The Time Machine

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Gödel's metric (1949) describes a homogeneous and rotating universe unveiling the existence of closed time-like curves (CTCs) which make it feasible to go on a journey into one's own past. In the first part of this paper we follow Gödel's initial work and its conclusions but we show that his metric as it stands does not represent a cosmological model. Introducing a simple conformal factor readily induces a pressure term that straightforwardly leads to a perfect fluid field equation. This term was wrongly interpreted by Gödel as the ad hoc cosmological constant he was forced to introduce in order for his solution to satisfy Einstein's field equations. The theory is now endowed with a physical meaning and the dynamics no longer apply to the space but to a fluid which can be acted upon. In the second part, we investigate the possibility of creating a time machine materialized by a specific warp drive device travelling along a Gödel closed curve. The third part is devoted to highlighting the properties resulting from our model and some conjectures as to reversed CTCs.

Introduction

In the science fiction novel *The Time Machine* by H. G. Wells (1895), an English scientist constructs a machine which allows him to travel back and forth in time. He used this device to visit the world far in the future but returned from his journey only a few hours after he has started it. The history of the scientific controversy about the possibility of time travel can be traced back to the ingenious logician Kurt Gödel. In a paper published in 1949 to honour his close friend Albert Eistein on the occasion of his 70th birthday, he described a homogeneous and rotating universe unveiling the existence of closed time-like curves (CTCs) which make it feasible to go on a journey into one's own past.

In the first part of this paper we follow Gödel's initial work and its conclusions but we show that his metric as it stands does not represent a cosmological model. Introducing a simple conformal factor readily induces a pressure term that straightforwardly leads to a perfect fluid field equation. This term was wrongly interpreted by Gödel as the *ad hoc* cosmological constant he was forced to introduce in order for his solution to satisfy Einstein's field equations. The theory is now endowed with a physical meaning and the dynamics no longer apply to the space but to a fluid which can be acted upon. In the second part, we investigate the possibility of creating a time machine materialized by a specific warp drive device traveling along a Gödel closed curve. The third part is devoted to highlighting the properties resulting from our model and some conjectures as to reversed CTCs.

Notations

Space-time indices: $\mu, \nu = 0, 1, 2, 3$;
Spatial indices: $a, b = 1, 2, 3$;
Space-time signature: -2 (unless otherwise specified);
Newton's constant of gravitation: G.

Part I

1 The Gödel universe

1.1 General considerations

In his original paper [1], Kurt Gödel has derived an exact solution to Einstein's field equation which describes a homogeneous and non-isotropic universe where matter takes the form of a shear free fluid. This metric exhibits a rotational symmetry which allows for the existence of closed time-like curves (previously called CTCs).

The Gödel space-time has a five dimensional group of isometries (G5) which is transitive (an action of a group is transitive on a manifold $(\mathrm{M}, \boldsymbol{g})$, if it can map any point of the manifold into any other point of this manifold. It admits a five dimensional Lie algebra of Killing vector fields generated by a time translation ∂_{ct}, two spatial translations ∂_x,∂_y plus two further Killing vector fields

$$\partial_z - y\,\partial_y\,, \quad 2\,e^x\,\partial_{ct} + y\,\partial_z + \left(e^{2x} - \frac{1}{2}\,y^2\,\partial_y\right).$$

The Weyl tensor of the standard Gödel solution has Petrov type D

$$C^{\alpha\beta}_{\mu\nu} = R^{\alpha\beta}_{\mu\nu} + \frac{1}{3}\,R\,\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} + 2\,\delta^{[\alpha}_{[\mu}R^{\beta]}_{\nu]}\,.$$

The presence of the non-vanishing Weyl tensor prevents the Gödel metric from being Euclidian, unlike the Friedmann-Lemaître-Robertson-Walker metric, which can be shown to reduced to a conformal euclidian metric implying that its Weyl tensor is zero [2].

The Gödel model is dismissed because it has a cosmological constant and also since its rotation would conflict with observational data. In what follows we are able to relax our demand that Gödel's metric be the description of our actual Universe which is expanding.

1.2 Gödel's metric

The classical Gödel line element is given in Cartesian coordinates by

$$ds^2 = a^2\left(c^2dt^2 + \frac{1}{2}\,e^{2x}dy^2 - 2\,e^x c\,dt\,dy - dx^2 - dz^2\right), \quad (1.1)$$

where $a > 0$ is a constant. The components of the metric tensor $g_{\mu\nu}$ and $g^{\mu\nu}$ are, respectively,

$$\left.\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} a^2 & 0 & a^2e^x & 0 \\ 0 & -a^2 & 0 & 0 \\ a^2e^x & 0 & \frac{1}{2}\,a^2e^{2x} & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \\ g^{\mu\nu} &= \begin{pmatrix} a^{-2} & 0 & 2a^{-2}e^{-x} & 0 \\ 0 & -a^{-2} & 0 & 0 \\ 2a^{-2}e^{-x} & 0 & -2a^{-2}e^{-2x} & 0 \\ 0 & 0 & 0 & -a^{-2} \end{pmatrix} \end{aligned}\right\}. \quad (1.2)$$

Owing to the fact that only $\partial_1 g_{22} \neq 0$ and $\partial_1 g_{02} \neq 0$, one easily computes

$$\Gamma^0_{01} = 1\,, \qquad \Gamma^0_{12} = \Gamma^1_{02} = \frac{1}{2}\,e^x,$$

$$\Gamma^1_{22} = \frac{1}{2}\,e^{2x}, \qquad \Gamma^2_{01} = -e^{-x}.$$

The Ricci tensor is very simplified

$$R_{\beta\gamma} = \partial_1\Gamma^1_{\beta\gamma} + \Gamma^1_{\beta\gamma} - \Gamma^\delta_{\alpha\beta}\Gamma^\alpha_{\delta\gamma}\,, \quad (1.3)$$

the components of which are reduced to

$$R_{00} = 1\,, \quad R_{22} = e^{2x}, \quad R_{02} = R_{20} = e^x,$$

therefore the Ricci scalar is

$$R = \frac{1}{a^2}\,.$$

The normalized unit vector $\boldsymbol{u}$ of matter has components

$$u^\mu = \left(a^{-1}, 0, 0, 0\right), \quad u_\mu = (a,\, 0,\, ae^x,\, 0)\,, \quad (1.4)$$

thus the Ricci tensor takes the formulation

$$R_{\mu\nu} = u_\mu u_\nu\, a^{-2} \quad (1.5)$$

and the Ricci scalar takes the form

$$R = u^\mu u_\mu = a^{-2}. \quad (1.6)$$

Since R is a constant, the field equations (with the x^0-lines as world lines of matter)

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = \frac{8\pi G}{c^4}\,\rho\, c^2 u_\mu u_\nu + \Lambda\, g_{\mu\nu} \quad (1.7)$$

are satisfied (for a given value of the density ρ), if we put

$$a^{-2} = \frac{8\pi G\rho}{c^2}\,, \quad (1.8)$$

$$\Lambda = -\frac{1}{2}\,R = \frac{1}{2a^2} = -\frac{4\pi G\rho}{c^2}\,. \quad (1.9)$$

The sign of the cosmological constant Λ here is the opposite of that occurring in Einstein's field equations. Bearing in mind that a is a constant, fine tuning the density of the universe with the cosmological constant and the Ricci scalar appears as a dubious result. It then becomes clear that such cosmological constraints are physically irrelevant.

2 Rotation of Gödel's model

As primarily assumed by Gödel, the stationary space-time of his model is homogeneous. For every point A of the manifold $(\mathrm{M}, \boldsymbol{g})$, there is a one-parameter group of transformations of M carrying A into itself. In addition, the manifold $(\mathrm{M}, \boldsymbol{g})$ is endowed with a rotational symmetry and the flow lines have a vorticity magnitude $\boldsymbol{\omega}$ orthogonal to $\boldsymbol{u}$.

2.1 Vorticity vector

Let u_α be a 4-unit vector everywhere tangent to the flow line on $(\mathrm{M}, \boldsymbol{g})$. The covariant derivative $u_{\alpha;\mu}$ of this time-like vector may be expressed in a invariant manner in terms of tensor fields which describe the kinematics of the congruence of curves generated by the velocity vector field u^α [3]

$$u_{(\alpha;\mu)} = \sigma_{\alpha\mu} + \omega_{\sigma\mu} + \frac{1}{3}\,\theta h_{\alpha\mu} + {}^*u_{(\alpha}u_{\mu)}\,, \quad (2.1)$$

where θ is the scalar expansion

$$\theta = u^\alpha_{;\alpha}\,, \quad (2.2)$$

and ${}^*u_\alpha$ is the 4-acceleration vector of the flow lines

$${}^*u_\alpha = u_{\alpha;\mu}\,u^\mu, \quad (2.3)$$

while $h_{\mu\nu}$ is the projection tensor determined as

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu\,.$$

Besides ${}^*u_\alpha$ and θ, one can define the vorticity tensor

$$\omega_{\alpha\mu} = h^\sigma_\alpha h^\nu_\mu\, {}^*u_{[\sigma;\nu]} = u_{[\alpha;\mu]} + {}^*u_{[\alpha}u_{\mu]}\,, \quad (2.4)$$

and also the quantity

$$\sigma_{\alpha\mu} = \theta_{\mu\nu} - \frac{1}{3}\,h_{\alpha\mu}\theta$$

which is the symmetric trace free shear tensor, where

$$\theta_{\mu\nu} = h^\sigma_\alpha h^\nu_\mu\, u_{(\sigma;\nu)}$$

is the expansion tensor.

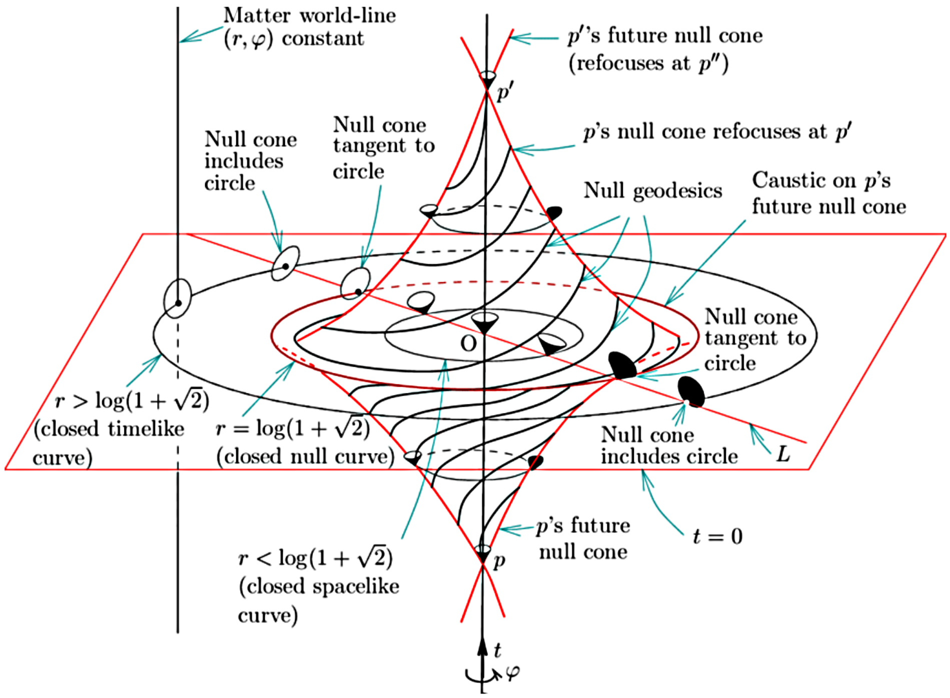

Fig. 1: With increasing $r > r_G$, the light cones continue to tip over and their opening angles increase until their future parts reach negative values of t'. Thus $\partial/\partial\phi$ becomes a timelike vector, and circles of constant r and t' are closed time-like curves.

Thus the components of the 4-vorticity vector $\boldsymbol{\omega}$ of the flow lines tangent to u^{μ} are expressed by

$$\omega^{\beta} = \frac{1}{6}\,\eta^{\beta\gamma\sigma\rho} u_{\gamma}\omega_{\sigma\rho}\,, \tag{2.5}$$

where $\eta^{\beta\gamma\sigma\rho}$ is the Levi-Civita tensor indicator

$$\eta^{\beta\gamma\sigma\rho} = \frac{\varepsilon^{\beta\gamma\sigma\rho}}{\sqrt{g}}\,.$$

The kinematic quantities $\omega_{\sigma\mu}$, ω_{μ} and ${}^{*}u_{\mu}$ are completely orthogonal to u^{μ}, i.e.,

$$\omega_{\sigma\mu}u^{\mu} = \omega_{\mu}u^{\mu} = {}^{*}u_{\mu}u^{\mu} = h_{\mu\nu}u^{\mu} = 0\,.$$

In the Gödel model the shear tensor is zero, therefore

$$\sigma_{\alpha\mu} = u_{(\alpha;\mu)} - \frac{1}{3}\,\theta h_{\alpha\mu} - \omega_{\sigma\mu} - {}^{*}u_{(\alpha}u_{\mu)} = 0 \tag{2.5bis}$$

(shear free flows of a perfect fluid in relation with the Weyl tensor have been extensively investigated by A. Barnes [4]).

Knowing that $\sqrt{g} = a^4\sqrt{\frac{1}{2}e^{2x}}$, we compute the contravariant components of the 4-vorticity vector $\boldsymbol{\omega}$

$$\omega^{\alpha} = \left(0,\,0,\,0,\,\frac{\sqrt{2}}{a^2}\right) \tag{2.6}$$

and we find

$$\boldsymbol{\omega} = \sqrt{g_{\alpha\beta}\,\omega^{\alpha}\omega^{\beta}} = \frac{\sqrt{2}}{a}\,. \tag{2.7}$$

Taking into account (1.8) the magnitude of this vector is

$$\boldsymbol{\omega} = \sqrt{\frac{1}{2}\left(\frac{8\pi G}{c^2}\right)\rho}\,. \tag{2.8}$$

2.2 Closed time-like curves

Following Gödel we introduce cylindrical coordinates (t', r, θ)

$$e^{x} = \cosh 2r + \cosh\phi\,\sinh 2r\,,$$

$$y\,e^{x} = \sqrt{2}\sinh\phi\,\sinh 2r\,,$$

$$\tan\frac{1}{2}\left[\phi + \left(ct - \frac{2t'}{2\sqrt{2}}\right)\right] = e^{-2r}\tan\frac{\phi}{2}$$

thus the Gödel metric reads now

$$ds^2 = 4a^2\Big[dt'^2 - dr^2 + \left(\sinh^4 r - \sinh^2 r\right)d\phi^2 + \\ + 2\sqrt{2}\sinh^2 r\,d\phi\,dt'\Big] \tag{2.9}$$

(with the inessential coordinate z suppressed).

In its original formulation, the Gödel universe describes a set of masses (stars and planets) rotating about arbitrary axes. The metric (2.9) manifests a rotational symmetry with respect to the axis t', and $r = 0$ since we clearly see that the components of the metric tensor do not depend ϕ.

For $r \geqslant 0$, we have $0 \leqslant \phi \leqslant 2\pi$. If a curve r_G is defined by $\sinh r = 1$ that is $r_G = \log\,(1 + \sqrt{2})$, then such a curve

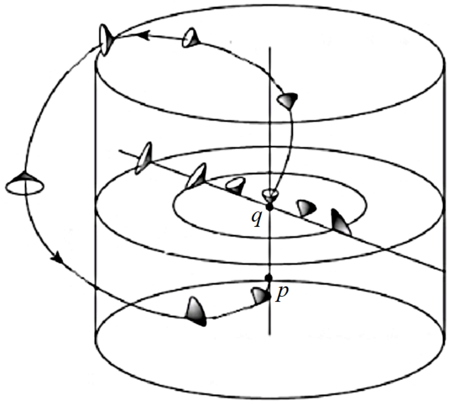

Fig. 2: The Gödel trajectory loops back in the past at p after crossing a Cauchy-like "horizon".

which materializes in the "plane" $t' = const$ is a closed light-like curve. The radius r_G referred to as the Gödel radius, thus induces a closed null curve where the light cones are tangent to the plane of constant t'.

With increasing $r > r_G$, the light cones continue to tip over and their opening angles increase until their future parts reach negative values of t'. Thus $\partial/\partial\phi$ becomes a timelike vector, and circles of constant r and t' are closed time-like curves (see Fig. 1). Starting from the centre of the axis at q, the Gödel trajectory loops back in the past at p after crossing a Cauchy-like "horizon" (see Fig. 2).

3 The Gödel model as a homogeneous perfect fluid

3.1 Reformulation of the Gödel metric

In our publication [5], we assumed that a is slightly space-time variable and we set

$$a^2 = e^{2U}. \tag{3.1}$$

The positive scalar $U(x)$ will be explained below. The Gödel metric thus becomes

$$ds^2 = e^{2U} \times \times \left(c^2dt^2 + \frac{1}{2}\, e^{2x}dy^2 - 2\, e^x\, c\,dt\,dy - dx^2 - dz^2\right). \tag{3.2}$$

We see that it is conformal to the Gödel metric with the constant $a = 1$

$$(ds^2)_G = c^2dt^2 + \frac{1}{2}\, e^{2x}\, dy^2 - 2\, e^x\, c\,dt\,dy - dx^2 - dz^2. \tag{3.3}$$

It is clear that this solution retains the properties related to CTCs of the initial Gödel metric (1.1).

3.2 Differential geodesic system

Let us consider the manifold $(\mathrm{M}, \boldsymbol{g})$ on which is defined a vector tangent to the curve C

$$^*x^\alpha = \frac{dx^\alpha}{d\zeta}\,,$$

where ζ is an affine parameter. In these local coordinates, we consider the scalar function $f(x^\alpha, {}^*x^\alpha)$, which is homogeneous and of first degree with respect to ${}^*x^\alpha$. To the curve C joining the points x_1 and x_2 one can always associate the integral $\mathcal{A}$ such that

$$\mathcal{A} = \int_{\zeta_1}^{\zeta_2} f(x^\alpha, {}^*x^\alpha)\, d\zeta = \int_{x_1}^{x_2} f(x^\alpha, {}^*x^\alpha)\, dx^\alpha. \tag{3.3}$$

We now want to evaluate the variation of $\mathcal{A}$ with respect to the points ζ_1 and ζ_2

$$\delta\mathcal{A} = f\,\delta\zeta_2 - f\,\delta\zeta_1 - \int_{\zeta_1}^{\zeta_2} \delta\, d\zeta\,.$$

Classically we know that

$$\int_{\zeta_1}^{\zeta_2} \delta\, d\zeta = \left(\frac{\partial f}{\partial\, {}^*x^\alpha}\right) \delta x^\alpha - \int_{\zeta_1}^{\zeta_2} E_\alpha\, \delta x^\alpha\, d\zeta\,,$$

where E_α is the first member of the Euler equation associated with the function f.

With E_α as the components of $\boldsymbol{E}$, we infer the expression

$$\delta\mathcal{A} = [w(\delta)]_{x_2} - [w(\delta)]_{x_1} - \int_{\zeta_1}^{\zeta_2} \boldsymbol{E}\,\delta x\, d\zeta\,, \tag{3.5}$$

where $w(\delta)$ has the form

$$w(\delta) = \frac{\partial f}{\partial\, {}^*x^\alpha}\, \delta x^\alpha - \frac{x^\alpha\, \partial f}{\partial\, {}^*x^\alpha - f}\, \delta\zeta\,.$$

Due to the homogeneity of f it reduces to

$$w(\delta) = \frac{\partial f}{\partial\, {}^*x^\alpha}\, \delta x^\alpha.$$

Let us apply the above results to the function

$$f = e^U\, \frac{ds}{d\zeta} = e^U \sqrt{g_{\alpha\beta}\, {}^*x^\alpha\, {}^*x^\beta}\,, \tag{3.6}$$

where e^U is defined everywhere on $(\mathrm{M}, \boldsymbol{g})$.

We first differentiate $f^2 = e^{2U}(g_{\alpha\beta}\, {}^*x^\alpha\, {}^*x^\beta)$ with respect to ${}^*x^\alpha$ and x^α

$$\frac{f\,\partial f}{\partial\, {}^*x^\alpha} = e^{2U} g_{\alpha\beta}\, {}^*x^\beta, \tag{3.7}$$

$$\frac{f\,\partial f}{\partial x^\alpha} = e^U \sqrt{g_{\beta\mu}\, {}^*x^\beta\, {}^*x^\mu} \times \times \left[\partial_\alpha e^U \sqrt{g_{\beta\mu}\, {}^*x^\beta\, {}^*x^\mu} + \frac{1}{2}\, e^U\, \partial_\alpha \left(g_{\beta\mu}\, {}^*x^\beta\, {}^*x^\mu\right)\right]. \tag{3.8}$$

We now choose s as the affine parameter ζ on the curve C, so the vector ${}^*x^\beta$ is here regarded as the 4-unit vector u^β tangent to C whose curvilinear abscissa is noted s. Equations (3.7) and (3.8) then reduce to the following

$$\frac{\partial f}{\partial\, {}^*x^\beta} = e^U u_\beta\,, \tag{3.9}$$

$$\frac{\partial f}{\partial x^\beta} = \partial_\beta e^U + \frac{1}{2}\, e^U \partial_\beta g_{\alpha\mu} u^\alpha u^\mu = \\ = \partial_\beta e^U + e^U \Gamma_{\alpha\beta,\mu} u^\alpha u^\mu , \tag{3.10}$$

where $\Gamma_{\alpha\beta,\mu}$ denote here the Christoffel symbols of the first kind. Expliciting the Euler equations $f(x^\alpha, du^\alpha)$

$$E_\beta = \frac{d}{ds}\frac{\partial f}{\partial u^\beta} - \frac{\partial f}{\partial x^\beta}\,, \tag{3.11}$$

we obtain

$$\begin{aligned} E_\beta &= \frac{d}{ds}\, e^U u^\beta - e^U \Gamma_{\alpha\beta,\mu} u^\alpha u^\mu - \partial_\beta e^U = \\ &= e^U \left(u^\mu \partial_\mu u_\beta - \Gamma_{\alpha\beta,\mu} u^\alpha u^\mu\right) - \partial_\alpha e^U \left(\delta^\alpha_\beta - u^\alpha u_\beta\right) \\ &= e^U \left[\left(u^\mu \nabla_\mu u_\beta\right) - \partial_\beta U - \partial_\alpha U \left(\delta^\alpha_\beta - u^\alpha u_\beta\right)\right]. \end{aligned} \tag{3.12}$$

Equation (3.5) becomes

$$\delta\mathcal{A} = [w(\delta)]_{x_2} - [w(\delta)]_{x_1} - \int_{x_1}^{x_2} \langle \boldsymbol{E}\,\delta x\rangle\, ds\,, \tag{3.13}$$

where locally we have $w(\delta) = e^U u_\alpha dx^\alpha$. When the curve C varies between two fixed points x_1 and x_2, the local variations $[w(\delta)]_{x_2}$ and $[w(\delta)]_{x_1}$ vanish.

Applying the variation principle to (3.13) simply leads to

$$\delta\mathcal{A} = -\int_{x_1}^{x_2} \langle \boldsymbol{E}\,\delta x\rangle\, ds = 0\,, \tag{3.14}$$

i.e., $\boldsymbol{E} = 0$, and since $e^U \neq 0$, we obtain

$$u^\mu \nabla_\mu u_\beta - \left(\delta^\alpha_\beta - u^\alpha u_\beta\right) \partial_\alpha U = 0\,. \tag{3.15}$$

The equation (3.15) is formally identical to the differential system obeyed by the flow lines of a perfect fluid of density ρ and pressure P with an equation of state $\rho = f(P)$ and where

$$U(x^\mu) = \int_{P_1}^{P_2} \frac{dP}{\rho c^2 + P}$$

accounts for the fluid indice [6]. Pressure P_1 and P_2 are referred to x_1 and x_2 [7, 8]; see Appendix. The resulting field equation is [9]

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = \frac{8\pi G}{c^4}\left[\left(\rho c^2 + P\right) u_\mu u_\nu - P g_{\mu\nu}\right]. \tag{3.16}$$

Here, the 4-unit vector u^μ of the fluid is the real 4-velocity defined in Gödel's metric (3.3):

$$u^\mu = (1, 0, 0, 0)\,, \quad u_\mu = (1, 0, e^x, 0)\,. \tag{3.17}$$

3.3 Fluid rotation in the framework of the Gödel model

The wave vector $k^\mu = dx^\mu/d\lambda$ determines the propagation of ligh rays tangent to the light cone (λ is a given parameter varying along those rays). The equation of propagation is here

$$\frac{dk^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu} k^\alpha k^\nu = 0\,.$$

Substituting $\partial_\mu \psi$ (here ψ is the eikonal) in this expression, one finds the eikonal equation

$$g^{\mu\nu} \partial_\mu \psi\, \partial_\nu \psi = 0\,.$$

Let us now examine the case of the light cone for closed lines, when the Gödel radius is reached. To this effect, we revert to the metric (2.9) which reads now

$$ds^2 = 4\, e^{2U}\Big[dt'^2 - dr^2 + \left(\sinh^4 r - \sinh^2 r\right) d\phi^2 + \\ + 2\sqrt{2}\, \sinh^2 r\, d\phi\, dt'\Big]. \tag{3.18}$$

The wave vectors k^μ tangent to the light cone follow les closed lines located to the plane orthogonal to the time axis $t' = const$: the integral U performed over the closed path has no endpoints

$$U(r) = \int \frac{dP}{\rho c^2 + P} + const. \tag{3.19}$$

Beyond r_G, the fluid trajectory does not loop up at the same point but in the past, and the magnitude of the time shift will depend on the pressure difference ΔP.It is now easy to compute the vorticity tensor $\omega_{\mu\nu}$ which is derived from u^μ

$$\omega_{\mu\nu} = \partial_\mu u_\nu - \partial_\nu u_\mu\,. \tag{3.20}$$

The components of the 4-vorticity vector $\boldsymbol{\omega}$ of the fluid flow lines are

$$\omega^\beta = \frac{1}{6}\, \eta^{\beta\gamma\sigma\rho} u_\gamma \omega_{\sigma\rho}\,.$$

For calculating the Levi-Civita tensor $\eta^{\beta\gamma\sigma\rho} = \varepsilon^{\beta\gamma\sigma\rho}/\sqrt{g}$, the $g_{\mu\nu}$ determinant is now $g = \frac{1}{2}\, e^{2x}$. A simple calculation leads to the Gödel rotation which remains constant

$$\boldsymbol{\omega} = \sqrt{g_{\alpha\beta}\, \omega^\alpha \omega^\beta} = \sqrt{2}\,. \tag{3.21}$$

We note that the Kretschmann scalar is still invariant

$$R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = 12\,\omega\,.$$

Part II

4 Warp drive

4.1 The (3 + 1) formalism or ADM technique

Arnowitt, Deser and Misner (ADM) suggested a technique which leads to decompose the space-time into a family of spacelike hypersurfaces and parametrized by the value of an arbitrarily chosen time coordinate x^0 [10]. This *foliation* displays a proper time element $d\tau$ between two nearby hypersurfaces labeled

$$x^0 = const, \qquad x^0 + dx^0 = const,$$

and the proper time element $cd\tau$ must be proportional to dx^0, thus we write

$$cd\tau = N(x^a, x^0)\, dx^0.$$

The line element corresponding to the hypersurfaces separation is therefore written in the form

$$(ds^2)_{\text{ADM}} = -N^2(dx^0)^2 + \\ + \, g_{ab}(N^a dx^0 + dx^a)(N^b dx^0 + dx^b). \tag{4.1}$$

In the ADM terminology, N is called the *lapse function*. Let us now evaluate the 3-vector whose spatial coordinates x^a are lying in the hypersurface $x^0 = const$, which is normal to it, on the second hypersurface $x^0 + dx^0 = const$, and where these coordinates now become $N^a dx^0$. The N^a vector is called the *shift vector*. The 4-metric tensor covariant and contravariant components $(g_{\alpha\beta})_{\text{ADM}}$ and $(g^{\alpha\beta})_{\text{ADM}}$ are

$$\left.\begin{aligned} (g_{\alpha\beta})_{\text{ADM}} &= \begin{pmatrix} -N^2 - N_a N_b\, g^{ab} & N_b \\ N_a & g_{ab} \end{pmatrix} \\ (g^{\alpha\beta})_{\text{ADM}} &= \begin{pmatrix} -N^{-2} & \dfrac{N^b}{N^2} \\ \dfrac{N^a}{N^2} & g^{ab} - \dfrac{N^a N^b}{N^2} \end{pmatrix} \end{aligned}\right\}. \tag{4.2}$$

The line element corresponding to the hypersurfaces separation is therefore written as

$$\begin{aligned} (ds^2)_{\text{ADM}} &= \\ &= -N^2(dx^0)^2 + g_{ab}(N^a dx^0 + dx^a)(N^b dx^0 + dx^b) \\ &= -N^2 + N_a N^a (dx^0)^2 + 2N_b\, dx^0 dx^b + g_{ab}\, dx^a dx^b, \end{aligned} \tag{4.3}$$

where g_{ab} is the 3-metric of the hypersurfaces. As a result, the hypersurfaces have a unit time-like normal with contravariant components

$$u^\alpha = N^{-1}(1, -N^a). \tag{4.4}$$

If the universe is approximated to the Minkowski space within an orthonormal coordinates frame of reference and where the fundamental 3-tensor satisfies $g^{ab} = \delta^{ab}$, the metric (4.3) becomes

$$(ds^2)_{\text{ADM}} = -(N^2 - N_a N^a)\, c^2 dt^2 + 2N^a dx\, cdt + dx^a dx^b, \tag{4.5}$$

$$(ds^2)_{\text{ADM}} = -N^2 dt^2 + (dx + N^a cdt)^2 + dy^2 + dz^2. \tag{4.5bis}$$

The Einstein action can be written in terms of the 4-metric tensor $(g_{\alpha\beta})_{\text{ADM}}$ according to [11] as follows

$$S_{\text{ADM}} = \int cdt \int N\left({}^{(3)}R - K^a_b K^b_a + K^2\right)\sqrt{{}^{(3)}g}\, d^3x + \\ + \text{boundary terms},$$

where $K^a_a K^b_b = K^2$, and ${}^{(3)}R$ is the 3-Ricci scalar and stands for the *intrinsic curvature* of the hypersurface

$$x^0 = const,$$

$$\sqrt{{}^{(3)}g} = \sqrt{\det\|g_{ab}\|} \;\longleftrightarrow\; \sqrt{{}^{(4)}{-g}} = N\sqrt{{}^{(3)}g}\,,$$

while

$$K_{ab} = (2N)^{-1}(-N_{a;b} - N_{b;a} + \partial_0\, g_{ab}) \tag{4.6}$$

represents the *extrinsic curvature*, and as such describes the manner in which the hypersurface $x^0 = const$ is embedded in the surrounding space-time. The rate of change of the 3-metric tensor g_{ab} with respect to the time label can be decomposed into "normal" and "tangential" contributions:

— The normal change is proportional to the extrinsic curvature $2K_{ab}/N$ of the hypersurface;

— The tangential change is given by the Lie derivative of g_{ab} along the shift vector N^a

$$\mathbf{L}_N\, g_{ab} = 2N_{(a;b)}\,. \tag{4.7}$$

With the choice of $N^a = 0$, we have a particular coordinate frame called *normal coordinates* according to which is called an *Eulerian gauge*.

Inspection shows that

$$K_{ab} = -u_{a;b}\,, \tag{4.8}$$

which is sometimes called the *second fundamental form* of the 3-space. Six of the ten Einstein equations imply for K^a_b to evolve according to

$$\frac{\partial K^a_b}{c\partial t}\mathbf{L}_N K^a_b = \nabla^a\nabla_b N + \\ + N\left[R^a_b + K^a_a K^a_b + 4\pi(T - C)\,\delta^a_b - \frac{8\pi G}{c^4}\,T^a_b\right], \tag{4.9}$$

$$C = T_{\alpha\beta}u^\alpha u^\beta, \tag{4.10}$$

where C is the matter energy density in the rest frame of normal congruence (time-like vector field) with $T = T^a_a$.

With the Gauss-Codazzi relations [12] we can express the Einstein tensor as a function of both the intrinsic and extrinsic curvatures. At this stage it is convenient to introduce the 3-momentum current density $I_a = -u_c T^c_a$. Thus, the remaining four equations finally form the so-called *constraint equations*

$$H = \frac{1}{2}\left({}^{(3)}R - K^a_b K^b_a + K^2\right) - \frac{8\pi G}{c^4}\, C = 0\,, \tag{4.11}$$

$$H_b = \nabla_a \left(K^a_b - K\,\delta^a_b \right) - \frac{8\pi G}{c^4}\, I_b = 0\,. \qquad (4.12)$$

Therefore, another way of writing (4.10) eventually leads to

$$C = \frac{c^4}{16\pi G}\left({}^{(3)}R - K_{ab}K^{ab} + K^2 \right). \qquad (4.13)$$

4.2 Alcubierre's theory

In 1994, M. Alcubierre showed that a superluminal velocity can be achieved without violating the laws of General Relativity [13]. He considered a perturbed space-time region likened to bubble (called "warp drive") which could transport a machine in a surfing mode: inside the bubble, the proper time element $d\tau$ is the coordinate time element dt measured by an external observer called "Eulerian". The motion is only achieved by the space wave, so that the occupant of the machine is at rest and would not suffer any acceleration nor time dilation in the displacement. This process requires a front contraction of the space and a rear expansion.

The distance of the machine centre located in the bubble

$$r_s(t) = \sqrt{\left[\, y - y_s(t)\right]^2 + x^2 + z^2}$$

varies until R_e, which is the external radius of the bubble. With respect to the distant observer the apparent velocity of the machine is

$$\mathrm{v}_s(t) = \frac{dy_s(t)}{dt}\,,$$

where $y_s(t)$ is the coordinate of the bubble's trajectory along the y-direction. Within the ADM formalism in the signature +2, the Alcubierre metric is defined on a flat space-time thus the lapse vectors and shift vectors reduce to

$$\left.\begin{aligned} &N = 1 \\ &N^1 = -\mathrm{v}_s(t) f(r_s, t) \\ &N^2 = N^3 = 0 \end{aligned}\right\}. \qquad (4.14)$$

The shape of the function $f(r_s, t)$ induces both a volume contraction and expansion ahead and behind of the bubble. This can be checked by using the scalar expansion $\theta = u^\alpha_{;\alpha}$

$$\theta = \mathrm{v}_s \frac{df}{(dy)_{\mathrm{Al}}}\,. \qquad (4.15)$$

Alcubierre chooses the following step function $f(r_s, t)$

$$f(r_s, t) = \frac{\tanh\sigma(r_s + R_e) - \tanh\sigma(r_s - R_e)}{2\tanh\,(\sigma R_e)}\,, \qquad (4.16)$$

where $R_e > 0$ is the external radius of the bubble, and σ is a "bump"parameter used to tune the wall thickness of the bubble: the larger the parameter σ, the greater the contained energy density, for its shell thickness decreases. Moreover, the absolute increase of σ means a faster approach of the condition

$$\lim_{\sigma\to\infty} f(r_s, t) = 1 \quad \text{for} \;\; r_s \in [-R_e, R_e]$$

and is 0 everywhere else.

Here the expansion scalar becomes

$$\theta = \partial_1 N^1 = -\operatorname{trace} K_{ab}\,.$$

With (4.16) one finally gets

$$\theta = \mathrm{v}_s \frac{df}{dr_s}\frac{y_s}{r_s}\,. \qquad (4.17)$$

The Natàrio warp drive evades the problem of contraction/expansion, by imposing the divergence free constraint to the shift vector $\nabla[\mathrm{v}_s^2 f^2(r_s, t)] = 0$ [14].

The distant observer is called *Eulerian* [15], and his 4-velocity relative to the bubble has components

$$(u^\alpha)_{\mathrm{E}} = [\,c,\, \mathrm{v}_s\, c\, f(r_s, t),\, 0,\, 0]\,, \qquad (4.18)$$

$$(u_\alpha)_{\mathrm{E}} = [-c, 0, 0, 0]\,. \qquad (4.18\text{bis})$$

The Eulerian observer is a special type of observer which refers to the Eulerian gauge defined above but with $N^1 \neq 0$, and as such, it follows timelike geodesic orthogonal to euclidean hypersurfaces. Such an observer starts out just inside the bubble shell at its first equator with zero initial velocity.

Once during his stay inside the bubble, this observer travels along a time-like curve $y = y_s(t)$ with a constant velocity nearing the machine local velocity $\mathrm{v}_s = dy_s/dt$. The Eulerian observer's velocity will always be less than the bubble's velocity unless $r_s = 0$, i.e. when this observer is at the centre of the machine located inside. After reaching the second region's equator, this observer decelerates and is left at rest while going out at the rear edge of the bubble. The Eulerian observer's velocity is needed to evaluate the energy density required to create the bubble (see below).

The Alcubierre metric is:

$$(ds^2)_{\mathrm{Al}} = -c^2dt^2 + \left[dy - \mathrm{v}_s f(r_s, t)\, c dt\right]^2 + dx^2 + dz^2 \quad (4.19)$$

or, in the framework of signature −2,

$$(ds^2)_{\mathrm{Al}} = c^2dt^2 - \left[dy - \mathrm{v}_s f(r_s, t)\, c dt\right]^2 - dx^2 - dz^2. \quad (4.19\text{bis})$$

Let us now write the Alcubierre metric in the equivalent form which puts in evidence the covariant components of the metric tensor

$$\begin{aligned}(ds^2)_{\mathrm{Al}} = &\left[(1 - \mathrm{v}_s^2 f^2(r_s, t)\right] c^2dt^2 + \\ &+ 2\,\mathrm{v}_s f(r_s, t)\, c dt dy - dx^2 - dz^2,\end{aligned} \qquad (4.20)$$

$$\left.\begin{aligned} &(g_{00})_{\mathrm{Al}} = \left[1 - \mathrm{v}_s^2 f^2(r_s, t)\right] \\ &(g_{01})_{\mathrm{Al}} = (g_{10})_{\mathrm{Al}} = 2\,\mathrm{v}_s f(r_s, t) \\ &(g_{11})_{\mathrm{Al}} = (g_{22})_{\mathrm{Al}} = (g_{33})_{\mathrm{Al}} = -1 \end{aligned}\right\}. \qquad (4.21)$$

With the components (4.21) the Einstein-Alcubierre tensor reads

$$(G^{\alpha\beta})_{\rm Al} = (R^{\alpha\beta})_{\rm Al} - \frac{1}{2}\,(g^{\alpha\beta})_{\rm Al}\,R\,, \tag{4.22}$$

$$(T^{\alpha\beta})_{\rm Al} = \frac{c^4}{8\pi G}\,(G^{\alpha\beta})_{\rm Al}\,. \tag{4.23}$$

The weak energy conditions stipulate

$$C_{\rm Al} = (T^{\alpha\beta})_{\rm Al}\,(u_\alpha)_{\rm E}\,(u_\beta)_{\rm E} \geqslant 0\,. \tag{4.24}$$

Considering (4.13), we see that in the Alcubierre space-time ${}^{(3)}R = 0$, hence

$$C_{\rm Al} = \frac{c^4}{16\pi G}\left(K^2 - K_{ab}\,K^{ab}\right), \tag{4.25}$$

$$C_{\rm Al} = \frac{c^4}{16\pi G}\times \times\left[(\partial_1 N^1)^2 - (\partial_1 N^1)^2 - 2\,(\partial_2 N^2)^2 - 2\,(\partial_3 N^1)^2\right], \tag{4.26}$$

$$(T^{00})_{\rm Al}\,(u_0)_{\rm E}\,(u_0)_{\rm E} = (T^{00})_{\rm Al} = -\frac{c^4}{32\pi G}\,{\rm v}_s^2\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right] < 0\,. \tag{4.27}$$

Taking into account (4.16), one eventually finds the energy tensor:

$$(T^{00})_{\rm Al} = -\frac{c^4}{32\pi G}\,{\rm v}_s^2\left(\frac{\partial f}{dr_s}\right)^2\frac{x^2+z^2}{r_s^2} < 0 \tag{4.28}$$

This expression is unfortunately negative as measured by the Eulerian observer and therefore it violates the weak energy conditions (WEC) [16]. Notwithstanding this violation, one is nevertheless forced to introduce a way to obtain a negative energy density. This possibility is examined below.

4.3 Nature of the negative energy

The machine has a shell whose thickness is: $R_e - R_i$, where R_e is the external radius while R_i is the inner radius. R_e coincides with the Alcubierre bubble which thus constitutes the whole machine contour. The mass has a charge μ circulating within the shell thus giving rise of a 4-current density $j^\alpha = \mu u^\alpha$. This current is coupled to a co-moving electromagnetic field with the 4-potential A^α, which yields the interacting energy-momentum tensor

$$(T^{\alpha\beta})_{\rm elec} = \frac{1}{4\pi}\left(\frac{1}{4}\,g^{\alpha\beta}F_{\gamma\delta}\,F^{\gamma\delta} + F^{\alpha\nu}F_{\nu\cdot}^{\cdot\beta}\right) + g^{\alpha\beta}\,j_\nu A^\nu - j^\alpha A^\beta,$$

and the extracted energy density is

$$(T^{00})_{\rm elec} = \frac{1}{4\pi}\left(\frac{1}{4}\,F_{\gamma\delta}\,F^{\gamma\delta} + F^{0\nu}F_{\nu\cdot}^{\cdot 0}\right) + j_\nu A^\nu - j^0 A^0. \tag{4.29}$$

Since we chose an orthonormal basis, we have

$$(T^{00})_{\rm elec} = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \frac{1}{4\pi}\,\Delta(\Phi\boldsymbol{E}), \tag{4.30}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are respectively the electric and magnetic field strengths derived from the Maxwell tensor

$$F_{\gamma\delta} = \partial_\gamma A_\delta - \partial_\delta A_\gamma$$

(we assume that the field potential $A^\alpha(\Phi, \boldsymbol{A})$ is given in the Lorentz gauge). The charge density is derived from

$$\Delta\boldsymbol{E} = 4\pi\mu, \tag{4.31}$$

which is just the time component of the 4-current density inferred from Maxwel's equations

$$\nabla_\alpha F^{\alpha\beta} = \frac{4\pi}{c}\;j^\beta. \tag{4.32)}$$

Therefore negative energy density may be shown explicitly by the interaction tensor

$$(T^{00})_{\rm elec-int} = \frac{1}{4\pi}\,\boldsymbol{E}\,\Delta\Phi + \mu\Phi, \tag{4.33}$$

$$(T^{00})_{\rm elec-int} = \frac{1}{4\pi}\left[\,-\Delta\Phi - \frac{1}{c}\,\partial_t\boldsymbol{A}\right]\Delta\Phi + \mu\Phi \tag{4.34}$$

since $\boldsymbol{E} = -\Delta\Phi - \frac{1}{c}\,\partial_t\boldsymbol{A}$.

In (4.34) the first term in the brackets is always negative. As to the last term, it is made negative when the time varying charge density μ and the scalar potential Φ are 180° out of phase (method reached by the use of phasors).

We now assume that the positive free radiative energy density

$$(T^{00})_{\rm elec-rad} = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2) \tag{4.35}$$

is confined within the machine, i.e., right to the inner side of the shell wall.

The interacting tensor $(T^{00})_{\rm elec-int}$ is set so as to exhibit its energy density part on the *external* side of the shell. Now, we see that negative energy production can be achieved with such a configuration. The higher the charge density and the higher the scalar potential, then the most effective negative energy density. The local field equations read

$$G_{\mu\beta} = \frac{8\pi G}{c^4}\left[\left(\rho c^2 + P\right)u_\mu u_\beta - P g_{\mu\beta} + (T_{\mu\beta})_{\rm elec}\right]. \tag{4.36}$$

The energy density level $(T^{00})_{\rm elec-int}$ is now remaining and is anticipated to be very huge. There is however a possible drastic reduction which adequately exploits the contribution of the electromagnetic field interacting with the charges.

4.4 The energy required for the propulsion

The machine is externally charged surrounded by a comoving electromagentic field. Thus, it follows the *Finsler geodesic* [17] provided that the ratio $\mu/\rho c^2$ remains constant along the trajectory

$$(ds)_{\rm shell} = ds + \frac{\mu}{\rho c^2}\,A_\alpha\,dx^\alpha, \quad ds = \sqrt{\eta_{\alpha\beta}\,dx^\alpha dx^\beta}\,. \tag{4.37}$$

Neglecting the non quadratic terms, the metric reads

$$(ds^2)_{\text{shell}} = ds^2 + \left(\frac{\mu}{\rho c^2} A_\alpha dx^\alpha\right)^2 . \tag{4.38}$$

For the energy density of the machine, the spatial components $(\mu/\rho c^2)A_\alpha dx^\alpha$ in (4.38) do not come into play. The interaction term reduces to its time component

$$\frac{\mu}{\rho c^2} A_0 dx^0 = \frac{\Phi\mu}{\rho c^2} cdt, \tag{4.39}$$

where Φ is the scalar potential.

If $g_{\alpha\beta}$ is approximated to the Minkowski tensor $\eta_{\alpha\beta}$, the metric (4.38) reads

$$ds^2 = \left(1 + \frac{\Phi\mu}{\rho c^2}\right)^2 c^2dt^2 - dz^2 - dx^2 - dy^2.$$

In this case, we notice that the time component of the metric tensor

$$g_{00} = \left(1 + \frac{\Phi\mu}{\rho c^2}\right)^2 \tag{4.41}$$

formally corresponds to the expression of the ADM formalism (signature −2)

$$M = (1 + N) , \tag{4.42}$$

where the lapse function is defined as

$$N = \frac{\Phi\mu}{\rho c^2} . \tag{4.43}$$

The Alcubierre metric (4.20) is now

$$\begin{aligned}(ds^2)_{\text{Al}} = \left[M^2 - \text{v}_s^2 f^2(r_s)\right] c^2dt^2 + \\ + 2\,\text{v}_s f(r_s)\, cdt\,dy - dx^2 - dz^2.\end{aligned} \tag{4.44}$$

The interaction term should be only function of r_s, R_e, σ, and of the thickness $(R_e - R_i)$, but not depending on the velocity v_s.

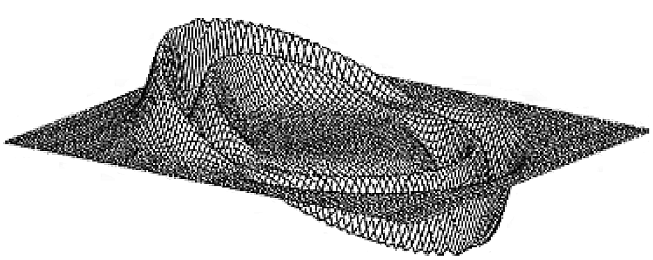

Fig. 3: 2D representation of the warped region propagating from left (expansion) to right (contraction). The groove corresponds to the shell thickness determined by the function N.

Here, our analysis is not too dissimilar to the approach detailed in [18, 19].

From the metric (4.43), it is now easy to derive the components of the Eulerian observer's velocity. We write

$$c^2 = c^2\left(M^2 - \text{v}_s^2 f^2\right)\left(\frac{dt}{d\tau}\right)^2 + 2\,\text{v}_s fc\left(\frac{dt}{d\tau}\right)\boldsymbol{u}_{\text{E}} - \boldsymbol{u}_{\text{E}}^2.$$

Travelling along a geodesic the observer "sees"

$$\frac{dt}{d\tau} = M^{-1}, \tag{4.45}$$

therefore

$$0 = \boldsymbol{u}_{\text{E}}^2\, 2\,\text{v}_s fcM^{-1}\boldsymbol{u}_{\text{E}} + \text{v}_s^2 f^2c^2M^{-2}. \tag{4.46}$$

Hence we find the velocity

$$\boldsymbol{u}_{\text{E}} = \text{v}_s fcM^{-1}, \tag{4.47}$$

the components of which are easy to compute

$$(u^\mu)_{\text{E}} = \left[cM^{-1},\, \text{v}_s fcM^{-1},\, 0,\, 0\right], \tag{4.48}$$

$$(u_\mu)_{\text{E}} = \left[cM,\, 0,\, 0,\, 0\right] . \tag{4.49}$$

By inserting M into (4.24), the expression

$$C_{\text{Al}} = (u_0)_{\text{E}}(u_0)_{\text{E}}(T^{00})_{\text{Al}} \tag{4.50}$$

leads to the new required energy density

$$(T^{00})_{\text{Al}} = -\frac{c^4}{32\pi G}\frac{\text{v}_s^2(x^2+z^2)}{M^4r_s^2}\left(\frac{df}{dr_s}\right)^2 . \tag{4.51}$$

Therefore we may choose the factor N (thereby M) arbitrarily large so as to substantially reduce the required energy density for the machine frame.

Looking at (4.43), the higher the charge and the potential, the lower the energy requirement. In the closed volume V of the machine shell one can inject a flow of electrons according to the constant ratios

$$\frac{\mu}{\rho} = \frac{\sum_{\text{V}} e}{\sum_{\text{V}} m} .$$

We see that the leptonic electron lightweight has the capacity to lower the negative energy even further.

The negative energy supply is fnally expressed by

$$\left[\Delta\Phi + \frac{1}{c}\,\partial_t \boldsymbol{A}\right]\Delta\Phi + \mu\Phi = \frac{c^4}{8G}\frac{\text{v}_s^2(x^2+z^2)}{\left(1+\frac{\Phi\mu}{\rho c^2}\right)^4 r_s^2}\left(\frac{df}{dr_s}\right)^2 . \tag{4.52}$$

Part III

5 The generalized Gödel metric

5.1 Dynamics of the fluid

The splitting shell/inner part of the spacecraft frame, is really the hallmark of the theory here. It implies that the proper time τ of the inner part of the machine is not affected by the term

N. We now set the machine to follow the trajectory $y_s(t)$ *tangential* to a CTC beyond the Gödel radius r_G. Hence, we may write down the Gödel-Alcubierre metric that was generalized (3.3) in the following form

$$ds^2 = e^{2U(1-f)} \left\{ \left[\left(1 + \frac{\Phi\mu}{\rho} \right)^2 - \mathrm{v}_s^2 f^2 \right] c^2 dt^2 - \right. - \left[f - \frac{1}{2}(1-f)\, e^{2x} \right] dy^2 - \left. - 2\left[\mathrm{v}_s f + (1-f)\, e^x \right] c\,dt\,dy - dx^2 - dz^2 \right\}. \tag{5.1}$$

The shell of the machine has a volume V and the total energy required for the propulsion along y is

$$E = -\int_{\mathrm{V}} \frac{c^4}{32\pi G} \frac{\mathrm{v}_s^2 (x^2+z^2)}{\left(1+\frac{\Phi\mu}{\rho c^2}\right)^4 r_s^2} \left(\frac{df}{dr_s} \right)^2 . \tag{5.2}$$

From the machine's perspective, the space-time can be regarded as *globally hyperbolic* since for $f = 1$, it is always defined by the metric (4.43) and the occupant of the machine will never know whether he moves along a CTC.

In the absence of charge, beyond R_e (i.e., where $R > R_e$ and $R \to \infty$), we have $f = 0$ implying $M = 1$ outside of the machine and we thus retrieve Gödel's original modified metric (3.3) in this case.

It is now easy to determine the acceleration of the flow lines carrying the machine.

Let us revert to equation (2.1bis)

$$\sigma_{\alpha\mu} = u_{(\alpha;\mu)} - \frac{1}{3}\theta h_{\alpha\mu} - \omega_{\sigma\mu} - {}^{*}u_{(\alpha} u_{\mu)} = 0, \tag{5.3}$$

$${}^{*}u_{(\alpha} u_{\mu)} = -\left(u_{[\alpha;\mu]} + {}^{*}u_{[\alpha} u_{\mu]} \right) - \frac{1}{3}\theta h_{\alpha\mu} + u_{\alpha;\mu}\,. \tag{5.4}$$

In our case, the scalar expansion is

$$\theta = \mathrm{v}_s \frac{df}{dr_s} \frac{y_s}{r_s},$$

see (4.17). In the equation (A7), see Appendix, we found

$${}^{*}u_\alpha = h_{\alpha\mu} \partial^\mu U,$$

therefore we have

$${}^{*}u_\mu h_{\alpha\mu} \partial^\mu U = -\left(u_{[\alpha;\mu]} + {}^{*}u_{[\alpha} u_{\mu]} \right) - - \frac{1}{3} h_{\alpha\mu} \mathrm{v}_s \frac{df}{dr_s} \frac{y_s}{r_s} + u_{(\alpha;\mu)}\,. \tag{5.5}$$

This equation is fundamental: it displays all elements related to the dynamics of the fluid described by the Gödel-Alcubierre metric (5.1): the pressure and density of the fluid as function of its rotation along a flow line subjected to the Alcubierre local deformation.

5.2 A thermodynamic aspect

Consider a fluid that consists of n particles in motion within a given region. The primary variables are:

— The particle current

$$I^\mu = n u^\mu\,; \tag{5.6}$$

— The energy-momentum $T^{\mu\nu}$ and the entropy flux S^μ. These quantities are conserved

$$T^{\mu\nu}_{\ ;\nu} = 0, \qquad I^\mu_{\ ;\mu} = 0.$$

In a relativistic case, the second law of thermodynamics requires

$$S^\mu_{\ ;\mu} \geqslant 0. \tag{5.7}$$

For equilibrium states we have

$$S^\mu = n s u^\mu, \tag{5.8}$$

where s is the entropy per particle. Denoting Q as the chemical potential and T the heat quantity of the medium, the Euler relation reads

$$n s = \frac{\rho + P}{\mathrm{T}} - \frac{\mathrm{Q}n}{\mathrm{T}}, \tag{5.9}$$

where ρ and P are respectively the density and pressure of the medium.

We also have the fundamental thermodynamic equation of Gibbs

$$\mathrm{T}\,ds = ds\,\frac{\rho}{n} + P\,d\left(\frac{1}{n}\right) \tag{5.10}$$

or

$$\mathrm{T}\,n\,ds = d\rho - \frac{\rho + P}{n} + dn. \tag{5.11}$$

From (5.9), we get

$$S^\mu = -\frac{\mathrm{Q}I^\mu}{\mathrm{T}} + \frac{(\rho + P)u^\mu}{\mathrm{T}}\,. \tag{5.12}$$

Since in the rest system, the matter energy flux must vanish, we have

$$u_\lambda T^{\lambda\mu} = \rho u^\mu \tag{5.13}$$

and thus, we find the following expression for the entropy vector in equilibrium

$$S^\mu = -\frac{\mathrm{Q}I^\mu}{\mathrm{T}} + \frac{u_\lambda T^{\lambda\mu}}{\mathrm{T}} + \frac{P u^\mu}{\mathrm{T}}\,. \tag{5.14}$$

Let us consider our machine moving along a Gödel trajectory. We obviously neglect the chemical potential of the machine's bodyframe as well as the pressure and the entropy vector reduces to

$$S^\mu = \frac{u_\lambda T^{\lambda\mu}}{\mathrm{T}}\,. \tag{5.15}$$

This vector must be measured by the Eulerian observer which travels along the trajectory tangent to u^λ and (5.15) becomes

$$(S^\mu)_{\mathrm{E}} = \frac{(u_\lambda)_{\mathrm{E}} (T^{\lambda\mu})_{\mathrm{Al}}}{\mathrm{T}}\,. \tag{5.16}$$

Keeping in mind our definition of the Eulerian velocity

$$(u^{\mu})_{\mathrm{E}} = \left[cM^{-1}, \mathrm{v}_s f cM^{-1}, 0, 0 \right], \qquad (5.17)$$

$$(u_{\lambda})_{\mathrm{E}} = [cM, 0, 0, 0], \qquad (5.18)$$

and since we are interested in the entropy scalar part we have

$$(S^{0})_{\mathrm{E}} = \frac{(u_0)_{\mathrm{E}}\,(T^{00})_{\mathrm{Al}}}{\mathrm{T}} \qquad (5.19)$$

with

$$(T^{00})_{\mathrm{Al}} = -\frac{c^4}{32\pi G}\,\frac{\mathrm{v}_s^2\,(y^2+z^2)}{M^4 r_s^2}\left(\frac{df}{dr_s}\right)^2, \qquad (5.20)$$

$$(u_0)_{\mathrm{E}} = cM. \qquad (5.21)$$

We clearly see that the entropy $(S^0)_{\mathrm{E}}$ of the system attached to the machine is seen negative with respect to the Eulerian observer which measures a "negentropy". While travelling to the past, the occupant of the machine experiences a positive entropy, i.e., he is ageing in his own proper time.

5.3 The voyage home

The above analysis has been extended to the backwards time travel as initially detailed by Gödel. Once this voyage is completed, the machine should be able to return to its own epoch. Therefore, the reversed oriented loop is obviously to be envisaged although it should be emphasized that it does not represent a future travel.

To this end it is useful to refer to our publication [20], where we recalled that the Einstein tensor is derived from the second Bianchi identity verified by the Riemann tensor $R_{\alpha\beta\nu\gamma}$. A particular form of the latter is described by the *Landau-Lifchitz superpotential*

$$H^{\alpha\beta\nu\gamma} = -g\left(g^{\alpha\nu}g^{\beta\gamma} - g^{\beta\nu}g^{\gamma\alpha}\right) \qquad (5.22)$$

The second order tensor

$$H^{\alpha\beta\nu\gamma}_{\ ,\beta\gamma} = \partial_\beta\left\{\partial_\gamma\left[-g\left(g^{\alpha\nu}g^{\beta\gamma} - g^{\beta\nu}g^{\gamma\alpha}\right)\right]\right\} \qquad (5.23)$$

is a special choice of the Ricci tensor in which all first derivatives of the metric tensor vanish at the considered point. The corresponding field equations read

$$H^{\alpha\beta\nu\gamma}_{\ ,\beta\gamma} = \frac{16\pi G}{c^4}\left[-g\left(T^{\alpha\nu} + t^{\alpha\nu}_{\mathrm{L-L}}\right)\right], \qquad (5.24)$$

where $T^{\alpha\nu}$ is the energy-momentum tensor of matter with its gravitational field described by the *Landau-Lifchitz energy-momentum pseudo-tensor* $t^{\alpha\nu}_{\mathrm{L-L}}$

$$\begin{aligned}(-g)\,t^{\alpha\nu}_{\mathrm{L-L}} = \frac{c^4}{16\pi G}\Big\{ &\boldsymbol{g}^{\alpha\nu}_{,\lambda}\boldsymbol{g}^{\lambda\mu}_{,\mu} - \boldsymbol{g}^{\alpha\lambda}_{,\lambda}\boldsymbol{g}^{\nu\mu}_{,\mu} + \frac{1}{2}\,g^{\alpha\nu}g_{\lambda\mu}\,\boldsymbol{g}^{\lambda\theta}_{,\rho}\boldsymbol{g}^{\rho\mu}_{,\theta} - \\ &- \left(g^{\alpha\lambda}g_{\mu\theta}\,\boldsymbol{g}^{\nu\theta}_{,\rho}\boldsymbol{g}^{\mu\rho}_{,\lambda} + g^{\nu\lambda}g_{\mu\theta}\,\boldsymbol{g}^{\alpha\theta}_{,\rho}\boldsymbol{g}^{\mu\rho}_{,\lambda}\right) + g_{\mu\lambda}\,g^{\theta\rho}\,\boldsymbol{g}^{\alpha\lambda}_{,\theta}\boldsymbol{g}^{\nu\mu}_{,\rho} + \\ &+ \frac{1}{8}\left(2g^{\alpha\lambda}g^{\nu\mu} - g^{\alpha\lambda}g^{\lambda\mu}\right)\left(2g_{\theta\rho}g_{\delta\tau} - g_{\rho\delta}g_{\theta\tau}\right)\boldsymbol{g}^{\theta\tau}_{,\lambda}\boldsymbol{g}^{\rho\delta}_{,\mu}\Big\},\end{aligned}$$

where $\boldsymbol{g}^{\alpha\nu} = \sqrt{-g}\,g^{\alpha\nu}$.

In this way, the right hand side of (5.24) is conserved

$$\partial_\nu\left[-g\left(T^{\alpha\nu} + t^{\alpha\nu}_{\mathrm{L-L}}\right)\right] = 0.$$

Besides equation (5.24), there exists a second field equation having the form

$$H^{\alpha\beta\nu\gamma}_{\ ,\gamma\alpha} = \frac{16\pi G}{c^4}\left[-g\left(T^{\beta\nu} + t^{\beta\nu}_{\mathrm{L-L}}\right)\right]. \qquad (5.25)$$

A quick inspection at (5.22), shows that field equations (5.24) and (5.25) differ from a sign and are linked by a common index but they are not necessarily symmetrical. The intertwined metrics are

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu, \qquad (5.26)$$

$${}^{(-)}ds^2 = -g_{\mu\lambda}\,dx^\mu dx^\lambda. \qquad (5.27)$$

In our case, the time coordinate $x^0 = ct'$ is chosen to be the cosmic time-axis of the expanding universe described by the positive metric (5.26). It is then pertinent to identify x^0 with the common index

$$ds^2 = g_{0\nu}\,dx^0 dx^\nu, \qquad (5.28)$$

$${}^{(-)}ds^2 = -g_{0\lambda}\,dx^0 dx^\lambda. \qquad (5.29)$$

The Gödel solution (3.18) corresponding to (5.27) can be expressed by

$$\begin{aligned}{}^{(-)}ds^2 = -4\,e^{2U}\Big[&dt'\,(2\sinh r - 1)^2 - dr^2 + \\ &+ \left(\sinh^4 r - \sinh^2 r\right)d\theta^2 + 2\sqrt{2}\,\sinh^2 r\,d\theta dt'\Big].\end{aligned} \qquad (5.30)$$

One notices that t' is negative which means that the trajectory of the machine derived from (5.30) is reversed with respect to the one resulting from the solution (3.18). Starting from the point p (see Fig. 2 on page 19) the machine reaches the Gödel radius r_G for $\sinh r = 1$, while it is still being governed by the equation (5.30). As soon as $r = 0$ after crossing the plane containing r_G, then ds^2 becomes positive again and reconnects to the cosmic time t' at the departure point q. In order to come back to its epoch, our machine can then legitimately exploit the second field equation whose solution is given by (5.30).

Conclusions

When Gödel introduced his metric, he was led to introduce a distinctive constant factor a in order to retranscript the field equations with a cosmological constant along with additional constraints. Our theory is free of all theses constraints and moreover, it provides a physical meaning to the a term.

The Gödel space-time is no longer a cosmological model but a limited domain wherein takes place the dynamics of a physical fluid which retains all basic properties related to closed time-like curves. The modified Gödel metric can be locally replicated and this fact naturally sheds new light on time travel possibility.

Our theory, which relies on the Alcubierre metric propulsion, has some similarities with the one suggested by B. Tippett and D. Tsang (University of British Columbia, Okanagal and MacGill University, Montreal, respectively). The essence of their paper is to describe an Alcubierre bubble which travels backwards and forwards along a loop in flat space-time [21]. In this geometry, the bubble is referred to as a *Traversable Achronal Retrograd Domain in Space-Time* or TARDIS, in short. The TARDIS is also an acronym for *Time and Relative Dimensions in Space*, a fictional hybrid of a time machine and spacecraft that appears in the British science fiction television series *Doctor Who* and its various spin-offs. See also [22].

Historically, it seems that the first model exhibiting CTCs was pioneered by the German mathematician C. Lanczos, assistant to Einstein in 1924 [23] and later re-discovered in 1937 in an improved form by the Dutch physicist W. J. Van Stockum [24].

A typical example of a time machine was first proposed in 1974 by the American F. J. Tipler, Prof. of Physics at the Tulane University, New Orleans [25]. It describes an infinitely long massive cylinder spinning along its longitudinal axis which gives rise to the *frame dragging effect*. If the rotation rate is fast enough the light cones of objects in cylinder's vicinity becomes tilted. Tipler claimed that a finite cylinder could also produce CTCs which was objected by S. Hawking who argued that any finite region would require negative energy and at the same time, vacuum fluctuation mechanism would impede any attempts to travel in time [26]. Several authors have however challenged this last conclusion and rejected Hawking's statement [27, 28].

At the same time, travelling backwards in time highlights many paradox problems. Among them is the well-known *grandfather paradox*: a person travelling to the past and causing the death of his ancestor beforehand is thus never born and it would not be possible for him to undertake such an act in the first place. In fact, you can not fix your issues by travelling back in time: you go back in time to prevent something that happened in the past and arrive just before the event. You race to stop it, yet in doing as such, directly or indirectly cause it to happen in the first place. This can be illustrated by the predestination paradox where a billiard ball is sent in the past through the time machine:

In other words one cannot change the past (at least major events): this is confirmed by the famous *Self Consistency Principle* introduced in 1990 by the physicist I. Novikov [29]. The principle asserts that if an event exists that would cause a paradox or any change to the past whatsoever, then the probability of that event is zero. This principle does not exclude the predestined fate of our history if some actions from the future would have marked the successive events of our evolution.

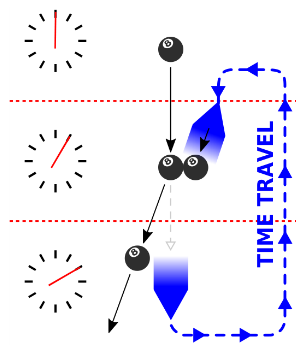

Fig. 4: The predestination paradox where a billiard ball is sent in the past through the time machine.

A typical example is the so-called *writer's paradox*, when the inventor of the time machine sends his calculations to a selected scientist in his own past. The question naturally arises: who could be this scientist?

Fig. 5: Albert Einstein and Kurt Gödel at the Princeton Institute for Advanced Study, Princeton, New Jersey, December 5, 1947. Photo by Oskar Morgenstern.

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Appendix

The 4-unit vector u^μ is normalized on $(M, \boldsymbol{g})$

$$g_{\mu\nu}\, u^\mu u^\nu = g^{\mu\nu} u_\mu u_\nu = 1.$$

By differentiating it we get

$$u^\nu\, \nabla_\mu u_\nu = 0\,. \tag{A1}$$

Let us define the vector L_ν by the relation

$$\nabla_\mu P\, \delta^\mu_\nu = \mathbf{r}\, L_\nu \tag{A2}$$

having set $\mathbf{r} = \rho c^2 + P$.

The conservation law for $T_{\mu\nu} = \mathbf{r}\, u_\mu u_\nu - P\, g_{\mu\nu}$ is expressed by $\nabla_\mu T^\mu_\nu = 0$ or

$$\nabla_\mu (\mathbf{r}\, u^\mu u_\nu) = \mathbf{r}\, L_\nu\,, \quad \nabla_\mu (\mathbf{r}\, u^\mu)\, u_\nu + \mathbf{r}\, u^\mu\, \nabla_\mu u_\nu = \mathbf{r}\, L_\nu\,. \tag{A3}$$

Multiplying through this relation with u^ν and taking into account (A1), after substituting it into (A3) and then dividing by $\mathbf{r}$, we obtain

$$u^\mu\, \nabla_\mu u_\nu = \left(g_{\mu\nu} - u_\mu u_\nu\right) L^\mu \tag{A4}$$

or

$$^*u_\nu = h_{\mu\nu} L^\mu. \tag{A5}$$

Setting $L_\nu = \partial_\nu U$, the equation (A5) takes the form $^*u_\nu = h_{\mu\nu}\, \partial^\mu U$ and (A2) reads

$$\left(\rho c^2 + P\right) L_\nu = \nabla_\mu P\, \delta^\mu_\nu\,, \quad L_\nu = \frac{\partial_\nu P}{\rho c^2 + P}\,.$$

As a result we find

$$U = \int_{P_1}^{P_2} \frac{dP}{\rho c^2 + P}\,.$$

The flow lines of the fluid everywhere tangent to the vector u^μ are determined by the differential system (3.15)

$$u^\mu\, \nabla_\mu u_\beta = \left(\delta^\alpha_\beta - u^\alpha u_\beta\right) \partial_\alpha U\,.$$

These flow lines are time-like geodesics of the conformal metric

$$\mathcal{A} = s' = \int_{S_1}^{S_2} e^U\, ds\,. \tag{A6}$$

The 4-vector

$$^*u_\nu = h_{\mu\nu}\, \partial^\mu U \tag{A7}$$

must be regarded as the 4-acceleration $^*u_\nu$ of the flow lines given by the pressure gradient orthogonal to those lines; see [30, p. 70].