delta^i_{jl}A^{jl} - \frac{1}{c^2}F_j A^{ij} + \Delta^l_{lj}A^{ij} - \frac{1}{c^2}F_j A^{ij} = {}^*\nabla_j A^{ij} - \frac{2}{c^2}F_j A^{ij}, \tag{120}$$

which completely coincides with the left-hand side of the obtained vector chr.inv.-Einstein equation (112), while the right-hand side of the equation is zero.

The *3rd and 4th Riemannian conditions* for the space metric of a rotating massive body follow from the obtained tensor chr.inv.-Einstein equation (113), re-written in the expanded component notation

$$2A_{1j}A_{1\cdot}^{\cdot j} - \frac{1}{c^2}F_1F_1 + {}^*\nabla_1 F_1 - c^2 C_{11} = 0\,, \tag{121}$$

$$2A_{1j}A_{2\cdot}^{\cdot j} - c^2 C_{12} = 0\,, \tag{122}$$

$$2A_{2j}A_{2\cdot}^{\cdot j} - c^2 C_{22} = 0\,, \tag{123}$$

$$2A_{3j}A_{3\cdot}^{\cdot j} - c^2 C_{33} = 0\,, \tag{124}$$

in accordance with the non-zero components of A_{ij}, F_i and C_{ik}, which we have calculated earlier (see above).

So, the *3rd Riemannian condition* follows from the first component (124). It says:

The 3rd Riemannian condition

In the space of a rotating massive body, the physically observable curvature of space in the radial direction $x^1 = r$ from the body is caused by both the physically observable rotation of space (the first term of the equation) and the physically observable divergence of the gravitational inertial force acting in the same radial direction (the second term):

$$2A_{13}A_{1\cdot}^{\cdot 3} + {}^*\widetilde{\nabla}_1 F_1 = c^2 C_{11}\,. \tag{125}$$

The *4th Riemannian condition* follows from the rest three non-zero components (122–124) of the tensor chr.inv.-Einstein equation:

The 4th Riemannian condition

In the space of a rotating massive body, the physically observable curvature of space in all other directions from the body, except for the radial direction $x^1 = r$ (in which the gravitational-inertial force acts), is caused only by the physically observable rotation of space:

$$\left.\begin{aligned} & 2A_{13}A_{2\cdot}^{\cdot 3} = c^2 C_{12} \\ & 2A_{23}A_{2\cdot}^{\cdot 3} = c^2 C_{22} \\ & 2\left(A_{31}A_{3\cdot}^{\cdot 1} + A_{32}A_{3\cdot}^{\cdot 2}\right) = c^2 C_{33} \end{aligned}\right\}. \tag{126}$$

P.S. It should be noted that the components A_{13} and A^{31} (59–62) of the chr.inv.-tensor of the angular velocity of rotation of space A_{ij} contain both terms determined only by the rotation of space and terms dependent on $r_g = 2GM/c^2$ (which includes the mass M of the attracting body). This is because the chr.inv.-tensor A_{ij} (18) by definition takes into account the effect of the acting gravitational inertial force F_i onto the tensor A_{ij}, thereby making A_{ij} a truly physically observable quantity dependent on the physical properties of space.

Therefore, when we say a "physically observable rotation of space" or a "physically observable quantity" in general, we mean a chronometrically invariant physical quantity, actually registered by the observer in his real measurements and, therefore, dependent on the physical properties of space.

Finally, summing up the results obtained in this Section of the present work, we can state the following:

Conclusion

Under the four Riemannian conditions deduced above, the space metric of a rotating massive body (10) that we have introduced and studied in this paper satisfies Einstein's field equations (thereby turning them into zero identities) and is therefore proven to be Riemannian and can be used in General Relativity.

The above conclusion has great significance for General Relativity, cosmology and astrophysics. This is because the introduced (and now proven) space metric of a rotating spherical body, approximated by a mass-point, is not only a new metric to General Relativity, which is an extension and replacement of the classical Schwarzschild mass-point metric (which does not take into account the rotation of space). The introduced space metric is the *main space metric in the Universe*, characterizing the physically observable field of any real cosmic body, be it a planet, star, galaxy or something else (since all real cosmic bodies rotate).

6 Deflection of light rays and mass-bearing particles in the space of a rotating massive body

In the previous study [2], we considered massless (light-like) and mass-bearing particles moving in the space of a rotating body, where the gravitational field created by the body was so weak that its influence on the moving particles could be neglected. The solutions obtained for the chronometrically invariant equations of motion of free massless and free mass-bearing particles in the space of a rotating body showed that their physically observable motion should deviate from a straight line due to the curvature of space caused by the rotation of space. In other words, the trajectories of light rays and mass-bearing particles should be deflected near a rotating body due to the curvature of space caused by its rotation.

These are two new fundamental effects of General Relativity, in addition to the deflection of light rays in the field of a gravitating body (known in Einstein's theory from the very beginning).

In the paper [2], the mentioned two new effects were calculated in the space metric of a rotating body, where $g_{00} = 1$, i.e., the gravitational potential is zero $\mathrm{w} = c^2(1 - \sqrt{g_{00}}) = 0$, in order to show these effects of the rotation of space in their "pure form" (i.e., in the absence of the gravitational field).

Now we are going to calculate these two new effects of General Relativity anew, now in the space of a rotating massive body, the metric of which (10) takes the gravitational field of the rotating body into account: the gravitational potential is $\mathrm{w} \neq 0$ and, hence, $g_{00} < 1$; for details, see the space metric (10) that we are considering. This, in contrast to the abstract case considered in the previous work [2], is a *real physical case*, since all real cosmic bodies in the Universe such as planets, stars, galaxies and something else not only rotate, but also have their own gravitational field.

So, let us begin. The chr.inv.-equations of motion are the physically observable chr.inv.-projections of the general covariant four-dimensional equations of motion onto the time line and the three-dimensional spatial section associated with a particular observer. Such projections are invariant throughout the spatial section of the observer (his physically observable three-dimensional space) and are expressed through the physical properties of his local reference space. A detailed derivation of the chr.inv.-equations of motion can be found in the monographs [7, 8], the first of which is devoted to free (geodesic) motion of particles, while the second is a study of non-geodesic motion.

The chr.inv.-equations of motion of a free mass-bearing particle have the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = 0, \tag{127}$$

$$\frac{d(m\mathrm{v}^i)}{d\tau} + 2m\left(D^i_k + A_{k\cdot}^{\cdot i}\right)\mathrm{v}^k - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0, \tag{128}$$

and the chr.inv.-equations of motion of a free massless (light-like) particle have the form

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i + \frac{\omega}{c^2} D_{ik} c^i c^k = 0, \tag{129}$$

$$\frac{d(\omega c^i)}{d\tau} + 2\omega\left(D^i_k + A_{k\cdot}^{\cdot i}\right)c^k - \omega F^i + \omega\Delta^i_{nk} c^n c^k = 0, \tag{130}$$

where the first (scalar) chr.inv.-equation of motion is the projection of the general covariant equations of motion onto the observer's time line, and the second (vector) chr.inv.-equation of motion is the projection onto his spatial section (his three-dimensional space).

Here m is the relativistic mass of the mass-bearing particle, ω is the relativistic frequency of the massless (light-like) particle, the physically observable time interval $d\tau$ (11) is expressed through the gravitational potential w (12) and the linear velocity of the rotation of space v_i (13) as

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right)dt - \frac{1}{c^2}\, v_i\, dx^i, \tag{131}$$

and the chr.inv.-vector of the physically observable velocity of the particle has the form

$$\mathrm{v}^i = \frac{dx^i}{d\tau}, \qquad \mathrm{v}_i\mathrm{v}^i = h_{ik}\mathrm{v}^i\mathrm{v}^k = \mathrm{v}^2,$$

which, in the case of massless (light-like) particles, transforms into the chr.inv.-vector of the physically observable velocity of light, for which $c_i c^i = h_{ik} c^i c^k = c^2 = const$ (despite the fact that its individual components c^i are variables depending on the properties of space).

Since the space of a rotating massive body, which we are considering, does not deform ($D_{ik} = 0$), then the chr.inv.-equations of motion simplify by vanishing D_{ik}. For a free mass-bearing particle they take the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i = 0, \tag{132}$$

$$\frac{d(m\mathrm{v}^i)}{d\tau} + 2mA_{k\cdot}^{\cdot i}\mathrm{v}^k - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0, \tag{133}$$

while for a massless (light-like) particle they become

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i = 0, \tag{134}$$

$$\frac{d(\omega c^i)}{d\tau} + 2\omega A_{k\cdot}^{\cdot i} c^k - \omega F^i + \omega\Delta^i_{nk} c^n c^k = 0. \tag{135}$$

6.1 Solving the chr.inv.-scalar equation of motion

Since the rotating massive body we are considering is not a gravitational collapsar, i.e., its physical radius r is much greater than its gravitational radius ($r \gg r_g$), then according to the formulae for F_i (55) and F^i (56) obtained for the field of a rotating massive body we have

$$F_1 = F^1 = -\frac{c^2 r_g}{2r^2} = -\frac{GM}{r^2}\,. \tag{136}$$

With this fact taken into account, the scalar equation of motion of a free mass-bearing particle (132), in the case when it travels along the radial direction $x^1 = r$ from the rotating massive body, takes the form

$$\frac{dm}{m} = -\frac{GM}{c^2}\frac{dr}{r^2}\,, \tag{137}$$

which is a simple differential equation $\frac{dy}{y} = -a\frac{dx}{x^2}$ or, which is the same, $d\ln m = -a\frac{dx}{x^2}$. It solves as $y = Ce^{a/x}$, where the integration constant C in this case is $C = m_{(r\,=\,r_0\,=\,0)} = m_0$. As a result, we obtain that the scalar equation of motion of a free mass-bearing particle (132) solves as

$$m = m_0\, e^{\frac{GM}{c^2 r}} \simeq m_0\left(1 + \frac{GM}{c^2 r}\right). \tag{138}$$

For example, according to the obtained solution, the mass of a body located on the Earth's surface ($M_\oplus = 5.97\times10^{27}$ gram, $r_\oplus = 6.37\times10^{8}$ cm) is greater than its mass, measured when the body was located at a distance of the Moon's orbit from the Earth ($r = 3.0\times10^{10}$ cm) by a value of $1.5\times10^{-11}\,m_0$ due to the greater magnitude of the Earth's gravitational field potential on the Earth's surface.

The scalar equation of motion of a free massless (light-like) particle (134), when it radially travels in space, solves in the same way. Its solution has the form

$$\omega = \omega_0\, e^{\frac{GM}{c^2 r}} \simeq \omega_0\left(1 + \frac{GM}{c^2 r}\right). \tag{139}$$

This solution means that photons gain an additional energy (and frequency) from the gravitational field. For example, a photon with a frequency ω_0 at the moment of emission from the surface of a star has a lower frequency $\omega < \omega_0$ (and energy) when it moves away from this star at some distance. The greater the gravitational field potential (i.e., the closer the photon is to the source of the gravitational field), the more the photon's frequency is redshifted. According to the above solution, the photon's redshift z in the field of a rotating massive body is determined as (where $r_0 < r_1$)

$$z = \frac{\omega_0 - \omega}{\omega} = e^{\frac{GM}{c^2 r_0} - \frac{GM}{c^2 r_1}} - 1 \simeq \frac{GM}{c^2 r_0} - \frac{GM}{c^2 r_1}\,. \quad (140)$$

So, by solving the chr.inv.-scalar equation of free mass-bearing and massless (light-like) particles we have deduced two effects. First, we have deduced the well-known relativistic effect of the decrease in the mass of a body with height above the Earth's surface (138). Second, we have deduced the gravitational redshift (140), which is also the effect of General Relativity, known from the very beginning and first registered in the spectra of white dwarfs.

Landau and Lifshitz derived these effects from the conservation of energy of a free particle travelling in a stationary gravitational field; for example, see [11, §88]. Zelmanov followed the same way of derivation. However, the new derivation method presented here, based on the integration of the chr.inv.-scalar geodesic equation, allows us to represent the mentioned effects as something not specifically related to the stationary gravitational field, but as general effects of General Relativity that can be calculated in any metric space.

Note that the chr.inv.-scalar equation of motion does not take the rotation of space into account. Therefore, the obtained solutions of the equation (and the effects following from them) coincide with the solutions in a space of the Schwarzschild's mass-point field (which does not rotate).

6.2 Solving the chr.inv.-vector equation of motion

Let us now solve the chr.inv.-vector equation of motion. For a free mass-bearing particle, radially travelling in the space of a rotating massive body, this is the equation (133), while for a massless particle this is the equation (135).

Since the chr.inv.-vector equation of motion depends on the tensor of the angular velocity of rotation of space A_{ik}, we expect that its solution will reveal new effects of General Relativity, previously unknown in the framework of the non-rotating Schwarzschild mass-point metric.

The chr.inv.-vector equations of motion are unsolvable in their general form (133) and (135), because they require substitution of the solutions for the particle's mass m (138) and frequency ω (139) obtained from the chr.inv.-scalar equations of motion, which in turn contain an exponential function of distance r (as a result, each term of the vector equations of motion would contain this complicated function).

Therefore, we will solve the chr.inv.-vector equations of motion in an approximation that the mass-bearing particle's mass m and the massless (light-like) particle's frequency ω remain constant during the travel. This approximation can be used in problems of motion near planets and stars, because, as shown above, the mass m_0 of a body located on the surface of the Earth is only $1.5\times10^{-11}\,m_0$ greater than its mass measured when the body was at the distance of the Moon.

In addition to the assumed approximations $m = const$ and $\omega = const$, we assume, as well as when we solved the scalar equations of motion above, that the rotating massive body that is the source of the gravitational field is not a gravitational collapsar ($r \gg r_g$), so the acting gravitational inertial force is expressed in the simplified form (136).

Moreover, to further simplify the vector equations of motion, we assume that the particle travels at a very high radial velocity v_1 in the equatorial plane along the radial axis $x^1 = r$ towards the origin of the coordinates (the body's centre). For example, it could be a particle falling from the near-Earth space in the equatorial plane onto the Earth's surface. In this case: a) the polar angle is $\theta = \frac{\pi}{2}$ and, therefore, $\cos\theta = 0$ and $\sin\theta = 1$, b) the velocities v^2 and v^3, with which the particle is deflected along the geographical latitudes and longitudes, are negligible compared to its radial velocity v^1.

Finally, we assume that the body that is the source of the field rotates (synchronously with its entire space) with slow linear velocities compared to the velocity of light.

Now we substitute into the chr.inv.-vector equations of motion (133) and (135) the components of the gravitational inertial force F_i (136), the tensor of the angular velocity of rotation of space A_{ik} (59–66), and also the inhomogeneity coefficients of space, a.k.a. the Christoffel symbols Δ^i_{nk} (77–83), which we have calculated above in this paper in accordance with the space metric of a rotating massive body. As a result, after using the above approximations, we obtain the vector equations of motion in component notation.

The resulting chr.inv.-vector equation of motion of a free mass-bearing particle, in component notation derived after some algebra, has the form

$$\left.\begin{aligned} &\frac{d\mathrm{v}^1}{d\tau} - 2\,\omega\, r\, \mathrm{v}^3 - r\,\mathrm{v}^2\mathrm{v}^2 - r\,\mathrm{v}^3\mathrm{v}^3 + \frac{GM}{r^2} = 0 \\ &\frac{d\mathrm{v}^2}{d\tau} + \frac{2}{r}\,\mathrm{v}^1\mathrm{v}^2 = 0 \\ &\frac{d\mathrm{v}^3}{d\tau} + \frac{2\,\omega}{r}\,\mathrm{v}^1 + \frac{2}{r}\,\mathrm{v}^1\mathrm{v}^3 = 0 \end{aligned}\right\}. \quad (141)$$

and for a massless (light-like) particle the resulting chr.inv.-vector equation of motion has the components

$$\left.\begin{aligned} &\frac{dc^1}{d\tau} - 2\,\omega\, r\, c^3 - r\,c^2c^2 - r\,c^3c^3 + \frac{GM}{r^2} = 0 \\ &\frac{dc^2}{d\tau} + \frac{2}{r}\,c^1c^2 = 0 \\ &\frac{dc^3}{d\tau} + \frac{2\,\omega}{r}\,c^1 + \frac{2}{r}\,c^1c^3 = 0 \end{aligned}\right\}. \quad (142)$$

As can be seen from the equations, the gravitational field of a rotating body makes a contribution in the form of only the last term in the first equation, i.e., it affects the motion of the particle only along the radial direction $x^1 = r$. On the contrary, the rotation field of this body makes a contribution to the motion of the particle both along the radial axis r and along the equatorial (longitudinal) coordinate axis φ and the latitudinal coordinate axis θ.

As is seen, the vector equations of motion for a mass-bearing particle and a massless (light-like) particle are identical. The only difference is that the equations for a massless (light-like) particle contain the physically observable velocity of light c^i instead of the mass-bearing particle's physically observable velocity v^i. For this reason, we will solve only the equation of motion of a mass-bearing particle (the solution for a massless particle will coincide).

The problem is that this system of differential equations is unsolvable even when considered in the above simplified form. Therefore, we will solve them using the *small parameter method*.

Namely, — we assume that the radially travelling particle gains only a very small increment or decrement α' to its initial numerical value v^1. This allows us to set $\mathrm{v}^1 = const$ in the third equation of the system, which is the equation of motion along the equatorial (longitudinal) axis φ, and in the second equation that is the equation of motion along the latitudinal axis θ. Then, using the obtained solutions of the third and second equations, we will solve the first equation (the equation of motion along the radial axis r) with respect to $\mathrm{v}^1 + \alpha'$, i.e., with respect to the small parameter α.

But even now, without solving the vector equations of motion, but only based on their general form given above, we see that three effects are possible, namely:

1. The deflection of a radially travelling particle along the geographic longitudes due to the influence of the rotation of space (the third equation);
2. The deflection of a radially travelling particle along the geographic latitudes due to the influence of the rotation of space (the second equation);
3. The acceleration or braking of a radially travelling particle in the radial direction due to both the gravitational field and the rotation of space (the first equation).

6.2.1 Solving the third vector equation of motion

The third equation is an equation of motion along the equatorial axis φ. This is a differential equation of the form

$$y' + a y + b = 0, \tag{143}$$

or, which is the same,

$$\varphi'' + a \varphi' + b = 0, \tag{144}$$

where the variable y and the constants used are

$$y = \mathrm{v}^3 = \frac{d\varphi}{d\tau}, \tag{145}$$

$$a = \frac{2}{r}\,\mathrm{v}^1 = const, \quad b = \frac{2\omega}{r}\,\mathrm{v}^1 = const. \tag{146}$$

The above equations (143) and (144) solve as

$$y = \frac{C}{e^{ax}} - \frac{b}{a}, \qquad \varphi = \frac{C_1}{e^{ax}} - \frac{bx}{a} + C_2. \tag{147}$$

Substituting the integration constants, calculated from the initial conditions $x = x_0 = 0$ and $y = y_0 = 0$,

$$C = \frac{b}{a} = \omega, \tag{148}$$

$$C_1 = -\frac{b}{a^2} = -\frac{\omega r}{2\,\mathrm{v}^1}, \quad C_2 = -C_1 = \frac{\omega r}{2\,\mathrm{v}^1}, \tag{149}$$

below we represent the above solutions of the equations (143) and (144) in their final form.

As a result, the obtained solution of the equation (143), which is the physically observable velocity $y = \mathrm{v}^3$ of the radially travelling particle along the equatorial axis φ at the point of arrival on the surface of the rotating body (onto which the particle was falling down from the cosmos along the radial direction r), takes the final form

$$\mathrm{v}^3 = -\omega + \omega e^{-\frac{2}{r}\mathrm{v}^1\tau}. \tag{150}$$

The first term here is the basic equatorial velocity of the particle, the cause of which is the shift of its equatorial coordinate φ towards negative numerical values due to the turn of the rotating massive body during the time of the particle's travel to the body's surface.

The second term is absent in the classical theory. This additional term reveals an additional velocity gained by the free falling mass-bearing particle along the equatorial coordinate φ (geographical longitudes) of the rotating massive body in the direction, opposite to its rotation.

In turn, the obtained solution for the equatorial coordinate φ of the particle's point of arrival, which is the solution of the equation (144), takes the final form as follows

$$\varphi = \varphi_0 - \omega\tau + \frac{\omega r}{2\,\mathrm{v}^1}\left(1 - e^{-\frac{2}{r}\mathrm{v}^1\tau}\right). \tag{151}$$

The first and second terms of the solution are known in the classical theory.

The third, additional term of this solution, unknown in the classical theory, reveals a deflection of the free falling mass-bearing particle along the equatorial coordinate φ (geographical longitudes) of the rotating massive body in the direction, opposite to its rotation.

Respectively, the solutions of the third vector equation of motion for a massless (light-like) particle, such as a photon, have the same form

$$c^3 = -\omega + \omega e^{-\frac{2}{r}c^1\tau}, \tag{152}$$

$$\varphi = \varphi_0 - \omega\tau + \frac{\omega r}{2c^1}\left(1 - e^{-\frac{2}{r}c^1\tau}\right), \tag{153}$$

where the mass-bearing particle's velocity is replaced with the physically observable velocity of light.*

These solutions show another new effect of the rotation of space, which is absent in the classical theory and is revealed by the second term of the solution (152) and the third term of the solution (153). This is an additional deflection of a light ray travelling towards the surface of a rotating massive body, which occurs along the equatorial coordinate φ (geographical longitudes) of the body in the direction, in which the body rotates.

Note that the solutions of the third vector equation of motion, which we have derived above in the field of a rotating massive body with a significant gravitational field, coincide with those derived earlier [2] in the field of a rotating body, the gravitational field can be neglected (i.e., in the absence of the gravitational field). This is because the acting gravitational force takes effect on only the first vector equation of motion (along the radial axis r), but is not included into the second and third vector equations of motion (along the latitudinal polar coordinates θ and the equatorial longitudinal coordinates φ).

For this reason, the numerical examples of the solutions will be identical to those calculated in the previous paper [2] in the absence of the gravitational field. Therefore, we now reproduce the examples here in short from [2].

Thus, the curvature of space caused by the rotation of the Earth around its axis ($\omega_\oplus = 1$ rev/day $= 1.16\times10^{-5}$ rev/sec, $r_\oplus = 6.37\times10^{8}$ cm) deflects a light ray arriving at the Earth's surface from the Moon ($\tau = 1$ sec) along the geographical longitudes φ in the direction of the Earth's rotation. The angle of deflection of the light ray is†

$$\Delta\varphi = \frac{\omega_\oplus r_\oplus}{2c^1}\left(1 - e^{-\frac{2}{r}c^1\tau}\right) \simeq 1.2\times10^{-7}\,\text{rev} \simeq 0.16'', \quad (154)$$

where the deflection of the light ray is mainly due to the first term, and the second term, depending on the travel time τ, is equal to 1.5×10^{-41} and, therefore, can be neglected.

The magnitude of this effect increases with the radius and rotation velocity of the cosmic body. Thus, a light ray arriving at the Sun ($\omega_\odot = 4.5\times10^{-7}$ rev/sec, $r_\odot = 7.0\times10^{10}$ cm) is deflected by the curvature of space caused by the Sun's rotation by an angle, the numerical value of which is

$$\Delta\varphi \simeq 5.3\times10^{-7}\,\text{rev} \simeq 0.68'', \quad (155)$$

the value of which is much larger in the case of a rapidly rotating star, such as Wolf-Rayet stars or neutron stars.

6.2.2 Solving the second vector equation of motion

The second vector equation of motion is an equation of motion along the geographical latitudes, where the latitudinal coordinate θ (polar angle) is measured from the North Pole. This is a differential equation of the form

$$y' + ay = 0, \quad (156)$$

or, with respect to the latitudinal coordinates θ,

$$\theta'' + a\theta' = 0, \quad (157)$$

where the variable y and the constant a are

$$y = \mathrm{v}^2 = \frac{d\theta}{d\tau}, \qquad a = \frac{2}{r}\,\mathrm{v}^1 = const. \quad (158)$$

These equations solve as

$$y = \frac{C}{e^{ax}}, \qquad \theta = \frac{C_1}{e^{ax}} + C_2, \quad (159)$$

where the integration constants are calculated from the initial conditions $x = x_0 = 0$ and $y = y_0 = 0$. They are $C = 0$, $C_1 = 0$ and $C_2 = \theta_0$.

Thus, the final solutions of the second vector equation of motion have the following form

$$\mathrm{v}^2 = 0, \qquad \theta = \theta_0, \quad (160)$$

which means that a particle travelling radially towards the surface of a massive rotating body is not deflected along the geographical latitudes.

6.2.3 Solving the first vector equation of motion

The first vector equation of motion is an equation of motion along the first (radial) coordinate axis r.

This equation contains contributions from both the rotation of space (the second term) and the gravitational field (the last term of the equation). Therefore, its solution will differ from the solution of the first equation of motion in the field of a rotating body, the gravitational field of which can be neglected (i.e., in the absence of the gravitational field).

Assume that the particle's velocity in the radial direction gains only a very small increment or decrement α' to its initial numerical value v^1. In other words, we assume $\mathrm{v}^1 = const$ and, therefore, solve the first vector equation of motion with respect to the sum $\mathrm{v}^1 + \alpha'$, i.e., with respect to the small parameter α.

Taking the obtained solutions $\mathrm{v}^3 = -\omega$ and $\mathrm{v}^2 = 0$ into account, the first vector equation of motion is reduced to

$$\frac{d\mathrm{v}^1}{d\tau} + \omega^2 r + \frac{GM}{r^2} = 0, \quad (161)$$

where r is the radius of the rotating body, and M is its mass. This is a differential equation having the form

$$y' + b = 0, \quad (162)$$

*Note that, despite the components of the physically observable velocity of light are variables depending on the properties of space, its square remains constant $c_i c^i = h_{ik} c^i c^k = c^2 = const$).

†In this case, the physically observable velocity of light has a negative numerical value of $c^1 = -3\times10^{10}$ cm/sec, since the velocity of light vector is directed towards the Earth, i.e., opposite to the radial coordinates r measured from the centre of the Earth.

or, with respect to the small parameter α,

$$\alpha'' + b = 0\,, \tag{163}$$

where the variable y and the constant b are

$$y = \alpha'\,, \qquad b = \omega^2 r + \frac{GM}{r^2} = const. \tag{164}$$

The above equations (162) and (163) solve as

$$y = C - bx\,, \qquad \alpha = -\frac{bx^2}{2} + C_2\,x + C_1\,, \tag{165}$$

where the integration constants, calculated from the initial conditions $x = x_0 = 0$, $\alpha = \alpha_0 = 0$ and $y = y_0 = 0$, are zero. As a result, the solutions of the equations (162) and (163) take their final form

$$\alpha' = -\omega^2 r\tau - \frac{GM}{r^2}\,\tau\,, \qquad \alpha = -\frac{\omega^2 r}{2}\,\tau^2 - \frac{GM}{2r^2}\,\tau^2. \tag{166}$$

The second terms in the solutions are the contribution of the gravitational field, created by the rotating massive body, which is the well-known effect of the classical theory. The terms reveal, respectively, the additional radial velocity gain by the falling particle (in the solution for α') and also the reduction of the distance travelled by the particle (in the solution for α), all due to the influence of the gravitational field attracting the particle to the rotating body.

However, the first terms in the solutions are absent in the classical theory. They show, respectively, the additional negative radial velocity (in the solution for α') and the stretching in the distance travelled by the particle (in the solution for α) due to the influence of the rotation of space of the gravitating body onto which the particle falls.

We see that here only the rotation of space produces a new effect of General Relativity in addition to the classical theory (i.e., the gravitational field of the rotating body does not produce a new additional effect).

In the absence of the gravitational field, the obtained solutions (166) coincide with those obtained in the previous paper [2] for a particle travelling towards a rotating body, the gravitational field of which can be neglected.

In fact, the new effect revealed by the first terms of the solutions (166) means that a mass-bearing particle or a light ray reaches a rotating massive body later due to the "stretching" of its path of travel due to the curvature of space caused by the rotation of space of the body, i.e., the mass-bearing particle or the light ray arrives at the rotating body with a time delay compared if the body did not rotate.

These new effects are the same for both mass-bearing and massless (light-like) particles. For example, the increment of the path length travelled by a light ray from the Moon to the Earth, and also the delay in its travel time are

$$\alpha = -\frac{\omega_{\oplus}^2\,r_{\oplus}}{2}\,\tau^2 \simeq -1.7\ \text{cm}, \tag{167}$$

$$\Delta\tau = \frac{\alpha}{c^1} \simeq 5.7\times10^{-11}\ \text{sec}, \tag{168}$$

and for a light ray that travelled from the Earth to the Sun the increment of the travelled path length and the delay in its travel time are

$$\alpha = -\frac{\omega_{\odot}^2\,r_{\odot}}{2}\,\tau^2 \simeq -6.6\times10^4\ \text{cm}, \tag{169}$$

$$\Delta\tau = \frac{\alpha}{c^1} \simeq 2.2\times10^{-6}\ \text{sec}, \tag{170}$$

which are the same as those calculated in the previous paper [2] in the field of a rotating body, the gravitational field of which can be neglected.

6.2.4 Conclusion

In concluding this Section of the present paper, let us formulate the two new effects of General Relativity calculated here in the field of a rotating massive body:

The 1st new effect of General Relativity

> A mass-bearing particle radially falling onto the surface of a rotating body gains an additional velocity, directed along the equatorial coordinate φ (geographical longitudes) of the body in the opposite direction of its rotation, thereby causing a deflection of the particle in the longitudinal direction φ.
>
> In addition, the radially falling mass-bearing particle arrives at the rotating body with a time delay compared if the body did not rotate.
>
> This happens due to the "stretching" of the rotating body's space along the equatorial coordinate φ (along the geographical longitudes) and the radial direction r (towards the body) as a result of the curvature of space, caused by its rotation (together with the body).

The 2nd new effect of General Relativity

> A light ray radially spreading towards the surface of a rotating body acquires an additional deflection upon arrival along the equatorial latitudinal coordinate φ of the body in the direction, in which the body rotates.
>
> In addition, the radially spreading light ray arrives at the rotating body with a time delay compared if the body did not rotate.
>
> This deflection of the light ray and the delay in its arrival at the rotating body occurs due to the "stretching" of the rotating body's space along the equatorial coordinate φ (along the geographical longitudes) and the radial direction r (towards the body), which are the result of the curvature of space, caused by its rotation (together with the body).

The physical origin of the new effects is obvious from our above calculation of the curvature of space, which we found to be caused by not only the gravitational field but also the rotation of space:

On the origin of the new effects

As has been found, the rotation of any body curves space in the direction of its rotation and to the centre of this body (the centre of rotation), thereby creating a "slope of the hill" slowing "down" along the equator in the direction, in which this body rotates, and also to the centre of this body.

In addition, the gravitational field created by the rotating body also curves space, making its own contribution in the form of the curvature of space towards the body's centre.

As a result, due to the created curvature of space, a mass-bearing particle or a light ray freely travelling towards a rotating massive body "rolls down the curvature hill" of space along the equator of the body in the direction of the body's rotation (the contribution of the rotation of space), and also "rolls" towards the centre of the body (the combined contribution of the rotation of space and the gravitational field).

7 Length stretching and time loss/gain in the space of a rotating massive body

According to the chronometrically invariant formalism, the three-dimensional physically observable chr.inv.-interval $d\sigma$ (14) and the physically observable time interval $d\tau$ (11)

$$d\sigma^2 = h_{ik}\,dx^i dx^k, \quad d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right)dt - \frac{1}{c^2}\,v_i\,dx^i \quad (171)$$

depend on the chr.inv.-metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2}\,v_i v_k$ (15), the gravitational field potential w (12) and the linear velocity of the rotation of space v_i (13). Thus, we can calculate $d\sigma$ and $d\tau$ in the space of any particular metric, for which we have previously calculated the quantities h_{ik}, w and v_i.

Let us now calculate the length of a rigid rod and the time interval in the field of a rotating massive body.

7.1 Length stretching

Let us substitute into the formula for $d\sigma$ the non-zero components h_{ik} (47) that we have calculated according to the space metric of a rotating massive body (10).

Thus, we obtain the physically observable length dl of a rigid rod, installed in stages along each of the coordinates

$$dl_r = \sqrt{h_{11}\,dr^2} = \frac{dr}{\sqrt{1-\frac{r_g}{r}}} = \frac{dl_0}{\sqrt{1-\frac{r_g}{r}}}\,, \quad (172)$$

$$dl_\theta = \sqrt{h_{22}\,d\theta^2} = r\,d\theta = dl_0\,, \quad (173)$$

$$dl_\varphi = \sqrt{h_{33}\,d\varphi^2} = \sqrt{1 + \frac{\omega^2 r^2 \sin^2\theta}{c^2}}\; r\sin\theta\,d\varphi = = \sqrt{1 + \frac{\omega^2 r^2 \sin^2\theta}{c^2}}\; dl_0\,, \quad (174)$$

where $dr = dl_0$ is the length of an elementary segment along the radial axis r, $r\,d\theta = dl_0$ is the length of an elementary arc along the latitudinal axis θ (the polar angle θ is measured from the North Pole), and $r\sin\theta\,d\varphi = dl_0$ is the length of an elementary arc along the equatorial latitudinal axis φ.

As is seen from the above calculation, a rigid rod located in the field of a rotating massive body (say, in the field of the Earth or the Sun) retains its original physically observable length dl_0, when installed along the geographical latitudes ($dl_\theta = dl_0$).

In contrast, when the rod installed in the position along the radial coordinate r, i.e., in the direction towards the centre of the rotating massive body (along its radius), its physically observable length dl_r is greater than its original length dl_0 by a small value δl_r

$$dl_r = \sqrt{h_{11}\,dr^2} = \frac{dl_0}{\sqrt{1-\frac{r_g}{r}}} \simeq \left(1 + \frac{r_g}{2r}\right)dl_0\,, \quad (175)$$

$$\delta l_r \simeq \frac{r_g}{2r}\,dl_0 \simeq \frac{1}{2}\,C\,r^2 dl_0\,, \quad (176)$$

which is determined by the curvature of space $C = \frac{r_g}{r^3}$ caused by the gravitational field of the rotating body. See the second term in the formula for the physically observable curvature C (106) of the space of a rotating massive body, which we have derived above in this paper.

And also, when the rod is installed in the position along the equatorial coordinate φ, i.e., in the direction along the geographical longitudes along which the massive body (say, the Earth or the Sun) rotates around its own axis, its physically observable length dl_φ is greater than its original length dl_0 by a small value δl_φ

$$dl_\varphi = \sqrt{1 + \frac{\omega^2 r^2 \sin^2\theta}{c^2}}\; dl_0 \simeq \left(1 + \frac{\omega^2 r^2 \sin^2\theta}{2c^2}\right)dl_0\,, \quad (177)$$

$$\delta l_\varphi \simeq \frac{\omega^2 r^2 \sin^2\theta}{2c^2}\,dl_0 \simeq \frac{1}{12}\,C\,r^2 \sin^2\varphi\,dl_0\,, \quad (178)$$

determined by the curvature of space $C = \frac{6\,\omega^2}{c^2}$ created by its rotation (together with the massive body) and is expressed with the first term in the formula for the physically observable curvature C (106), which we have derived in this paper.

As a result of the above derivation, we obtain the 3rd new effect of General Relativity:

The 3rd new effect of General Relativity

A rigid rod installed along the radial coordinate in the field of a rotating massive body (i.e., in the direction to the body's centre) acquires an additional length. This additional length is determined by the curvature of the body's space caused by its gravitational field.

In addition, if the rod is installed along the equatorial coordinate φ (i.e., along the geographical longitudes of the body), then its length acquires an additional length determined by the curvature of the body's space caused by its rotation.

This effect of length stretching of a rod in the field of a rotating massive body is due to the "stretching" of the body's space along the radial direction r (towards the body) caused by its gravitational field, and along the equatorial coordinate φ (along the geographical longitudes), caused by the rotation of the body's space (together with the body).

In other words, a rod in the field of a rotating massive body is "stretched" together with the "stretching" of the coordinate grid of space in the radial and equatorial directions. The "stretching" of the grid of space in the radial direction occurs due to the curvature of the body's space (the funnel of space) in this direction, caused by its gravitational field. Whereas the "stretching" of the coordinate grid of space along the equatorial coordinates is caused by the curvature of the body's space due to its rotation in this direction.

For example, the length stretching of a rod installed at the equator of the Earth ($\omega_\oplus = 1$ rev/day $= 7.27\times10^{-5}$ sec^{-1}, $r_\oplus = 6.37\times10^{8}$ cm) in the direction along the longitudinal axis φ, i.e., along the equator, has a numerical value of

$$\delta l_\varphi \simeq \frac{\omega_\oplus^2 r_\oplus^2 \sin^2\theta}{2c^2}\, dl_0 \simeq 1.2\times10^{-12}\, dl_0 \qquad (179)$$

of the original length dl_0 of the rod.

The length stretching of a rod installed vertically on the Earth's surface, has a numerical value of

$$dl_r \simeq \frac{r_{g\oplus}}{2r_\oplus}\, dl_0 \simeq 7.0\times10^{-10}\, dl_0\,. \qquad (180)$$

This length stretching effect is maximum at the equator, where the curvature and "stretching" of the Earth's space caused by the Earth's gravitational field are maximum (since the Earth is oblate towards the equator), and the curvature and "stretching" of the Earth's space caused by the Earth's rotation are also maximum. This length stretching effect decreases towards the geographical poles, where the length stretching caused by the rotation of the Earth's space vanishes (since $\sin\theta = 0$ at the poles), and the length stretching caused by the gravitational field is a little lesser than at the equator.

7.2 Time loss/gain

Let us now substitute into the general formula for the physically observable interval $d\tau$ the gravitational potential w (54) and the linear velocity of the rotation of space $v_3 = \omega r^2 \sin^2\theta$ (45) that we have calculated above in this paper among the other characteristic of the space metric of a rotating massive body (10).

Thus, we obtain the physically observable time interval $d\tau$, which will be registered by an observer travelling along the equatorial direction φ (i.e., along the geographical longitudes) in the space of a rotating massive body

$$d\tau = \sqrt{1-\frac{r_g}{r}}\, dt - \frac{1}{c^2}\, v_3\, u^3\, dt = \sqrt{1-\frac{r_g}{r}}\, dt - \frac{\omega r^2 \sin^2\theta}{c^2}\, u^3\, dt\,, \qquad (181)$$

where $u^3 = \frac{d\varphi}{dt}$ is the coordinate velocity of the observer in the equatorial direction $x^3 = \varphi$, along which he travels.

The first term in this formula determines the known effect of time loss due to the curvature of the body's space $C = \frac{r_g}{r^3}$ caused by its the gravitational field: the stronger the gravitational field (the closer the observer is to a massive body), the shorter the time intervals registered by him

$$d\tau = \sqrt{1-\frac{r_g}{r}}\, dt \simeq \left(1-\frac{r_g}{2r}\right) dt\,, \qquad (182)$$

$$\delta\tau \simeq -\frac{r_g}{2r}\, dt \simeq -\frac{1}{2}\, C r^2 dt\,. \qquad (183)$$

In other words, this is the known effect of the classical theory: the higher the observer is above the surface of a massive body, the weaker the curvature of the body's space and, consequently, the shorter the time intervals that the observer records.

However, the second term of (181) is absent in the classical theory. This term reveals the increment of the physically observable time, which is due to the curvature of the body's space $C = \frac{6\omega^2}{c^2}$ caused by its rotation (together with the massive body itself)

$$\delta\tau = -\frac{\omega r^2 \sin^2\theta}{c^2}\, u^3 dt = -\frac{C r^2 \sin^2\theta}{6\omega}\, u^3 dt\,. \qquad (184)$$

The sign of this effect depends on the direction, in which the observer travels with respect to the rotation of space, i.e., on the sign of the observer's coordinate velocity u^3 (he travels along the equatorial axis $x^3 = \varphi$).

As a result, based on the second term in the obtained solution, we obtain the 4th new effect of General Relativity in addition to those three explained above. This effect says:

The 4th new effect of General Relativity

A clock on board an airplane (or a spacecraft) flying in the field of a rotating massive body in the same direction in which the body's space rotates (together with the body itself) should register a time loss depending on the airplane's (or a spacecraft's) velocity and the rotation velocity of the body's space.

In contrast, a clock on board an airplane (or a spacecraft) flying in the direction, opposite to the body's space rotation should register a time increment, as well depending on the airplane's (or a spacecraft) velocity and the velocity, with which the body's space rotates.

This effect of time loss/gain in the field of a rotating massive body is due to the "stretching" of the body's space along the equatorial direction φ (along the geographical longitudes), caused by the rotation of the body's space along this axis. When, say, an airplane flies towards the Earth's rotation, the magnitude of the total rotation of space registered on its board is less than the proper rotation of the Earth's space at the point of departure/arrival and, therefore, the "stretching" (and curvature) of space registered on board the airplane is also less. In contrast, when an airplane flies backwards the Earth's space rotation, the clock on its board registers a time increment due to the greater magnitude of the total rotation and, therefore, greater "stretching" (and curvature) of space.

For example, consider a typical commercial flight travelling at 10 000 m along the Earth's equator ($\omega_{\oplus} = 1$ rev/day $= 7.27 \times 10^{-5}$ sec^{-1}, $r_{\oplus} = 6.37 \times 10^{8}$ cm) at a typical cruising speed of 800 km/hour, which means a flight time around the globe of $t \simeq 1.8 \times 10^{5}$ sec. Since the planet Earth rotates from West to East, the above 800 km/hour mean that the airplane's velocity is $u^3 = +3.5 \times 10^{-5}$ sec^{-1} when flying Eastward and $u^3 = -3.5 \times 10^{-5}$ sec^{-1} when flying Westward.

Then, according to the second term (184) in the obtained solution for $d\tau$ (181) we have obtained in the field of a rotating massive body, a clock installed on board the airplane should register a time loss of

$$\delta\tau_{\text{East}} = -\frac{\omega_{\oplus} r_{\oplus}^2 \sin^2\theta}{c^2}\, u^3 t \simeq -210 \text{ nanosec}, \qquad (185)$$

when flying to the East (i.e., in the same direction, in which the Earth's space rotates), and also a time increment

$$\delta\tau_{\text{West}} = +\frac{\omega_{\oplus} r_{\oplus}^2 \sin^2\theta}{c^2}\, u^3 t \simeq +210 \text{ nanosec}, \qquad (186)$$

when flying to the West (i.e., oppositely to the rotation of the Earth's space).*

This effect is maximum at the equator (where the curvature of the Earth's space caused by its rotation is maximum and, therefore, space is maximally "stretched") and decreases towards the poles, where $\sin\theta = 0$ and, therefore, this effect vanishes.

This effect was first registered in the "around-the-world-clock experiment", conducted in 1971 by Joseph C. Hafele and Richard E. Keating [12–14] and then repeated in 2005 by the UK's National Measurement Laboratory [15], despite the fact that they did not know about the chronometrically invariant formalism and the effects caused by the rotation of space; I discussed this issue in extensive friendly correspondence with Joseph C. Hafele in the last years of his life, before he passed away in 2014 [16]. Their flights took place in the Northern Hemisphere (not at the equator) and at different altitudes. In addition, the results of their measurements were affected by the relativistic addition of the airplane's velocity to the Earth's rotation velocity when flying Eastward (and subtraction when flying Westward), as well as the decrease in the Earth's gravitational potential with flight altitude. That is their measurement results were not purely the effect of the rotation of space. The total effect registered in the Hafele-Keating experiment was a time loss of -59 ± 10 nanoseconds when flying Eastward and a time increment of $+273 \pm 7$ nanoseconds when flying Westward, which fits well with our above calculation of the new effect due to the rotation of space, if we take into account the relativistic addition of the airplane's velocity to the Earth's rotation velocity when flying Eastward and subtraction when flying Westward.

8 Conclusion

The main contribution of this paper is introducing and proving the space metric of a rotating massive body, approximated by a mass-point. This is a new space metric to General Relativity, the main purpose of which is to be a modern extension and replacement of the classical Schwarzschild mass-point metric (since in the space of the Schwarzschild metric a massive body creating gravitational field does not rotate).

We have proven that the introduced space metric of a rotating massive body satisfies Einstein's field equations, and also derived the Riemann conditions under which this occurs. Therefore, the introduced metric can be legitimately used in General Relativity.

We have calculated all known physically observable properties of space determined by the introduced metric of a rotating massive body, including the physically observable curvature of space. And here is what is especially interesting: we have found that the curvature of space is caused not only by the gravitational field filling it, but also by the rotation of space (together with the massive body). Based on this theoretical discovery, we have predicted and calculated four new effects of General Relativity:

1. Deflection along the equatorial coordinate and time delay of mass-bearing particles falling onto a rotating massive body, which is due to the "stretching" (curvature) of space, caused by its rotation (together with the body itself);
2. Deflection along the equatorial coordinate and time delay of light rays spreading to a rotating massive body, which is due to the "stretching" (curvature) of space, caused by its rotation;
3. Length stretching of a rod installed along the radial and equatorial coordinates in the field of a rotating massive body due to the "stretching" (curvature) of space in these directions, caused by its rotation;

*The calculated numerical values are the same as those calculated in the previous paper [3] in the absence of the gravitational field, since the gravitational field produces an individual effect, expressed by the first term of the obtained solution for $d\tau$ (181).

4. The loss of time in a clock travelling in the direction of the body's space rotation, which is due to the increase in the "stretching" (curvature) of space in the direction of its rotation, and accordingly the increment of time when the clock travels oppositely to the rotation of space.

All real cosmic bodies in the Universe rotate. Therefore, the introduced and proved space metric is the *main space metric in the Universe*, characterizing the field of any real cosmic body, be it a planet, star, galaxy or something else.

Feel free to use this new metric instead of the classical Schwarzschild metric to solve problems in General Relativity and astrophysics, if you have the necessary mathematical skills and wishes to do so, of course.

Submitted on September 28, 2024

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