Calculating Riemannian Conditions for the Space Metrics Used in General Relativity

Dmitri Rabounski

Puschino, Moscow Region, Russia. E-mail: rabounski@yahoo.com

Here we explain Riemannian conditions — the particular conditions, under which the space (space-time) metric under consideration satisfies Einstein's field equations, and therefore becomes Riemannian. We show that the Riemannian conditions depend only on the presence of a gravitational field, rotation of the space, its deformation and curvature, and not the particular formulae for these factors. Then, we deduce the Riemannian conditions for the classical space metrics most used in General Relativity.

1 Problem statement

Many different space-time metrics can be devised by the human mind. But not all of these metrics can be used in the General Theory of Relativity, because not all of these metrics are Riemannian. Numerous experiments confirming the effects of General Relativity commencing in 1919 (Eddington) had proven that the four-dimensional pseudo-Riemannian space is the basic space-time of the Universe. Therefore, today we have every reason to assert that the geometry of the basic space-time of the Universe is Riemannian. The said means that any space (space-time) metric used in General Relativity must belong to the family of Riemannian spaces.

There are well-known formal conditions under which a space is Riemannian:

- the metric of this space must have the Riemann square form $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ that is determined by the Riemann fundamental metric tensor $g_{\alpha\beta}$;
- the metric must be invariant ($ds^2 = inv$) everywhere in the space;
- the metric must also satisfy Einstein's field equations

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa\, T_{\alpha\beta} + \lambda g_{\alpha\beta}\,, \qquad (1)$$

  which are a form of the 2nd Bianchi identities satisfied in any Riemannian space, and are the relation connecting the Ricci curvature tensor $R_{\alpha\beta}$, the fundamental metric tensor $g_{\alpha\beta}$ multiplied by the curvature scalar R, and the energy-momentum tensor $T_{\alpha\beta}$ of a distributed medium that fills the space, whereas the scalar λ determines the density of the physical vacuum (which is also known as the λ-field).

The third condition means the following. If we substitute the components of the particular fundamental metric tensor $g_{\alpha\beta}$ (taken from the formula of a particular space metric) into the left-hand side of the field equations, and the components of the energy-momentum tensor of the "space filler" into the right-hand side of the equations, and then the left-hand side of the field equations becomes the same as the right-hand side (i.e., the field equations vanish), then this particular metric satisfies the field equations, and therefore the space described by this particular metric is Riemannian.

For a minor number of space metrics, the field equations are satisfied if the given space is empty (the right-hand side of the equations is zero). In this case, the terms on the left-hand side of the equations vanish each other, and therefore the space is Riemannian without distributed matter or any additional conditions.

However, for most other space metrics the left-hand side cannot be reduced to zero and must therefore be equalized with the right-hand side (distributed matter). i.e., the space metric satisfies the field equations only in the presence of distributed matter, say, a dust, a gas, an electromagnetic field or something else. What kind of distributed matter fills the space is determined by its specific energy-momentum tensor, with which the right-hand side of the field equations equalizes the left-hand-side. But even in this case not many space metrics are proven to be Riemannian and therefore are used in General Relativity.

To correct this situation and increase the number of space metrics used by relativists, the author introduced *Riemannian conditions* in 2022 [1]. The principle of the Riemannian conditions was then applied in [2], where the author deduced and proved the space metric of a rotating massive body in emptiness (the rotating Schwarzschild mass-point metric), and in [3], where the author first obtained mathematically correct solutions for rotating black holes based on this metric.

The mathematical essence of the Riemannian conditions is simple. Let us have a Riemannian-like space metric that does not satisfy the field equations both in emptiness (i.e., with the zero right-hand side of the equations) and in the presence of distributed matter (i.e., with the energy-momentum of a distributed medium on the right-hand side of the equations). Then, if the metric satisfies the first two Riemann requirements (see above), i.e., it is a Riemannian-like metric, we can always find particular mathematical conditions under which the left-hand side and the right-hand side of the field equations equalize each other. As a result, we will have a space metric which in general has not been proven to be Riemannian, but in the particular case under the specific Riemannian conditions the metric has been proven to be Riemannian, and therefore can be used in General Relativity.

But, despite the obvious advantage of the Riemannian conditions, there is a "minor problem". They are calculated via the field equations, and the equations contain the curvature tensor, which therefore must necessarily be calculated for any space metric under consideration. Calculating the space curvature is a non-trivial task for even a well skilled relativist. This takes weeks of hard work, taking into account four or five necessary re-calculations to be sure in the obtained result, which may not match due to calculation errors, which needs new re-calculations, etc.

However, when the problem of calculating the Riemannian conditions was considered in more detail, the author discovered an interesting circumstance:

> The sense of the Riemannian conditions is seen better, if the properties specific to the space under consideration (they are on the left-hand side of the field equations) as well as the components of the energy-momentum tensor of the distributed matter on the right-hand side (if the space is non-empty) are not expressed in the form, specific to a particular space metric, but are presented in their general form without detail.

Therefore, before doing anything that requires introducing a new space metric, it makes sense to find the Riemannian conditions for the classical metrics introduced and proven from the very beginning. This is what will be now done in this article.

As in all previous studies by the author, here we use the *mathematical apparatus of chronometric invariants*, which was introduced in 1944 by Abraham Zelmanov and is also known as the *chronometrically invariant formalism*. Chronometrically invariant quantities (or chronometric invariants, in short) are the projections of four-dimensional general covariant quantities onto the three-dimensional spatial section belonging to a real observer (his observable three-dimensional space) and onto his own time line. The projection process follows the geometric structure of the space-time itself thanks to special projection operators specific to the four-dimensional pseudo-Riemannian space. Thus, the resulting time and spatial projections are invariant throughout the entire spatial section belonging to the observer (his observable space), and also depend on its geometric and physical properties. Therefore, chronometric invariants are physically observable quantities in the space-time of General Relativity.

To avoid overloading this article, the author refers anyone curious about the details of this mathematical formalism to Zelmanov's original publications [4–6] or to the most comprehensive (to date) compendium of it, collected and published in 2023 [7].

We need here only the chr.inv.-projections of the general covariant field equations (1). They are derived as for any symmetric 2nd-rank tensor (since the field equations are 2nd-rank tensor equations) and represent: the projection onto the observer's time line (the first, scalar equation), the projection onto his space (the third, tensor equation), and the so-called "mixed space-time projection" (the second, vector equation). They were deduced by Zelmanov in 1941–1944, and were called by him the *chr.inv.-Einstein equations*

$$\left.\begin{aligned}
&\frac{{}^*\partial D}{\partial t} + D_{jl}D^{jl} + A_{jl}A^{lj} + {}^*\nabla_j F^j - \frac{1}{c^2}F_jF^j = \\
&\qquad\qquad = -\frac{\varkappa}{2}\left(\rho c^2 + U\right) + \lambda c^2 \\
&{}^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}F_jA^{ij} = \varkappa J^i \\
&\frac{{}^*\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D^j_k + A_{k\cdot}^{\cdot j}\right) + DD_{ik} + 3A_{ij}A_{k\cdot}^{\cdot j} - \\
&\qquad - \frac{1}{c^2}F_iF_k + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - c^2C_{ik} = \\
&\qquad = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right) + \lambda c^2 h_{ik}
\end{aligned}\right\}. \tag{2}$$

They connect physically observable properties of the observer's reference space (presented on the left-hand side) with physically observable properties of a distributed matter that fills the space (the right-hand side).

The physically observable properties of the observer's reference space are expressed on the left-hand side of the equations with the four chr.inv.-quantities. The chr.inv.-vector of the acting gravitational inertial force F_i is created by the gradient of the gravitational potential (i.e., the gradient of g_{00}) and the non-stationarity of the linear velocity with which the space rotates (determined by g_{0i}), i.e., the non-stationarity of g_{0i}. The chr.inv.-antisymmetric tensor A_{ik} of the angular velocity of the space (if it rotates) is the vortex of the linear rotational velocities of the space (i.e., the vortex of g_{0i}) with correction for the acting gravitational inertial force F_i. The chr.inv.-tensor D_{ik} of the deformation rates of the space (if it deforms) is due to the non-stationarity of the chr.inv.-metric tensor h_{ik}. In turn, h_{ik} is determined via g_{ik} with correction for g_{0i} and g_{00}, and has all properties of the fundamental metric tensor $g_{\alpha\beta}$ throughout the entire spatial section belonging to the observer (his observable space). The chr.inv.-curvature tensor C_{ik} is a combination of the second and first chr.inv.-spatial derivatives of the chr.inv.-metric tensor h_{ik}. The sign ${}^*\partial$ denotes chr.inv.-differentiation, ${}^*\nabla_i$ is the sign of chr.inv.-derivative (which is similar to general covariant derivative, but is based on chr.inv.-quantities), and ${}^*\nabla_i - \frac{1}{c^2}F_i = {}^*\widetilde{\nabla}_i$ is chr.inv.-physical derivative, which takes into account the fact that the flow of time is different on the opposite walls of an elementary volume; see the compendium of chronometric invariants [7, p.18–19] for detail.

On the right-hand side of the chr.inv.-Einstein equations, you can find the chr.inv.-projections of the energy-momentum tensor $T_{\alpha\beta}$ of a distributed medium that fills the space

$$\rho = \frac{T_{00}}{g_{00}}, \qquad J^i = \frac{c\,T_0^i}{\sqrt{g_{00}}}, \qquad U^{ik} = c^2 T^{ik}, \tag{3}$$

which are the chr.inv.-scalar ρ (the phys.observ.-mass density of the medium), the chr.inv.-vector J^i of the phys.observ.-momentum density of the medium, and the chr.inv.-tensor U^{ik} (the phys.observ.-stress tensor of the medium). You can also find there the scalar term λ, which determines the density of the physical vacuum (λ-field).

Based on the definitions of F_i, A_{ik}, and D_{ik} via the respective components of the fundamental metric tensor $g_{\alpha\beta}$, we can obtain the Riemannian conditions for any space metric.

Namely, — just looking at the formula for any metric, we see whether $g_{00} \neq 1$, $g_{0i} \neq 0$, and $g_{ik} = f(t)$ exist in this particular space, and thus which of the factors F_i, A_{ik}, and D_{ik} are non-zero in it (the curvature C_{ik} remains non-zero in any case, since it is based on the spatial derivatives of the chr.inv.-metric tensor h_{ik}). Then, removing the zero factors on the left-hand side of the chr.inv.-Einstein equations (2), we immediately obtain the Riemannian conditions for this space.

2 The Riemannian conditions for spaces containing only a gravitational field

2.1 Schwarzschild's mass-point metric. This metric describes an empty spherical space (i.e., there is no distributed matter), wherein a gravitational field and curvature are created by a spherical mass approximated by a mass-point. The mass-point metric has the form

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{4}$$

where r is the distance from the mass M, $r_g = \frac{2GM}{c^2}$ is the gravitational radius of the mass, G is the constant of gravitation, and $x^1 = r$, $x^2 = \varphi$, $x^3 = \theta$ are spherical coordinates.

Just looking at this formula, we see that $g_{00} \neq 1$, $g_{0i} = 0$ and $g_{ik} \neq f(t)$, and therefore $F_i \neq 0$, but $A_{ik} = 0$ and $D_{ik} = 0$ in this space. Therefore, the right-hand side of the chr.inv.-Einstein equations (2) is zero for the mass-point metric, and the Riemannian conditions have the form

$$\left.\begin{aligned} {}^*\nabla_j F^j &= \frac{1}{c^2} F_j F^j \\ {}^*\nabla_i F_k - \frac{1}{c^2} F_i F_k &= c^2 C_{ik}\,, \quad i = k \end{aligned}\right\}, \tag{5}$$

where $\frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2} F_i F_k = {}^*\nabla_i F_k - \frac{1}{c^2} F_i F_k$ as well as for any spherically symmetric field, since it has just one non-zero component $F_i = F_1$. They mean the following:

The 1st Riemannian condition

In a space of the mass-point metric, the observable field of the acting gravitational force F_i is inhomogeneous (i.e., ${}^*\nabla_j F^j \neq 0$) due to only the spherically symmetric structure of this force field.

The 2nd Riemannian condition

The observable curvature C_{ik} of such a space is created by the strength ${}^*\nabla_i F_k$ of the gravitational force field, minus a correction for the spherically symmetric structure of this force field.

2.2 De Sitter's space metric. It describes a spherical space that does not rotate or deform, does not contain islands of substance, but is filled with a spherical homogeneous distribution of the physical vacuum (λ-field), which creates a gravitational field and curvature. This metric has the form

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{6}$$

In this metric, $g_{00} \neq 1$, $g_{0i} = 0$, $g_{ik} \neq f(t)$, and therefore $F_i \neq 0$, $A_{ik} = 0$, $D_{ik} = 0$. The right-hand side of the chr.inv.-Einstein equations (2) is non-zero (since $\lambda \neq 0$), and therefore the Riemannian conditions have the form

$$\left.\begin{aligned} {}^*\nabla_j F^j &= \frac{1}{c^2} F_j F^j + \lambda c^2 \\ {}^*\nabla_i F_k - \frac{1}{c^2} F_i F_k - \lambda c^2 h_{ik} &= c^2 C_{ik} \end{aligned}\right\}. \tag{7}$$

They have the following physical sense:

The 1st Riemannian condition

In a de Sitter space, the observable gravitational force field is inhomogeneous (${}^*\nabla_j F^j \neq 0$) due to the spherically symmetric structure of this force field, with a constant correction for the density of the physical vacuum that fills the space (since $\lambda = const$).

The 2nd Riemannian condition

In a de Sitter space, the observable curvature C_{ik} is created by the strength ${}^*\nabla_i F_k$ of the gravitational force field, minus corrections determined by the spherically symmetric structure of this force field and by the density of the physical vacuum (λ-field).

2.3 Schwazschild's metric of a liquid sphere. It describes the space inside a sphere filled with ideal liquid

$$ds^2 = \frac{1}{4}\left(3\sqrt{1 - \frac{\varkappa\rho a^2}{3}} - \sqrt{1 - \frac{\varkappa\rho r^2}{3}}\right)^2 c^2 dt^2 - \frac{dr^2}{1 - \frac{\varkappa\rho r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{8}$$

where a is the sphere's radius, and $\rho = \frac{M}{V} = \frac{3M}{4\pi a^3}$ is the density of the liquid. According to this formula, $g_{00} \neq 1$, $g_{0i} = 0$, $g_{ik} \neq f(t)$, and therefore $F_i \neq 0$, $A_{ik} = 0$, $D_{ik} = 0$.

Substituting the above into the chr.inv.-Einstein equations (2), and since $U_{ik} = p h_{ik}$ and $U = 3p$ for any non-viscous medium possessing a pressure [10], including ideal liquid, we obtain the Riemannian conditions for the above metric

$$\left.\begin{aligned} {}^*\nabla_j F^j &= \frac{1}{c^2}\, F_j F^j - \frac{\varkappa}{2}\,(\rho c^2 + 3p) \\ {}^*\nabla_i F_k - \frac{1}{c^2}\, F_i F_k - \frac{\varkappa}{2}\,(\rho c^2 - p)\, h_{ik} &= c^2 C_{ik} \end{aligned}\right\}, \tag{9}$$

the physical sense of which means the following:

The 1st Riemannian condition

In the space inside a sphere filled with ideal liquid, the observable gravitational force field is inhomogeneous (${}^*\nabla_j F^j \neq 0$) due to the spherically symmetric structure of this force field, with corrections for the density ρ and pressure p of the liquid.

The 2nd Riemannian condition

The observable curvature C_{ik} of the space inside a liquid sphere is created by the strength ${}^*\nabla_i F_k$ of the gravitational force field, minus a correction for the spherically symmetric structure of this force field, and with corrections for the density ρ and pressure p of the liquid that fills this space.

2.4 Reissner-Nordström's metric. There is also another type of spherical spaces having $F_i \neq 0$, but $A_{ik} = 0$ and $D_{ik} = 0$. It is described by Reissner-Nordström's space metric

$$ds^2 = \left(1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}\right) c^2 dt^2 - \\ - \frac{dr^2}{1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{10}$$

which is an extension of the mass-point metric for the case, where the spherical mass is electrically charged. Therefore, such a space is filled with a spherically symmetric electrostatic field. In this metric, q is the electric charge of this point mass, $r_q^2 = \frac{Gq^2}{4\pi\varepsilon_0 c^4}$, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's force constant.

Since $F_i \neq 0$ and taking into account that, as is known, the trace of the energy-momentum tensor for any electromagnetic field is always $U = \varrho c^2$, we obtain the Riemannian conditions for Reissner-Nordström's space metric

$$\left.\begin{aligned} {}^*\nabla_j F^j &= \frac{1}{c^2}\, F_j F^j - \varkappa \varrho c^2 \\ {}^*\nabla_i F_k - \frac{1}{c^2}\, F_i F_k - \varkappa\, U_{ik} &= c^2 C_{ik} \end{aligned}\right\}, \tag{11}$$

which mean the following:

The 1st Riemannian condition

In the space of an electrically charged point mass, the observable gravitational force field is inhomogeneous (${}^*\nabla_j F^j \neq 0$) due to the spherically symmetric structure of this force field, minus a correction determined by the density ϱ of the electrostatic field.

The 2nd Riemannian condition

The observable curvature C_{ik} of the space is created by the strength ${}^*\nabla_i F_k$ of the gravitational force field, minus corrections for the spherically symmetric structure of this force field and for the electric field stress U_{ik}.

3 The Riemannian conditions for spaces free of gravitational field, rotation and deformation

A typical example of such spaces is a finite spherical space of Einstein's metric

$$ds^2 = c^2 dt^2 - \frac{dr^2}{1 - \lambda r^2} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{12}$$

where obviously $g_{00} = 1$, $g_{0i} = 0$ and $g_{ik} \neq f(t)$ (which means $F_i = 0$, $A_{ik} = 0$ and $D_{ik} = 0$), and the observable curvature is created by a homogeneous distribution of ideal liquid and the physical vacuum (λ-field) that together fill the space.

By substituting $F_i = 0$, $A_{ik} = 0$ and $D_{ik} = 0$ into the chr.inv.-Einstein equations (2), and since $U_{ik} = p h_{ik}$ and $U = 3p$ for ideal liquid, we obtain the Riemannian conditions for Einstein's space metric

$$\left.\begin{aligned} \frac{\varkappa}{2}\,(\rho c^2 + 3p) &= \lambda c^2 \\ \frac{\varkappa}{2}\,(p - \rho c^2)\, h_{ik} - \lambda c^2 h_{ik} &= c^2 C_{ik} \end{aligned}\right\} \tag{13}$$

These conditions mean the following:

The 1st Riemannian condition

In a space of Einstein's metric, the density ρ and pressure p in the ideal liquid that fills the space are balanced by the density of the physical vacuum, which also fills it (since $\lambda \neq 0$).

The 2nd Riemannian condition

In a space of Einstein's metric, the observable curvature C_{ik} is created by the pressure p in the ideal liquid that fills the space, minus corrections determined by the density ρ of the liquid and the density of the physical vacuum (since $\lambda \neq 0$).

4 The Riemannian conditions for rotating spaces

4.1 Gödel's space metric. It was introduced in the form

$$ds^2 = a^2\left[(d\tilde{x}^0)^2 + 2e^{\tilde{x}^1} d\tilde{x}^0 d\tilde{x}^2 - \\ - (d\tilde{x}^1)^2 + \frac{e^{2\tilde{x}^1}}{2}(d\tilde{x}^2)^2 - (d\tilde{x}^3)^2\right], \tag{14}$$

expressed with the special dimensionless Cartesian coordinates $d\tilde{x}^0 = \frac{1}{a}\,dx^0$, $d\tilde{x}^1 = \frac{1}{a}\,dx^1$, $d\tilde{x}^2 = \frac{1}{a}\,dx^2$, $d\tilde{x}^3 = \frac{1}{a}\,dx^3$, which contain a new world-constant $a = const > 0$ [cm] that Gödel introduced as

$$a^2 = \frac{c^2}{8\pi G\rho} = -\frac{1}{2\lambda}\,. \tag{15}$$

When transformed to the ordinary Cartesian coordinates $dx^0 = c\,dt$, dx^1, dx^2, dx^3, Gödel's metric takes the form

$$ds^2 = c^2 dt^2 + 2\,e^{\frac{x^1}{a}} c\,dt\,dx^2 - \\ - (dx^1)^2 + \frac{e^{\frac{2x^1}{a}}}{2}(dx^2)^2 - (dx^3)^2, \qquad (16)$$

which is more suitable for analysis. As is seen from the above metric, a space of Gödel's metric does not contain a gravitational field ($g_{00} = 1$, $F_i = 0$), does not deform ($g_{ik} \neq f(t)$, $D_{ik} = 0$), but rotates ($g_{02} \neq 0$, $A_{ik} \neq 0$) in such a way that its time-like curves are self-closed.

Gödel spaces are filled with ideal dust (its density is ρ) and the physical vacuum (λ-field), with which Einstein's field equations are satisfied for this metric. By substituting $F_i = 0$, $A_{ik} \neq 0$ and $D_{ik} = 0$ into the chr.inv.-Einstein equations (2), we obtain the Riemannian conditions for Gödel's space metric

$$\left.\begin{aligned} & A_{jl}A^{jl} = \frac{\varkappa}{2}\rho c^2 - \lambda c^2 \\ & {}^*\nabla_j A^{ij} = 0 \\ & 2A_{ij}A_{k\cdot}^{\cdot j} - \frac{\varkappa}{2}\rho c^2 h_{ik} - \lambda c^2 h_{ik} = c^2 C_{ik} \end{aligned}\right\}, \qquad (17)$$

where we take into account that ideal dust is a pressureless medium ($p = 0$), and its energy-momentum tensor has only one non-zero physically observable component determining the density ρ of this substance (i.e., $J^i = 0$ and $U_{ik} = 0$).

The obtained Riemannian conditions mean the following:

The 1st Riemannian condition

In a Gödel space, the value of its observable angular rotation velocity A_{ik} is determined by the density ρ of the ideal dust that fills this space, minus a correction for the density of the physical vacuum (since $\lambda \neq 0$).

The 2nd Riemannian condition

The observable rotation field of a Gödel space is homogeneous (${}^*\nabla_j A^{ij} = 0$).

The 3rd Riemannian condition

The observable curvature C_{ik} of a Gödel space is created by its rotation field A_{ik}, minus corrections determined by the density ρ of the dust and the density of the physical vacuum (λ-field).

4.2 Other rotating spaces are also conceivable. Earlier, the author introduced "rotating modifications" of the classical space metrics: the simplest rotating space metric, the rotating mass-point space metric, the rotating de Sitter space metric, and the metric inside a rotating liquid sphere [1]. This list was then expanded by the rotating Reissner-Nordström space metric [3] and can now be added by the rotating Einstein space metric. Together they form a *new family of rotating space metrics* for use in General Relativity.

All these new space metrics were derived from the classical ones in the same way. Assume that the space of one of the classical metrics rotates along the φ-axis (i.e., along the geographical longitudes) with the linear velocity $v_3 = \omega r^2 \sin^2\theta$, where $\omega = const$ is the angular velocity of this rotation. Then, according to the definition of v_i [4–7],

$$v_3 = \omega r^2 \sin^2\theta = -\frac{c\,g_{03}}{\sqrt{g_{00}}}\,, \quad g_{03} = -\frac{\omega r^2 \sin^2\theta}{c}\sqrt{g_{00}}\,, \qquad (18)$$

and the metric of this space acquires a corresponding additional term determined by g_{03}.

Note that, since the classical space metrics satisfy the field equations, the space rotation field A_{ik} introduced to the left-hand side of the equations must be balanced by an electromagnetic field added to the right-hand side (because both these fields have a similar structure). This circumstance was not clearly explained in the previous articles [1–3], where these new metrics were introduced.

Let us now obtain Riemannian conditions for each of the rotating space metrics — specific physical conditions under which these metrics satisfy Einstein's field equations and can therefore be used in General Relativity.

4.2.1 The simplest rotating space metric. It has the form

$$ds^2 = c^2 dt^2 - 2\omega r^2 \sin^2\theta\, dt\, d\varphi - \\ - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \qquad (19)$$

This metric cannot be considered without distributed matter, otherwise the scalar chr.inv.-Einstein equation would become nonsense $A_{jl}A^{lj} = 0$ (the space does not rotate). Therefore, in [1] this space was considered filled with an electromagmetic field (since it can exist in the absence of its sources such as electric charges or currents). In this case, $U = \varrho c^2$, and the metric (19) satisfies the chr.inv.-Einstein equations (2) under the Riemannian conditions

$$\left.\begin{aligned} & A_{jl}A^{jl} = \varkappa \varrho c^2 \\ & {}^*\nabla_j A^{ij} = -\varkappa J^i \\ & 2A_{ij}A_{k\cdot}^{\cdot j} - \varkappa U_{ik} = c^2 C_{ik} \end{aligned}\right\}, \qquad (20)$$

which have the following physical sense:

The 1st Riemannian condition

In a space of the simplest rotating metric, which has no masses, and is filled with only an electromagnetic field without sources (i.e., without electric charges and currents), the value of its observable angular rotation velocity A_{ik} is determined by the density ϱ of the source-free electromagnetic field ("space filler").

The 2nd Riemannian condition

The observable field of rotations of a simplest rotating space is inhomogeneous (${}^*\nabla_j A^{ij} \neq 0$) due to an electromagnetic radiation presented in it (since $J^i \neq 0$ for any emitting medium, e.g., the electromagnetic field).

The 3rd Riemannian condition

The observable curvature C_{ik} of a simplest rotating space is created by its rotation field A_{ik}, minus a correction for the electromagnetic field stress U_{ik}.

4.2.2 The rotating mass-point space metric. It has the following form

$$ds^2 = \left(1-\frac{r_g}{r}\right)c^2dt^2 - 2\omega r^2\sin^2\theta\sqrt{1-\frac{r_g}{r}}\,dt\,d\varphi - \\ - \frac{dr^2}{1-\frac{r_g}{r}} - r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right), \quad (21)$$

from which we immediately obtain the Riemannian conditions, under which this metric satisfies the chr.inv.-Einstein equations (2). They have the form

$$\left.\begin{aligned} &{}^*\nabla_j F^j = \frac{1}{c^2}F_jF^j + A_{jl}A^{jl} - \varkappa\varrho c^2 \\ &{}^*\nabla_j A^{ij} = \frac{2}{c^2}F_jA^{ij} - \varkappa J^i \\ &{}^*\nabla_i F_k - \frac{1}{c^2}F_iF_k + 2A_{ij}A_{k\cdot}^{\cdot j} - \varkappa U_{ik} = c^2C_{ik} \end{aligned}\right\}. \quad (22)$$

Here $A_{jl}A^{jl} = \varkappa\varrho c^2$ in the 1st condition, because the terms containing the gravitational force F_i in the 1st condition remains the same as those in the non-rotating mass-point metric, and therefore they satisfy themselves. But $2A_{ij}A_{k\cdot}^{\cdot j} \neq \varkappa U_{ik}$ in the 3rd condition, because the space curvature C_{ik} depends on the angular rotation velocity of the space. Therefore, when we add the space rotation term $2A_{ij}A_{k\cdot}^{\cdot j}$ to the left-hand side of the 3rd field equation and the electromagnetic field stress U_{ik} to the right-hand side, the space rotation affects the space curvature C_{ik} (creating additional curvature, see the derivations in [1,2] for example). So the added U_{ik} compensates not only for the added space rotation term $2A_{ij}A_{k\cdot}^{\cdot j}$, but also for the extra curvature in C_{ik} caused by the space rotation.

The obtained Riemannian conditions mean the following:

The 1st Riemannian condition

In the space of a rotating point mass, the observable gravitational force field is inhomogeneous (${}^*\nabla_j F^j \neq 0$) due to the spherically symmetric structure of this force field. And the value of the observable angular rotation velocity A_{ik} of this space is determined by the density ϱ of the source-free electromagnetic field that fills it (since $A_{jl}A^{jl} = \varkappa\varrho c^2$).

The 2nd Riemannian condition

The observable rotation field of such a space is inhomogeneous (${}^*\nabla_j A^{ij} \neq 0$) due to the spherically symmetric structure of the gravitational force field F_i superimposed onto this rotation field A_{ik}, and due to an electromagnetic radiation presented in this space (since $J^i \neq 0$ for any emitting medium).

The 3rd Riemannian condition

The observable curvature C_{ik} of such a space is created by the combined effect of the strength ${}^*\nabla_i F_k$ of the gravitational force field and the rotation field A_{ik} of this space, minus corrections for the spherically symmetric structure of the force field F_i and for the electromagnetic field stress U_{ik}.

4.2.3 The rotating de Sitter space metric. It has the form

$$ds^2 = \left(1-\frac{\lambda r^2}{3}\right)c^2dt^2 - 2\omega r^2\sin^2\theta\sqrt{1-\frac{\lambda r^2}{3}}\,dt\,d\varphi - \\ - \frac{dr^2}{1-\frac{\lambda r^2}{3}} - r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right), \quad (23)$$

thus, the Riemannian conditions, under which this metric satisfies the chr.inv.-Einstein equations (2), have the form

$$\left.\begin{aligned} &{}^*\nabla_j F^j = \frac{1}{c^2}F_jF^j + A_{jl}A^{jl} - \varkappa\varrho c^2 + \lambda c^2 \\ &{}^*\nabla_j A^{ij} = \frac{2}{c^2}F_jA^{ij} - \varkappa J^i \\ &{}^*\nabla_i F_k - \frac{1}{c^2}F_iF_k + 2A_{ij}A_{k\cdot}^{\cdot j} - \\ &\qquad - \varkappa U_{ik} - \lambda c^2 h_{ik} = c^2C_{ik} \end{aligned}\right\}, \quad (24)$$

where $A_{jl}A^{jl} = \varkappa\varrho c^2$ in the 1st condition, because the terms containing F_i and the λ-term in the 1st condition do not depend on A_{ik}, and therefore they satisfy themselves as in a non-rotating de Sitter space. However $2A_{ij}A_{k\cdot}^{\cdot j} \neq \varkappa U_{ik}$ in the 3rd condition due to the reason explained above for the rotating mass-point space metric.

They have the following physical sense:

The 1st Riemannian condition

In a rotating de Sitter space, the observable gravitational force field is inhomogeneous (${}^*\nabla_j F^j \neq 0$) due to the spherically symmetric structure of this force field with a constant correction for the density of the physical vacuum that fills the space (since $\lambda = const$). And, since $A_{jl}A^{jl} = \varkappa\varrho c^2$, the value of the observable angular rotation velocity A_{ik} of this space is determined by the density ϱ of the source-free electromagnetic field that fills it.

The 2nd Riemannian condition

The observable rotation field of such a space is inhomogeneous (${}^*\nabla_j A^{ij} \neq 0$) due to the spherically symmetric structure of the gravitational force field F_i superimposed onto this rotation field A_{ik}, and due to an electromagnetic radiation presented therein (since $J^i \neq 0$).

The 3rd Riemannian condition

The observable curvature C_{ik} of a rotating de Sitter space is created by the combined effect of the strength

$^*\nabla_i F_k$ of the gravitational force field and the rotation field A_{ik} of this space, minus corrections determined by the spherically symmetric structure of the force field F_i, the density of the physical vacuum (λ-field), and also the electromagnetic field stress U_{ik}.

4.2.4 The metric of a rotating liquid sphere. The interior space metric of a rotating sphere filled with ideal liquid is

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 c^2dt^2 - \\ - \omega r^2\sin^2\theta\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right) dt\,d\varphi - \\ - \frac{dr^2}{1-\frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right). \quad (25)$$

Based on the chr.inv.-Einstein equations (2), we obtain the Riemannian conditions under which this metric is valid

$$\left.\begin{aligned} &^*\nabla_j F^j = \frac{1}{c^2}F_jF^j + A_{jl}A^{jl} - \varkappa\varrho c^2 - \frac{\varkappa}{2}\left(\rho c^2+3p\right) \\ &^*\nabla_j A^{ij} = \frac{2}{c^2}F_jA^{ij} - \varkappa J^i \\ &^*\nabla_i F_k - \frac{1}{c^2}F_iF_k + 2A_{ij}A_{k\cdot}^{\cdot j} - \varkappa U_{ik} - \\ &\qquad - \frac{\varkappa}{2}\left(\rho c^2 - p\right)h_{ik} = c^2C_{ik} \end{aligned}\right\}, \quad (26)$$

where $A_{jl}A^{jl} = \varkappa\varrho c^2$ in the 1st condition, but $2A_{ij}A_{k\cdot}^{\cdot j} \neq \varkappa U_{ik}$ in the 3rd condition due to the reasons explained above.

They have the following physical sense:

The 1st Riemannian condition

The observable gravitational force field inside a rotating sphere filled with ideal liquid is inhomogeneous ($^*\nabla_j F^j \neq 0$) due to the spherically symmetric structure of this force field with corrections for the density ρ and pressure p in the liquid. And, since $A_{jl}A^{jl} = \varkappa\varrho c^2$, the value of the observable angular rotation velocity A_{ik} of this space is determined by the density ϱ of the source-free electromagnetic field that fills this space along with the liquid.

The 2nd Riemannian condition

The observable rotation field of such a space is inhomogeneous ($^*\nabla_j A^{ij} \neq 0$) due to the spherically symmetric structure of the gravitational force field F_i superimposed onto this rotation field A_{ik}, and due to an electromagnetic radiation presented therein (since $J^i \neq 0$).

The 3rd Riemannian condition

The observable curvature C_{ik} of such a space is created by the combined effect of the strength $^*\nabla_i F_k$ of the gravitational force field and the rotation field A_{ik} of this space, minus corrections determined by the spherically symmetric structure of the force field F_i, the density ρ and pressure p in the liquid, and also the electromagnetic field stress U_{ik}.

4.2.5 The rotating Reissner-Nordström space metric. It has the following form

$$ds^2 = \left(1-\frac{r_g}{r}+\frac{r_q^2}{r^2}\right)c^2dt^2 - \\ - 2\omega r^2\sin^2\theta\sqrt{1-\frac{r_g}{r}+\frac{r_q^2}{r^2}}\,dt\,d\varphi - \\ - \frac{dr^2}{1-\frac{r_g}{r}+\frac{r_q^2}{r^2}} - r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right), \quad (27)$$

and describes the space of an electrically charged rotating spherical body approximated by a mass-point. Thus, based on the chr.inv.-Einstein equations (2), we obtain the Riemannian conditions under which this metric is valid

$$\left.\begin{aligned} &^*\nabla_j F^j = \frac{1}{c^2}F_jF^j + A_{jl}A^{jl} - \varkappa\left(\varrho+\dot{\varrho}\right)c^2 \\ &^*\nabla_j A^{ij} = \frac{2}{c^2}F_jA^{ij} - \varkappa J^i \\ &^*\nabla_i F_k - \frac{1}{c^2}F_iF_k + 2A_{ij}A_{k\cdot}^{\cdot j} - \varkappa\left(U_{ik}+\dot{U}_{ik}\right) = c^2C_{ik} \end{aligned}\right\}, \quad (28)$$

where $\dot{\varrho}$ and $\dot{U}_{ik}$ are the density and stress of the electrostatic field created by the electrically charged point mass, while ϱ, J^i, U_{ik} characterize the external electromagnetic field without sources (electric charges and currents), which is also presented in the space. In addition, $A_{jl}A^{jl} = \varkappa\varrho c^2$ in the 1st condition and $2A_{ij}A_{k\cdot}^{\cdot j} \neq \varkappa U_{ik}$ in the 3rd condition due to the previously explained reasons.

These conditions physically mean the following:

The 1st Riemannian condition

In the space of an electrically charged rotating point mass, the observable gravitational force field is inhomogeneous ($^*\nabla_j F^j \neq 0$) due to the spherically symmetric structure of this force field, minus a correction for the density $\dot{\varrho}$ of the electrostatic field created by the charged mass. And, since $A_{jl}A^{jl} = \varkappa\varrho c^2$, the value of the observable angular rotation velocity A_{ik} of this space is determined by the density ϱ of the source-free electromagnetic field that fills it along with the electrostatic field of the charged mass.

The 2nd Riemannian condition

The observable rotation field of such a space is inhomogeneous ($^*\nabla_j A^{ij} \neq 0$) due to the spherically symmetric structure of the gravitational force field F_i superimposed onto this rotation field A_{ik}, and due to an electromagnetic radiation presented therein (since $J^i \neq 0$).

The 3rd Riemannian condition

The observable curvature C_{ik} of such a space is created by the combined effect of the strength ${}^*\nabla_i F_k$ of the gravitational force field and the rotation field A_{ik} of this space, minus corrections determined by the spherically symmetric structure of the force field F_i, the stress $\dot{U}_{ik}$ of the electrostatic field created by the charged mass, and also the stress U_{ik} of the source-free electromagnetic field that fills this space.

4.2.6 The rotating Einstein space metric. It is a "rotating modification" of Einstein's space metric (12)

$$ds^2 = c^2 dt^2 - 2\omega r^2 \sin^2\theta\, dt\, d\varphi - \frac{dr^2}{1-\lambda r^2} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \qquad (29)$$

Since $U_{ik} = p h_{ik}$ and $U = 3p$ in an ideal liquid, and also $U = \varrho c^2$ in an electromagnetic field, the Riemannian conditions under which the metric (29) is valid have the form

$$\left.\begin{aligned} &A_{jl}A^{jl} = \varkappa\varrho c^2 + \frac{\varkappa}{2}\left(\rho c^2 + 3p\right) - \lambda c^2 \\ &{}^*\nabla_j A^{ij} = -\varkappa J^i \\ &2A_{ij}A_{k\cdot}^{\cdot j} - \varkappa U_{ik} + \frac{\varkappa}{2}\left(p - \rho c^2\right) h_{ik} - \lambda c^2 h_{ik} = c^2 C_{ik} \end{aligned}\right\}, \qquad (30)$$

where $A_{jl}A^{jl} = \varkappa\varrho c^2$ in the 1st condition and $2A_{ij}A_{k\cdot}^{\cdot j} \neq \varkappa U_{ik}$ in the 3rd condition as was explained before.

The physical sense of the conditions is as follows:

The 1st Riemannian condition

In a space of the rotating Einstein metric, the density ρ and pressure p in the ideal liquid that fills this space are balanced by the density of the physical vacuum, which also fills it (since $\lambda \neq 0$). And since $A_{jl}A^{jl} = \varkappa\varrho c^2$, the value of the observable angular rotation velocity A_{ik} of this space is determined by the density ϱ of the source-free electromagnetic field that fills it along with the liquid and the physical vacuum (λ-field).

The 2nd Riemannian condition

The observable rotation field of such a space is inhomogeneous (${}^*\nabla_j A^{ij} \neq 0$) due to an electromagnetic radiation presented in this space (since $J^i \neq 0$).

The 3rd Riemannian condition

The observable curvature C_{ik} of such a space is created by the combined effect of the strength ${}^*\nabla_i F_k$ of the gravitational force field, the rotation field A_{ik} of this space, and the pressure p in the ideal liquid that fills the space, minus corrections for the density ρ of this liquid, the density of the physical vacuum (since $\lambda \neq 0$), and also the stress U_{ik} of the source-free electromagnetic field that fills this space along with the liquid and the physical vacuum.

5 The Riemannian conditions for deforming spaces

5.1 Friedmann's space metric. This is the most known and often considered as the solely deforming space metric

$$ds^2 = c^2 dt^2 - \frac{dr^2}{1-\kappa\frac{r^2}{R^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (31)$$

because its particular case, where the space expands, is the basis of Lemaître's cosmological model of the expanding Universe, in the framework of which Lemaître in 1927 first explained the observed cosmological redshift.

It is obvious that a space of Friedmann's metric is free of a gravitational field ($g_{00} = 1$, $F_i = 0$) and rotation ($g_{0i} = 0$, $A_{ik} = 0$), but is deforming ($D_{ik} \neq 0$) due to $g_{11} = f(t)$ in the metric. It may expand, compress, or oscillate. Such a space can be empty, or be homogeneously filled with ideal liquid (or ideal dust) along with the physical vacuum (λ-field), or be filled with one of these media.

Here $R = R(t)$ is the curvature radius of the space, which depends on time since the space is deforming (its volume is expanding or compressing). While $\kappa = 0, \pm 1$ is the so-called "curvature factor": with $\kappa = -1$ the three-dimensional space has the hyperbolic (open) geometry, with $\kappa = 0$ its geometry is flat, and with $\kappa = +1$ its geometry is elliptic (closed).

Friedmann's metric is usually considered in the form

$$ds^2 = c^2 dt^2 - R^2\left[\frac{d\tilde{r}^2}{1-\kappa\tilde{r}^2} + \tilde{r}^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right], \qquad (32)$$

expressed with a "homogeneous" radial coordinate $\tilde{r}$, which is the radial coordinate r divided by the curvature radius R, whose scales change accordingly during expansion or compression of the space, and therefore the homogeneous radial coordinate $\tilde{r}$ does not change its scale with time.

5.1.1 An empty Friedmann space. The Riemannian conditions, under which Friedmann's metric (32) is valid without distributed matter, have the form

$$\left.\begin{aligned} &D_{jl}D^{jl} = -\frac{{}^*\partial D}{\partial t} \\ &{}^*\nabla_j D^{ij} = {}^*\nabla_j\left(h^{ij}D\right) \\ &\frac{{}^*\partial D_{ik}}{\partial t} - D_{ij}D_k^j + D D_{ik} = c^2 C_{ik} \end{aligned}\right\}, \qquad (33)$$

where the trace $D = h^{ik}D_{ik} = h_{ik}D^{ik}$ of the chr.inv.-space deformation rate tensor D_{ik} shows how much the space is compressed or stretched in all directions at once, i.e., the quantity D is responsible for changing the volume of the space without changing its shape.

The 1st equation plays the key rôle in the analysis regarding the particular conditions under which Friedmann's space metric satisfies Einstein's field equations.

The Riemannan conditions obtained means the following:

THE 1ST RIEMANNIAN CONDITION

A space of Friedmann's metric without distributed matter deforms only non-stationarily (otherwise this condition would become nonsense — the space does not deform). Moreover, only a decelerating deforming Friedmann space is conceivable without distributed matter: in this case, the deformation rate D decreases with time (which indicates a decelerating deforming space), and therefore the always positive left-hand side of the condition is balanced by the positive right-hand side.

THE 2ND RIEMANNIAN CONDITION

An empty Friedmann space deforms everywhere and always homogeneously and isotropic: the field D_{ik} exactly corresponds to the observable space metric h_{ik}.

THE 3RD RIEMANNIAN CONDITION

The observable curvature C_{ik} of an empty Friedmann space is created by only the field of its non-stationary deformations.

5.1.2 A non-empty Friedmann space. In this case, the 2nd condition is the same ($J^i = 0$ in ideal liquid and dust, since these are non-emitting media), and the 1st and 3rd conditions acquire terms determined by distributed matter. If a Friedmann universe is filled with ideal liquid ($U_{ik} = p h_{ik}$, $U = 3p$) and the physical vacuum ($\lambda \neq 0$), then the Riemannian conditions under which Friedmann's metric (32) is valid, are

$$\left.\begin{aligned} & D_{jl} D^{jl} = -\frac{{}^*\partial D}{\partial t} - \frac{\varkappa}{2}\left(\rho c^2 + 3p\right) + \lambda c^2 \\ & {}^*\nabla_j D^{ij} = {}^*\nabla_j \left(h^{ij} D\right) \\ & \frac{{}^*\partial D_{ik}}{\partial t} - D_{ij} D^j_k + D D_{ik} + \\ & \qquad - \frac{\varkappa}{2}\left(\rho c^2 - p\right) h_{ik} - \lambda c^2 h_{ik} = c^2 C_{ik} \end{aligned}\right\}, \tag{34}$$

whereas in a Friedmann universe filled with ideal dust ($\rho \neq 0$, $p = 0$, $U_{ik} = 0$, $U = 0$) and the physical vacuum, the Riemannian conditions have the form

$$\left.\begin{aligned} & D_{jl} D^{jl} = -\frac{{}^*\partial D}{\partial t} - \frac{\varkappa}{2}\rho c^2 + \lambda c^2 \\ & {}^*\nabla_j D^{ij} = {}^*\nabla_j \left(h^{ij} D\right) \\ & \frac{{}^*\partial D_{ik}}{\partial t} - D_{ij} D^j_k + D D_{ik} - \\ & \qquad - \frac{\varkappa}{2}\rho c^2 h_{ik} - \lambda c^2 h_{ik} = c^2 C_{ik} \end{aligned}\right\}, \tag{35}$$

where in both cases, in the absence of the physical vacuum, we can merely place $\lambda = 0$. Their sense is as follows:

THE 1ST RIEMANNIAN CONDITION

A space of Friedmann's metric, which is filled with distributed matter, deforms stationarily (i.e., with a constant deformation rate $D = const$), if the density of the physical vacuum (λ-field) overcomes the effects of the density ρ and pressure p in the medium that fills this space: in this case the always positive left-hand side of the condition is balanced by the positive sum of the right-hand side terms. A non-empty Friedmann space deforms with acceleration (in this case, its deformation rate D increases with time), if the negative term created by the accelerating deformation as well as the negative effects of the density ρ and pressure p in the medium are compensated by the greater effect of the density of the physical vacuum. In the absence of the physical vacuum (i.e., when $\lambda = 0$), a non-empty Friedmann space deforms only with deceleration (similarly to an empty Friedmann space).

THE 2ND RIEMANNIAN CONDITION

A Friedmann space filled with distributed matter deforms everywhere and always homogeneously and isotropic: the field D_{ik} exactly corresponds to the observable space metric h_{ik} (as well as in the case of an empty Friedmann space).

THE 3RD RIEMANNIAN CONDITION

The observable curvature C_{ik} of a non-empty Friedmann space is created by the field of its deformations with corrections depending on the properties of the distributed media that fill it.

5.2 The rotating Friedmann space metric. This metric

$$\begin{aligned} ds^2 = c^2 dt^2 - R^2 \Bigg[& 2\,\omega \tilde{r}^2 \sin^2\theta \, dt \, d\varphi + \\ & + \frac{d\tilde{r}^2}{1 - \kappa \tilde{r}^2} + \tilde{r}^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\Bigg] \end{aligned} \tag{36}$$

is introduced in the same way as the rotating metrics considered above. Here $g_{03} \neq 0$, and therefore $A_{ik} \neq 0$. Using $F_i = 0$, $A_{ik} \neq 0$, and $D_{ik} \neq 0$ in the chr.inv.-Einstein equations (2), we obtain the Riemannan conditions for this metric.

To understand the next step, it is important to note that Friedmann's metric is proved using a different method than all other space metrics used in General Relativity.

All other space metrics are proved purely algebraically. This means that when proving a certain space metric, we substitute the specific components g_{00}, g_{0i}, and g_{ik} from its formula into the left-hand side of the chr.inv.-Einstein equations (2), i.e., into the formulae for the physically observable characteristics of this space (since they are formulated in terms of these components). Then, if, after reducing similar terms, the left-hand side coincides with the right-hand side, or the sum of all terms on the left-hand side is zero, and the sum of all terms on the right-hand side is zero, then this space metric has been proven.

But this algebraic method obviously does not work for proving Friedmann's space metric (32). Everyone is able to ascertain this by himself.

Therefore, Alexander Friedmann used his own original method to prove the space metric he introduced. He expressed the left-hand side of the field equations with the curvature radius R according to the space metric (32). Thereby, based on the scalar and tensor field equations, he obtained two differential equations with respect to R, which contain $\ddot{R}$, $\dot{R}$, R and a free term without the curvature radius (which is non-zero, if the space is non-empty, and contains the right-hand side characteristics of distributed matter). Solving these differential equations gives a particular function $R = R(t)$ under which this metric is valid.

What the above means is that when we add a term containing rotation ($g_{0i} \neq 0$) to Friedmann's space metric (32), as we did in the case of the rotating Friedmann metric (36), and hence add non-zero terms containing A_{ik} to the left-hand side of the chr.inv.-Einstein equations, the Friedmann differential equations (and their solution) are not changed in principle, but only the coefficients of the variables in them are changed. The proof remains the same, and so we do not need to add a "compensating" term or terms to the right-hand side of the field equations (unlike what we did when we considered the field equations for other classical space metrics).

As a result, we can consider the field equations for both an empty rotating Friedmann space and a non-empty rotating Friedmann space simply by using the non-zero A_{ik} terms on the left-hand side.

5.2.1 A rotating empty Friedmann space. In this case, the Riemannian conditions under which the rotating Friedmann metric (36) is valid have the form

$$\left.\begin{aligned} & D_{jl}D^{jl} = -\frac{{}^*\partial D}{\partial t} + A_{jl}A^{jl} \\ & {}^*\nabla_j\left(h^{ij}D\right) - {}^*\nabla_j D^{ij} = {}^*\nabla_j A^{ij} \\ & \frac{{}^*\partial D_{ik}}{\partial t} - D_{ij}D^j_k + D D_{ik} - \\ & \qquad - D_{ij}A_{k\cdot}^{\cdot j} - A_{ij}D^j_k + 2A_{ij}A_{k\cdot}^{\cdot j} = c^2 C_{ik} \end{aligned}\right\}, \tag{37}$$

which mean the following:

The 1st Riemannian condition

A space of the rotating Friedmann metric in the absence of distributed matter can deform stationarily, decelerate and accelerate. If a rotating empty Friedmann space deforms stationarily (i.e., with a constant deformation rate), then the square of its deformation rate tensor D_{ik} is equal to the square of its angular rotation velocity A_{ik}, and its rotation is stationary as well. If a rotating empty Friedmann space deforms with an acceleration (i.e., its deformation rate D increases with time), then this acceleration is compensated by a greater value of the observable angular rotation velocity A_{ik} of this space. If a rotating empty Friedmann space deforms with a deceleration, then its deformation rate D decreases with time, and therefore its angular rotation velocity A_{ik} is lesser than that in the stationary deforming space.

The 2nd Riemannian condition

The deformation field D_{ik} of a rotating empty Friedmann space deviates from homogeneity and isotropy exactly in accordance with the field of its rotation A_{ik}.

The 3rd Riemannian condition

The observable curvature C_{ik} of a rotating empty Friedmann space is created by the field of its deformations and the field of its rotation.

5.2.2 A rotating non-empty Friedmann space. If such a universe is filled with ideal liquid ($U_{ik} = p h_{ik}$, $U = 3p$) and the physical vacuum ($\lambda \neq 0$), then the Riemannian conditions under which the rotating Friedmann metric (36) is valid have the following form

$$\left.\begin{aligned} & D_{jl}D^{jl} = -\frac{{}^*\partial D}{\partial t} + A_{jl}A^{jl} - \frac{\varkappa}{2}\left(\rho c^2 + 3p\right) + \lambda c^2 \\ & {}^*\nabla_j\left(h^{ij}D\right) - {}^*\nabla_j D^{ij} = {}^*\nabla_j A^{ij} \\ & \frac{{}^*\partial D_{ik}}{\partial t} - D_{ij}D^j_k + D D_{ik} - D_{ij}A_{k\cdot}^{\cdot j} - A_{ij}D^j_k + \\ & \qquad + 2A_{ij}A_{k\cdot}^{\cdot j} - \frac{\varkappa}{2}\left(\rho c^2 - p\right)h_{ik} - \lambda c^2 h_{ik} = c^2 C_{ik} \end{aligned}\right\}, \tag{38}$$

and if such a universe is filled with ideal dust ($\rho \neq 0$, $p = 0$, $U_{ik} = 0$, $U = 0$) and the physical vacuum, the Riemannian conditions have the form

$$\left.\begin{aligned} & D_{jl}D^{jl} = -\frac{{}^*\partial D}{\partial t} + A_{jl}A^{jl} - \frac{\varkappa}{2}\rho c^2 + \lambda c^2 \\ & {}^*\nabla_j\left(h^{ij}D\right) - {}^*\nabla_j D^{ij} = {}^*\nabla_j A^{ij} \\ & \frac{{}^*\partial D_{ik}}{\partial t} - D_{ij}D^j_k + D D_{ik} - D_{ij}A_{k\cdot}^{\cdot j} - A_{ij}D^j_k + \\ & \qquad + 2A_{ij}A_{k\cdot}^{\cdot j} - \frac{\varkappa}{2}\rho c^2 h_{ik} - \lambda c^2 h_{ik} = c^2 C_{ik} \end{aligned}\right\}, \tag{39}$$

where $\lambda = 0$ in the absence of the physical vacuum.

It is interesting that these Riemannian conditions do not change the physical sense that we have obtained above for a rotating Friedmann universe, when it is empty. Indeed, new terms appeared in the conditions due to the presence of distributed matter. But these new terms play the rôle of only corrections that disturb the main effect created by the deformation and rotation of the background space (in contrast to a non-rotating Friedmann universe, considered above, where distributed matter plays a decisive rôle).

5.3 Conclusions regarding a Friedmann universe. First fact: modern astronomical estimates indicate that the average density of substance in the Universe is $\rho < 10^{-29}$ gram/cm^3, and $\lambda < 10^{-56}$ cm^{-2}. Therefore, on "cosmological scales" the distributed matter in the Universe is so rarefied that the terms $\frac{1}{2}\varkappa\rho c^2 < 10^{-35}$ sec^{-2} and $\lambda c^2 < 10^{-35}$ sec^{-2} in the Riemannian conditions can be neglected compared to the deformation rate. This means that the background Friedmann space of the observable Universe must be considered empty.

Second fact: astronomical evidence of a non-linear increase in the redshift of very distant galaxies, registered over the past two decades, clearly shows that our Universe is expanding with an increasing linear velocity.

Third, in 2011–2012 the author showed that the cosmological redshift has a non-linear (exponential) formula in a constant-deformation expanding Friedmann space, i.e., when the linear expansion velocity $\dot{R}$ increases according to the increasing curvature radius R so that the deformation rate remains constant $D = \frac{3\dot{R}}{R} = const$. This theoretical result, obtained using a new method — by solving the scalar geodesic equation for particles travelling in an expanding Friedmann universe [11–13], was then resumed in the article [14].

This theoretical result, together with the modern astronomical data as above, leads us to the conclusion that the background space of our Universe is a *constant-deformation Friedmann space without distributed matter*.

However, an empty Friedmann space cannot be a constant-deformation space. This is according to the 1st Riemannian condition in such a space, which is the first equation in (33). Otherwise, the 1st Riemannian condition in (33) would be nonsense — the space does not deform, since the square of the deformation rate tensor $D_{jl}D^{jl}$ on the left-hand side of the equation becomes zero due to the right-hand side, which is zero because the deformation rate D is constant.

This leads the author to the only possible conclusion:

> Astronomical evidence of the negligible average density of matter in the Universe and the non-linear cosmological redshift indicate that the background space of the observable Universe is a rotating empty Friedmann space, i.e., a space of the rotating Friedmann metric (36), which is valid (satisfies Einstein's field equations) under the Riemannian conditions (37).

But in this case, according to the 1st Riemannian condition in (37), we must assume that the background space of our Universe possesses a fundamental global rotation with a significant angular velocity Ω, the square of which is equal to the square of the space deformation rates $A_{jl}A^{jl} = D_{jl}D^{jl}$, and, consequently, the field of the space deformation rates exactly repeats the field of the angular rotational velocities of the space. The author will devote another study to this problem, going beyond the formal derivation and analysis of Riemannian conditions for various space metrics (which was done in this article).

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